ECE 2C, notes set 12: Sampling and Analog-Digital Conversion

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Goals of this note set

Temporal and frequency relationships

involved with

conversion from continuous - time to sampled - time signals

and

analog - digital coversion

Sampling: Turning Signals into Numbers

We want to store a music file on our computer:

The music is 1000 seconds (16 minutes, 40 seconds) long.

Let us approximate our range of hearing as DC - 20kHz

(good ears: ~ 20Hz - 20kHz).

How many #s must we store on our hard drive?

Remember the Fourier Sine/Cosine Series:

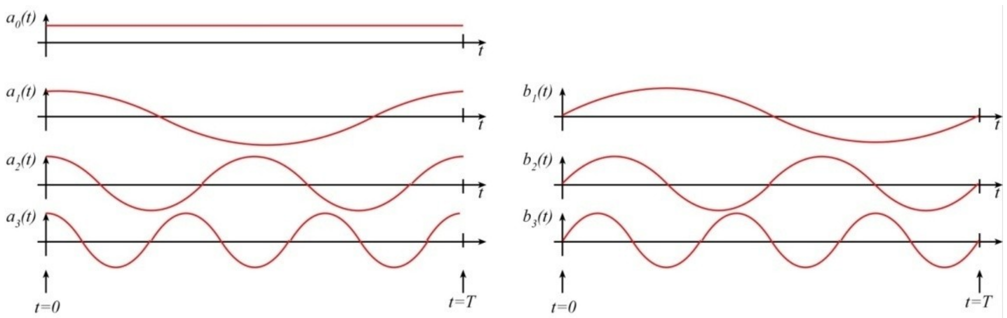
$$V(t) = a_0 + a_1\sqrt{2}\cdot\cos\left(1\cdot\frac{2\pi}{T}\cdot t\right) + a_2\sqrt{2}\cdot\cos\left(2\cdot\frac{2\pi}{T}\cdot t\right) + a_3\sqrt{2}\cdot\cos\left(3\cdot\frac{2\pi}{T}\cdot t\right) + \dots$$

$$\dots + b_1\sqrt{2}\cdot\sin\left(1\cdot\frac{2\pi}{T}\cdot t\right) + b_2\sqrt{2}\cdot\sin\left(2\cdot\frac{2\pi}{T}\cdot t\right) + b_3\sqrt{2}\cdot\sin\left(3\cdot\frac{2\pi}{T}\cdot t\right) + \dots$$

$$a_0 = \frac{1}{T}\int_0^T V(t)dt$$

$$a_n = \frac{1}{T}\int_0^T V(t)\sqrt{2}\cdot\cos\left(n\cdot\frac{2\pi}{T}\cdot t\right)dt$$

$$b_n = \frac{1}{T}\int_0^T V(t)\sqrt{2}\cdot\sin\left(n\cdot\frac{2\pi}{T}\cdot t\right)dt$$



Our waveform is 10^3 seconds long.

Our signal : sine waves at $0\text{Hz}, 1\cdot10^{-3}\text{Hz}, 2\cdot10^{-3}\text{Hz}, \dots, 20\text{kHz}$

Our signal : cosine waves at $0\text{Hz}, 1\cdot10^{-3}\text{Hz}, 2\cdot10^{-3}\text{Hz}, \dots, 20\text{kHz}$

$\rightarrow$ We need $(20\text{kHz}/10^{-3}\text{Hz}) = 2\cdot10^7$ sinsoidal coefficients (a_n)

$\rightarrow$ We also need $2\cdot10^7$ cosinsoidal coefficients (b_n)

We need $4\cdot10^7$ numbers (coefficients) to describe our music file.

Nyquist's Sampling Theorem: 1st viewpoint

If we have a signal of time duration T.

If its bandwidth is limited between DC and f_{high}.

Then we need $N_{sample} = 2 f_{high} T$ numbers (measurements)

to describe the signal

- So -

If we have a signal

whose bandwidth is limited between DC and f_{high}.

Then we need $f_{sample} = 2 f_{high}$ samples per unit time

to describe the signal

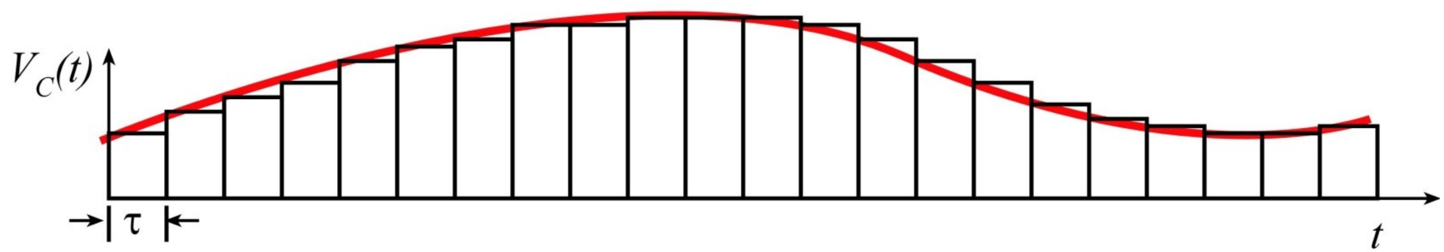
For a DC - 20 kHz signal, we must sample at 40KHz; samples every 25 μs.

