ECE594I Notes set 5:
More Math: Random Processes

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References and Citations:

Sources / Citations :

Kittel and Kroemer : Thermal Physics

Van der Ziel : Noise in Solid - State Devices

Papoulis : Probability and Random Variables (hard, comprehensive)

Peyton Z. Peebles : Probability, Random Variables, Random Signal Principles (introductory)

Wozencraft & Jacobs : Principles of Communications Engineering.

Motchenbaker : Low Noise Electronic Design

Information theory lecture notes : Thomas Cover, Stanford, circa 1982

Probability lecture notes : Martin Hellman, Stanford, circa 1982

National Semiconductor Linear Applications Notes : Noise in circuits.

Suggested references for study.

Van der Ziel, Wozencraft & Jacobs, Peebles, Kittel and Kroemer

Papers by Fukui (device noise), Smith & Personik (optical receiver design)

National Semi. App. Notes (!)

Cover and Williams : Elements of Information Theory

random processes

Random Processes

Draw a set of graphs, on separate sheets of paper,
of functions of voltage vs. time.
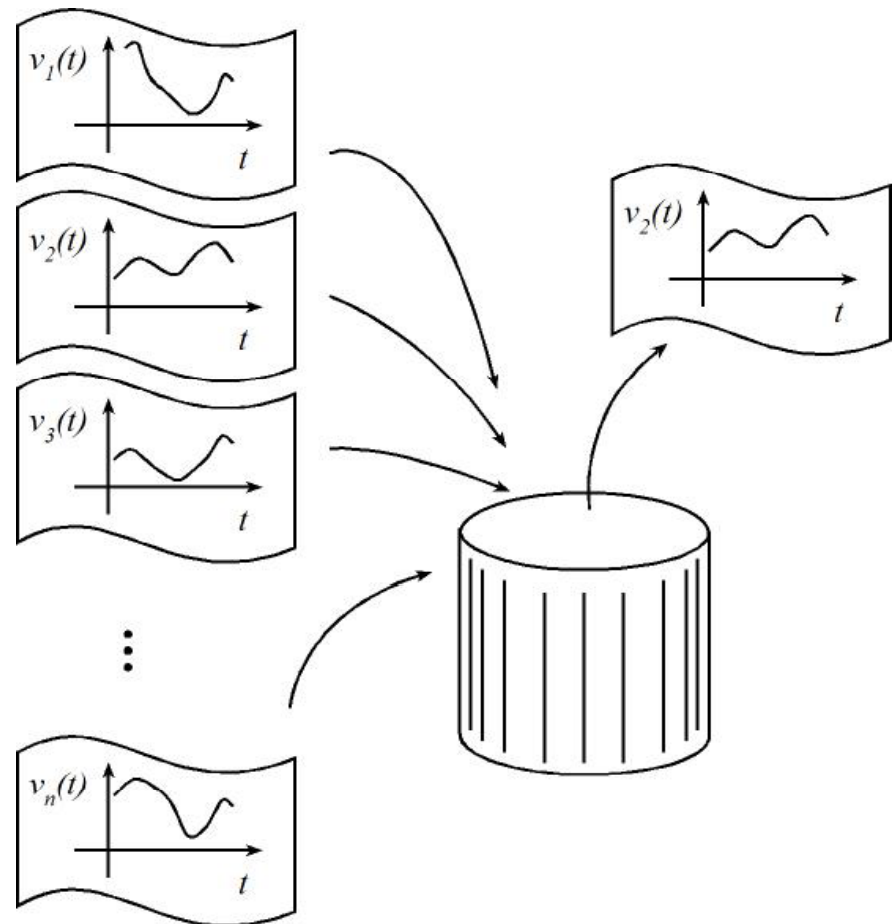
Put them into a garbage can.

This garbage can is called
the probabilit y sample space.

Pick out one sheet at random.
This is our random function of time.

The random process is $V(t)$.
The particular outcome is v(t)

