# ECE137A, Notes Set 14: Fourier Series and Transforms

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# Goals of this note set:

Remember how to represent sine waves as phasors $\leftrightarrow e^{j\omega t}$

Understand what a Fourier series *is*.

Understand what a Fourier transform *is*.

...where they come from

...and how they work

Review how to use Fourier transforms to find circuit transient response.

# Fourier Transforms and Fourier Series

Why ?

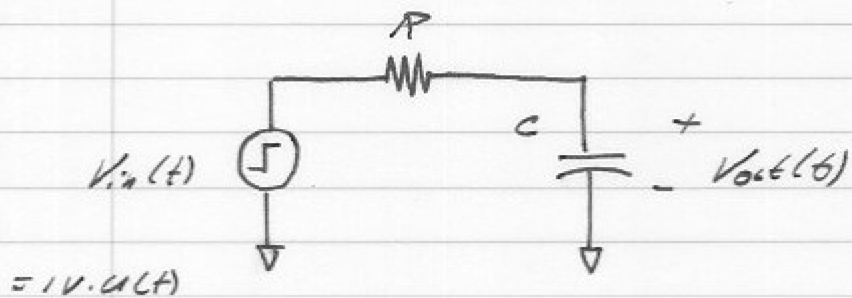
We must often solve ordinary and partial differential equations.

We need efficient tools to do this.

We want better understanding and intuition.

# Solving Simple Differential Equations

A simple first-order circuit:




$\tau = RC$

How do we solve for $v_{out}(t)$ given $v_{in}(t)$?

First method, which can be laborious,

is to directly solve the differential equation.

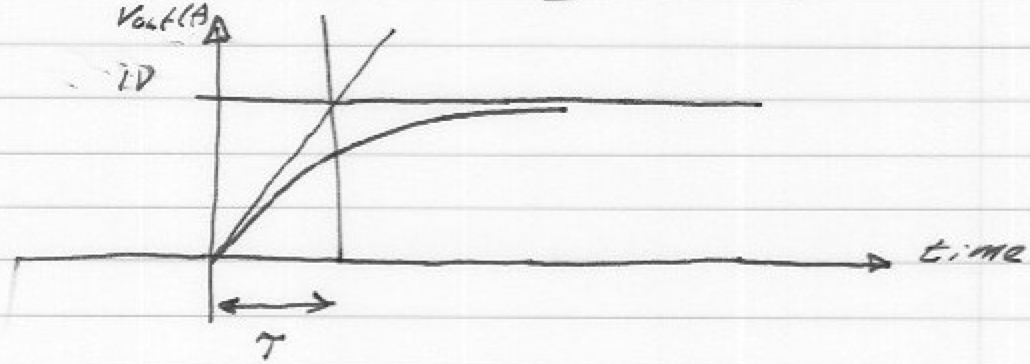
# Solving Simple Differential Equations

From Nodal analysis,

$$v_i(t) = v_o(t) + \tau \cdot \partial v_o(t)/\partial t$$

2 pages work ⇐ !

$$v_o(t) = u(t) \cdot 1V \cdot \left[1 - e^{-t/\tau}\right]$$




We need better ways to solve such problems

# Phasors

If $c = a + jb = \|c\| e^{j\theta_c}$

then $\mathrm{Re}\left[ c \cdot e^{j\omega t} \right] = \mathrm{Re}\left[ \|c\| e^{j\theta_c} e^{j\omega t} \right]$

$= \|c\| \cos(\omega t + \theta_c)$

so the complex # $c$ represents the amplitude and phase of a cosine wave.

# Differential Equations: Easy to Solve with Sinewaves

Differential equations are easy to solve with sinewaves.

$$(d/dt)(e^{j\omega t}) = j\omega(e^{j\omega t})$$

"$e^{j\omega t}$ is the eigenfunction of the derivative operation"

so given $\tau\, \partial v_o(t)/\partial t + v_o(t) = v_{in}(t)$

and given $v_o(t) = v_o e^{j\omega t}$ and $v_i(t) = v_i e^{j\omega t}$

$$\Rightarrow \quad j\omega\tau [v_o] + v_o = v_{in}$$

so

$$\boxed{v_o = \frac{v_{in}}{1 + j\omega\tau}}$$

# Sinewaves and Complex Exponentials

Why work with $e^{j\omega t}$ ? What is it ?

Euler identity: $e^{j\omega t} = \cos(\omega t) + j\sin(\omega t)$

$$e^{j\omega t} = \cos(\omega t) + j\sin(\omega t) \qquad e^{j\omega t} = \cos(\omega t) + j\sin(\omega t)$$

$$e^{-j\omega t} = \cos(\omega t) - j\sin(\omega t) \qquad e^{-j\omega t} = \cos(\omega t) - j\sin(\omega t)$$

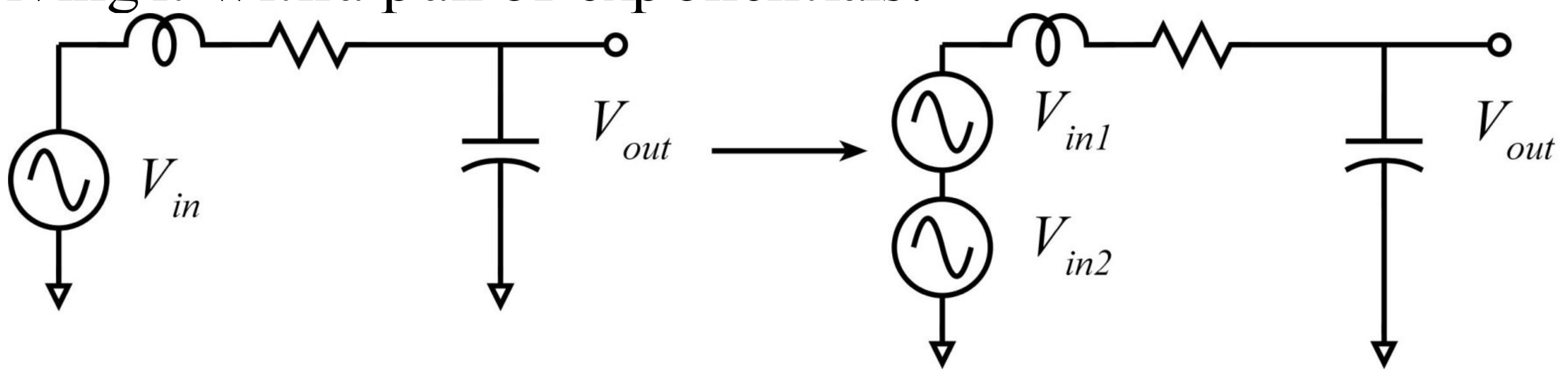
$$e^{j\omega t} + e^{-j\omega t} = 2\cos(\omega t) \qquad e^{j\omega t} - e^{-j\omega t} = 2j\sin(\omega t)$$

$$\cos(\omega t) = (e^{j\omega t} + e^{-j\omega t})/2 \qquad \sin(\omega t) = (e^{j\omega t} - e^{-j\omega t})/2j$$

