ECE 2C, notes set 9: Second-Order Circuits

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Goals:

2nd - Order Systems

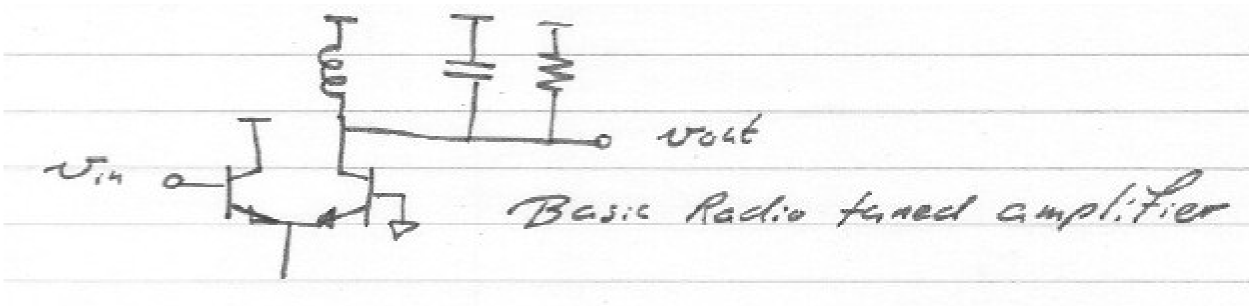
Analysis

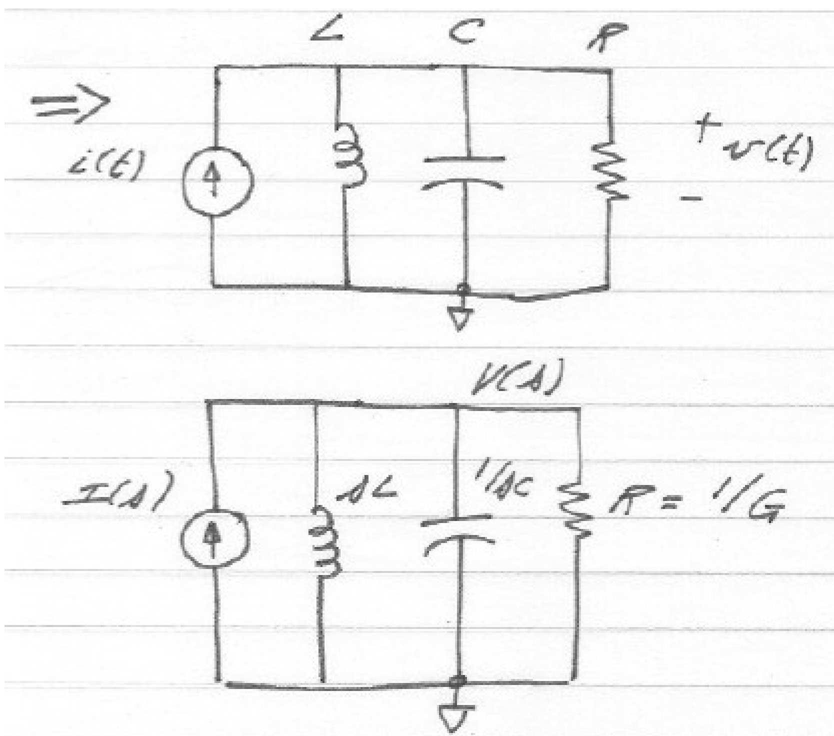
Transfer function in $s-$domain.

Frequency response analysis

Transient analysis

Parallel RLC Resonance





Parallel RLC Resonance: Analysis

$$V[sC + 1/sL + G] = I$$

$$V/I = \frac{1}{sC + 1/sL + G} = \frac{sL}{1 + sL/R + s^2LC}$$

put in standard ("canonical") form

$$\frac{V(s)}{I(s)} = Z(s) = R \cdot \frac{(2\zeta/\omega_n)s}{s^2/\omega_n^2 + (2\zeta/\omega_n)s + 1}$$

ω_n = the natural resonant frequency

$\alpha = \zeta\omega_n$ = the exponential damping coefficient

ζ = the damping factor

Equating terms for this specific problem only

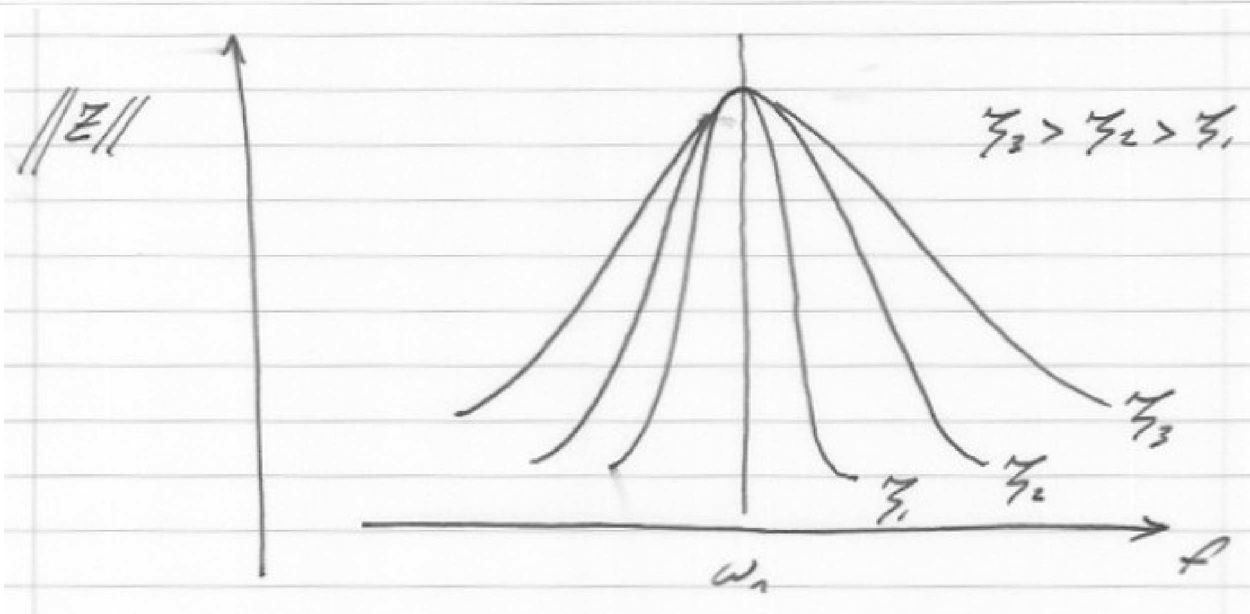
$\omega_n = 1/\sqrt{LC}$, $\alpha = 1/2RC$

$$\zeta = \frac{1}{2R}\sqrt{\frac{L}{C}}$$

Parallel RLC Resonance: Frequency Reponse

response to signal $v_0 e^{j\omega_0 t}$ → $s = j\omega + 0$

$$\frac{V(j\omega)}{I(j\omega)} = Z(j\omega) = R\,\frac{j\omega(2\zeta/\omega_n)}{1 + j\omega(2\zeta/\omega_n) - \omega^2/\omega_n^2}$$