Time Averages vs. Sample Space Averages

Recall the definion of the expectation of a function g(X) of a random variable X
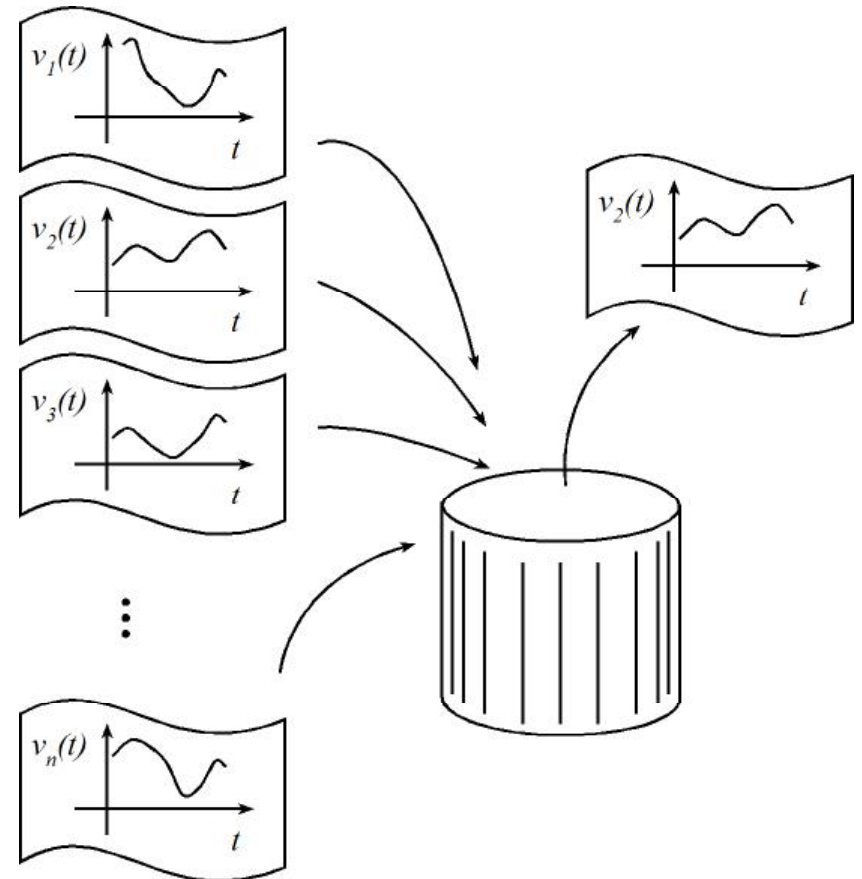
$$E[g(x)] = \int_{-\infty}^{+\infty} g(x) f_X(x) dx = \overline{g}$$

$\overline{g}$ is the $*$ average value $*$ of g, where the average is over the sample space.

With our random process definition,
we can define an average over the
 sample space at some particular time t_1 :

$$E[g(v(t_1))] = \int_{-\infty}^{+\infty} g(v(t_1)) f_V(v(t_1)) d(v(t_1))$$

We can also define an average of any
one outcome function over time :

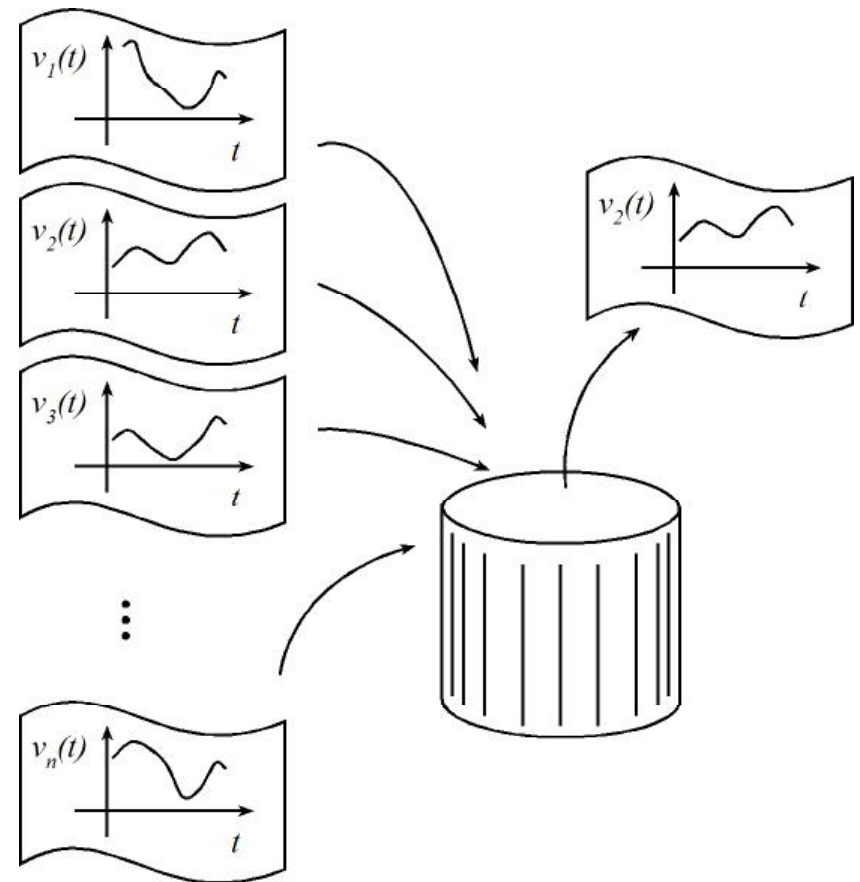$$A[g(v_i(t))] = \lim_{T \to \infty} \frac{1}{T} \int_{-T/2}^{+T/2} g(v_i(t)) dt$$

What is Random in a Random Process ?

We somehow have the expectation
that the signal " varies randomly with time".

Yet, in our definition, it is the selection
of the sample which is random.