# Sinewaves and Complex Exponentials

$$\cos(\omega t) = (e^{j\omega t} + e^{-j\omega t})/2 \quad \text{and} \quad \sin(\omega t) = (e^{j\omega t} - e^{-j\omega t})/2j$$

Driving a circuit with a cosine or sine wave is the same as driving it with a pair of exponentials.




$$V_{in} = V_0 \cos(\omega t + \theta)$$

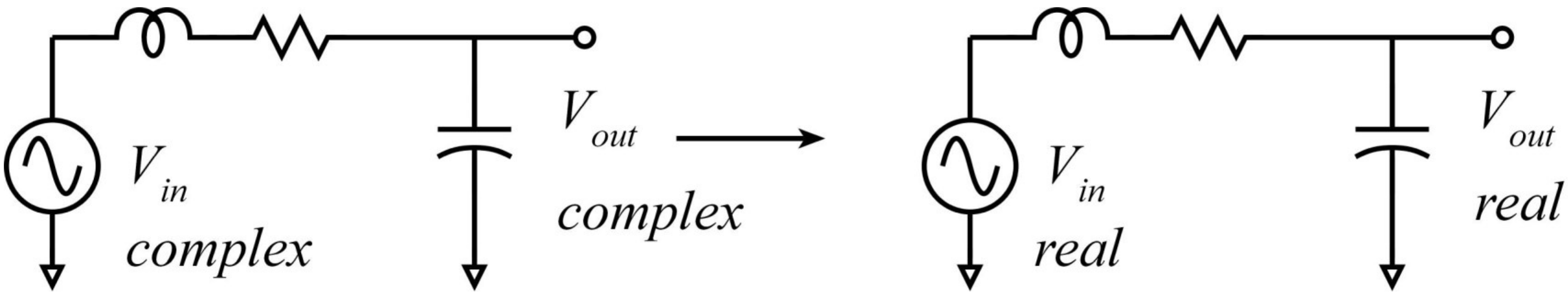
$$V_{in1} = (V_0/2)\exp(j\omega t + j\theta)$$

$$V_{in2} = (V_0/2)\exp(-j\omega t - j\theta)$$

Solve the problem by superpositon, separately solving for $V_{out}$ with $V_1$ applied and $V_{out}$ with $V_1$ applied.

# Complex Exponentials: Ignoring the Real Part

$$\text{Real part of } e^{j\omega t} = \mathrm{Re}\{e^{j\omega t}\} = \mathrm{Re}\{\cos(\omega t) + j\sin(\omega t)\} = \cos(\omega t)$$




$$\begin{aligned} V_{in}(t) &= V_{in}\exp(j\omega t) \\ &= \| V_{in} \| \exp(j\theta_{V_{in}})\exp(j\omega t) \\ V_{out}(t) &= V_{out}\exp(j\omega t) \\ &= \| V_{in} \| \exp(j\theta_{V_{out}})\exp(j\omega t) \end{aligned}$$

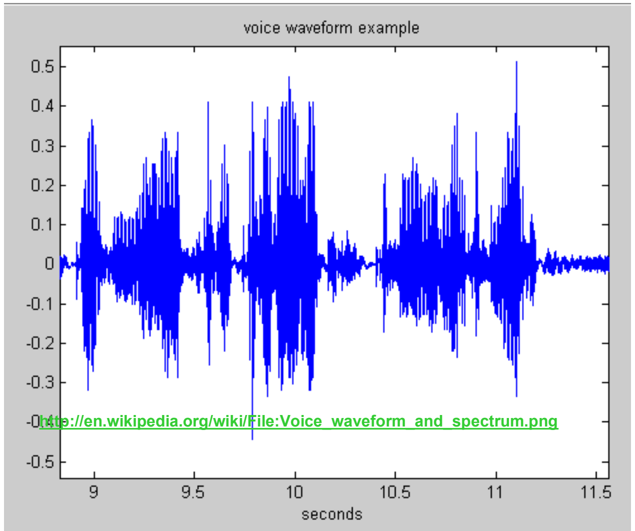
$$\begin{aligned} V_{in}(t) &= \mathrm{Re}\{V_{in}\exp(j\omega t)\} \\ &= \| V_{in} \| \cos(\omega t + j\theta_{V_{in}}) \\ V_{out}(t) &= \mathrm{Re}\{V_{out}\exp(j\omega t)\} \\ &= \| V_{out} \| \cos(\omega t + j\theta_{V_{out}}) \end{aligned}$$

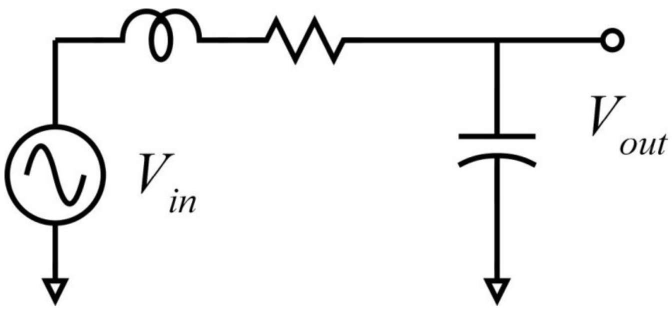
Apply the complex input, ignore the imaginary part of the output.

Proof 1: Principle of superposition.

Proof 2: Working though steps from previous page of notes.