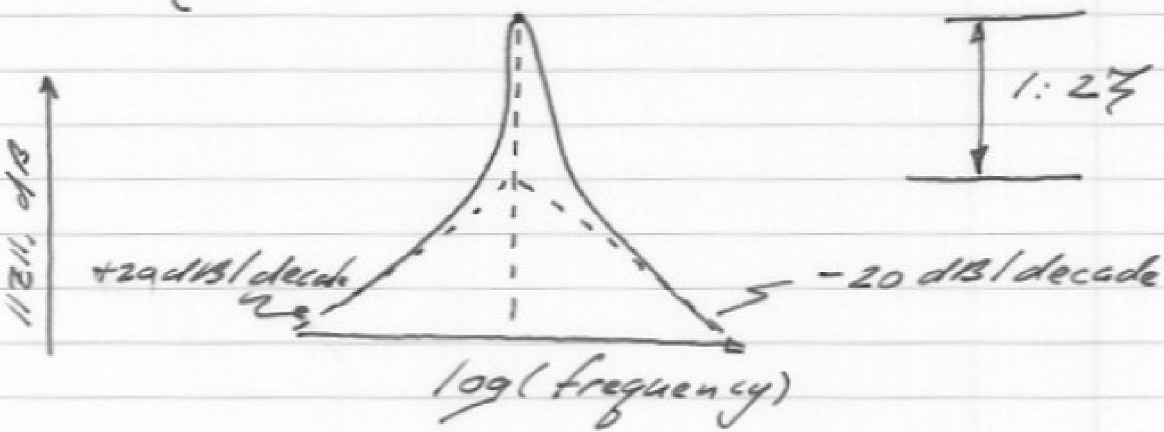


The larger ζ is, the wider the resonant peak in frequency.

Bode Plot of Resonance

$$Z(j\omega) = R\,\frac{j\omega(2\zeta/\omega_n)}{1 + j\omega(2\zeta/\omega_n) - \omega^2/\omega_n^2}$$

$$= \begin{cases} R\cdot(j\omega/\omega_n)2\zeta & \text{for } \omega \ll \omega_n \\ R\cdot 1 & \text{for } \omega = \omega_n \\ R\cdot(\omega_n/j\omega)2\zeta & \text{for } \omega \gg \omega_n \end{cases}$$



- Consider what will happen to the curve as we vary ζ

- Note asymptotes of ± 20 dB/decade and the $2\zeta:1$ resonant peak.

Poles, Zeros

$$V(s)/I(s) = Z(s) = R\,\frac{s\,(2\zeta/\omega_n)}{s^2/\omega_n^2 + s\,(2\zeta/\omega_n) + 1}$$

re-arrange into 2 useful forms:

$$Z(s) = R\cdot\frac{s\,(2\zeta\omega_n)}{(s + \cancel{2}\,\zeta\omega_n)^2 + \omega_d^2}$$

$$\omega_d^2 = (1-\zeta^2)\,\omega_n^2$$

$$\zeta\omega_n = \alpha$$

this form for inverse LaPlace transform

Poles, Zeros

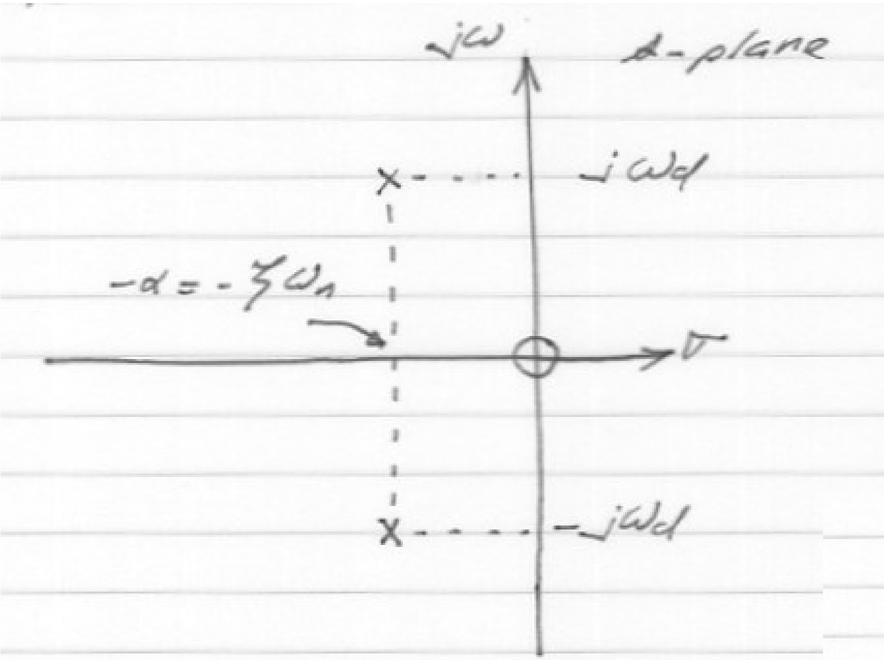
$$V(s)/I(s) = Z(s) = R\,\frac{s(2\zeta/\omega_n)}{s^2/\omega_n^2 + s(2\zeta/\omega_n) + 1}$$