Ergodicity addresses this discrepanc y.
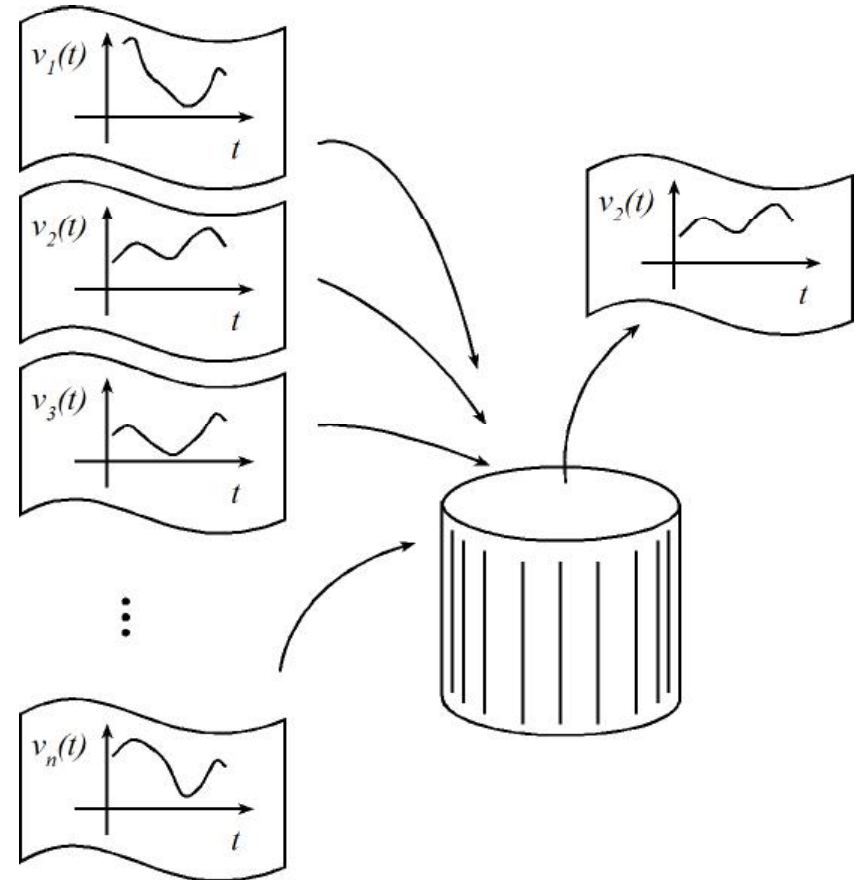
Its implications are suprising.

Ergodic Random Processes

An Ergodic random process has
averages over time equal to
averages over the statistical sample space

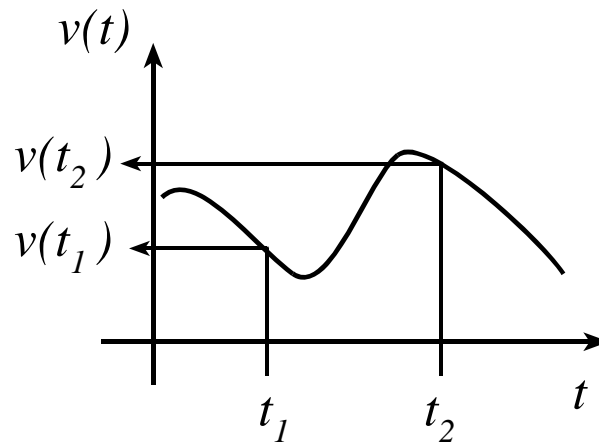$$E\big[g(v(t_1))\big] = A\big[g(v_i(t))\big] \text{ for all } i, \text{ all } t_1$$

In some sense, we have made
"random variation with time"
equivalent to
"random variation over the sample space"

Time Samples of Random Processes

With time samples at times t_1 and t_2

the random process $V(t)$ has values $V(t_1)$ and $V(t_2)$.

$V(t_1)$ and $V(t_2)$ have some joint probability distribution.

They might (or might not) be jointly Gaussian.

Random Waveforms are Random Vectors

Using Nyquist's sampling theorem,

if a random signal is bandlimited,

and if we pick regularly - spaced time samples $t_1...t_n$,

we convert our random process into a random vector.