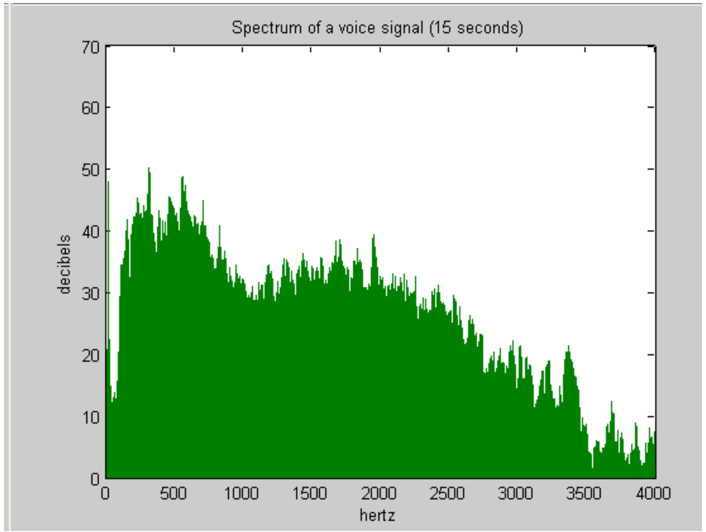
# Representing signals as sets of sinewaves




This input signal is not a sinewave...

...but we can represent it as a sum of sinewaves.




How do we do this?

# Understanding Fourier Transforms and Series

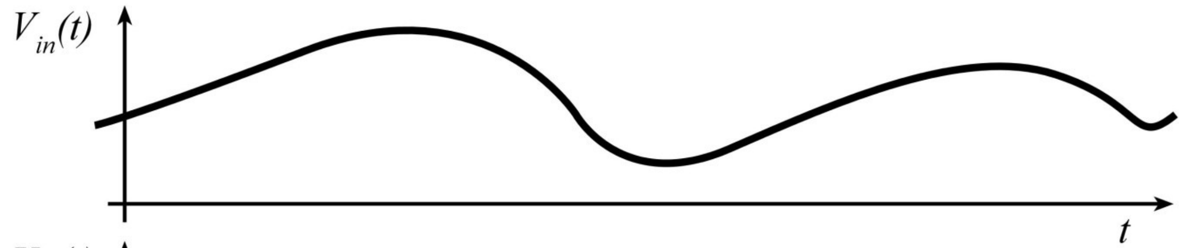
Students often have the Fourier Transform memorized, while having little sense of what it is or why it is done.

Fourier series and transform: writing a signal as a sum of sinewaves (or complex exponentials).

How do we do this?

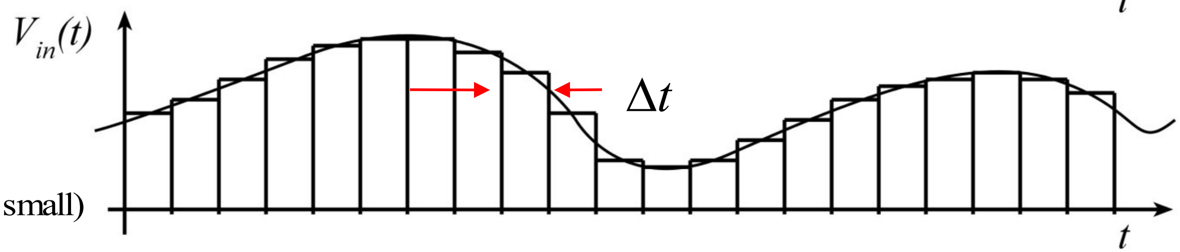
# Understanding Fourier: Signals are Vectors

Take an input signal

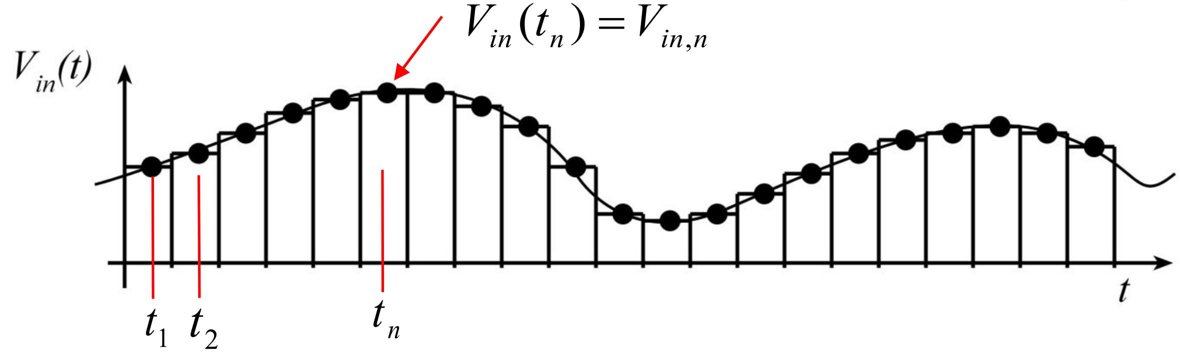



Approximate by a set of discrete steps.

(same method as integration : we can make $\Delta t$ very small)




$V_{in}$ becomes a list of numbers $V_{in}(t_1), V_{in}(t_2), \ldots, V_{in}(t_N)$.



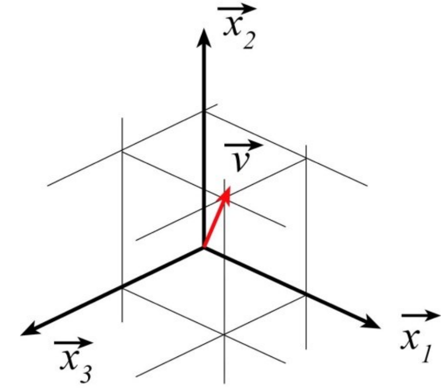

A vector is a list of numbers. So, $V_{in}(t)$ is a vector. $\vec{V}_{in} = \begin{bmatrix} V_{in}(t_1) \\ V_{in}(t_2) \\ \vdots \\ V_{in}(t_N) \end{bmatrix}$

# Writing Vectors as Sums of Basis Vectors

The vector $\vec{V}$ is a sum of the basis vectors $\vec{x}_1, \vec{x}_2, \vec{x}_3$

$$\vec{V} = V_{x1}\vec{x}_1 + V_{x3}\vec{x}_2 + V_{x3}\vec{x}_3$$

$V_{x1}$ is simply the projection of $\vec{V}$ in the $\vec{x}_1$ - direction.