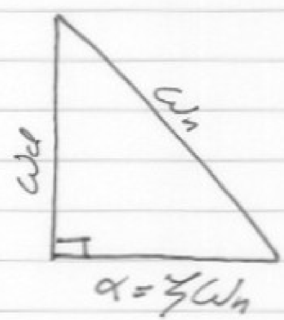
$$Z(s) = R\cdot\frac{s(2\zeta\omega_n)}{(s+\zeta\omega_n - j\omega_d)(s+\zeta\omega_n + j\omega_d)}$$

$$= R\,\frac{\overset{\text{zero at DC.}}{s}(2\zeta\omega_n)}{\underbrace{(s-s_{p1})(s-s_{p2})}_{\text{pole frequencies}}}$$

$$s_{p1,2} = -\zeta\omega_n \pm j\omega_d$$

Root Locus





$$\omega_d^2 + (\zeta\omega_n)^2 = \omega_n^2$$

$$\Downarrow$$

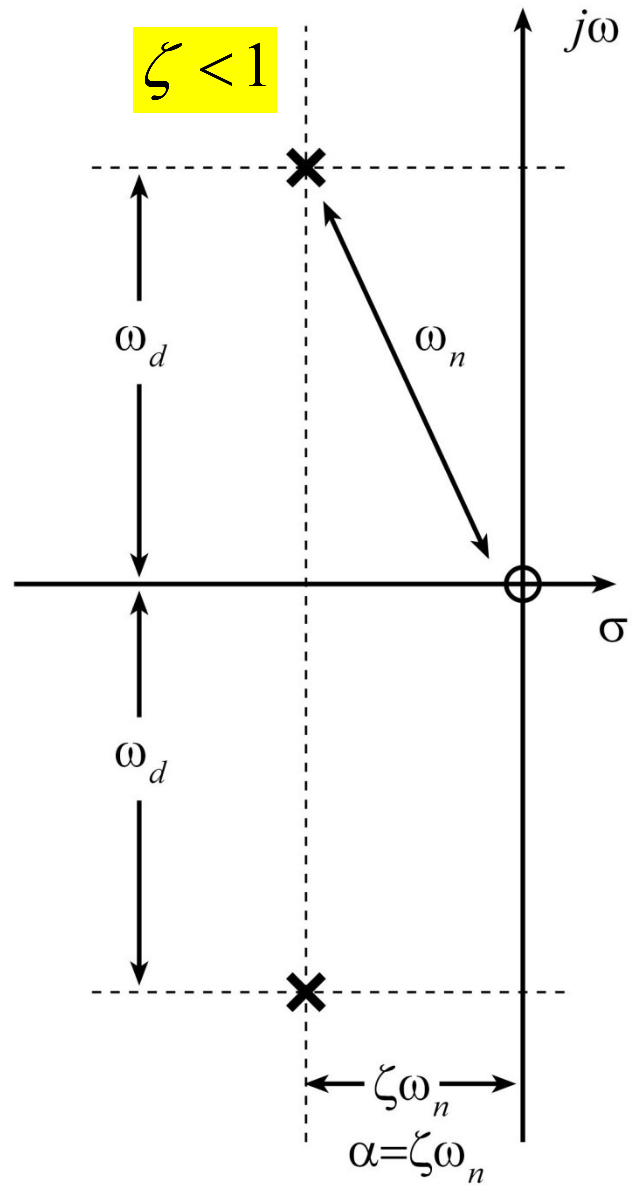
$$\omega_d = \omega_n\sqrt{1-\zeta^2}$$

Very important

Root Locus: For Underdamped System

$$s_{p1,2} = -\zeta\omega_n \pm j\omega_d$$

where $\omega_d = \omega_n \cdot \sqrt{1-\zeta^2}$



$$H(s) = \frac{s(2\zeta/\omega_n)}{1 + s(2\zeta/\omega_n) + s^2/\omega_n^2}$$

Root Locus: For Overdamped System

$$s_{p1,2} = -\zeta\omega_n \pm j\omega_d$$

but if $\zeta > 1$,

then $\omega_d = \omega_n \cdot \sqrt{1-\zeta^2}$ is complex.

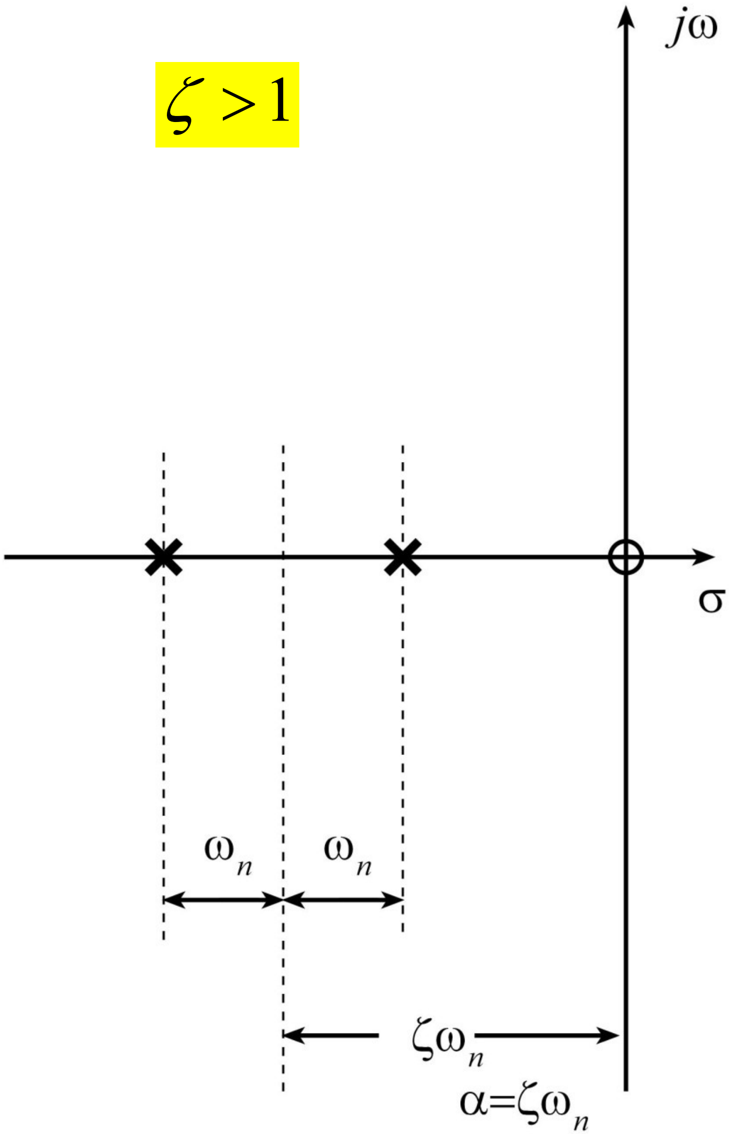
If $\zeta > 1$, then use

$$j\omega_d = j\omega_n \cdot \sqrt{1-\zeta^2} = \omega_n \cdot \sqrt{\zeta^2 - 1}$$

