ECE137A, Notes Set 15:
LaPlace Transforms

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Goals of this note set:

Understand what a LaPlace transform* is *.

....and where it comes from.

Remember how to use them to find circuit transient response.

LaPlace Tranforms are slightly modified Fourier Transforms.

The LaPlace Transform

Start with the Fourier transform

$$F(j\omega) = \int_{-\infty}^{+\infty} f(t) e^{-j\omega t} dt$$

$$f(t) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} F(j\omega) e^{j\omega t} d\omega$$

The LaPlace Transform

Now - restrict ourselves to signals which are zero for $t < 0$ ← strict inequality!

$$\left[F(j\omega) = \int_{0^-}^{+\infty} f(t)\, e^{-j\omega t}\, dt \right]$$

limit of "0^-" means that a δ- function occurring at exactly $t=0$ is included in the integral.

Multiply our Function with an Decaying Exponential:

Now consider $g(t) = f(t)e^{-\sigma t}$

$$G(j\omega) = \int_{0^-}^{\infty} g(t)e^{-j\omega t}\,dt$$

$$= \int_{0^-}^{+\infty} f(t)e^{-\sigma t}e^{-j\omega t}\,dt$$

$$\Rightarrow \quad G(j\omega) = F(\sigma + j\omega)$$

Multiply our Function with an Decaying Exponential:

Also: $f(t)e^{-\sigma t} = g(t) = \dfrac{1}{2\pi}\displaystyle\int_{-\infty}^{+\infty} G(j\omega)e^{j\omega t}\,d\omega$

$$= \frac{1}{2\pi}\int_{-\infty}^{+\infty} F(\sigma + j\omega)e^{j\omega t}\,d\omega$$

$$\Rightarrow f(t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} F(\sigma + j\omega)e^{\sigma t}e^{j\omega t}\,d\omega$$

Fourier Transform of G(t)=F(t)*exp(-σt)

so:

$$F(\sigma+j\omega)=\int_{0^-}^{+\infty}\left[f(t)e^{-\sigma t}\right]e^{-j\omega t}\,dt$$

$$f(t)=\frac{e^{\sigma t}}{2\pi}\int_{-\infty}^{+\infty}F(\sigma+j\omega)\,e^{j\omega t}\,d\omega$$

Fourier transform of $f(t)e^{-\sigma t}$

LaPlace Transform=Fourier Transform of G(t)=F(t)*exp(-σt)

If we write $s = \sigma + j\omega$

$$F(s) = \int_{0^-}^{+\infty} f(t)\, e^{-st}\, dt$$

$$f(t) = \frac{1}{2\pi j} \int_{\sigma - j\infty}^{\sigma + j\infty} F(s)\, e^{st}\, ds$$

The LaPlace transform

Please note that a LaPlace transform represents a sum of sinewaves of different frequencies $e^{j\omega t}$ all multiplied by a decaying exponential $e^{\sigma t}$ of one fixed decay rate σ

Who Gets Credit ?

Laplace transform - Wikip

en.wikipedia.org/wiki/Laplace_transform#History

Português
Română
Русский
Simple English
Slovenščina
Српски / srpski
Basa Sunda
Suomi
Svenska
ไทย
Türkçe

History [edit]

The Laplace transform is named after mathematician and astronomer Pierre-Simon Laplace, who used a similar transform (now called z transform) in his work on probability theory. The current widespread use of the transform came about soon after World War II although it had been used in the 19th century by Abel, Lerch, Heaviside and Bromwich. The older history of similar transforms is as follows. From 1744, Leonhard Euler investigated integrals of the form

$$z = \int X(x) e^{ax}\, dx \quad \text{and} \quad z = \int X(x) x^A\, dx$$

as solutions of differential equations but did not pursue the matter very far.[2] Joseph Louis Lagrange was an admirer of Euler and, in his work on integrating probability density functions, investigated expressions of the form

Why use the LaPlace Transform ?