To find $V_{x1}$, take the dot product

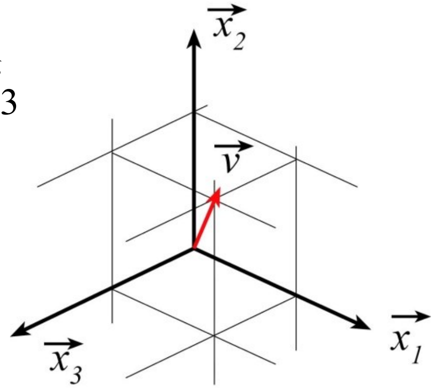
$$\left\langle \vec{V} \,\middle|\, \vec{x}_1 \right\rangle = V_{x1}\left\langle \vec{x}_1 \,\middle|\, \vec{x}_1 \right\rangle + V_{x3}\left\langle \vec{x}_2 \,\middle|\, \vec{x}_1 \right\rangle + V_{x3}\left\langle \vec{x}_3 \,\middle|\, \vec{x}_1 \right\rangle$$

but $\left\langle \vec{x}_1 \middle| \vec{x}_1 \right\rangle = 1$, $\left\langle \vec{x}_2 \middle| \vec{x}_1 \right\rangle = 0$ , and $\left\langle \vec{x}_3 \middle| \vec{x}_1 \right\rangle = 0$, so $\left\langle \vec{V} \middle| \vec{x}_1 \right\rangle = V_{x1}$

$$\rightarrow \vec{V} = \left\langle \vec{V} \,\middle|\, \vec{x}_1 \right\rangle \vec{x}_1 + \left\langle \vec{V} \,\middle|\, \vec{x}_2 \right\rangle \vec{x}_2 + \left\langle \vec{V} \,\middle|\, \vec{x}_3 \right\rangle \vec{x}_3$$

# Vector Projection

$\vec{V}$ is made up of a sum of basis vectors $\vec{x}_1, \vec{x}_2, \vec{x}_3$




$$\vec{V} = V_{x1}\vec{x}_1 + V_{x3}\vec{x}_2 + V_{x3}\vec{x}_3$$

$V_{x1}$ is "how much $\vec{V}$ is made from $\vec{x}_1$ ".

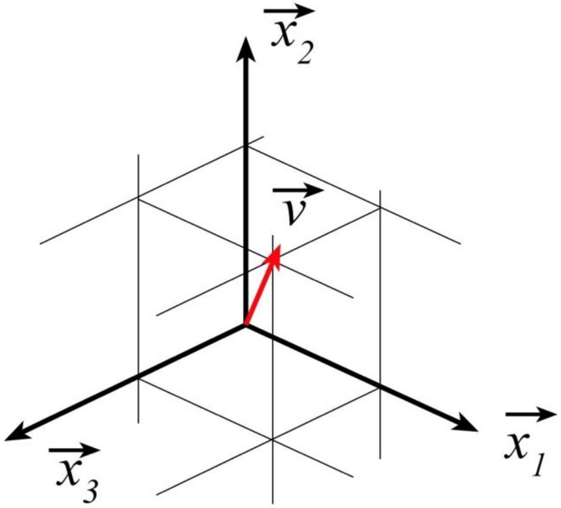
To find this, project $\vec{V}$ onto $\vec{x}_1$ (dot product)

$$V_{x1} = \left\langle \vec{V} \,\middle|\, \vec{x}_1 \right\rangle$$

# Change of Basis Vectors

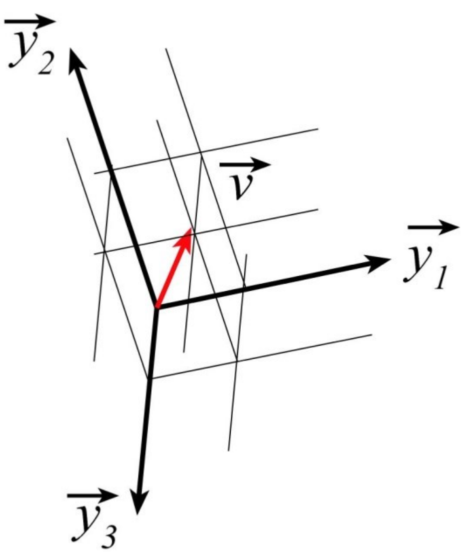
The vector $\vec{V}$ is a sum of the basis vectors $\vec{x}_1, \vec{x}_2, \vec{x}_3$

$$\vec{V} = \langle \vec{V} \,|\, \vec{x}_1 \rangle \vec{x}_1 + \langle \vec{V} \,|\, \vec{x}_2 \rangle \vec{x}_2 + \langle \vec{V} \,|\, \vec{x}_3 \rangle \vec{x}_3$$



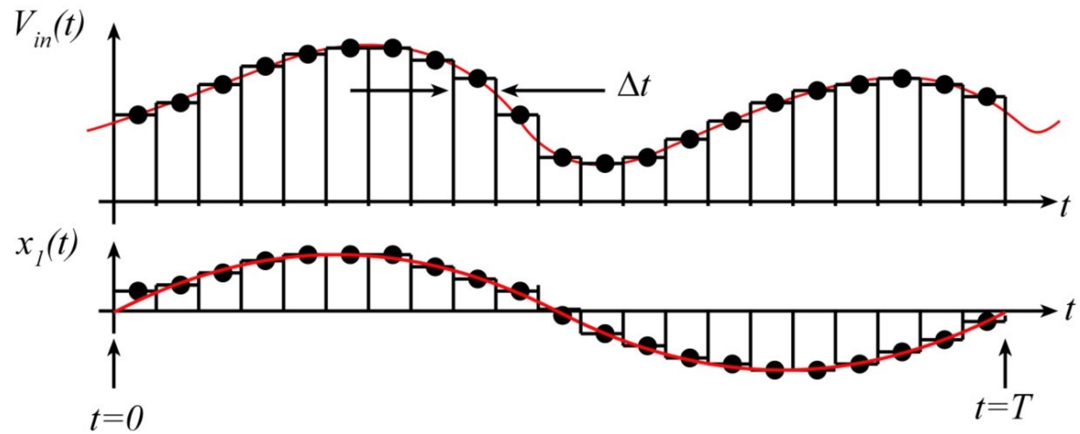

The vector $\vec{V}$ is also a sum of the basis vectors $\vec{y}_1, \vec{y}_2, \vec{y}_3$

$$\vec{V} = \langle \vec{V} \,|\, \vec{y}_1 \rangle \vec{y}_1 + \langle \vec{V} \,|\, \vec{y}_2 \rangle \vec{y}_2 + \langle \vec{V} \,|\, \vec{y}_3 \rangle \vec{y}_3$$




We can represent a vector using more than one set of basis vectors.