We can thus analyze random signals using

vector analysis and geometry.
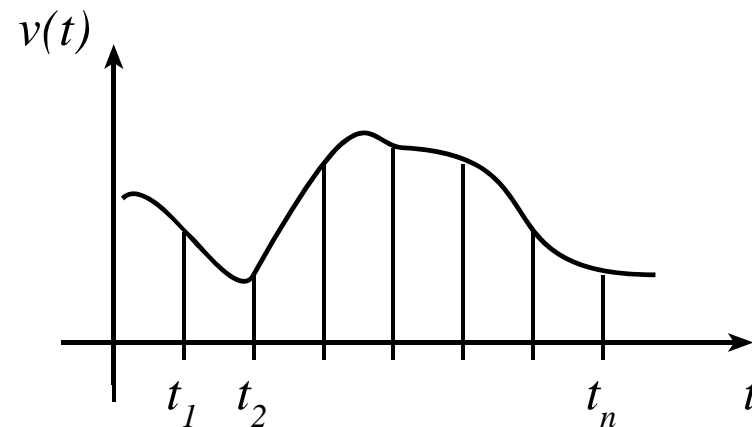
Stationary Random Processes

The statistics of a stationary process do not vary with time.
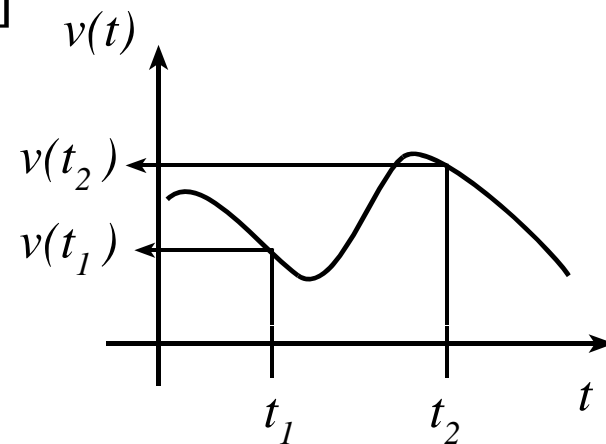
N^{th} − order stationarity :

$$E\big[f\big(V(t_1),V(t_2),...,V(t_n)\big)\big]= E\big[f\big(V(t_1+\tau),V(t_2+\tau),...,V(t_n+\tau)\big)\big]$$

..and lower orders

2nd − order stationarity :

$$E\big[f\big(V(t_1),V(t_2)\big)\big]= E\big[f\big(V(t_1+\tau),V(t_2+\tau)\big)\big]$$

lower orders $\rightarrow E\big[f\big(V(t_1)\big)\big]= E\big[f\big(V(t_1+\tau)\big)\big]$

Restrictions on the random processes we consider

We will make following restrictions to make analysis tractable:

The process will be **Ergodic**.

The process will be **stationary** to any order: all statistical properties are independent of time. Many common processes are not stationary, including integrated white noise and 1/f noise.

The process will be **Jointly Gaussian**. This means that if the values of a random process $X(t)$ are sampled at times t_1, t_2, etc, to form random variables $X_1=X(t_1)$, etc, then X_1, X_2, etc. are a jointly Gaussian random variable.

In nature, many random processes result from the sum of a vast number of small underlying random processses. From the central limit theorem, such processes can frequently be expected to be Jointly Gaussian.

Variation of a random process with time

For the random process X(t), look at $X_1=X(t_1)$ and $X_2=X(t_2)$.

$$R_{X_1X_2} = E[X_1X_2] = \int\limits_{-\infty}^{+\infty}\int\limits_{-\infty}^{+\infty} x_1x_2 \bullet f_{X_1X_2}(x_1,x_2)dx_1dx_2$$

To compute this we need to know the joint probability distribution. We have assumed a Gaussian process. The above is called the Autocorrellation function. IF the process is stationary, it is a function only of $(t_1-t_2)=$tau, and hence

$$R_{XX}(\tau) = E[X(t)X(t+\tau)]$$

this is the autocorrellation function. It describes how rapidly a random voltage varies with time….

PLEASE recall we are assuming zero-mean random processes (DC bias subtracted). Thus the autocorrelation and the auto-covariance are the same

Variation of a random process with time

Note that $R_{XX}(0) = E[X(t)X(t)] = \sigma_X^2$ gives the variance of the random process.

The autocorrelation function gives us variance of the random process and the correlation between its values for two moments in time. If the process is Gaussian, this is enough to completely describe the process.

Narrow autocorrelation:
Fast variation
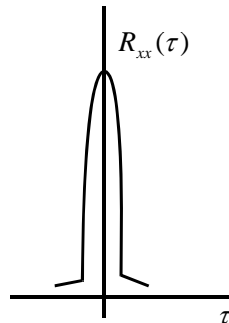
Broad autocorrelation
Slow variation
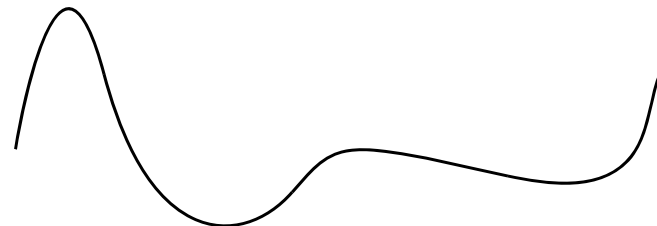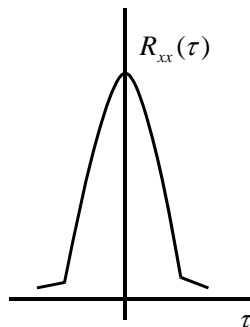
Autocorrelation is an Estimate of the Variation with Time

If random variables X and Y are **Jointly Gaussian**, and have **zero mean**, then knowledge of the value y of the outome of Y results in a best estimate of X as follows:

$$E[X|Y=y] = \langle X|Y=y \rangle = \frac{R_{XY}}{\sigma_Y^2} y$$

"The expected value of the random variable X, given that the random variable Y has value y is ..."

