ECE594I Notes set 6: Thermal Noise

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References and Citations:

Sources / Citations :

Kittel and Kroemer : Thermal Physics

Van der Ziel : Noise in Solid - State Devices

Papoulis : Probability and Random Variables (hard, comprehensive)

Peyton Z. Peebles : Probability, Random Variables, Random Signal Principles (introductory)

Wozencraft & Jacobs : Principles of Communications Engineering.

Motchenbaker : Low Noise Electronic Design

Information theory lecture notes : Thomas Cover, Stanford, circa 1982

Probability lecture notes : Martin Hellman, Stanford, circa 1982

National Semiconductor Linear Applications Notes : Noise in circuits.

Suggested references for study.

Van der Ziel, Wozencraft & Jacobs, Peebles, Kittel and Kroemer

Papers by Fukui (device noise), Smith & Personik (optical receiver design)

National Semi. App. Notes (!)

Cover and Williams : Elements of Information Theory

Boltzmann Law

But $k\ln(g)=S \Rightarrow \quad g=\exp\{S/k\}$

$$\frac{P(\varepsilon_1)}{P(\varepsilon_1)}=\frac{g_R(E_0-\varepsilon_1)}{g_R(E_0-\varepsilon_2)}=\frac{\exp\{S_R(E_0-\varepsilon_1)/k\}}{\exp\{S_R(E_0-\varepsilon_2)/k\}}=\exp\left\{\frac{S_R(E_0-\varepsilon_1)-S_R(E_0-\varepsilon_2)}{k}\right\}$$

But $S_R(E_0-\varepsilon_1)=S_R(E_0)-\varepsilon_1\dfrac{\partial S_R(E_0)}{\partial E}-O(\varepsilon_1^2)-\ldots \quad =S_R(E_0)-\dfrac{\varepsilon_1}{T_R}-\ldots$

If the reservoir is big, the deriviatives are small, and 1st - order is enough :

$$\frac{P(\varepsilon_1)}{P(\varepsilon_2)}=\exp\left\{\frac{\varepsilon_2-\varepsilon_1}{kT}\right\}$$

Important : this is the probabilty of a sub - system with 1 degree of freedom ($g=1$) being in a particular state, not the probability distribution of Energy. These differ because states are not uniformly distributed in energy.