Motivations:

1) Convergence: Fourier transforms dont always work

$f(t) = e^{\alpha t} u(t)$ has a Fourier integral which blows up for $\alpha > 0$

Its laplace transform integral is ok if we pick $\sigma > \alpha$

2) Main reason: allows ready examination of circuit responses to inputs of the form $e^{-\sigma t} e^{j\omega t}$, a natural and common signal

Transforms of Derivative and Integrals

Transforms of derivatives

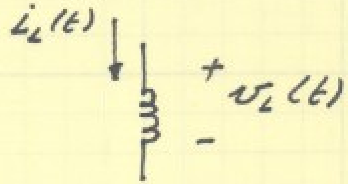
$$\mathcal{L}\left[(d/dt)f(t)\right] = sF(s) - f(0^-)$$

$f(t=0^-)$ = value just before $t=0$

Transforms of integrals

$$\mathcal{L}\left[\int_{0^-}^{t} f(\tau)\,d\tau\right] = F(s)/s$$

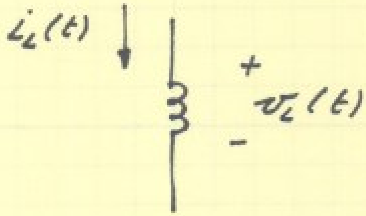
Applications to Circuit Elements: Inductor



$$v_L(t) = L \cdot \frac{\partial i_L(t)}{\partial t}$$

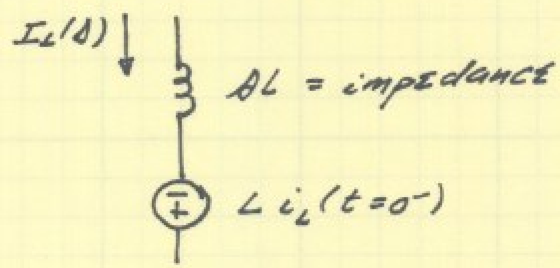
$$\Rightarrow V_L(s) = sL\, I_L(s) - L\, i_L(t=0^-)$$

time domain circuit model



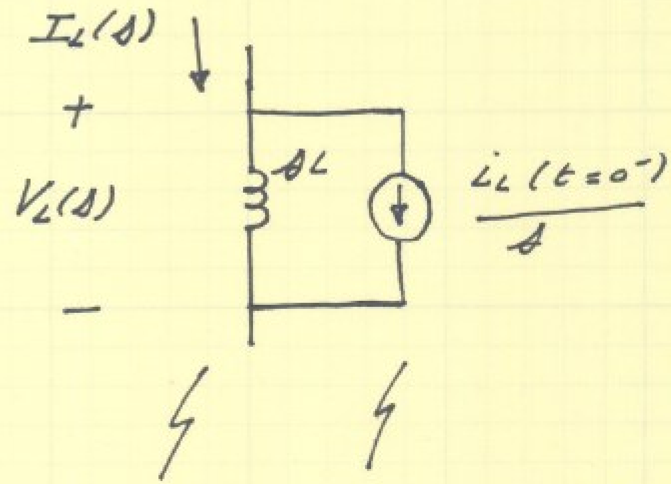
$i_L(t=0^-)$ given

Laplace domain circuit model



Applications to Circuit Elements: Inductor

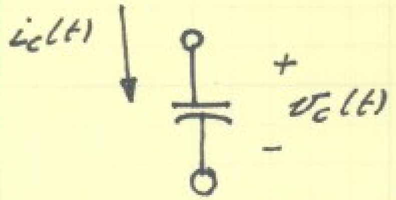
Representation is clearer after a Norton transformation:



impedance   initial condition.

Applications to Circuit Elements: Capacitor

Capacitor in Laplace domain:

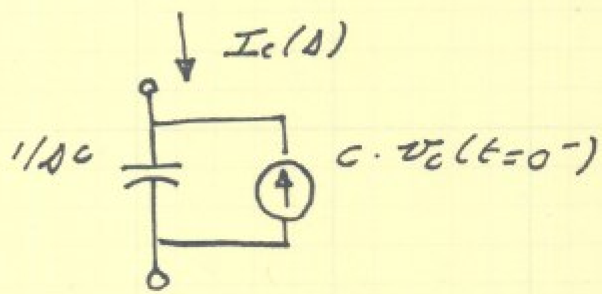


$$i_c(t) = C\,\partial v_c/\partial t$$

$$I_c(s) = sCV_c(s) - Cv_c(t=0^-)$$

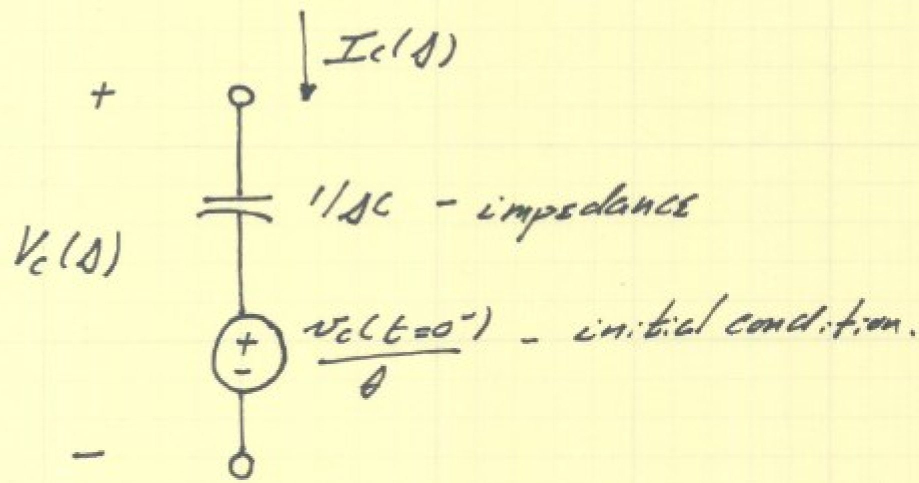
$v_c(t=0^-)$ given

Laplace-domain circuit model:



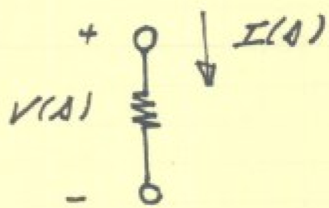
Applications to Circuit Elements: Capacitor

Again, this becomes clearer after a Thevenin transformation:



Applications to Circuit Elements: Resistors

resistors:



$$v(t) = R \cdot i(t)$$

$$V(s) = \mathcal{L}[v(t)] = \mathcal{L}[R \cdot i(t)] = R \cdot I(s)$$

this is trivial...

Applications to Circuit Elements: Independent Sources

independent sources:

time domain

$$\mathcal{L}[v_0] = \int_{0^-}^{+\infty} v_0 e^{-st}\,dt$$

$$= v_0/s \qquad (!)$$

Laplace domain

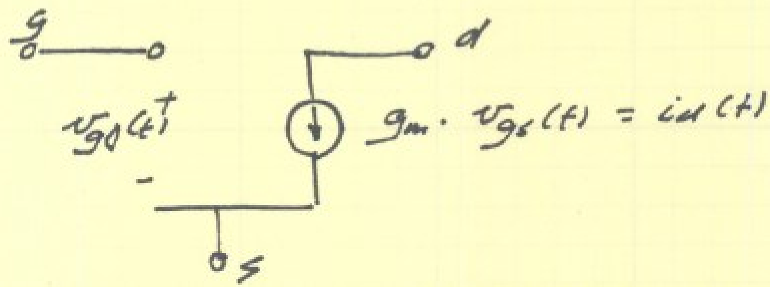
similarly:

time domain

Laplace domain.

Applications to Circuit Elements: *De*pendent Sources

but, don't make a common (and silly) mistake with controlled (dependent) generators:



$$I_d(s) = \mathcal{L}\left[g_m \cdot v_{gs}(t)\right] = g_m \cdot \mathcal{L}\left[v_{gs}(t)\right]$$