Hence, the autocorrellation function tells us the degree to which the signal at time t is related to the signal at time $t + \tau$

A narrow autocorrelation is indicative of a quickly-varying random process

Power spectral densities

The autocorrellation function describes how a random process evolves with time.
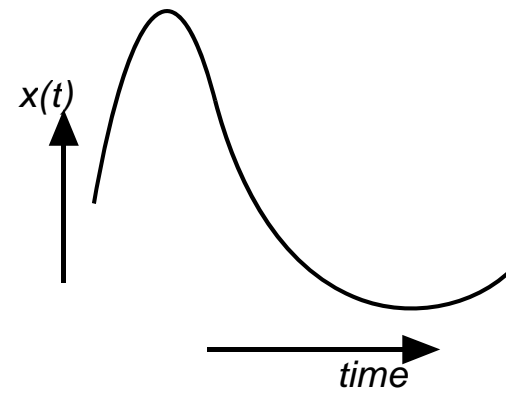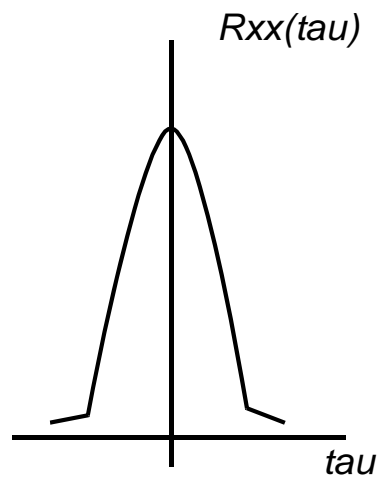
Find its Fourier transform:

$$S_{XX}(\omega) = \int\limits_{-\infty}^{+\infty} R_{XX}(\tau)\exp(-j\omega\tau)d\tau$$

This is called the power spectral density of the signal.

Remembering the usual Fourier transform relationships, if the power spectrum is broad, the autocorrellation function is narrow, and the signal varies rapidly--it has content at high frequencies, and the voltages of any two points are strongly related only if the two points are close together in time.

If the power spectrum is narrow, the autocorrellation function is broad, and the signal varies slowly--it has content only at low frequencies and the voltages of any two points are strongly related unless if the two points are broadly separated in time.

Power spectral densities
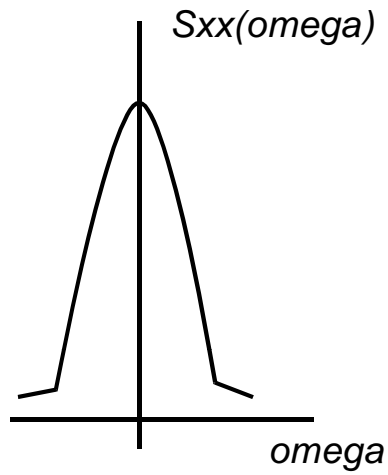
Power Spectral Densities

Recall that the power spectral density is
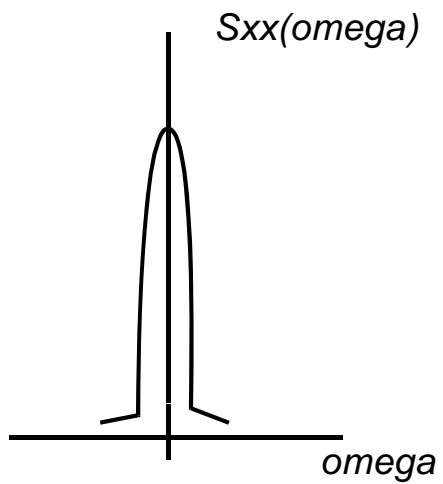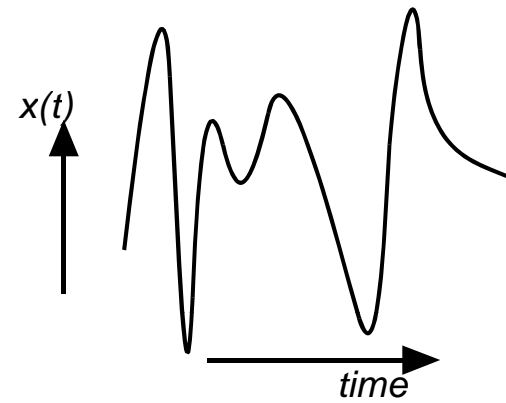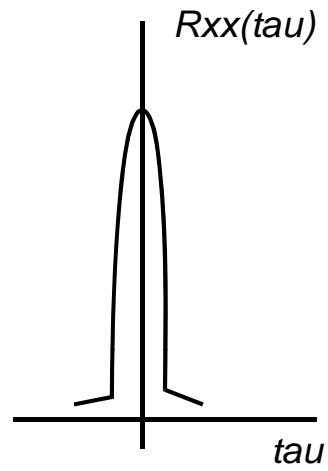
the Fourier transform of the autocorrelation function

$$S_{XX}(\omega) = \int_{-\infty}^{+\infty} R_{XX}(\tau)\exp(-j\omega\tau)d\tau$$