So

$$s_{p1,2} = -\zeta\omega_n \pm j\omega_d$$
$$= -\zeta\omega_n \pm \omega_n \cdot \sqrt{\zeta^2 - 1}$$

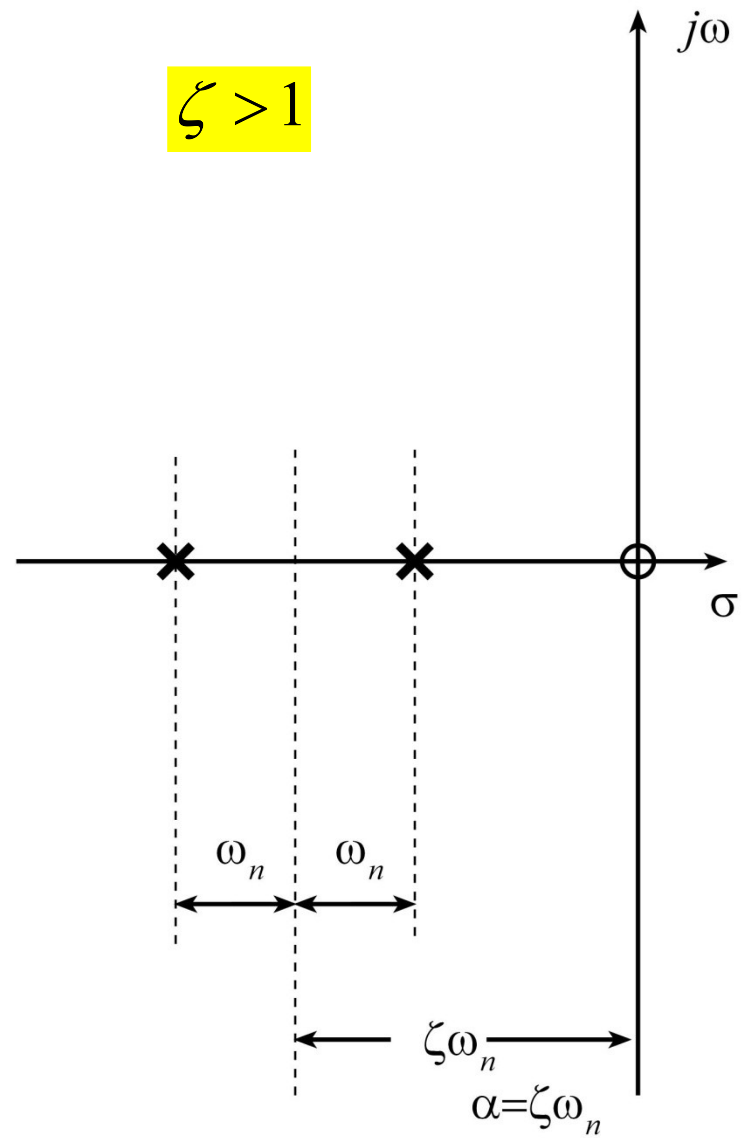
$\zeta > 1$



Root Locus: For Overdamped System

$$s_{p1,2} = -\zeta\omega_n \pm j\omega_d$$
$$= -\zeta\omega_n \pm \omega_n \cdot \sqrt{\zeta^2 - 1}$$

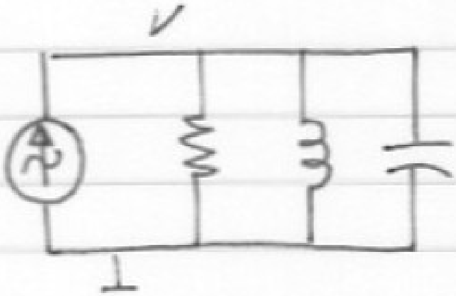
$\zeta > 1$



$$H(s) = \frac{s(2\zeta/\omega_n)}{1 + s(2\zeta/\omega_n) + s^2/\omega_n^2}$$

Quality factor of an Energy Storage Element

$$Q = 2\pi \cdot \frac{\text{Maximum energy stored}}{\text{energy dissipated in one cycle}}$$



$I = I_0 \cos \omega_n t$, $\omega_n = 1/\sqrt{LC}$

$v = \underbrace{R I_0}_{V_0} \cos \omega_n t$ because $\omega = \omega_n$

Energy in Capacitor

$$E_C = 1/2\, C v^2 = 1/2\, C V_0^2 \cos^2(\omega_n t)$$

Energy in inductor

$$I_L = (1/L)\int v(t)\,dt = (V_0/L)(\omega_n)^{-1} \sin(\omega_n t)$$

$$E_L = 1/2\, L I_L^2 = (V_0^2/2L\omega_n^2)\sin^2(\omega_n t) = 1/2\, C V_0^2 \sin^2(\omega_n t)$$

Quality factor of an Energy Storage Element

Total Energy stored: $E_T = E_C + E_L = 1/2\, C V_0^2$!

Energy Dissipated in 1 cycle

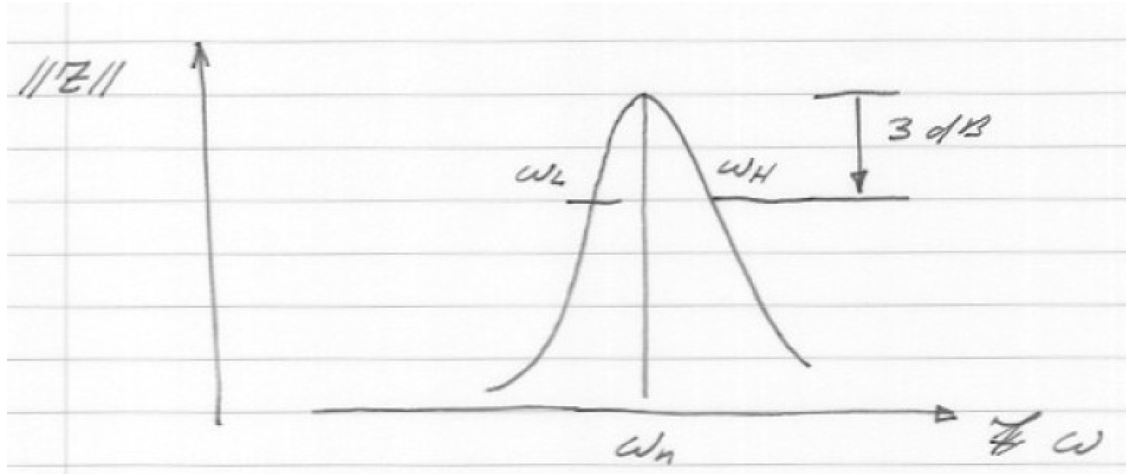
$$E = P \cdot T = (V_0^2/2R)\left(\frac{1}{f}\right) = (V_0^2/2R)\left(\frac{2\pi}{\omega_n}\right)$$

