Optimum Pre-Detection Signal Processing
(the "matched filter" concept)

Maximum Signal-to-Noise Ratio (Intuitive Derivation)

- Intuitively, detection in the presence of noise has limits imposed by physics (especially thermodynamics). To understand these limits, suppose we are attempting to detect the RF pulse plotted in Fig. 1 of waveform $x(t) = A_X \sin(\omega t + \phi)$, carrier frequency $\omega = 2\pi\nu$, and pulse duration T_P .
- Assume that we are trying to detect this pulse in the presence of AWGN having power spectral density S_P such that $<(\Delta P)^2> = S_P \Delta\nu$. In this case, the RF signal-to-noise ratio is

$$\left(SNR\right)_{\max} = \frac{\bar{P}}{\sqrt{<\left(\Delta P\right)^2>}} = \frac{\bar{P}\cdot T_p}{S_P \Delta\nu \cdot T_p} \approx \frac{U_p}{S_P \Delta\nu \cdot T_p} \qquad (1)$$

Where U_P is the electrical pulse energy. If we assume that the pulse is sampled consistent with the Nyquist condition, then the sample rate f_S should be matched to the pulse width

$$f_s = 1/T_P, \qquad (2)$$

and should be twice the twice the RF instantaneous bandwidth

$$f_S = 2\Delta\nu \ . \qquad (3)$$

Substitution of these last two into the SNR expression yields

$$\left(SNR\right)_{\max} \approx \frac{2U_p}{S_P} \equiv \frac{2U_p}{N_0} \qquad (4)$$

where N_0 is another way of writing the power spectral density (following the convention in communications theory). This is a factor of two higher than might be expected intuitively because it assumes implicitly that we have precise knowledge of the phase and amplitude of a signal, not just one or the other.

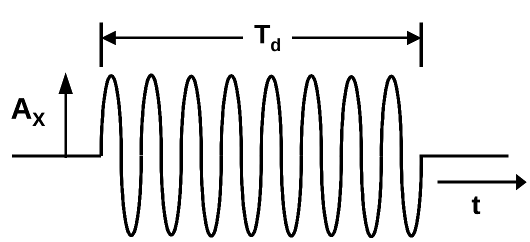

Fig. 1.

Maximum Signal-to-Noise Ratio (Rigorous Derivation)

A. Signal

As in the analysis of RF "detection" in the previous section, there are intuitive ways to derive important theorems in RF sensors, and there are more rigorous ways. This Professor believes students should see both because the intuitive ways are the ones easiest to remember and extend (at least qualitatively) to other concepts, but the rigorous ways are more precise and necessary in real systems engineering.

On the subject of optimum signal-to-noise ratio, the rigorous technique is based on methods of linear signal processing. We start by assuming the RF "front-end" of the receiver is a linear network having a known system transfer function H(ω) and impulse response function h(t) related by

$$\mathrm{H}(\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} h(t)e^{-j\omega t}dt \tag{5}$$

For a given signal x(t) at the input or output, waveform is related to the Fourier amplitudes X(ω) of the power spectral density by the Fourier transform pair

$$X(\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} x(t)e^{-j\omega t}dt \quad \text{and} \tag{6}$$

Linear systems theory teaches us that the input and output Fourier amplitudes are related by

$$X_{out}(\omega) = X_{in}(\omega)\cdot H(\omega) \tag{7}$$

Combining the last two

$$x_{out}(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X_{out}(\omega)e^{j\omega t}d\omega = \frac{1}{2\pi}\int_{-\infty}^{\infty} X_{in}(\omega)H(\omega)e^{j\omega t}d\omega \tag{8}$$