The inverse transform holds, so that

$$R_{XX}(\tau) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} S_{XX}(\omega)\exp(j\omega\tau)d\omega$$
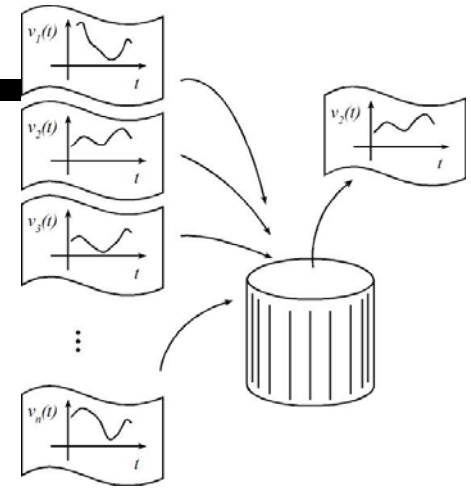
specifically,

$$R_{XX}(0) = \sigma_X^2 = \frac{1}{2\pi}\int_{-\infty}^{+\infty} S_{XX}(\omega)d\omega$$

So, if σ_X^2 is called the power in the process, then integrating

the power spectral density will give us the power.

This is the justification for the term, "power spectral density"

Back To Ergodicity (1)

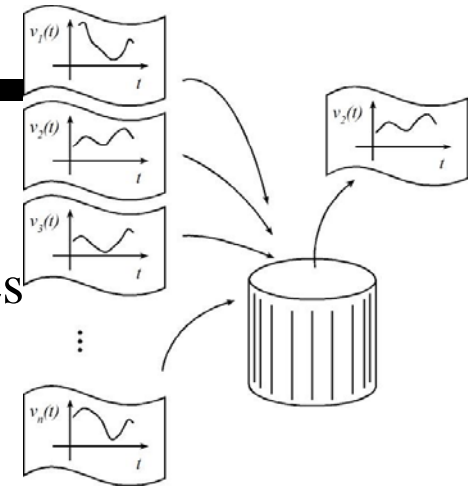Statistical autocorrelation (average over sample space) :

$$R_{XX}(\tau) = E[X(t)X(t+\tau)].$$

Time autocorrelation function (note subscripts) of a particular outcome $x_i(t)$ of the random process $X(t)$.

$$R_{x_i x_i}(\tau) = A[x_i(t)x_i(t+\tau)] = \lim_{T \to \infty} \frac{1}{T} \int_{-T/2}^{T/2} x_i(t)x_i(t+\tau)dt$$

If $X(t)$ is Ergodic, then $R_{x_i x_i}(\tau) = R_{XX}(\tau)$ for * all * outcomes $x_i(t)$.

Back To Ergodicity (2)

If $X(t)$ is Ergodic, then $R_{x_i x_i}(\tau) = R_{XX}(\tau)$ for *all* outcomes

$$R_{x_i x_i}(\tau) = A[x_i(t)x_i(t+\tau)] = R_{XX}(\tau)$$

But $R_{x_i x_i}(\tau) = \mathcal{F}^{-1}\{S_{x_i x_i}(j\omega)\} = \mathcal{F}^{-1}\{X_i(j\omega)X_i^*(j\omega)\}$

where (notation problems) $X_i(j\omega)$ is the Fourier transform of $x_i(t)$.

So, every outcome $x_i(t)$ has the same Fourier Magnitude $\|X(j\omega)\|$.

Each outcome $x_i(t)$ likely has a different Fourier phase.

Correlated Random Processes

Two processes can be statistically related.

Consider two random processes X(t) and Y(t).