Partition Function

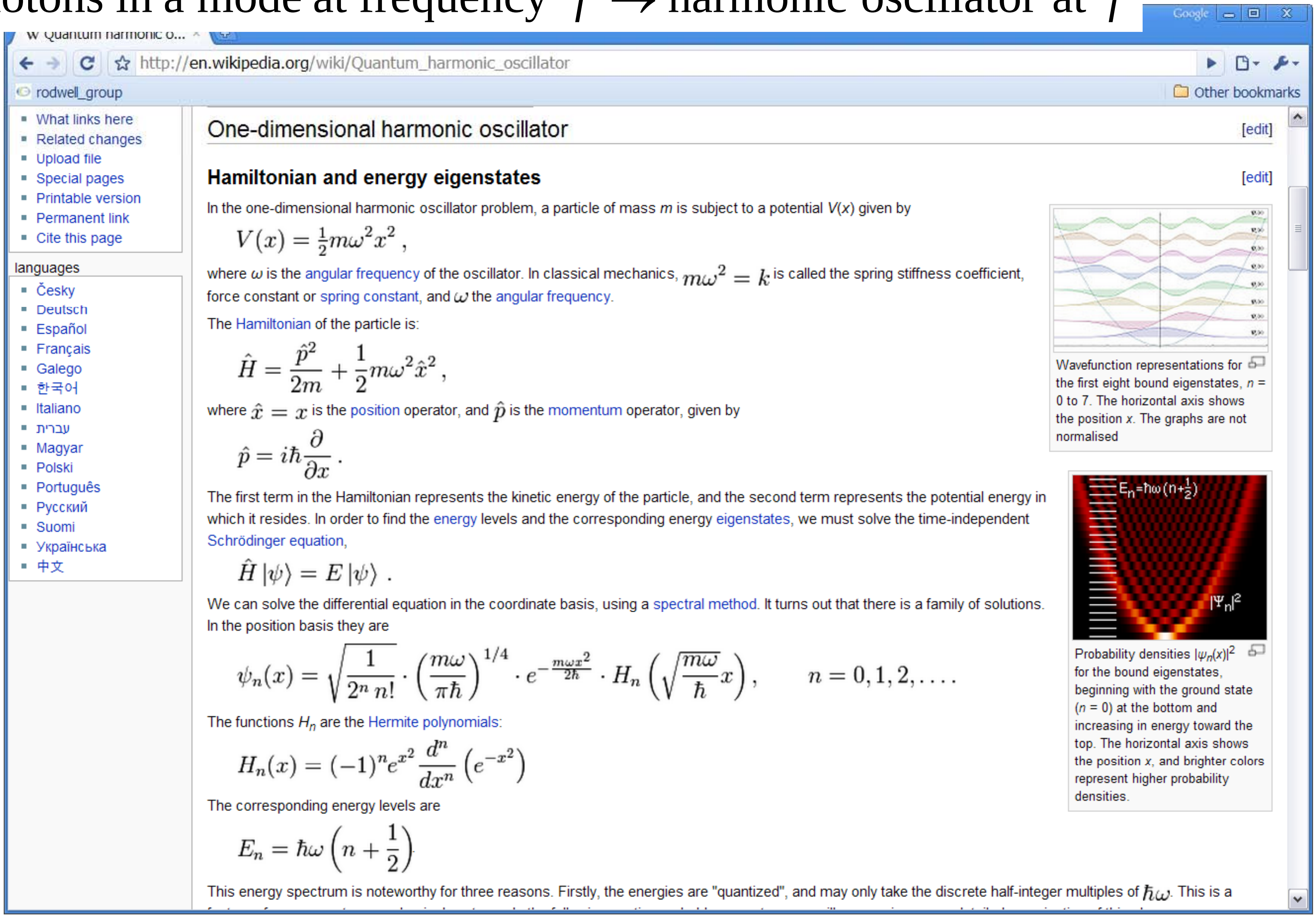
If the system S has allowed states s, then

$$P(\varepsilon_i) = \frac{\exp(-\varepsilon_i / kT)}{Z}$$

where Z = partition function $= \sum_s \exp(-E(s)/kT)$

Background: Harmonic Oscillator

Photons in a mode at frequency $f \rightarrow$ harmonic oscillator at f



One-dimensional harmonic oscillator [edit]

Hamiltonian and energy eigenstates [edit]

In the one-dimensional harmonic oscillator problem, a particle of mass m is subject to a potential $V(x)$ given by

$$V(x) = \tfrac{1}{2} m\omega^2 x^2 \,,$$

where ω is the angular frequency of the oscillator. In classical mechanics, $m\omega^2 = k$ is called the spring stiffness coefficient, force constant or spring constant, and ω the angular frequency.

The Hamiltonian of the particle is:

$$\hat{H} = \frac{\hat{p}^2}{2m} + \frac{1}{2} m\omega^2 \hat{x}^2 \,,$$

where $\hat{x} = x$ is the position operator, and $\hat{p}$ is the momentum operator, given by

$$\hat{p} = i\hbar \frac{\partial}{\partial x} \,.$$

The first term in the Hamiltonian represents the kinetic energy of the particle, and the second term represents the potential energy in which it resides. In order to find the energy levels and the corresponding energy eigenstates, we must solve the time-independent Schrödinger equation,

$$\hat{H} \left| \psi \right\rangle = E \left| \psi \right\rangle \,.$$

We can solve the differential equation in the coordinate basis, using a spectral method. It turns out that there is a family of solutions. In the position basis they are

$$\psi_n(x) = \sqrt{\frac{1}{2^n\, n!}} \cdot \left(\frac{m\omega}{\pi \hbar}\right)^{1/4} \cdot e^{-\frac{m\omega x^2}{2\hbar}} \cdot H_n\left(\sqrt{\frac{m\omega}{\hbar}}\, x\right), \qquad n = 0, 1, 2, \ldots.$$