# Understanding Fourier: Continuous time



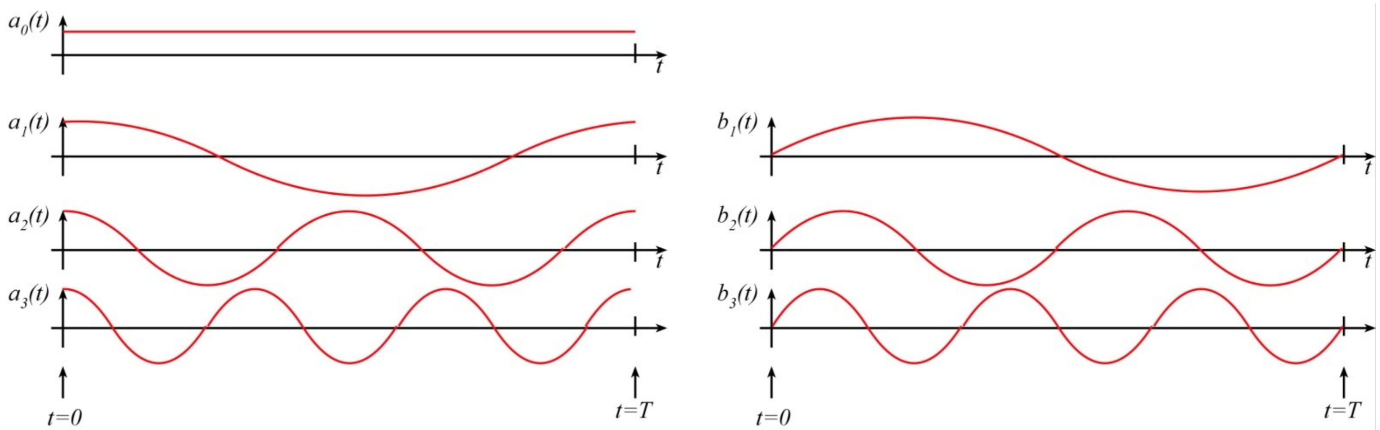

What is $\langle \vec{V} \,|\, \vec{x}_1 \rangle$?

$$\langle \vec{V} \,|\, \vec{x}_1 \rangle = V(t_1)x_1(t_1) + V(t_2)x_1(t_2) + \ldots + V(t_N)x_1(t_N)$$

If we make $\Delta t \to 0$, this becomes an integral.

$$\langle V(t) \,|\, x_1(t) \rangle = \frac{1}{T}\int_0^T V(t_1)x_1(t_1)dt$$

# Understanding Fourier: Our Set of Basis Vectors



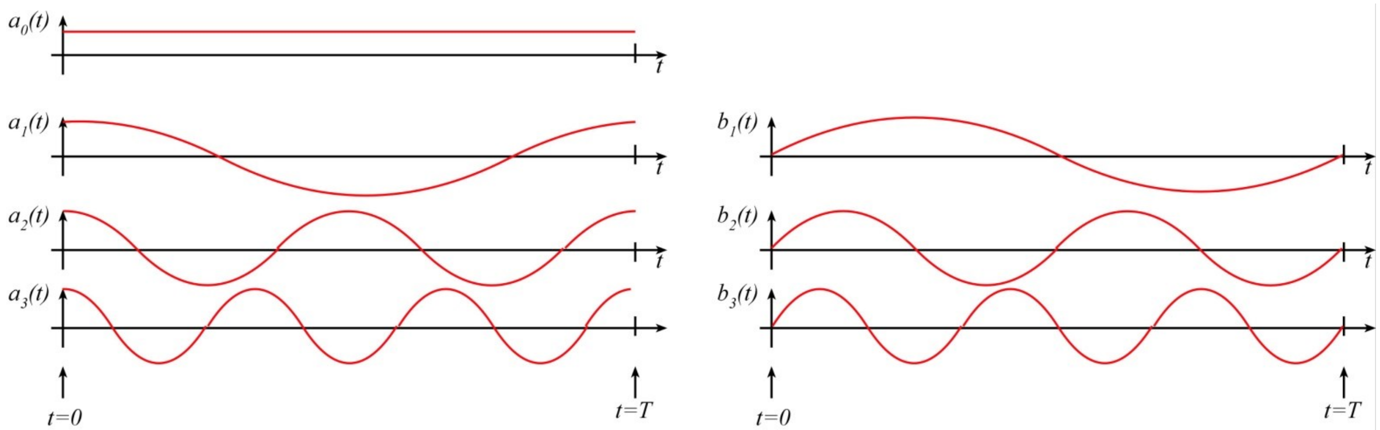

$$a_0(t) = 1$$

$$a_1(t) = \sqrt{2} \cdot \cos\left(1 \cdot \frac{2\pi}{T} \cdot t\right) \qquad b_1(t) = \sqrt{2} \cdot \sin\left(1 \cdot \frac{2\pi}{T} \cdot t\right)$$

$$a_2(t) = \sqrt{2} \cdot \cos\left(2 \cdot \frac{2\pi}{T} \cdot t\right) \qquad b_2(t) = \sqrt{2} \cdot \sin\left(2 \cdot \frac{2\pi}{T} \cdot t\right)$$

$$a_n(t) = \sqrt{2} \cdot \cos\left(n \cdot \frac{2\pi}{T} \cdot t\right) \qquad b_n(t) = \sqrt{2} \cdot \sin\left(n \cdot \frac{2\pi}{T} \cdot t\right)$$