Define the cross - correllation function of the processes

$$R_{XY}(\tau) = E[X(t)Y(t+\tau)]$$

They will have a cross - spectral density as follows :

$$S_{XY}(\omega) = \int_{-\infty}^{+\infty} R_{XY}(\tau)\exp(-j\omega\tau)d\tau$$

and therefore $R_{XY}(\tau) = \dfrac{1}{2\pi}\displaystyle\int_{-\infty}^{+\infty} S_{XY}(\omega)\exp(j\omega\tau)d\omega$

Single-Sided Hz-based Spectral Densities

Double - Sided Spectral Densities

$$R_{XX}(\tau) = E[X(t)X(t+\tau)] = \frac{1}{2\pi} \int_{-\infty}^{+\infty} S_{XX}(j\omega)\exp(j\omega\tau)d\omega$$

$$S_{XX}(j\omega) = \int_{-\infty}^{+\infty} R_{XX}(\tau)\exp(-j\omega\tau)d\tau$$

Single - Sided Hz - based Spectral Densities

$$R_{XX}(\tau) = E[X(t)X(t+\tau)] = \frac{1}{2} \int_{-\infty}^{+\infty} \tilde{S}_{XX}(jf)\exp(j2\pi f\tau)df$$

$$\tilde{S}_{XX}(jf) = 2\int_{-\infty}^{+\infty} R_{XX}(\tau)\exp(-j2\pi f\tau)d\tau$$

Single-Sided Hz-based Spectral Densities- Why ?

Why this notation ?

The signal power in the bandwidth $\{f_{\text{low}}, f_{\text{high}}\}$

$$\text{Power} = \frac{1}{2} \int_{-f_{\text{high}}}^{-f_{\text{low}}} \tilde{S}_{XX}(jf)\,df + \frac{1}{2} \int_{f_{\text{low}}}^{f_{\text{high}}} \tilde{S}_{XX}(jf)\,df = \int_{f_{\text{low}}}^{f_{\text{high}}} \tilde{S}_{XX}(jf)\,df$$

$\rightarrow \tilde{S}_{XX}(jf)$ is directly the Watts of signal power

per Hz of signal bandwidth

at frequencie s lying close to the frequency f.

Single-Sided Hz-based Cross Spectral Densities

Double - Sided Cross Spectral Densities

$$R_{XY}(\tau) = E[X(t)Y(t+\tau)] = \frac{1}{2\pi}\int_{-\infty}^{+\infty} S_{XY}(j\omega)\exp(j\omega\tau)d\omega$$

$$S_{XY}(j\omega) = \int_{-\infty}^{+\infty} R_{XY}(\tau)\exp(-j\omega\tau)d\tau$$

Single - Sided Hz - based Cross Spectral Densities

$$R_{XY}(\tau) = E[X(t)Y(t+\tau)] = \frac{1}{2}\int_{-\infty}^{+\infty} \tilde{S}_{XY}(jf)\exp(j2\pi f\tau)df$$

$$\tilde{S}_{XY}(jf) = 2\int_{-\infty}^{+\infty} R_{XY}(\tau)\exp(-j2\pi f\tau)d\tau$$

$\tilde{S}_{XY}(jf)$ is also often written as $\dfrac{d}{df}\langle XY \rangle$
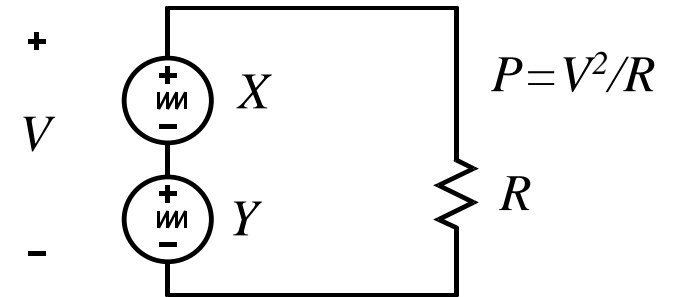
Example: Cross Spectral Densities

$$V(t) = X(t) + Y(t)$$

$$R_{VV}(\tau) = E\big[(X(t) + Y(t))(X(t+\tau) + Y(t+\tau))\big]$$
$$= R_{XX}(\tau) + R_{YY}(\tau) + R_{XY}(\tau) + R_{YX}(\tau)$$

$P = V^2/R$

R

$$S_{VV}(j\omega) = S_{XX}(j\omega) + S_{YY}(j\omega) + S_{XY}(j\omega) + S_{XY}^{*}(j\omega)$$
$$= S_{XX}(j\omega) + S_{YY}(j\omega) + 2 \cdot \mathrm{Re}\{S_{XY}(j\omega)\}$$

Or, in single - sided spectral densities

$$\tilde{S}_{VV}(jf) = \tilde{S}_{XX}(jf) + \tilde{S}_{YY}(jf) + 2 \cdot \mathrm{Re}\{\tilde{S}_{XY}(jf)\}$$
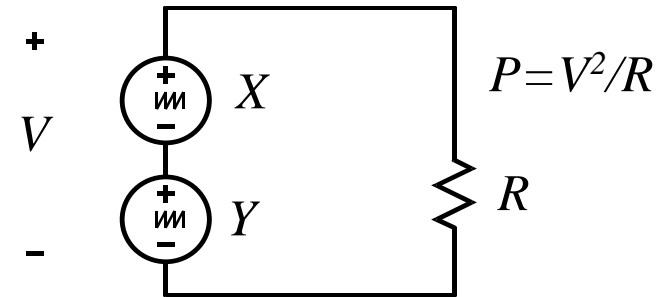
Example: Cross Spectral Densities

The Power $P = V^2(t)/R$ has expected value

$$E[V(t)V(t)/R] = R_{VV}(0)/R$$

And in the bandwidth between f_{low} and f_{high},

$$\overline{P} = \int_{f_{low}}^{f_{high}} \tilde{S}_{VV}(jf)df \ldots$$

Integrating with respect to frequency (over whatever bandwidth is relevant) gives the total (expected) power dissipated in R.