Nyquist's theorem proven by counting Fourier components.

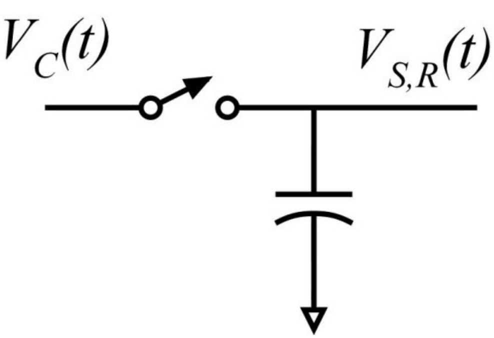
Sampling in the time domain.

We start with some analog continuous-time signal $V_C(t)$

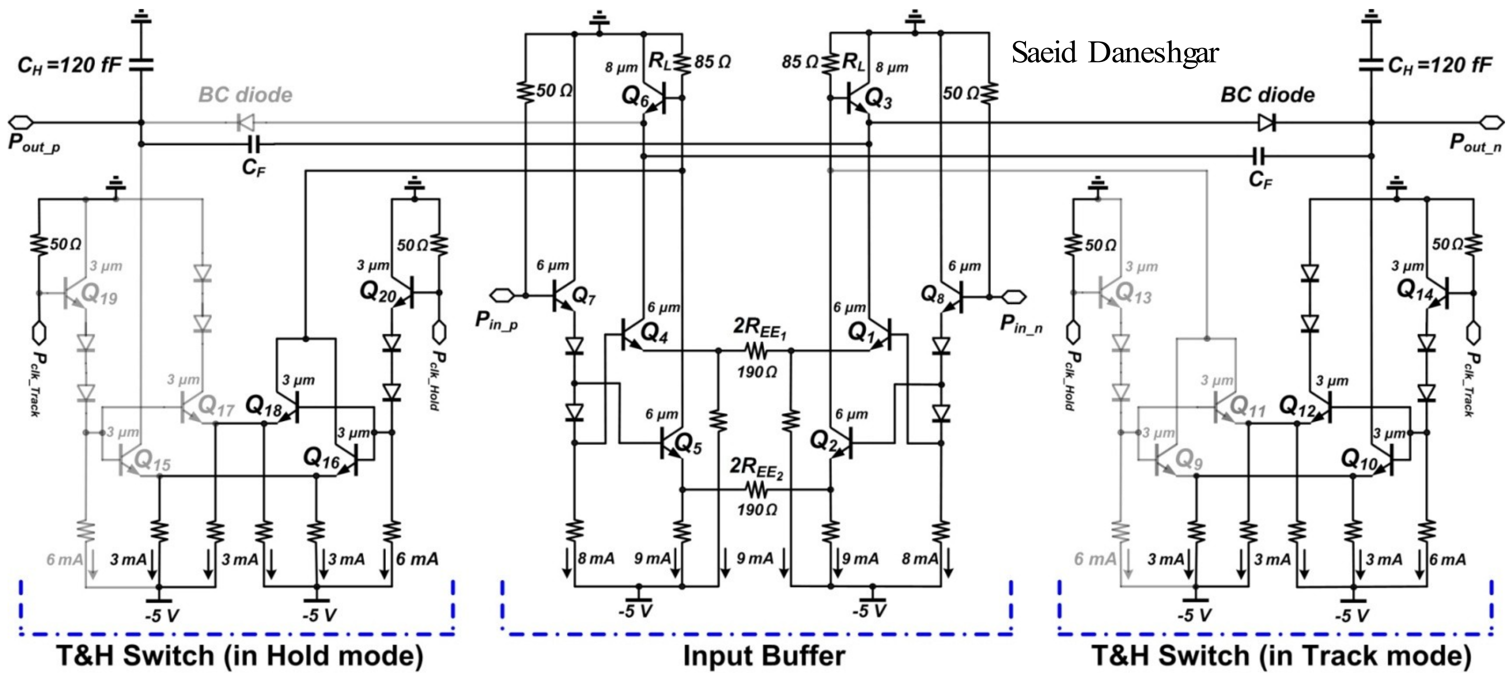
We then sample it with a sample-hold gate, producing signal $V_{S,R}(t)$