# Understanding Fourier: Our Set of Basis Vectors



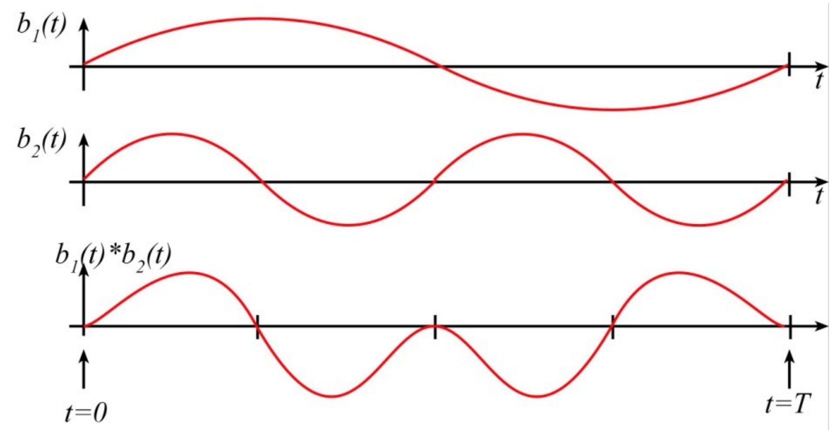

Are they of unit length ?

$$\langle a_0(t) | a_0(t) \rangle = \frac{1}{T}\int_0^T 1 \cdot dt = 1$$

$$\langle a_N(t) | a_N(t) \rangle = \frac{1}{T}\int_0^T 2 \cdot \cos^2\left( n \cdot \frac{2\pi}{T} \cdot t \right) dt = \ldots = 1$$

$$\langle b_N(t) | b_N(t) \rangle = \frac{1}{T}\int_0^T 2 \cdot \sin^2\left( n \cdot \frac{2\pi}{T} \cdot t \right) dt = \ldots = 1$$

# Understanding Fourier: Our Set of Basis Vectors




Are these functions perpendicular to each other?

Here is sketched the product $b_1(t)b_2(t)$.

Its integral between $t=0$ and $t=T$ is clearly zero.

Sketch out other combinations yourself.

$$\langle a_2(t) | a_3(t) \rangle = \langle b_1(t) | a_3(t) \rangle = etc = 0.$$

The set of functions $a_0(t), a_n(t), b_n(t)$ are a set of basis vectors.

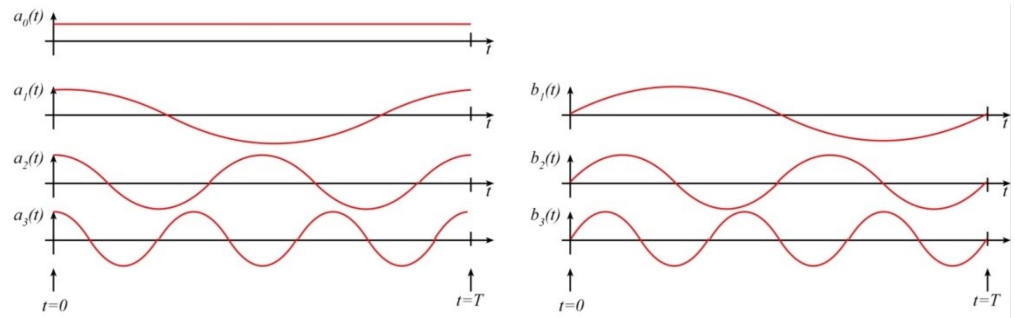
# We Have Just Derived the Fourier Series

$$a_0(t) = 1$$

$$a_1(t) = \sqrt{2}\cdot\cos\left(1\cdot\frac{2\pi}{T}\cdot t\right) \qquad b_1(t) = \sqrt{2}\cdot\sin\left(1\cdot\frac{2\pi}{T}\cdot t\right)$$

$$a_2(t) = \sqrt{2}\cdot\cos\left(2\cdot\frac{2\pi}{T}\cdot t\right) \qquad b_2(t) = \sqrt{2}\cdot\sin\left(2\cdot\frac{2\pi}{T}\cdot t\right)$$

$$a_n(t) = \sqrt{2}\cdot\cos\left(n\cdot\frac{2\pi}{T}\cdot t\right) \qquad b_n(t) = \sqrt{2}\cdot\sin\left(n\cdot\frac{2\pi}{T}\cdot t\right)$$




$$V(t) = \langle V(t)|a_0(t)\rangle a_0(t) + \langle V(t)|a_1(t)\rangle a_1(t) + \langle V(t)|a_2(t)\rangle a_2(t) + \ldots$$

$$\ldots + \langle V(t)|b_1(t)\rangle b_1(t) + \langle V(t)|b_2(t)\rangle b_2(t) + \ldots$$

$$\langle V(t)|a_N(t)\rangle = \frac{1}{T}\int_0^T V(t)a_N(t)\,dt$$

$$\langle V(t)|b_N(t)\rangle = \frac{1}{T}\int_0^T V(t)b_N(t)\,dt$$

# We Have Just Derived the Fourier Series

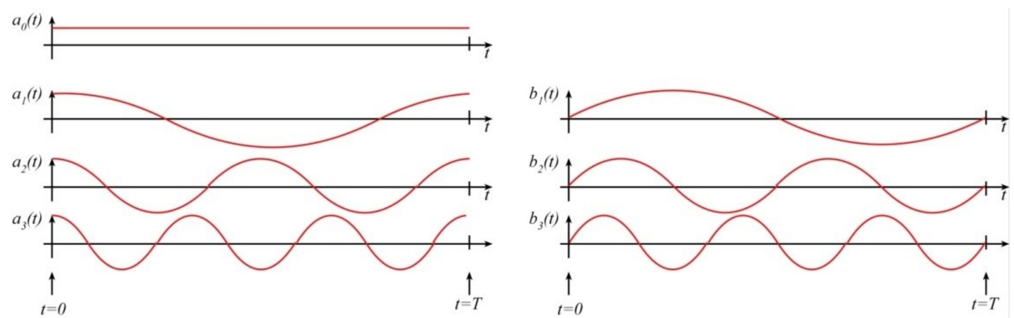
$$V(t) = a_0 + a_1\sqrt{2}\cdot\cos\left(1\cdot\frac{2\pi}{T}\cdot t\right) + a_2\sqrt{2}\cdot\cos\left(2\cdot\frac{2\pi}{T}\cdot t\right) + a_3\sqrt{2}\cdot\cos\left(3\cdot\frac{2\pi}{T}\cdot t\right) + \dots$$