Q factor.

$$Q = \frac{2\pi\,(1/2\, C V_0^2)}{2\pi\,(1/2\, V_0^2/R\omega_n)} = CR\omega_n = R\sqrt{\frac{L}{C}} = \frac{1}{2\zeta}$$

$$\boxed{Q = \omega_n/2\alpha = 1/2\zeta}$$

-3 dB (Half-Power) Bandwidth



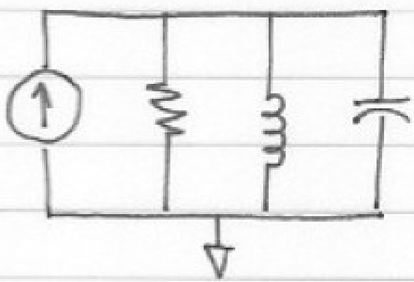
After some math:

$$\omega_L = \omega_n\left[\sqrt{1+\zeta^2} - \zeta\right]$$

$$\omega_H = \omega_n\left[\sqrt{1+\zeta^2} + \zeta\right]$$

3 dB bandwidth $= \omega_H - \omega_L = 2\zeta\omega_n = \omega_n / Q$

Impulse Response



$$i(t) = q_o \cdot \delta(t)$$

$$V(s)/I(s) = R\,\frac{s(2\zeta\omega_n)}{(s+\zeta\omega_n)^2+\omega_d^2}$$

but $I(s) = q_o$

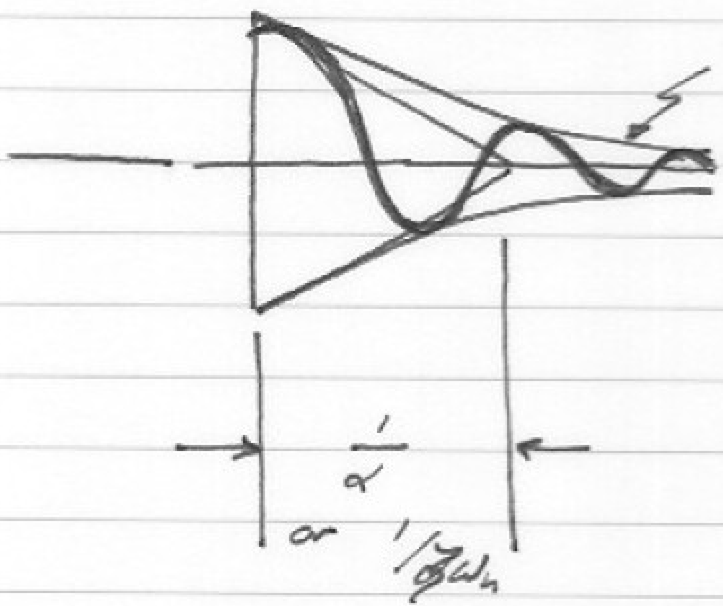
$$\Rightarrow V(s) = q_o R\,\frac{s(2\zeta\omega_n)}{(s+\zeta\omega_n)^2+\omega_d^2}$$

$$= q_o R 2\zeta\omega_n\left[\frac{s+\zeta\omega_n}{(s+\zeta\omega_n)^2+\omega_d^2}+\frac{(-\zeta\omega_n/\omega_d)\cdot\omega_d}{(s+\zeta\omega_n)^2+\omega_d^2}\right]$$

but $(\omega_n/\omega_d) = 1/\sqrt{1-\zeta^2}$

Impulse Response

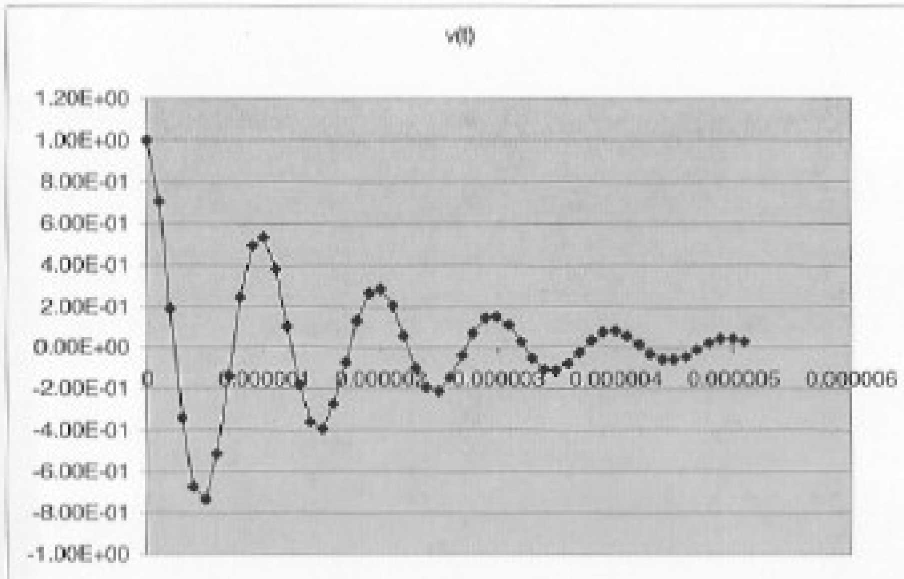
$$v(t) = q_o R(2\zeta\omega_n)\left[e^{-\zeta\omega_n t}\right]\left[\cos(\omega_d t) - \frac{\zeta}{\sqrt{1-\zeta^2}}\sin(\omega_d t)\right]$$