The functions H_n are the Hermite polynomials:

$$H_n(x) = (-1)^n e^{x^2} \frac{d^n}{dx^n}\left(e^{-x^2}\right)$$

The corresponding energy levels are

$$E_n = \hbar\omega\left(n + \frac{1}{2}\right)$$

This energy spectrum is noteworthy for three reasons. Firstly, the energies are "quantized", and may only take the discrete half-integer multiples of $\hbar\omega$. This is a

Wavefunction representations for the first eight bound eigenstates, n = 0 to 7. The horizontal axis shows the position x. The graphs are not normalised

Probability densities $|\psi_n(x)|^2$ for the bound eigenstates, beginning with the ground state (n = 0) at the bottom and increasing in energy toward the top. The horizontal axis shows the position x, and brighter colors represent higher probability densities.

Energy of Photons in Some Mode

Electromagnetic mode with frequency ω.

Allowed energies of the state :

$$E_S = (1/2 + s)\hbar\omega = (1/2 + s)hf \quad \text{where } s \text{ is an integer.}$$

Partition function :

$$Z = \sum_{states} \exp\{-E_s / kT\}$$

Energy of Photons in Some Mode

$$Z = \sum_{states} \exp\left\{\frac{-(s+1/2)\hbar\omega}{kT}\right\} = \sum_{s=0}^{+\infty} \exp\left\{\frac{-\hbar\omega}{2kT}\right\} \cdot \exp\left\{\frac{-s\hbar\omega}{kT}\right\}$$

$$= \exp\left\{\frac{-\hbar\omega}{2kT}\right\} \frac{1}{1-\exp\{-\hbar\omega/kT\}}$$

Probability of occupancy of a state s:

$$P(s) = \frac{\exp(-E(s)/kT)}{Z} = \exp\left(-s\frac{\hbar\omega}{kT}\right) \cdot \exp\left(-\frac{\hbar\omega}{2kT}\right) \cdot \frac{1-\exp\left\{\frac{-\hbar\omega}{kT}\right\}}{\exp\left\{\frac{-\hbar\omega}{2kT}\right\}}$$

$$= \exp\left(-s\frac{\hbar\omega}{kT}\right) \cdot \left(1-\exp\{-\hbar\omega/kT\}\right)$$

This is the probabilty of having $(s+1/2)$ photons in mode $\hbar\omega$.

Energy of Photons in Some Mode

$$P(s) = \exp\left(- s\frac{\hbar\omega}{kT} \right) \cdot \left(1 - \exp\{-\hbar\omega / kT\}\right)$$

Expected value of s:

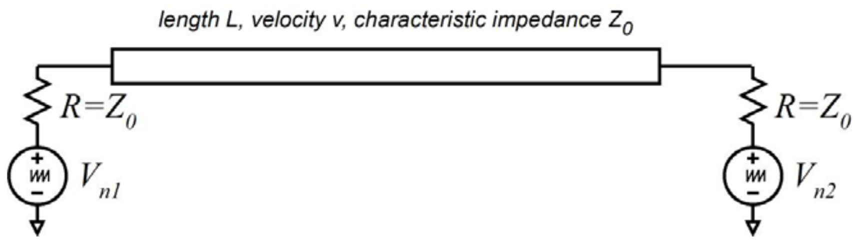
$$E[s] = \sum_{s=0}^{\infty} s \cdot P(s) = \text{(skip steps)} = \frac{1}{\exp\left(\frac{\hbar\omega}{kT}\right) - 1}$$

But $E = \hbar\omega(s + 1/2)$, so

$$\langle E \rangle = \hbar\omega\left[\langle s \rangle + 1/2\right] = \frac{\hbar\omega}{2} + \frac{\hbar\omega}{\exp\left(\frac{\hbar\omega}{kT}\right) - 1} \rightarrow kT \text{ for } kT << \hbar\omega$$

This is the averge mode energy at frequency $\hbar\omega$.