$$\dots + b_1\sqrt{2}\cdot\sin\left(1\cdot\frac{2\pi}{T}\cdot t\right) + b_2\sqrt{2}\cdot\sin\left(2\cdot\frac{2\pi}{T}\cdot t\right) + b_3\sqrt{2}\cdot\sin\left(3\cdot\frac{2\pi}{T}\cdot t\right) + \dots$$

$$a_0 = \frac{1}{T}\int_0^T V(t)\,dt$$

$$a_n = \frac{1}{T}\int_0^T V(t)\sqrt{2}\cdot\cos\left(n\cdot\frac{2\pi}{T}\cdot t\right)dt$$

$$b_n = \frac{1}{T}\int_0^T V(t)\sqrt{2}\cdot\sin\left(n\cdot\frac{2\pi}{T}\cdot t\right)dt$$




Warning: the form of these equations, though correct, is different from those in many textbooks.

# Fourier Series: What does it mean ?

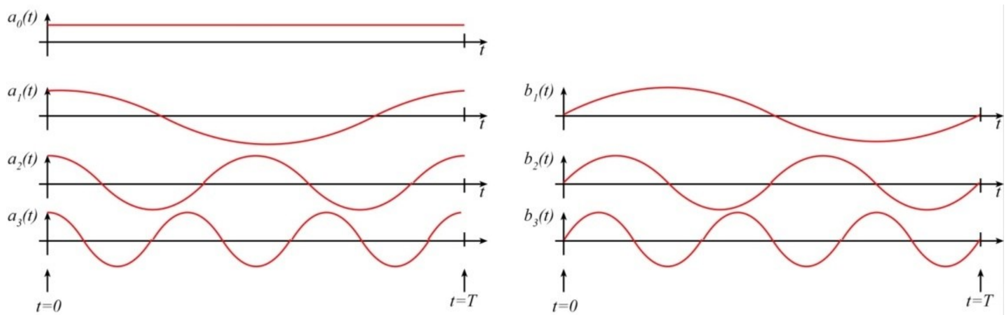
$$V(t) = a_0 + a_1\sqrt{2}\cdot\cos\left(1\cdot\frac{2\pi}{T}\cdot t\right) + a_2\sqrt{2}\cdot\cos\left(2\cdot\frac{2\pi}{T}\cdot t\right) + a_3\sqrt{2}\cdot\cos\left(3\cdot\frac{2\pi}{T}\cdot t\right) + \dots$$

$$\dots + b_1\sqrt{2}\cdot\sin\left(1\cdot\frac{2\pi}{T}\cdot t\right) + b_2\sqrt{2}\cdot\sin\left(2\cdot\frac{2\pi}{T}\cdot t\right) + b_3\sqrt{2}\cdot\sin\left(3\cdot\frac{2\pi}{T}\cdot t\right) + \dots$$

$$a_0 = \frac{1}{T}\int_0^T V(t)dt$$

$$a_n = \frac{1}{T}\int_0^T V(t)\sqrt{2}\cdot\cos\left(n\cdot\frac{2\pi}{T}\cdot t\right)dt$$

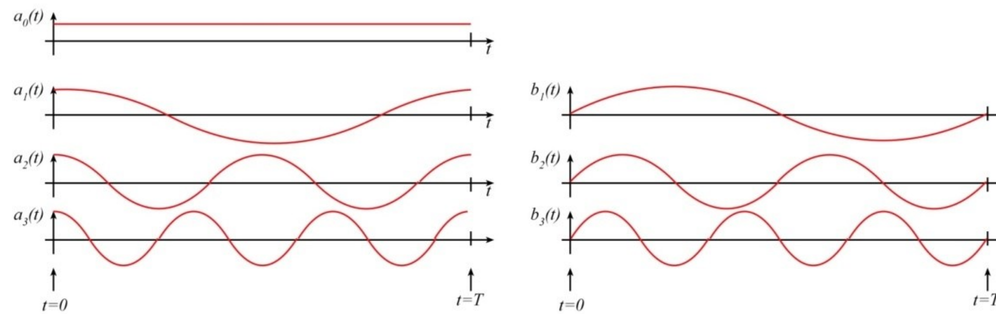
$$b_n = \frac{1}{T}\int_0^T V(t)\sqrt{2}\cdot\sin\left(n\cdot\frac{2\pi}{T}\cdot t\right)dt$$




We write a signal as a sum of DC, sinewaves and cosine waves.

The magnitude of the signal's component at any frequency (sine or cosine) is simply the dot product (projection integral) of the signal and the sine wave or cosine wave at that frequency.

# Complex Exponential Fourier Series




$$\text{Dot product for complex functions:} \left\langle a(t) \middle| b(t) \right\rangle = \frac{1}{T}\int_0^T a(t) b^*(t)\,dt$$

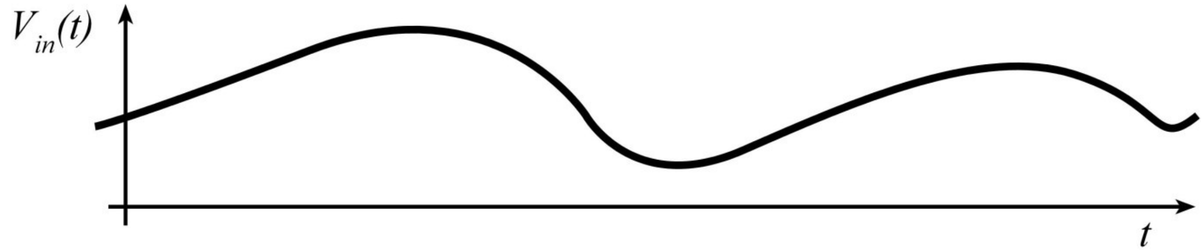
$$\text{Euler identity:} \; e^{j\omega t} = \cos(\omega t) + \sin(\omega t)$$