This is an idealized sample-hold gate



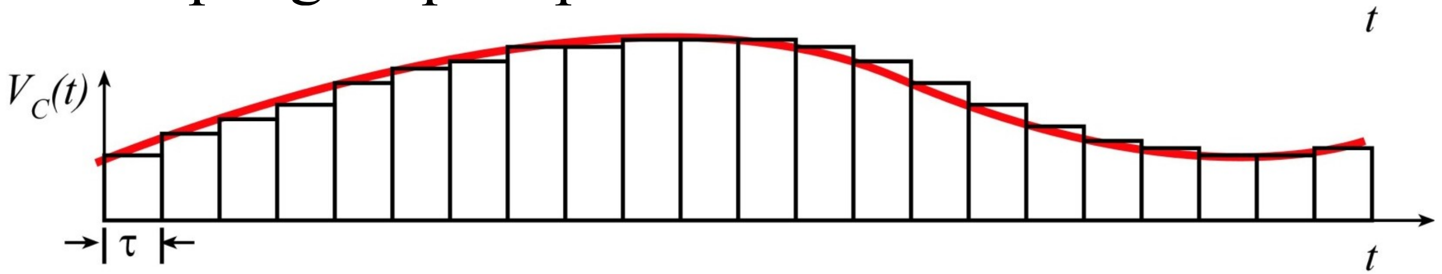
This is a Real Sample-Hold Gate



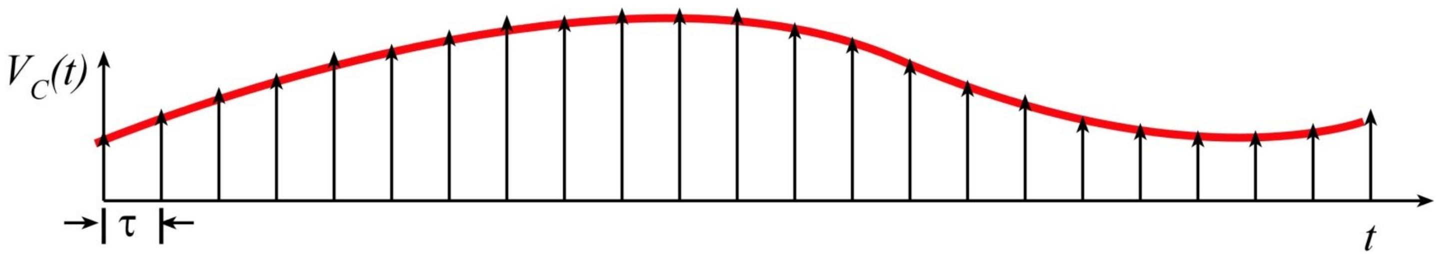
To learn about this: senior-level IC design sequence

Real vs Ideal Sampling (1)

Real Sampling : square pulses



Ideal Sampling : impulses (makes the math easier)

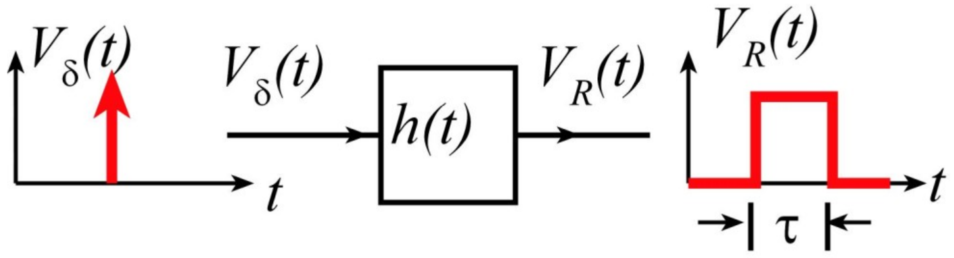


How do we relate these?

Real vs Ideal Sampling (2)

Suppose we have a filter $h(t)$,

whose impulse response is a square pulse of duration τ.



Impulse response: $h(t) = \dfrac{1}{\tau}\operatorname{rect}(t/\tau)$

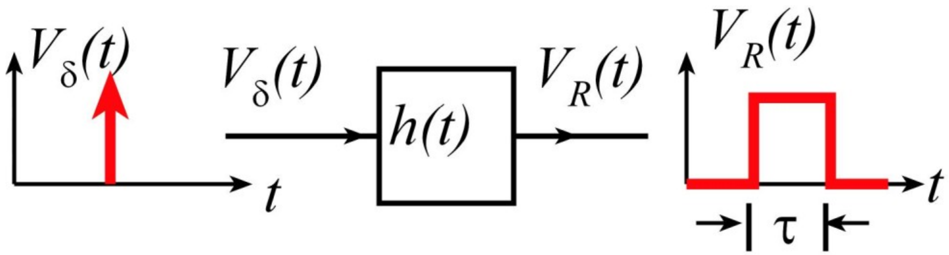
Transfer Function:

$$H(j\omega) = \int_{-\infty}^{+\infty} h(t)\exp(-j\omega t)dt = \frac{1}{\tau}\int_{-\tau/2}^{+\tau/2}\exp(-j\omega t)dt = \frac{1}{-j\omega\tau}\exp(-j\omega t)\Big|_{-\tau/2}^{+\tau/2}$$

$$H(j\omega) = \frac{1}{j\omega\tau}\left(e^{+j\omega\tau/2} - e^{-j\omega\tau/2}\right) = \frac{\sin(\omega\tau/2)}{(\omega\tau/2)}$$