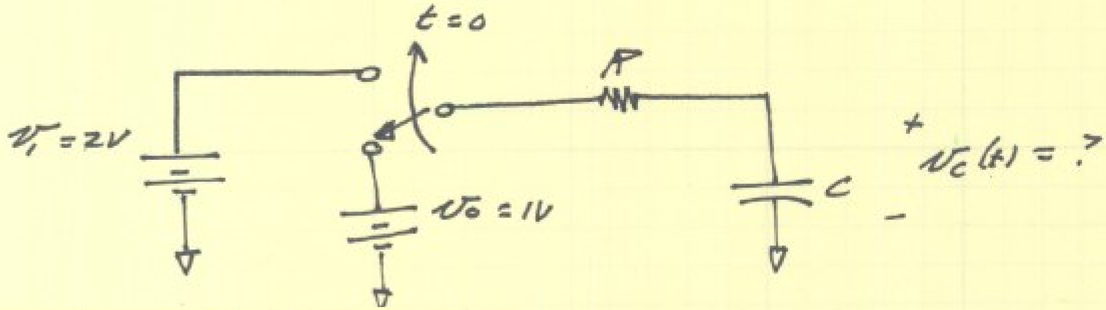
$$= g_m \cdot V_{gs}(s)$$

$I_d(s) = g_m \cdot V_{gs}(s)$ correct

$g_m \dfrac{V_{gs}(s)}{s}$ incorrect

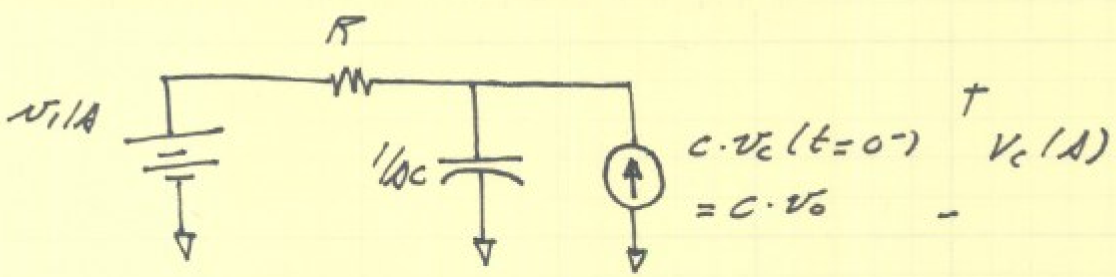
Example Problem---with Initial Conditions

Apply our LaPlace domain Models to an initial condition problem:



First: note that $v_C(t=0^-) = v_0$

LaPlace-domain circuit for $t>0$ is then:



Example Problem---with Initial Conditions

Solve by Superposition

$$V_c(s) = \frac{v_1}{s}\,\frac{1/sC}{r + sC} + C \cdot v_c(0^-)\left(\frac{1}{sC} \,\|\, R\right)$$

but $v_c(0^-) = v_0$

$$V_c(s) = \frac{v_1 - v_0}{s}\left(\frac{1}{1 + sRC}\right) + \frac{v_0}{s}$$

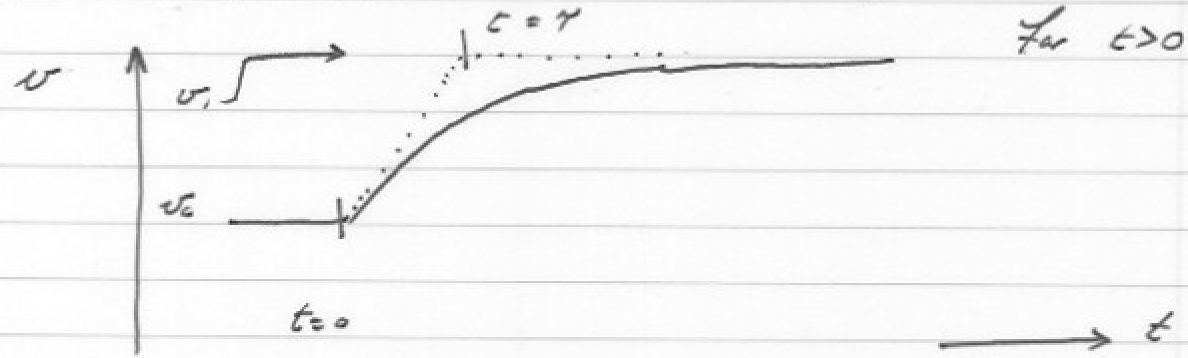
$$= \frac{(v_1 - v_0)}{s}\left[\frac{1}{s} - \frac{\tau}{1 + s\tau}\right] + \frac{v_0}{s}$$

$\tau = RC$

Example Problem---with Initial Conditions

$$v_o(t) = (v_1 - v_0)\cdot u(t)\cdot\left(1 - e^{-t/\tau}\right) + v_0\, u(t)$$

for $t>0$



Note that we *could not drop* the initial conditions. Often the I.C. are zero, hence we can...

For Reference: LaPlace Transform Pairs

Linearity

$$\mathcal{L}[a\,f(t) + b\,g(t)] = a\,F(s) + b\,G(s)$$

Exponential Function

$$\mathcal{L}[u(t)e^{at}] = \int_0^\infty e^{at}e^{-st}\,dt = \frac{1}{s-a}$$

Since $s = \sigma + j\omega$, integral converges only for $\sigma > a$

Writing a decaying exponential differently:

$$\mathcal{L}\left[u(t)\left(\frac{1}{\tau}\right)e^{-t/\tau}\right] = \frac{1}{1+s\tau}$$

For Reference: LaPlace Transform Pairs

Complex Sinusoid

$$\mathcal{L}[u(t)e^{j\omega_0 t}] = \frac{1}{s - j\omega_0}$$

$$\mathcal{L}[u(t)e^{-\alpha t}e^{j\omega_0 t}] = \frac{1}{s + \alpha - j\omega_0}$$

