ECE594I Notes set 3: Stat. Thero. : Recap / key points.

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References and Citations:

Sources / Citations :

Kittel and Kroemer : Thermal Physics

Van der Ziel : Noise in Solid - State Devices

Papoulis : Probability and Random Variables (hard, comprehensive)

Peyton Z. Peebles : Probability, Random Variables, Random Signal Principles (introductory)

Wozencraft & Jacobs : Principles of Communications Engineering.

Motchenbaker : Low Noise Electronic Design

Information theory lecture notes : Thomas Cover, Stanford, circa 1982

Probability lecture notes : Martin Hellman, Stanford, circa 1982

National Semiconductor Linear Applications Notes : Noise in circuits.

Suggested references for study.

Van der Ziel, Wozencraft & Jacobs, Peebles, Kittel and Kroemer

Papers by Fukui (device noise), Smith & Personik (optical receiver design)

National Semi. App. Notes (!)

Cover and Williams : Elements of Information Theory

Statistical Thermodynamics

Why discuss ?

Temperature

Boltzmann distribution

Material needed to derive thermal noise

carrier velocity distributions

similarly to information theory.

Thermal distributions

Consider a system of N particles.

Each particle at energy $E_0 = 0$ or energy $E_1 \neq 0$.

Suppose system has total energy $E_T = m_1 E_1$,

where m_1 is the # of particles in state E_1.

Total # of {ways, arrangemen ts, configurat ions} to have energy E_T is :

$$g(N, E_T = m_1 E_1) = \binom{N}{m_1} = \frac{N!}{m_1!(N - m_1)!}$$

We need to approximat e this...

Recall: Convergence of Binomial upon Gaussian

N Bernoulli trials :

$$P(k \text{ successes} \mid N \text{ trials}) = P(k \mid N) = \binom{N}{k} p^k q^{N-k}$$

If (Papoulis 1965, p. 66) $npq \gg 1$

and if $\mid k - np \mid \sim O\left(\sqrt{npq}\right)$ or less, then :

$$p_n(k) \cong \frac{1}{\sqrt{2\pi}\sqrt{npq}} \exp\left[-\frac{(k-np)^2}{2(npq)}\right]$$

This is a Gaussian of mean np and variance npq.

of System configurations.

How can we use this ?

Binomial distribution :

$$P(m \text{ successes} \mid N \text{ trials}) = P(m \mid N) = \binom{N}{m} p^m q^{N-m}$$

If $p = q = 1/2$, then $P(m \mid N) = \binom{N}{m} p^m q^{N-m} = \binom{N}{m} * (1/2)^N$

Hence $g(N, m) = \binom{N}{m} = P(m \mid N) \cdot 2^N$.

...i.e. we have related the multiplicity function

to the Binomial distribution function.

of System configurations.

$$g(N,m) = \binom{N}{m} = P(m \mid N) \cdot 2^N.$$

Now use Binomial $\rightarrow$ Gaussian limit

$$p_N(m) \cong \frac{1}{\sqrt{2\pi}\sqrt{Npq}} \exp\left[-\frac{(m-Np)^2}{2(Npq)}\right]$$

$$= \frac{1}{\sqrt{2\pi}\sqrt{N/4}} \exp\left[-\frac{(m-N/2)^2}{2(N/4)}\right]$$

$$\text{So}: \ g(N,m) = \frac{2^N}{\sqrt{2\pi}\sqrt{N/4}} \exp\left[-\frac{(m-N/2)^2}{2(N/4)}\right]$$

of System configurations.

$$g(N,m) \cong 2^N \frac{1}{\sqrt{2\pi} \cdot \sigma_M} \exp\left[-\frac{1}{2} \frac{(m - \overline{m})^2}{\sigma_M^2} \right]$$