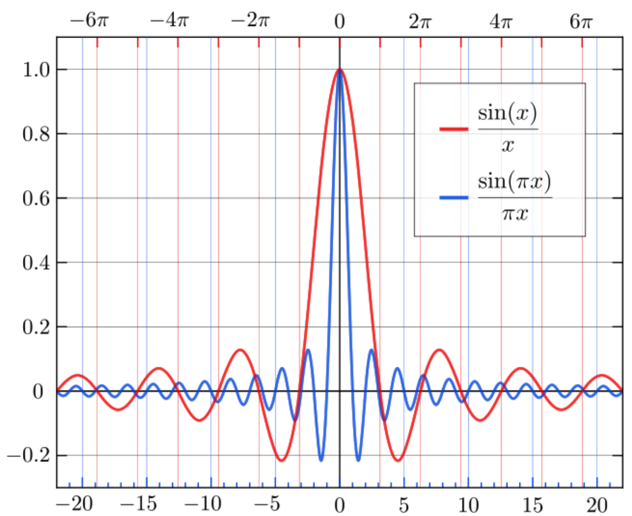
Real vs Ideal Sampling (3)

Filter:



$$\text{Impulse response: } h(t) = \frac{1}{\tau}\text{rect}(t/\tau)$$

$$\text{Transfer Function: } H(j\omega) = \frac{\sin(\omega\tau/2)}{(\omega\tau/2)}$$

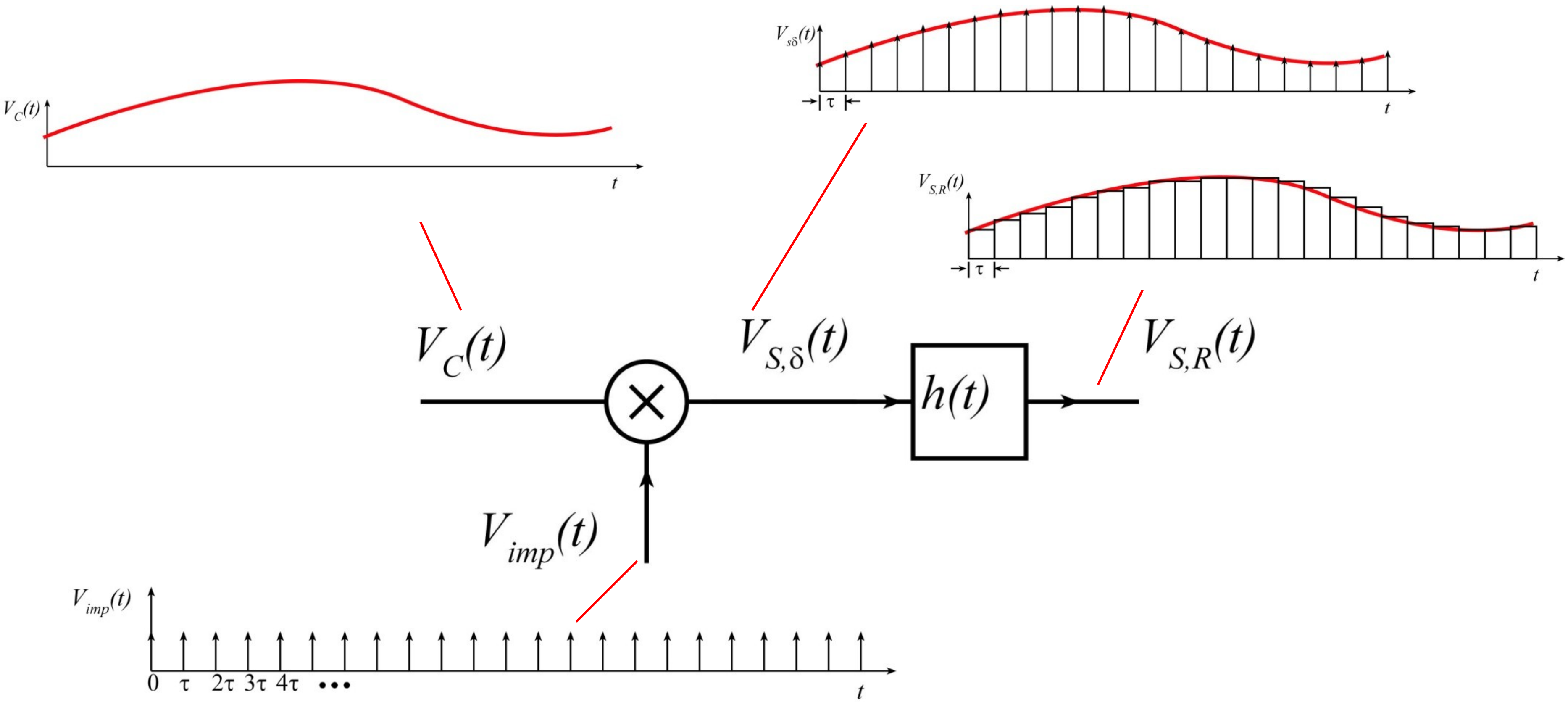


http://en.wikipedia.org/wiki/File:Si_sinc.svg

Filter converts impulses into square pulses.