Note that the cross-spectral density is relevant.

$+$

V

X

Y

$-$

$P=V^2/R$

R

Our Notation for Spectral Densities and Correlations

	Random Process	Outcome
function of time	$V(t)$	$v(t)$
function of frequency	$V(jf), V(j\omega)$	$v(jf), v(j\omega)$
autocorrelation function	$R_{VV}(\tau) = E[V(t)V(t+\tau)]$	$R_{vv}(\tau) = A[v(t)v(t+\tau)]$

power spectral density

$$\begin{cases} S_{VV}(j\omega) = \mathcal{F}[R_{VV}(\tau)] \\ \tilde{S}_{VV}(jf) = 2S_{VV}(j\omega/2\pi) \end{cases}$$

$$\begin{cases} S_{vv}(j\omega) = \mathcal{F}[R_{vv}(\tau)] \\ S_{vv}(j\omega) = v(j\omega)v^*(j\omega) \\ \tilde{S}_{vv}(jf) = 2S_{vv}(j\omega/2\pi) \end{cases}$$

crosscorrelation function $R_{XY}(\tau) = E[X(t)Y(t+\tau)]$ $R_{xy}(\tau) = A[v(t)y(t+\tau)]$

cross spectral density

$$\begin{cases} S_{XY}(j\omega) = \mathcal{F}[R_{XY}(\tau)] \\ \tilde{S}_{XY}(jf) = 2S_{XY}(j\omega/2\pi) \end{cases}$$

$$\begin{cases} S_{xy}(j\omega) = \mathcal{F}[R_{xy}(\tau)] \\ S_{xy}(j\omega) = x(j\omega)y^*(j\omega) \\ \tilde{S}_{xy}(jf) = 2S_{xy}(j\omega/2\pi) \end{cases}$$

When context makes it clear whether $v = v(t)$ or $v = v(j\omega)$, we can simply write v.

For stationary ergodic processes

$$S_{VV}(j\omega) = S_{vv}(j\omega) = v(j\omega)v^*(j\omega) \text{ and } S_{XY}(j\omega) = S_{xy}(j\omega) = x(j\omega)y^*(j\omega)$$

Example: Noise passing through filters & linear electrical networks

If the filter has impulse response $h(t)$ and transfer function $h(j\omega)$,

then for any $v_{in}(t) \rightarrow v_{out}(t)$, $v_{out}(j\omega) = h(j\omega)v_{in}(j\omega)$

So

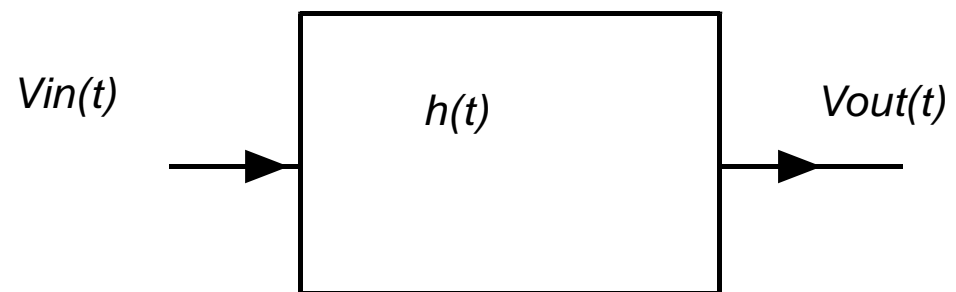$$v_{out}(j\omega)v_{out}^*(j\omega) = h(j\omega)v_{in}(j\omega)h^*(j\omega)v_{in}^*(j\omega)$$

$$S_{v_{out}v_{out}}(j\omega) = \|h(j\omega)\|^2 S_{v_{in}v_{in}}(j\omega)$$

$$S_{V_{out}V_{out}}(j\omega) = \|h(j\omega)\|^2 S_{V_{in}V_{in}}(j\omega)$$

$$v_{out}(j\omega)v_{in}^*(j\omega) = h(j\omega)v_{in}(j\omega)v_{in}^*(j\omega)$$

$$S_{v_{out}v_{in}}(j\omega) = h(j\omega)S_{v_{in}v_{in}}(j\omega)$$

$$S_{V_{out}V_{in}}(j\omega) = h(j\omega)S_{V_{in}V_{in}}(j\omega)$$

Vin(t) | h(t) | Vout(t)

It is trivial to change to single - sided Hz - based spectral densities.