$$\text{Basis functions:} \; c_n(t) = \exp(jn\omega_0 t) \quad \text{where} \; \omega_0 = 2\pi / T$$

$$V(t) = \sum_{n=-\infty}^{n=-\infty} \left\langle V(t) \middle| c_n(t) \right\rangle c_n(t) = \sum_{n=-\infty}^{n=-\infty} c_n \exp(jn\omega_0 t)$$

$$\text{where} \; c_n = \frac{1}{T}\int_0^T V(t) \exp(-jn\omega_0 t)\,dt$$

# Fourier Integrals / Fourier Transforms



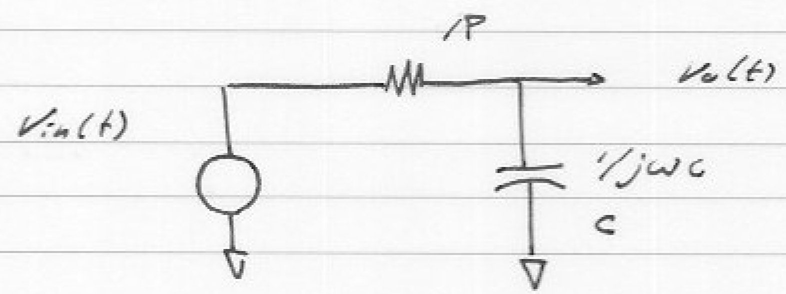

If we take the time limit T of the function to infinity,

the sum of frequencies becomes an integral. $\langle a(t) | b(t) \rangle = \dfrac{1}{T}\int_0^T a(t) b^*(t) dt$

$$V(t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} \langle V(t) | \exp(j\omega t) \rangle \exp(j\omega t) d\omega = \frac{1}{2\pi}\int_{-\infty}^{+\infty} F(j\omega) \exp(j\omega t) d\omega$$

where $F(j\omega) = \langle V(t) | \exp(j\omega t) \rangle = \displaystyle\int_{-\infty}^{+\infty} V(t) \exp(-j\omega t) dt$

# Solving Circuits Using Fourier Transforms




we know that if $v_{in}(t) = v_i e^{j\omega t}$

$v_o(t) = v_o e^{j\omega t} = v_i \cdot H(j\omega) \cdot e^{j\omega t}$

what If $V_{in}(t)$ is not a sine wave?

write $V_{in}(t)$ as a sum of sinewaves!

# Solving Circuits Using Fourier Transforms

$$V_{in}(t) = v_{i1} e^{j\omega_1 t} + v_{i2} e^{j\omega_2 t} + \ldots$$

usually an infinite # of sinewaves is needed:

$$= \int_{-\infty}^{+\infty} \frac{F(j\omega)}{2\pi} e^{j\omega t} d\omega$$

An integral is just an infinite sum – over $\omega \rightarrow$ "$d\omega$"

– of sinewaves of different frequency $e^{j\omega t}$

– each having different amplitude and

phase: $\frac{F(j\omega)}{2\pi}$

# Solving Circuits Using Fourier Transforms

So we can represent

$$v_{in}(t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} V_{in}(j\omega)e^{j\omega t}\,d\omega$$

("Inverse Fourier Transform")

as a sum of sinewaves, each of whose amplitude and phase $V_{in}(j\omega)$ are found by projecting $v_{in}(t)$ against that particular sinewave

$$V_{in}(j\omega) = \int_{-\infty}^{+\infty} v_{in}(t)e^{-j\omega t}\,dt$$

("Fourier transform")

# Solving Circuits Using Fourier Transforms

So, our method of solving transient problems:

1) Convert $V_{in}(t)$ into a sum of sinusoids $e^{j\omega t}$

$$V_{in}(j\omega) = \langle v_{in}(t) | e^{j\omega t} \rangle$$

"Take the Fourier transform"

2) Compute the response of the circuit to a sinewave $e^{j\omega t}$

$$v_o / v_i = H(j\omega) \Longleftrightarrow$$

"Compute the transfer function"

# Solving Circuits Using Fourier Transforms

3) Multiply the amplitude and phase of each sinewave which makes up $v_{in}(t)$ by this transfer function

4) Add these responses up to find $v_{out}(t)$ "take the inverse fourier transform"

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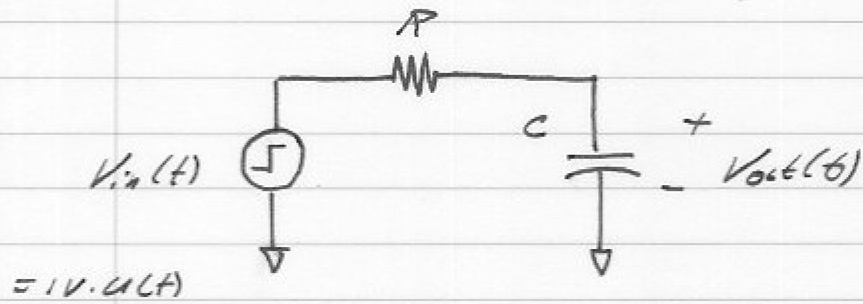


$\tau = RC$

$$v_i(t) = v_o(t) + \tau\, \partial v_o(t)/\partial t$$

$$V_i(j\omega) = \int_{-\infty}^{+\infty} v_i(t)\, e^{-j\omega t}\, dt \qquad \text{known}$$

$$V_o(j\omega) = \int_{-\infty}^{+\infty} v_o(t)\, e^{-j\omega t}\, dt; \qquad \text{unknown}$$

# Solving Circuits Using Fourier Transforms

For any "sinusoidal" ($e^{j\omega t}$) component of the signal

$$V_i(j\omega) = V_0(j\omega) + j\omega\tau V_0(j\omega)$$

$$\rightarrow \frac{V_0(j\omega)}{V_i(j\omega)} = H(j\omega) = \frac{1}{1 + j\omega\tau}$$

hence

$$v_{out}(t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} \underbrace{\frac{V_i(j\omega)}{1+j\omega\tau} e^{j\omega t}}_{\text{Sum of all responses.}} d\omega$$