Filter has frequency reponse $\dfrac{\sin(\omega\tau/2)}{(\omega\tau/2)}$

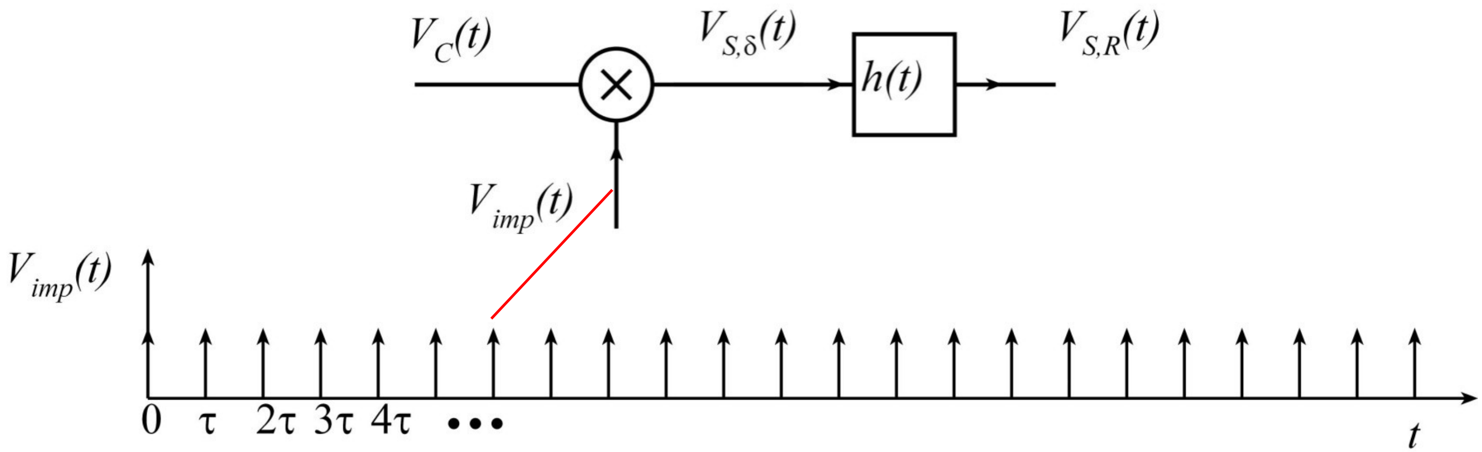
Real vs Ideal Sampling (4)



We first multiply signal against sequence of impulses: $v_{S,\delta}(t) = v_C(t) \cdot v_{imp}(t) / V_0$

We then filter the sampled signal: $V_{S,R}(j\omega) = V_{S,\delta}(j\omega) \cdot H(j\omega)$

Spectrum of the impulse train (1)



$$v_{imp}(t) = V_0\tau\cdot\delta(t) + V_0\tau\cdot\delta(t-\tau) + V_0\tau\cdot\delta(t-2\tau) + \ldots = V_0\tau\sum_{K=-\infty}^{K=+\infty}\delta(t-K\tau)$$

Because $v_{imp}(t)$ is periodic with period τ, we can write this as a Fourier series :

$$v_{imp}(t) = \sum_{n=-\infty}^{n=+\infty} c_n e^{jn\omega_0 t} \quad \text{where } \omega_0 = 2\pi/\tau$$

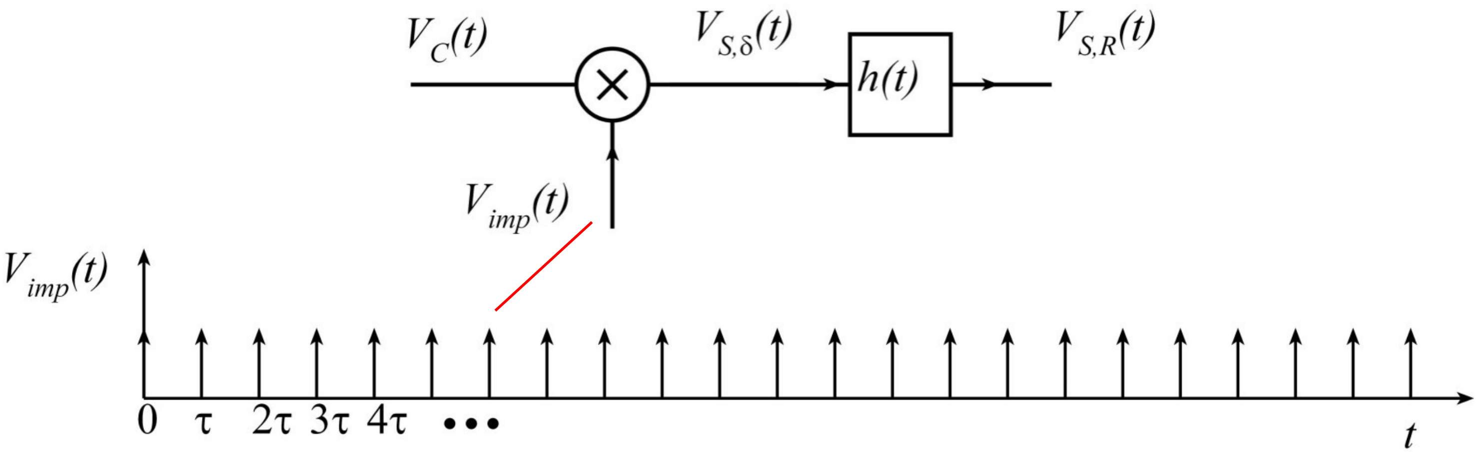
where

$$c_n = \frac{1}{\tau}\int_{-\tau/2}^{\tau/2} v_{imp}(t)e^{-jn\omega_0 t} = \frac{1}{\tau}\int_{-\tau/2}^{\tau/2} V_0\tau\cdot\delta(t)e^{-jn\omega_0 t} = \frac{V_0\tau}{\tau}e^{-jn\omega_0(0)} = V_0$$