Nyquist's Noise Derivation (from Van der Ziel)



Transmission-line with matched resistors. Temperature T.

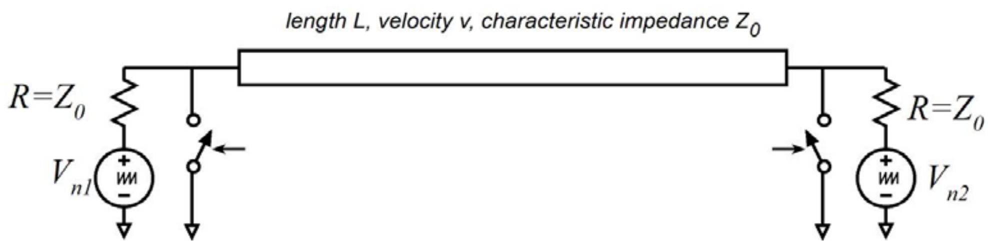
Each resistor has a thermal noise voltage. This causes a voltage wave to flow left->right (V_{n1}) and right-left (V_{n2}).

Define P_{av} = power available from each resistor in frequency interval Δf. From this, $\tilde{S}_{V_{n1}V_{n1}}(jf) = \tilde{S}_{V_{n2}V_{n2}}(jf) = 4R \cdot P_{av} / \Delta f$.

Power on line

$$P_{Line} = 2 \cdot P_{av} \cdot l / v$$

Nyquist's Noise Derivation (from Van der Ziel)



Trap this propagating radiation by closing 2 switches.

With switches closed, allowed frequencies are $n(v/2l)$

allowed frequencies within considered bandwidth Δf :

$$\# = \Delta f \cdot n(2l/v)$$

Nyquist's Noise Derivation (from Van der Ziel)

$$\text{Energy in line} = (\text{Power from resistors}) \cdot (\text{propagation time})$$

$$= P_{AV} \cdot (2l/v)$$

$$\text{Energy in line} = (\#\text{ modes in } \Delta f) \cdot (\text{energy per mode})$$

$$= \Delta f \cdot (2l/v) \cdot \left[\frac{\hbar\omega}{2} + \frac{\hbar\omega}{\exp(\hbar\omega/kT) - 1} \right]$$

So

$$P_{AV} = \Delta f \cdot \left[\frac{\hbar\omega}{2} + \frac{\hbar\omega}{\exp(\hbar\omega/kT) - 1} \right] = \Delta f \cdot \left[\frac{hf}{2} + \frac{hf}{\exp(hf/kT) - 1} \right]$$

And since $\tilde{S}_{V_{n1}V_{n1}}(jf) = \tilde{S}_{V_{n2}V_{n2}}(jf) = 4R \cdot P_{av} / \Delta f$

$$\tilde{S}_{V_{n1}V_{n1}}(jf) = 4R \cdot \left[\frac{hf}{2} + \frac{hf}{\exp(hf/kT) - 1} \right]$$

Nyquist's Noise Derivation (from Van der Ziel)

$$P_{AV} = \Delta f \cdot \left[\frac{hf}{2} + \frac{hf}{\exp\left(hf / kT\right) - 1} \right]$$

$$\frac{dP_{AV}}{df} = \left[\frac{hf}{2} + \frac{hf}{\exp\left(hf / kT\right) - 1} \right]$$

$$\tilde{S}_{V_{n1}V_{n1}}(jf) = 4R \cdot \left[\frac{hf}{2} + \frac{hf}{\exp\left(hf / kT\right) - 1} \right]$$

$$k = 1.38 \cdot 10^{-23} \text{ J/K}, \; h = 6.6 \cdot 10^{-34} \text{ J} \cdot \text{s}, \; \hbar = 1.06 \cdot 10^{-34} \text{ J} \cdot \text{s}$$

Comment about Noise Derivation

Another derivation uses an RLC resonator.
The physics is simpler, the math more complex.