where $\sigma_M = N^{1/2}/2$ and $\overline{m} = N/2$

g = Multiplicity function

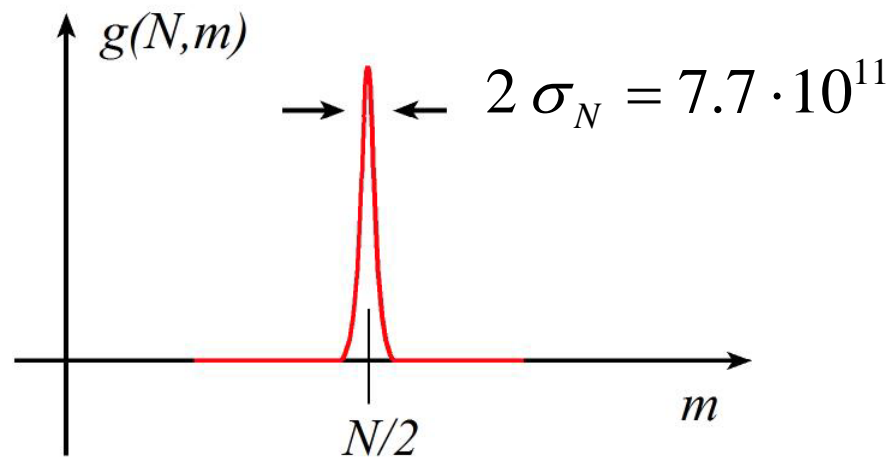= # ways system of N particles can have energy mE_1.

g is * not * a probability distribution.

g is approximately Gaussian with mean $N/2$ and variance $N/2$.

of System configurations.distributions

$$g(N,m) \cong 2^N \frac{1}{\sqrt{2\pi} \cdot \sigma_M} \exp\left[-\frac{1}{2} \frac{(m-\overline{m})^2}{\sigma_M^2}\right] \quad \text{where } \sigma_M = N^{1/2}/2 \text{ and } \overline{m} = N/2$$

Given a system with $6.02 \cdot 10^{22}$ particles, $\sigma_N = \sqrt{N}/2 = 3.9 \cdot 10^{11}$

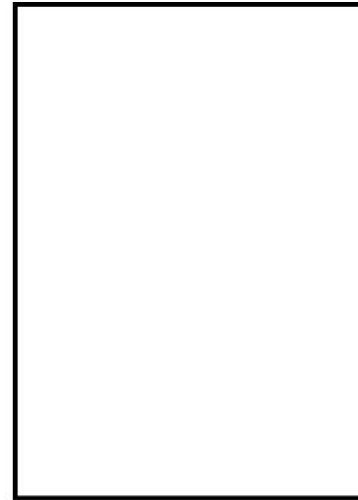$2\sigma_N = 7.7 \cdot 10^{11}$

Multiplicity function is * very * narrow

Coupled Systems

Now consider two such systems :

N_1 particles
m_1 in state $E=E_1$
(others in state $E=0$)
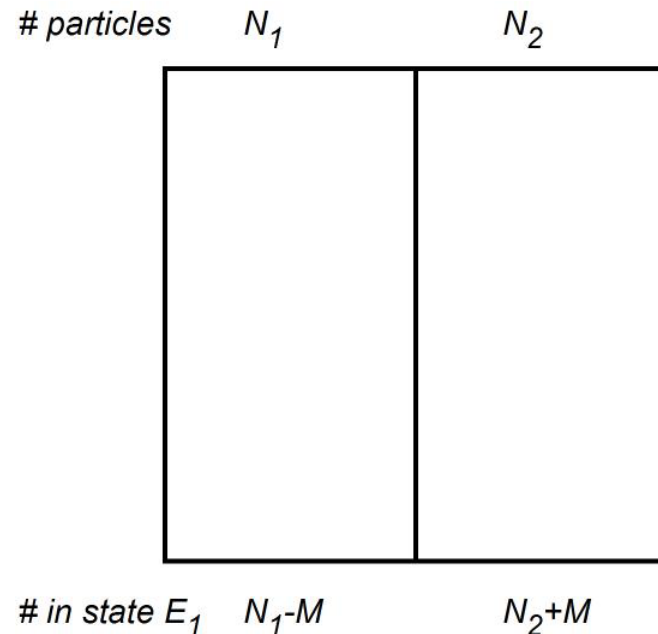$E_{T1}=m_1 E_1$

N_2 particles
m_2 in state $E=E_1$
(others in state $E=0$)
$E_{T2}=m_2 E_1$

If the 2 systems remain separated, then the multiplicity functions

(# of ways of having these Energies) are $g_1(N_1,m_1)$ and $g_2(N_2,m_2)$

Overall multiplicity function : $g_T(N_1,N_2,m_1,m_2) = g_1(N_1,m_1)\, g_2(N_2,m_2)$