So :

$$v_{imp}(t) = V_0\sum_{n=-\infty}^{n=+\infty} e^{jn\omega_0 t} = \ldots + V_0 e^{-j2\omega_0 t} + V_0 e^{-j\omega_0 t} + V_0 + V_0 e^{+j\omega_0 t} + V_0 e^{+j2\omega_0 t}\ldots$$

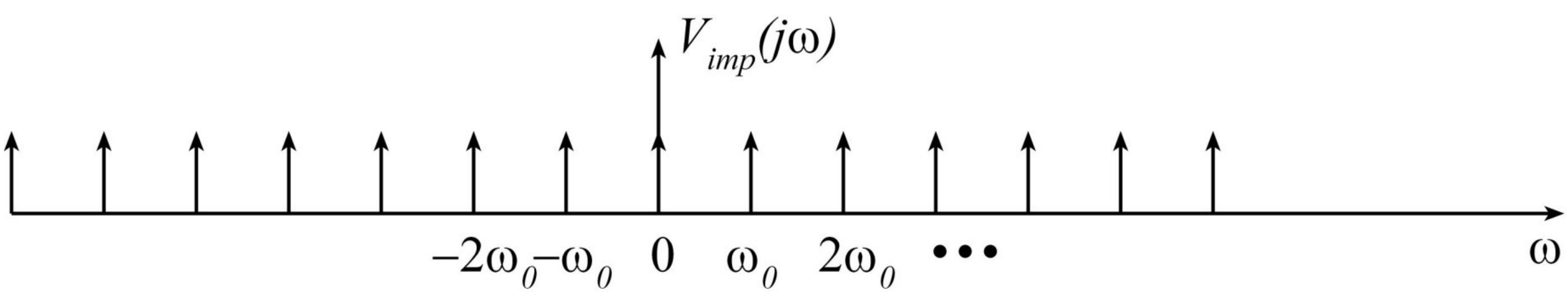
Spectrum of the impulse train (2)



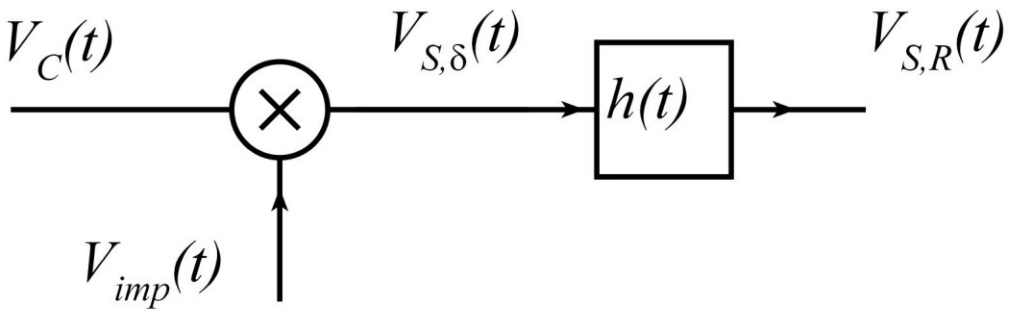
$$v_{imp}(t) = V_0\tau \cdot \delta(t) + V_0\tau \cdot \delta(t-\tau) + V_0\tau \cdot \delta(t-2\tau) + \ldots$$

$$= \ldots + V_0 e^{-j2\omega_0 t} + V_0 e^{-j\omega_0 t} + V_0 + V_0 e^{+j\omega_0 t} + V_0 e^{+j2\omega_0 t} \ldots$$

This is a spectrum with tones at $\ldots, -3\omega_0, -2\omega_0, -\omega_0, DC, \omega_0, 2\omega_0, 3\omega_0, \ldots$



Spectrum of the Sampled Signal (1)



The Fourier transform of the sampled signal is therefore
the convolution of the spectrum of the input
and spectrum of the impulse train

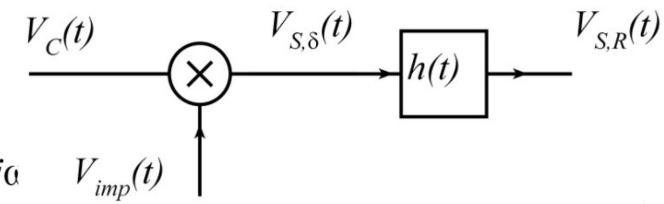
$$v_{S,\delta}(t) = v_C(t) \cdot v_{imp}(t) / V_0$$