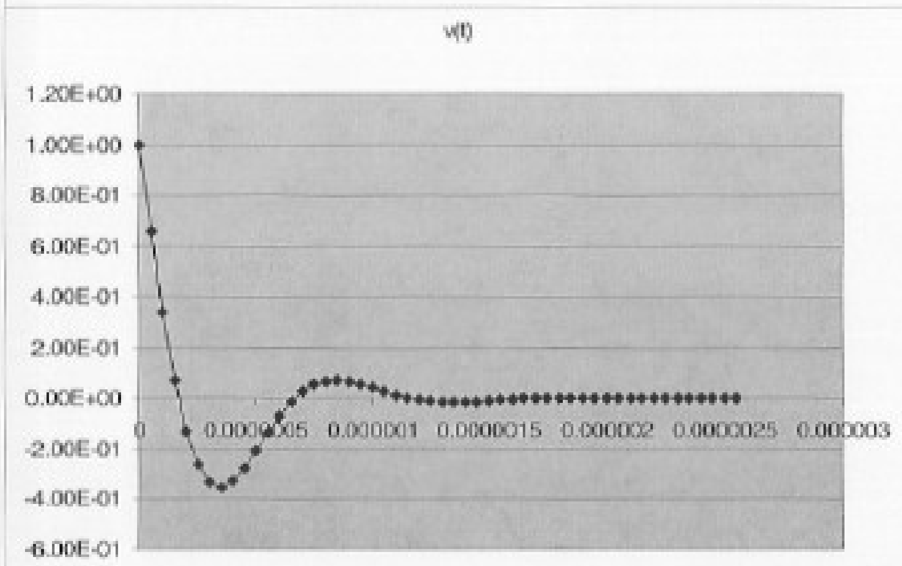
- rings at a frequency ω_d

- decays at a rate $\exp(-\alpha t)$

Impulse Response

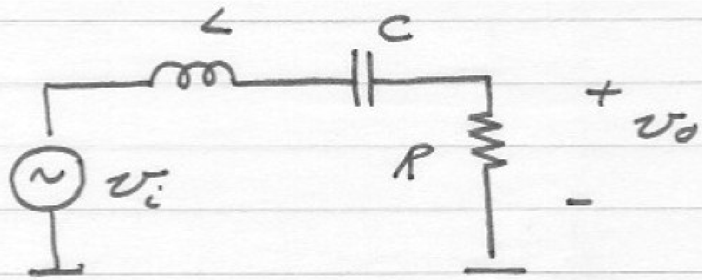


$\zeta = 0.1$



$\zeta = 0.5$

Series RLC Circuit



$$\frac{v_o}{v_i} = \frac{R}{R + sL + 1/sC} = \frac{sRC}{1 + sRC + s^2 LC}$$

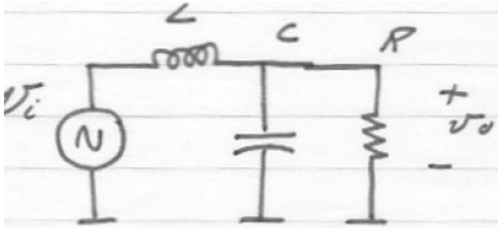
$$= \frac{s \cdot 2\zeta/\omega_n}{1 + s \cdot 2\zeta/\omega_n + s^2/\omega_n^2}$$

Canonical form.

$$\omega_n = 1/\sqrt{LC}$$

$$2\zeta/\omega_n = 2\zeta\left[\sqrt{LC}\right] = RC \Rightarrow \zeta = \frac{R}{2}\sqrt{\frac{C}{L}}$$

Another Resonant Circuit



$$v_o/r + v_o sC + v_o/sL - v_i/sL = 0$$

$$\Rightarrow \frac{v_o}{v_i} = \frac{1/sL}{1/R + sC + 1/sL}$$

$$= \frac{1}{1 + s(L/R) + s^2 LC}$$

$$= \frac{1}{1 + (2\zeta/\omega_n)s + s^2/\omega_n^2}$$

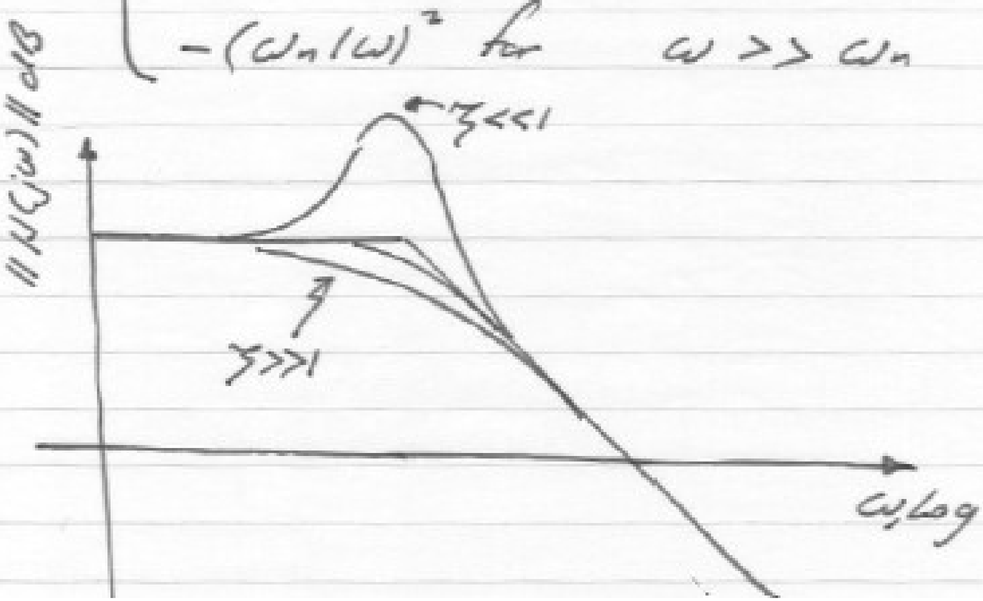
$$\omega_n = 1/\sqrt{LC}$$

$$\zeta = \frac{1}{2R}\sqrt{\frac{L}{C}}$$

Another Resonant Circuit

$$H(j\omega) = \frac{1}{1 + (2\zeta/\omega_n)\cdot j\omega - \omega^2/\omega_n^2}$$

$$= \begin{cases} 1 & \text{for} \quad \omega \ll \omega_n \\ 1/2\zeta & \text{for} \quad \omega = \omega_n \\ -(\omega_n/\omega)^2 & \text{for} \quad \omega \gg \omega_n \end{cases}$$



The case $\zeta \geq 1/\sqrt{2}$ gives no peaking

Peaking gets stronger for small ζ.