Coupled Systems

Now bring the 2 systems in contact and let then exchange energy :

particles N_1 N_2

in state E_1 N_1-M N_2+M

If the energy exchange $1 \rightarrow 2$ is mE_1, then

$$g_T(N_1, N_2, m_1, m_2, m) = g_1(N_1, m_1 - m) \cdot g_2(N_2, m_2 + m)$$

Coupled Systems

$$g_T(N_1, N_2, m_1, m_2, m) = g_1(N_1, m_1 - m) \cdot g_2(N_2, m_2 + m)$$

but $g_1(N_1, m_1 - m) = $ Gaussian with mean $N_1/2$ and variance $N_1/4$

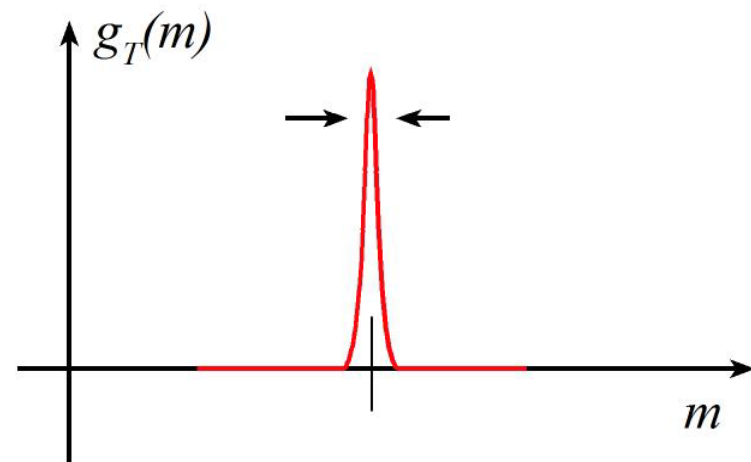and similarly for $g_2(N_2, m_2 + m)$, so :

$$g_T(N_1, N_2, m_1, m_2, m)$$

$$= \frac{2^{N_1 + N_2}}{2\pi \cdot \sqrt{N_1 N_2 / 16}} \exp\left[\frac{-2(m_1 - m - N_1/2)^2}{N_1} - \frac{2(m_2 + m - N_2/2)^2}{N_2} \right]$$

This is a Gaussian function of m with a variance of $\sigma_m^2 = \left(\dfrac{4}{N_1} + \dfrac{4}{N_2} \right)^{-1}$.

$\rightarrow$ peaked over a very narrow range, $\pm \sigma_m$, of m.

Becuase σ_M is so narrow, it is likely with probability $(1 - \varepsilon)$ that the final configuration of the coupled system is that which gives the largest g_T.

Entropy and Temperature

Define Entropy : $S \equiv k \cdot \ln(g_T)$ where $k = $ Boltzmann's constant

choice of k is arbitrary.

Maximizing $g_T(N_1, N_2, m_1, m_2, m) = g_1(N_1, m_1 - m) \cdot g_2(N_2, m_2 + m)$

correspond s to setting $\dfrac{\partial}{\partial m} \ln(g_1) = \dfrac{\partial}{\partial m} \ln(g_2)$.

Hence $\dfrac{\partial}{\partial m} S_1(N_1, m_1) = \dfrac{\partial}{\partial m} S_2(N_2, m_2)$.

Since $E = mE_1$, we have : $\dfrac{\partial}{\partial E} S_1(E) = \dfrac{\partial}{\partial E} S_2(E_2)$.

Define (temperatu re)$^{-1} = \dfrac{\partial}{\partial E} S(E)$, from which : $T_1 = T_2$

Entropy and Temperature: Summary

Bring two systems into contact, and they will exchange energy until

$T_1 = T_2$ "temperatur es are equal"

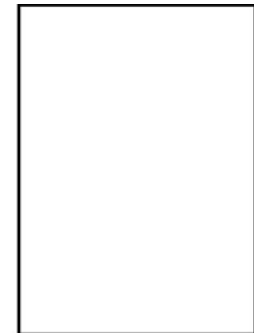where $\dfrac{1}{T} \equiv \dfrac{\partial}{\partial E} S(E)$ derivative of Entropy with Energy.