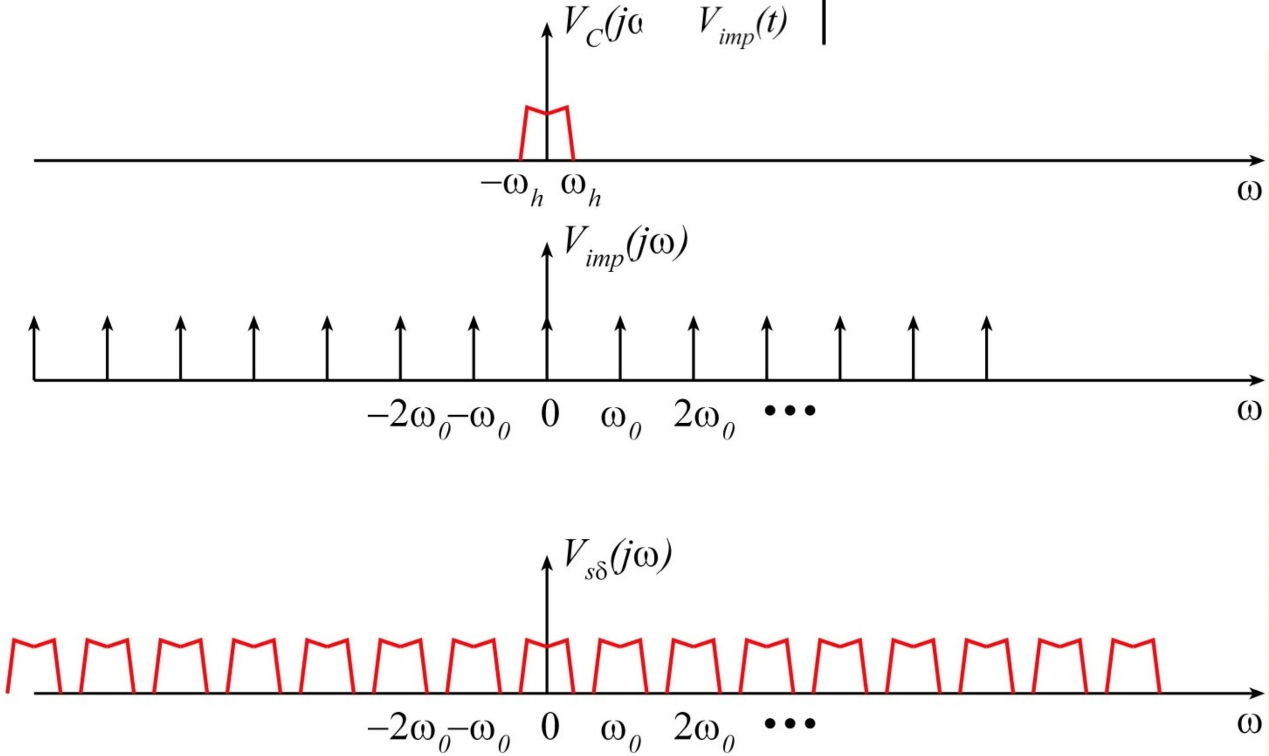
$$V_{S,\delta}(j\omega) = \frac{V_c(j\omega) * V_{imp}(j\omega)}{2\pi V_0} = \frac{\text{convolution of } V_c(j\omega) \text{ and } V_{imp}(j\omega)}{2\pi V_0}$$

$$= \frac{1}{2\pi V_0} \int_{-\infty}^{\infty} V_c(j\omega') \cdot V_{imp}(j\omega - j\omega') d(j\omega')$$

Spectrum of the Sampled Signal (2)

$$V_{S,\delta}(j\omega) = \frac{\text{convolution of } V_c(j\omega) \text{ and } V_{imp}(j\omega)}{2\pi V_0}$$