Impulse Response

$$\frac{v_o(s)}{v_i(s)} = \frac{1}{1 + (2\zeta/\omega_n)s + s^2/\omega_n^2}$$

so if $v_i(t) = V_0 \cdot \tau \cdot \delta(t) \rightarrow V_i(s) = V_0 \cdot \tau$

Hence: $$V_o(s) = \frac{V_0 \cdot \tau \omega_n^2}{\omega_n^2 + 2\zeta\omega_n s + s^2}$$

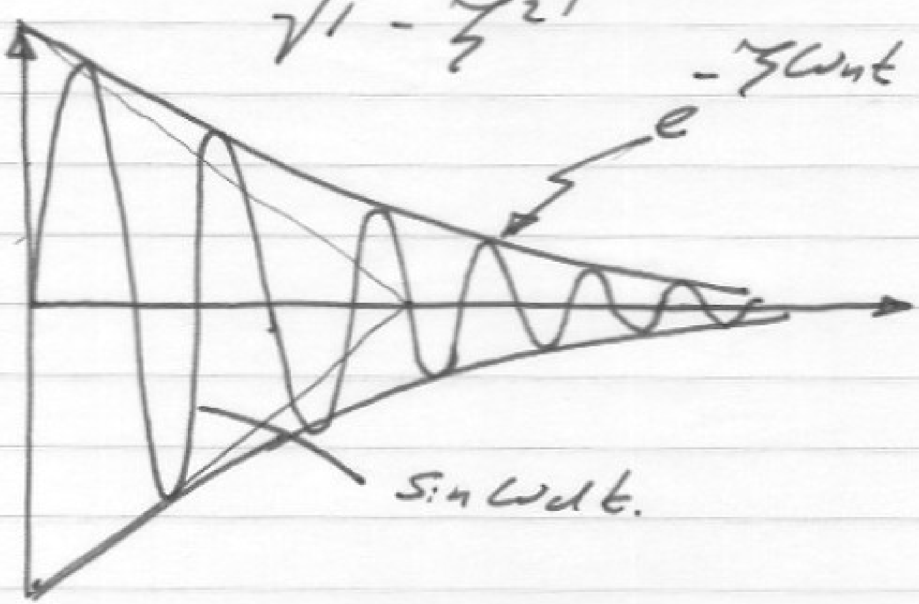
$$= \frac{V_0 \tau \omega_n^2}{(s + \zeta\omega_n)^2 + \omega_d^2}$$

Impulse Response

So: $v_{out}(t) = \dfrac{V_0 T}{\omega_d} \omega_n^2 \cdot e^{-\zeta \omega_n t} \cdot \sin(\omega_d \cdot t)$

but since: $\omega_d = \omega_n \cdot \sqrt{1-\zeta^2}$

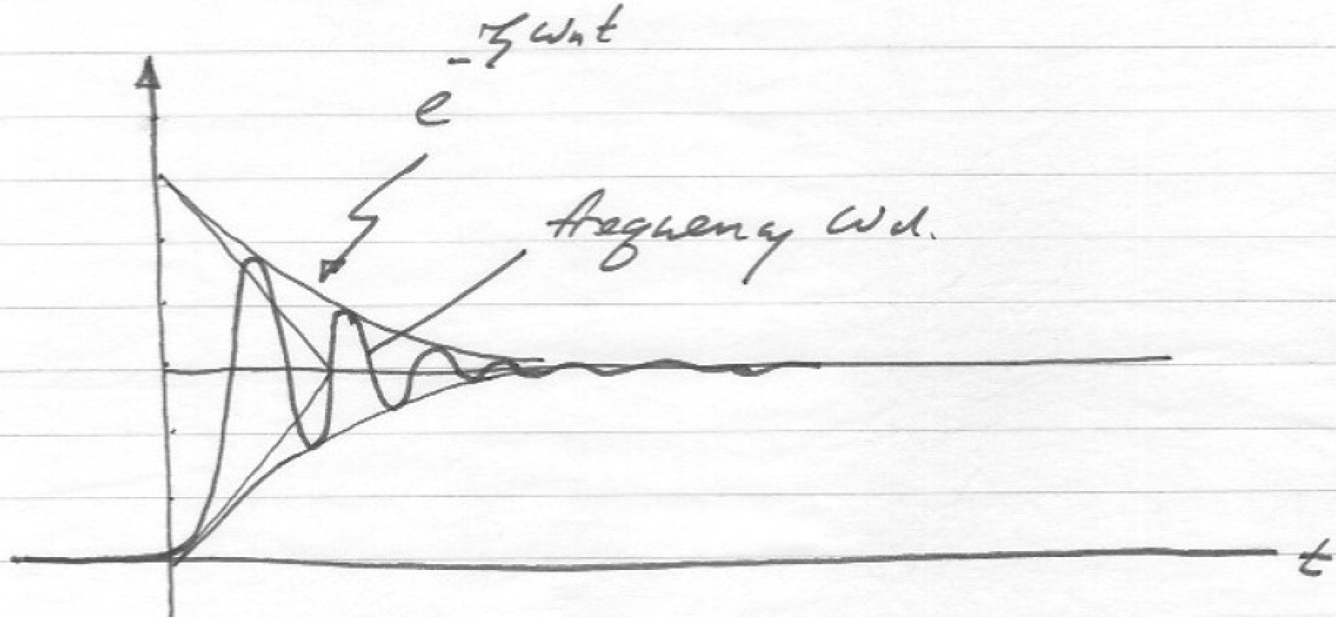
$\Rightarrow v_{out}(t) = \dfrac{V_0 \cdot T \cdot \omega_n}{\sqrt{1-\zeta^2}} e^{-\zeta \omega_n t} \sin(\omega_d t)$