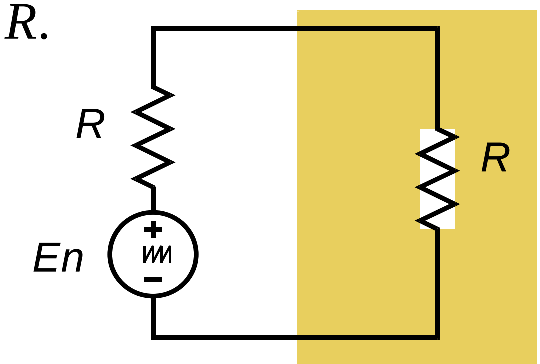
In both cases we have the same underlying difficulty.
In coupling the resonator to resistors, the resonator linewidth becomes non-zero, and allowed frequencies extend over some small bandwidth, rather than being restricted to the single frequency of a quantum harmonic oscillator.

Available Thermal Noise Power

Maximum power transfer : load R matched to generator R.

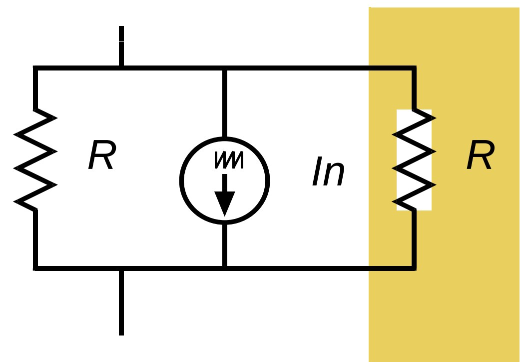
With matched load, voltage across load is $E_N / 2$

With matched load, current through load is $I_N / 2$



Given that

$$\tilde{S}_{E_n E_n}(jf) = 4kTR \text{ or } \tilde{S}_{I_n I_n}(jf) = \frac{4kT}{R} \quad \Rightarrow \quad \frac{dP_{load}}{df} = kT$$



P_{load} is the maximum (the available) noise power, hence

$$\frac{dP_{available,noise}}{df} = kT$$

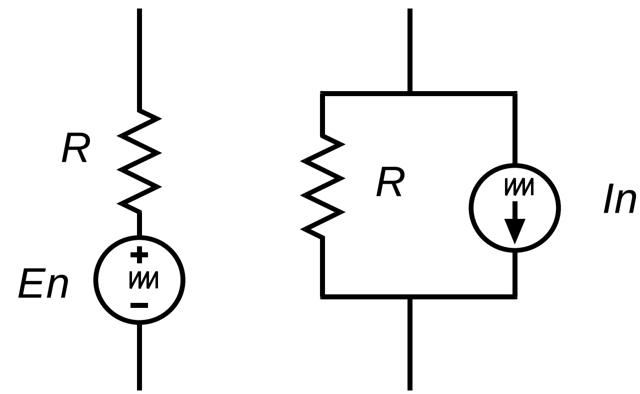
All resistors have equal available noise power.

Any compononent under thermal equilibrium (no bias) follows this law.

Thermal Noise

$$\tilde{S}_{E_nE_n}(jf) = 4R*\left[\frac{hf}{2}+\frac{hf}{\exp(hf/kT)-1}\right]$$

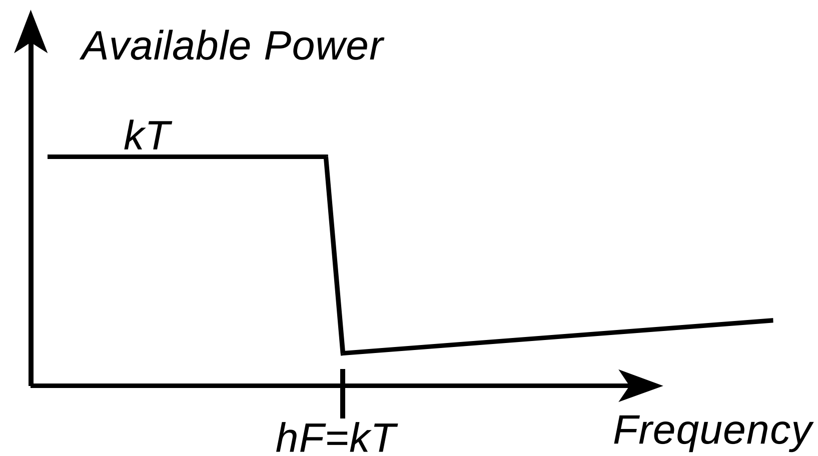
$$\tilde{S}_{I_nI_n}(jf) = \frac{4}{R}*\left[\frac{hf}{2}+\frac{hf}{\exp(hf/kT)-1}\right]$$