and : $S \equiv k \cdot \ln\big(g(N,E)\big)$ Entropy

and $g(N,E)$ Multiplicity

     # of arrangemen ts of a system of N particles having total energy E.

N_1 particles
m_1 in state $E=E_1$
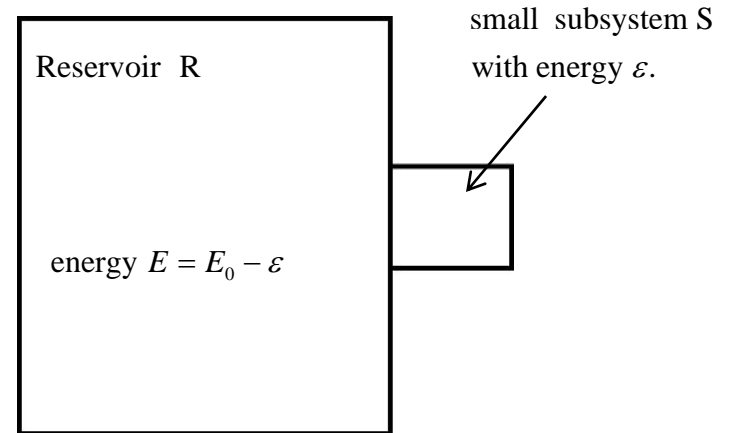(others in state $E=0$)
$E_{T1}=m_1E_1$

N_2 particles
m_2 in state $E=E_1$
(others in state $E=0$)
$E_{T2}=m_2E_1$

Boltzmann Law

Now examine the probability distribution
of energy of some particle which can
be in one of several states :

Reservoir R

small subsystem S
with energy ε.

energy $E = E_0 - \varepsilon$

We consider a large system R (reservoir)
in thermal contact with a small subsystem S.

What are the probabilities of various states of the subsystem ?

Boltzmann Law

We are specifying a particular state of the subsystem

$\rightarrow$ multiplicity (g) of this state is 1.

states of reservoir corresponding to S being in a state of energy E :

$g_R(E_0 - \varepsilon)$

If we assume that all reservoir states are equally probable :

$P(S$ in state of energy $\varepsilon) \propto g_R(E_0 - \varepsilon)$

So : $\dfrac{P(\varepsilon_1)}{P(\varepsilon_1)} = \dfrac{g_R(E_0 - \varepsilon_1)}{g_R(E_0 - \varepsilon_2)}$

$P(\varepsilon_i)$ is the probability of occupancy of a * state * of energy ε_i.

Boltzmann Law

But $k \ln(g) = S \Rightarrow g = \exp\{S/k\}$

$$\frac{P(\varepsilon_1)}{P(\varepsilon_1)} = \frac{g_R(E_0 - \varepsilon_1)}{g_R(E_0 - \varepsilon_2)} = \frac{\exp\{S_R(E_0 - \varepsilon_1)/k\}}{\exp\{S_R(E_0 - \varepsilon_2)/k\}} = \exp\left\{\frac{S_R(E_0 - \varepsilon_1) - S_R(E_0 - \varepsilon_2)}{k}\right\}$$

But $S_R(E_0 - \varepsilon_1) = S_R(E_0) - \varepsilon_1 \dfrac{\partial S_R(E_0)}{\partial E} - O(\varepsilon_1^2) - \ldots \quad = S_R(E_0) - \dfrac{\varepsilon_1}{T_R} - \ldots$

If the reservoir is big, the deriviatives are small, and 1st - order is enough :

$$\frac{P(\varepsilon_1)}{P(\varepsilon_2)} = \exp\left\{\frac{\varepsilon_2 - \varepsilon_1}{kT}\right\}$$

Important : this is the probabilty of a sub - system with 1 degree
of freedom ($g = 1$) being in a particular state, not the probability
distribution of Energy. These differ because states are not
uniformly distributed in energy.

Partition Function

(I don't recollect whether we need this for the present course)

If the system S has allowed states s, then

$$P(\varepsilon_i) = \frac{\exp(-\varepsilon_i / kT)}{Z}$$

where Z = partition function = $\displaystyle\sum_s \exp(-E(s)/kT)$
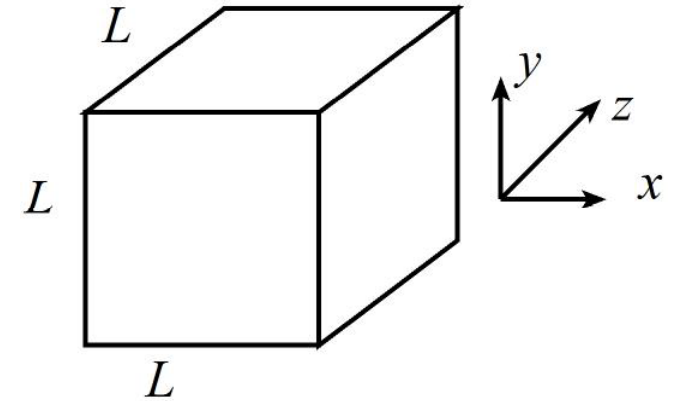
Boltzmann Velocity Distribution

Consider velocity distribution of a gas.