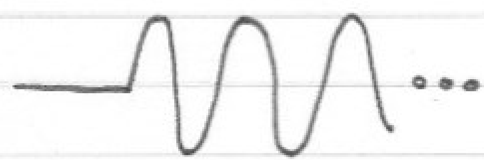
For Reference: LaPlace Transform Pairs

<u>Sine waves starting at $t=0$</u>

using $\mathcal{L}[u(t)e^{j\omega_0 t}] = \dots$ and

$$\sin \omega_0 t = \frac{e^{j\omega_0 t} - e^{-j\omega_0 t}}{2j}$$

$$\mathcal{L}[u(t)\sin \omega_0 t] = \frac{\omega_0}{s^2+\omega_0^2}$$

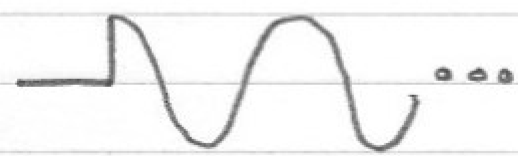


For Reference: LaPlace Transform Pairs

Cosine waves starting at $t=0$

$$\cos\omega_0 t = \frac{e^{j\omega_0 t}+e^{-j\omega_0 t}}{2}$$

$$\mathcal{L}[u(t)\cos\omega_0 t] = \frac{s}{\omega_0^2+s^2}$$



For Reference: LaPlace Transform Pairs

<u>Function times an exponential</u>

$$\mathcal{L}\left[e^{-\alpha t} f(t)\right] = \int_{0^-}^{\infty} e^{-\alpha t} f(t)\, e^{-st}\, dt$$

$$= \int_{0^-}^{\infty} f(t)\, e^{-(s+\alpha)t}\, dt$$

but $F(s) = \int_{\cancel{-\infty}\,0^-}^{\infty} f(t)\, e^{st}\, dt$

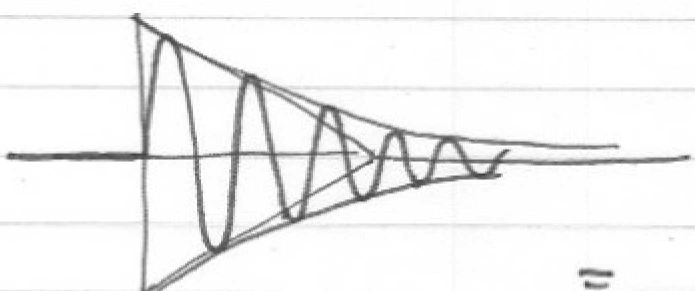
so $\mathcal{L}\left[e^{-\alpha t} f(t)\right] = F(s+\alpha)$

For Reference: LaPlace Transform Pairs

Function times an exponential

using the above ...

$$\mathcal{L}\left[e^{-\alpha t}\sin\omega_0 t\, u(t)\right] = \frac{\omega_0}{(s+\alpha)^2+\omega_0^2}$$

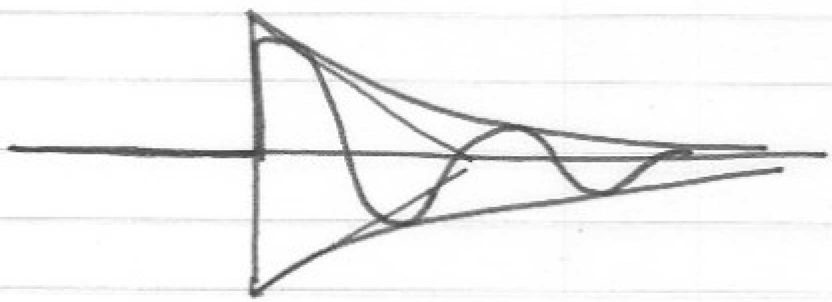


$$= \frac{\omega_0}{(s+\alpha+j\omega_0)(s+\alpha-j\omega_0)}$$

For Reference: LaPlace Transform Pairs

Function times an exponential

$$\mathcal{L}[e^{-\alpha t}\cos\omega_0 t\ u(t)] = \frac{s+\alpha}{(s+\alpha)^2+\omega_0^2}$$



$$= \frac{s+\alpha}{(s+\alpha+j\omega_0)(s+\alpha-j\omega_0)}$$

For Reference: LaPlace Transform Pairs

Step function

$$\mathcal{L}[u(t)] = 1/s$$

...and finally...

what about

$$\mathcal{L}[te^{-\alpha t}] \; ?$$

well $$\mathcal{L}[t] = \int_{0^+}^{\infty} t e^{st} \, dt = 1/s^2$$

so

$$\mathcal{L}[te^{-\alpha t}] = \frac{1}{(s+\alpha)^2}$$