For $hf << kT$ these become

$$\tilde{S}_{E_nE_n}(jf) = 4kTR$$

$$\tilde{S}_{I_nI_n}(jf) = \frac{4kT}{R}$$

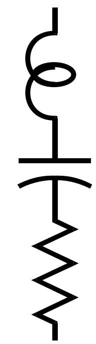


Noise from any impedance under thermal equilibrium

For any component or complex network under thermal equilibrium
(no energy supply)

$$\frac{dP_{available,noise}}{df} = kT$$

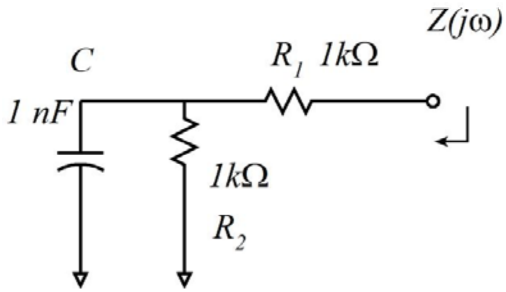
$$\Rightarrow \tilde{S}_{E_nE_n}(jf) = 4kT\,\mathrm{Re}(Z) \text{ or } \tilde{S}_{I_nI_n}(jf) = 4kT\,\mathrm{Re}(Y)$$



This follows from the 2nd law of thermodynamics.
This allows quick noise calculation of complex passive networks
This allows quick noise calculation of antennas.

Biased semiconductor devices are NOT in thermal equilibrium.

Noise from any impedance: Example



First method to calculate noise

$$Z(j\omega) = R_1 + \frac{R_2}{1 + j\omega R_2 C} = R_1 + \frac{R_2(1 - j\omega R_2 C)}{(1 + j\omega R_2 C)(1 - j\omega R_2 C)}$$

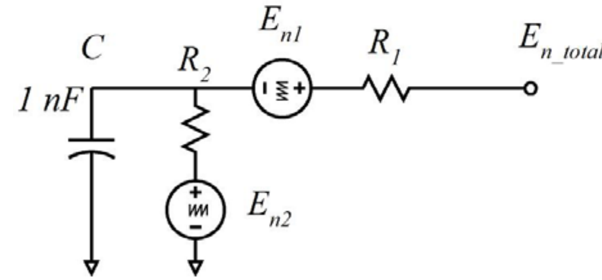
$$= R_1 + \frac{R_2(1 - j\omega R_2 C)}{1 + \omega^2 R_2^2 C^2} = R_1 + \frac{R_2}{1 + \omega^2 R_2^2 C^2} - \frac{j\omega R_2 C}{1 + \omega^2 R_2^2 C^2}$$

$Z(j\omega) = R(j\omega) + jX(j\omega)$ where $R(j\omega) = R_1 + \dfrac{R_2}{1 + \omega^2 R_2^2 C^2}$

$$\tilde{S}_{V_n V_n}(jf) = 4kTR(j\omega)$$

$$= 4kT \cdot \left[R_1 + \frac{R_2}{1 + \omega^2 R_2^2 C^2} \right]$$

Noise from any impedance: Example



Second method to calculate noise

$$e_{n,total}(j\omega) = e_{n1}(j\omega) + e_{n2}(j\omega)\frac{1}{1+j\omega R_2 C}$$

$$e_{n,total}e^*_{n,total} = \left(e_{n1} + \frac{e_{n2}}{1+j\omega R_2 C}\right)\cdot\left(e_{n1} + \frac{e_{n2}}{1+j\omega R_2 C}\right)^*$$

$$= e_{n1}e^*_{n1} + \frac{e_{n2}e^*_{n2}}{1+\omega^2 R_2^2 C^2} + \frac{e_{n1}e^*_{n2}}{1-j\omega R_2 C} + \frac{e_{n2}e^*_{n1}}{1+j\omega R_2 C}$$