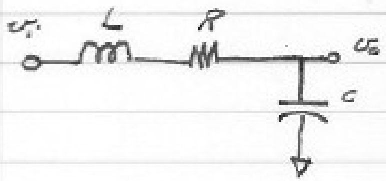
Recall Step Response: Not Derived Here Again

$$v_o(t) = V_h \cdot u(t)$$

$$\times \left[1 - e^{-\zeta \omega_n t} \left(\cos \omega_d t + \zeta \sqrt{1-\zeta^2} \sin \omega_d t \right) \right]$$

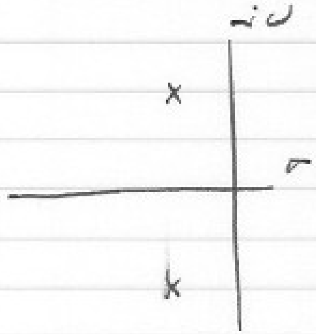


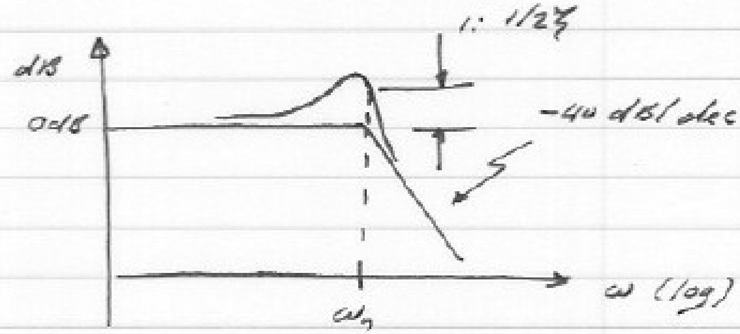
Low-Pass, 2nd-Order Filters



$$\frac{v_o}{v_i} = \frac{1/sC}{1/sC + R + sL} = \frac{1}{1 + sRC + s^2LC}$$

$$= \frac{1}{1 + s(2\zeta/\omega_n) + s^2/\omega_n^2}$$



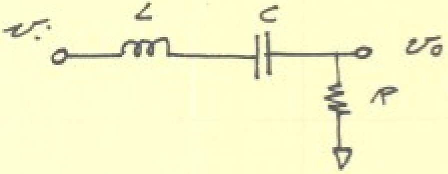


$$\frac{v_o}{v_i} = \begin{cases} 1 & \omega \ll \omega_n \\ 1/2\zeta & \omega = \omega_n \\ \omega_n^2/\omega^2 & \omega \gg \omega_n \end{cases}$$

Low Pass form

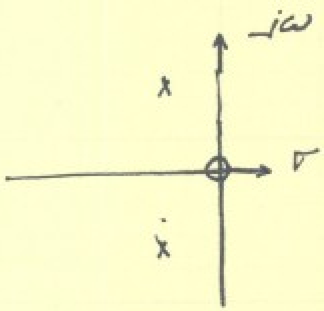
~~one~~ no zero, 2 complex poles

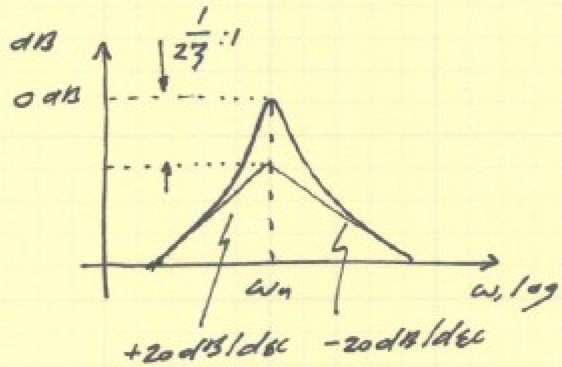
Band-Pass 2nd-Order Filters



$$\frac{v_o}{v_i}=\frac{R}{1/sC+R+sL}=\frac{sRC}{1+sRC+s^2LC}$$

$$=\frac{s(2\zeta/\omega_n)}{1+s(2\zeta/\omega_n)+s^2/\omega_n^2}$$



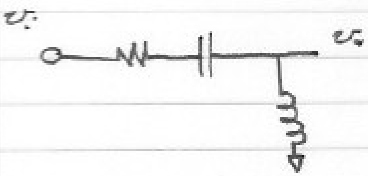


$$v_o/v_i=\begin{cases}2\zeta\cdot\omega/\omega_n & (+20\text{ dB/dec}) \text{ for } \omega\ll\omega_n\\ 1 & \text{for } \omega=\omega_n\\ 2\zeta\cdot\omega_n/\omega & (-20\text{ dB/dec}) \text{ for } \omega\gg\omega_n\end{cases}$$

band pass form of "resonator"

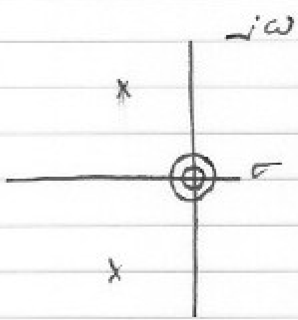
one zero, two complex poles.

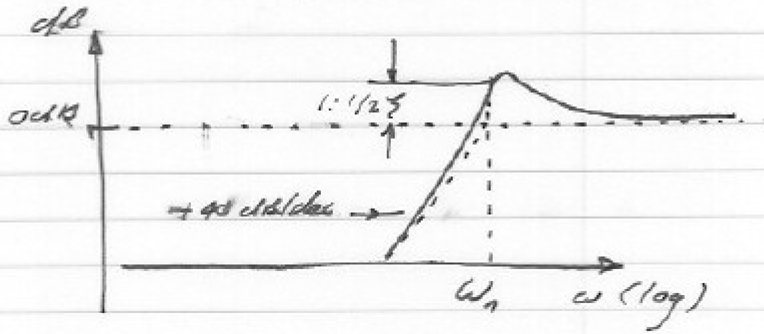
High-Pass 2nd-Order Filters



$$\frac{v_o}{v_i}=\frac{sL}{1/sC+R+sL}=\frac{s^2LC}{1+sRC+s^2LC}$$

$$=\frac{s^2/\omega_n^2}{1+s(2\zeta/\omega_n)+s^2/\omega_n^2}$$





$$v_o/v_i=\begin{cases}\omega^2/\omega_n^2 & (+40\ \text{dB/dec})\ \text{for}\ \omega\ll\omega_n\\ 1/2\zeta & \omega=\omega_n\\ 1 & \omega\gg\omega_n\end{cases}$$

High pass form

two zeros, 2 complex poles