But by the convolution theorem, the Fourier transform of the product equals the convolution of the Fourier transform pairs:

$$x_{out}(t) = \int_{-\infty}^{\infty} x_{in}(t-\tau)h(\tau)d\tau \tag{9}$$

In general t can be defined for any time $-\infty < t < \infty$. Here, we want to specify t to a sampling time t_S related to the sampling frequency by $f_S = 1/t_S$. Hence,

$$x_{out}(t_S) = \int_{0}^{t_S} x_{in}(t_S-\tau)h(\tau)d\tau \tag{10}$$

B. Noise

The assumption of a linear network and AWGN noise allows us to write immediately,

$$<(\Delta x)^2> = S_X \Delta v \equiv S_X \Delta v_N \equiv S_X \cdot \frac{1}{2}\int_{-\infty}^{\infty} \left|H(\omega)\right|^2 dv \tag{11}$$

where the factor ½ assumes H (ω) is two sided (i.e., negative frequencies are allowable). and Δv_N is called the noise bandwidth. But by Parseval's theorem, Fourier transform pairs are related by

$$\int_{-\infty}^{\infty} \left|H(\omega)\right|^2 dv = \int_0^{\infty} \left|h(\tau)\right|^2 d\tau \tag{12}$$

so that

$$<(\Delta x)^2> = S_x \cdot \frac{1}{2}\int_{-\infty}^{\infty} \left|h(\tau)\right|^2 d\tau \tag{13}$$

C. Signal-to-Noise Ratio

Combining the results of the previous two sections, we have

$$SNR = \frac{\left|x_{out}(t_S)\right|^2}{S_x \Delta v_N} = \frac{\left|\int_0^{t_S} x_{in}(t_S-\tau)h(\tau)\,d\tau\right|^2}{\frac{1}{2} S_x \int_0^{\infty} \left|h(\tau)\right|^2 d\tau} \tag{14}$$

But by the famous Schwarz inequality, we can write

$$\left|\int_0^{t_S} x_{in}(t_S-\tau)h(\tau)\,d\tau\right|^2 \le \int_0^{t_S} \left|x_{in}(t_S-\tau)\right|^2 d\tau \cdot \int_0^{t_S} \left|h(\tau)\right|^2 d\tau \tag{15}$$

Equality occurs if and only if

$$h(\tau) = C \cdot x_{in}(t_S - \tau) \tag{16}$$

This leads to a very useful bound on the SNR of

$$SNR \le \frac{\int_0^{t_S} \left|x_{in}(t_S-\tau)\right|^2 d\tau \int_0^{t_S} \left|h(\tau)\right|^2 d\tau}{\frac{1}{2} S_x \int_0^{\infty} \left|h(\tau)\right|^2 d\tau} = \frac{\int_0^{t_S} \left|x_{in}(t_S-\tau)\right|^2 d\tau}{\frac{1}{2} S_X} \tag{17}$$

By recognizing the relation $U_P = \int_0^{t_S} \left[x_{in}(t_S-\tau)\right]^2 d\tau$, we get the elegant result (again)

$$SNR \le \frac{2U_X}{S_X} \tag{18}$$

D. Matched-filter condition

We now recognize (16), $h(t) = C \cdot x_{in}(t_S - t)$, as the definition of the optimum system impulse response function. As written, it is just the time reversal of the input waveform. From linear response theory, we can write the equivalent transfer function

$$H(\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} h(t)e^{-j\omega t}dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} C \cdot x_{in}(t_S - t)e^{-j\omega t}dt \tag{19}$$
$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} C \cdot x_{in}(t_S - t)e^{-j\omega t}e^{j\omega(t_S - t_S)}dt = C \cdot X^*(\omega)e^{-j\omega t_S}$$

where the last step is recognized as phase conjugation. Eqns (16) and (19) are regarded as equivalent descriptions of optimum linear RF signal processing called "matched filtering." Substitution of the "matched-filter" $h(t)$ back in to the system transfer convolution leads to

$$x_{out}(t) = \int_0^{\infty} x_{in}(t-\tau)h(\tau)d\tau = C \cdot \int_0^{\infty} x_{in}(t-\tau)x_{in}(t_S - \tau)d\tau \tag{20}$$

This expression suggests a powerful signal processing strategy: multiply the input signal by a time-shifted replica of itself. This is the basis for *correlative* signal processing, now commonly used in radar and communications receivers alike.