But $e_{n1}e^*_{n2} = e_{n1}e^*_{n2} = 0$ because the processes are independent

$$\tilde{S}_{E_{n,total}E_{n,total}}(jf) = \tilde{S}_{E_{n,1}E_{n,1}}(jf) + \frac{\tilde{S}_{E_{n,2}E_{n,2}}(jf)}{1+(2\pi f)^2 R_2^2 C^2}$$

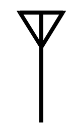
$$= 4kTR_1 + \frac{4kTR_2}{1+(2\pi f)^2 R_2^2 C^2}$$

....same answer.

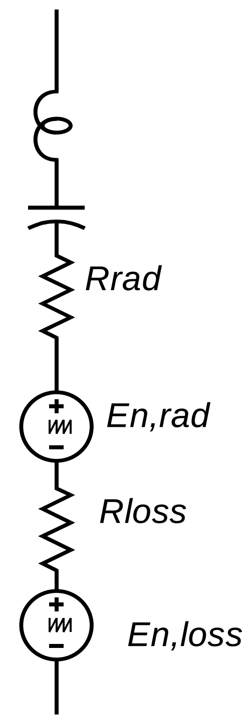
Noise from an Antenna

$$\frac{dP_{available,noise}}{df} = kT \Rightarrow \tilde{S}_{E_n E_n}(jf) = 4kT\,\mathrm{Re}(Z)$$

The antenna has both Ohmic and radiation resistances.



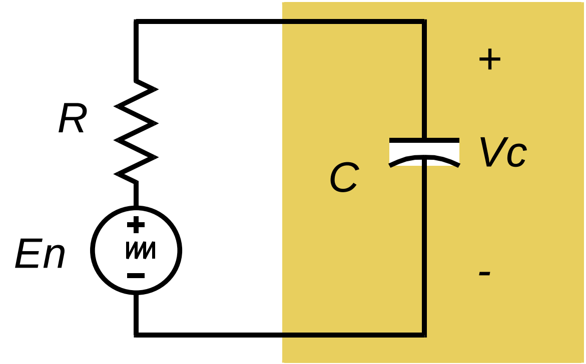
The Ohmic resistance has a noise voltage of spectral density $4kT_{ambient}R_{Ohmic}$, where $T_{ambient}$ is the physical antenna temperature



By the 2nd law, the radiation resistance has a noise voltage of spectral density $4kT_{field}R_{rad}$, where T_{field} is the average temperature of the region from which the antenna receives signal power

Inter-galactic space is at 3.8 Kelvin....

Noise on a capacitor



From

$$\tilde{S}_{E_nE_n}(jf) = 4kTR \text{ or } \tilde{S}_{I_nI_n}(jf) = \frac{4kT}{R}$$

We find that

$$\tilde{S}_{V_cV_c}(jf) = \left(\frac{1}{1+j2\pi fRC}\right)\left(\frac{1}{1+j2\pi fRC}\right)^* \tilde{S}_{E_nE_n}(jf)$$

$$= \left(\frac{1}{1+4\pi^2 f^2 R^2 C^2}\right)\tilde{S}_{E_nE_n}(jf)$$

So the mean stored Capacitor energy is

$$(1/2)C \cdot E[V_cV_c] = \int_0^\infty \left(\frac{1}{1+4\pi^2 f^2 R^2 C^2}\right)\tilde{S}_{E_nE_n}(jf)df = kT/2$$

This also follows directly from the Boltzmann law.