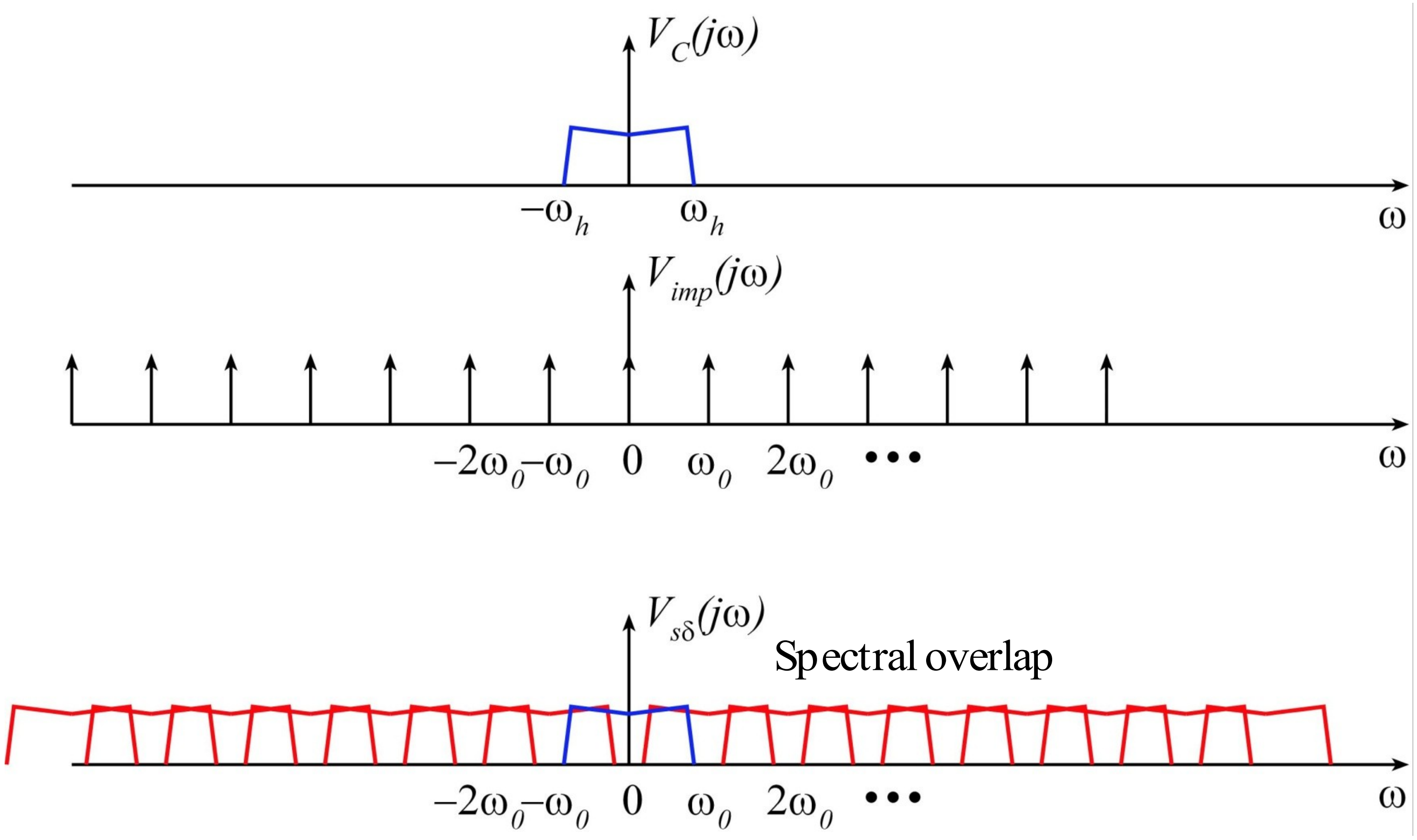


The signal $V_C(t)$ must have bandwidth ω_H less than $\omega_0 / 2$ if the spectra are not to overlap

$\omega_0 = \omega_{sample} > 2\,\omega_{high}$ or $f_{sample} > 2\,f_{high}$ if the spectra are not to overlap.

This is, again, the Nyquist sampling criterion

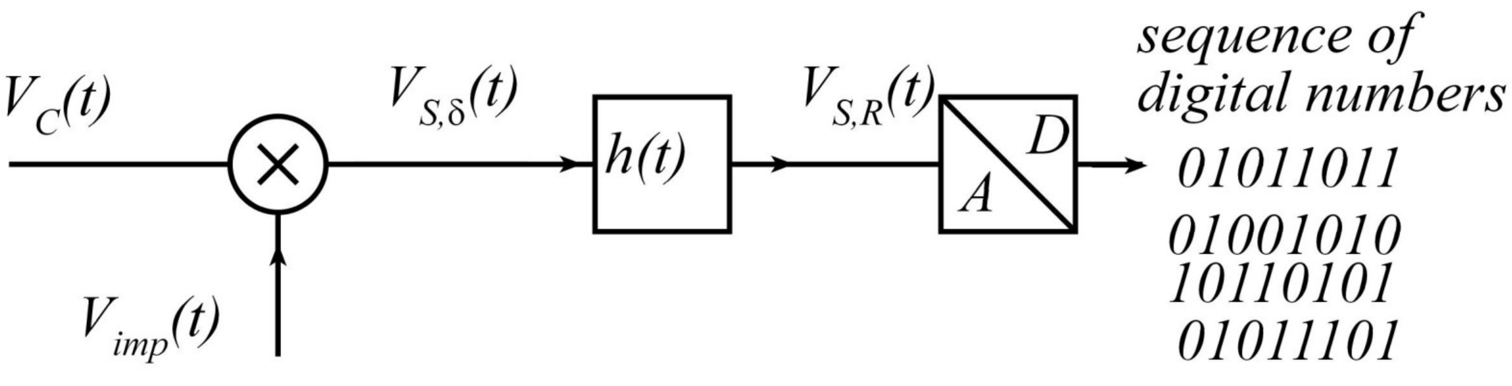
Spectral Overlap (Aliasing) if sampling is too slow



Must have $f_{sample} > 2f_{high}$ if the spectra are not to overlap.

This is, again, the Nyquist sampling criterion

Analog-Digital Conversion



Sampling is then followed by analog - digital coversion.

This generates an N - bit binary representation of each sample.

More bits : greater precision