Single particle in a box :

Allowed kinetic energies :

$$E(n_x, n_y, n_z) = \frac{\hbar^2}{2m} \cdot \left(\frac{\pi}{L}\right)^2 \cdot \left(n_x^2 + n_y^2 + n_z^2\right).$$

$$(m/2)(v_x^2 + v_y^2 + v_z^2) = E$$

If the (particle + box) is in equilibrium with a reservoir of temperature T, then

$$\frac{P(\text{atom in state of energy } \varepsilon_2)}{P(\text{atom in state of energy } \varepsilon_1)} = \exp\left\{\frac{\varepsilon_1 - \varepsilon_2}{kT}\right\}$$

Note : available states are spaced in equal increments of (n_x, n_y, n_z),

hence in equal increments of velocity (v_x, v_y, z_z)*, and not in

equal increments in energy.

* Because $v = \hbar k / m = \hbar \pi n / Lm$

Boltzmann Velocity Distribution

$$P(n_x < N_x \leq n_x + \delta n_x) = \delta n_x \cdot C \cdot \exp\left\{\frac{-\hbar^2\pi^2 n_x^2}{2mL^2}\frac{1}{kT}\right\}$$

Change variables from N_x to v_x :

$$P(v_x < V_x \leq v_x + \delta v_x) = \delta v_x \cdot D \cdot \exp\left\{\frac{-mv_x^2}{2kT}\right\}$$

$$= \delta v_x \cdot \frac{1}{\sqrt{2\pi}}\sqrt{\frac{m}{kT}}\cdot \exp\left\{\frac{-mv_x^2}{2kT}\right\}$$

i.e., $f_{V_x}(v_x)$ is exponential in $E(v_x)/kT$

but $f_{E(V_x)}(E(v_x))$ is not exponential in E/kT

Boltzmann Velocity Distribution

The distribution function of v_x is :

$$f_{V_x}(v_x) = \delta v_x \cdot \frac{1}{\sqrt{2\pi}} \sqrt{\frac{m}{kT}} \cdot \exp\left\{\frac{-mv_x^2}{2kT}\right\}$$

This is a Gaussian with zero mean and variance $\sigma_{V_x}^2 = kT/m$.

$$E\left[V_x^2\right] = \sigma_{V_x}^2 = kT/m.$$

$$E\left[\frac{mV_x^2}{2}\right] = \frac{kT}{m} \cdot \frac{m}{2} = \frac{kT}{2}$$

$$E\left[x\text{ - component of kinetic energy}\right] = kT/2$$

Boltzmann Velocity Distribution

Now consider instead the distribution function of

the kinetic energy $E_x = (1/2)mv_x^2$

$$f_E(e) = \frac{f_{V_x}\left(v_x = \sqrt{2E/m}\right) + f_{V_x}\left(v_x = -\sqrt{2E/m}\right)}{2\sqrt{Em/2}}$$

$$= \frac{1}{\sqrt{2\pi}}\sqrt{\frac{m}{kT}}\sqrt{\frac{2}{Em}}\exp\left\{\frac{-m}{2kT}\frac{2E}{m}\right\}$$

$$= \frac{1}{\sqrt{\pi}}\frac{1}{\sqrt{kT}}\frac{1}{\sqrt{E}}\exp\left\{\frac{-E}{kT}\right\}$$

This is troubling, as it appears to be in conflict with the

Boltzmann law, but - no - wait :

Boltzmann Velocity Distribution

Where is the conflict ?

This physical example forces us to think hard about the definition
of a distribution function :

$f_X(x_1) \cdot \varepsilon$ = probability that the R.V. X is found
within ε of x_1.

ε in what units ?   In units of x of course.

Stat thermo give us the probabilities of a * state * being occupied,
and the states are * not * distributed uniformly in energy.

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