ECE594I Notes set 7: Shot Noise

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References and Citations:

Sources / Citations :

Kittel and Kroemer : Thermal Physics

Van der Ziel : Noise in Solid - State Devices

Papoulis : Probability and Random Variables (hard, comprehensive)

Peyton Z. Peebles : Probability, Random Variables, Random Signal Principles (introductory)

Wozencraft & Jacobs : Principles of Communications Engineering.

Motchenbaker : Low Noise Electronic Design

Information theory lecture notes : Thomas Cover, Stanford, circa 1982

Probability lecture notes : Martin Hellman, Stanford, circa 1982

National Semiconductor Linear Applications Notes : Noise in circuits.

Suggested references for study.

Van der Ziel, Wozencraft & Jacobs, Peebles, Kittel and Kroemer

Papers by Fukui (device noise), Smith & Personik (optical receiver design)

National Semi. App. Notes (!)

Cover and Williams : Elements of Information Theory

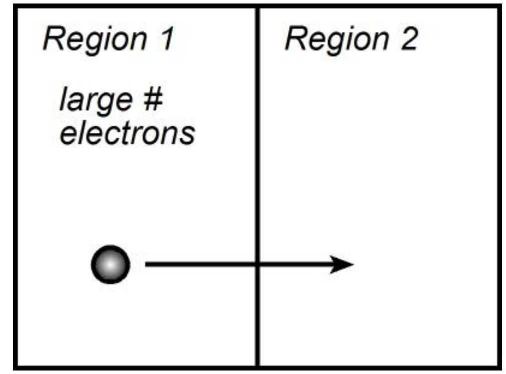
Shot Noise (idealized)

Focus first on idealized process

with independent emission events.

We will relate this, with some approximations,

to carrier transport in semiconductors.



Suppose that, observed in some small time period τ_1,

there is a probability p_1 of emission from region $1 \rightarrow 2$.

average electron flux $= \bar{r} = p_1 / \tau_1$

average DC current $= I = q \cdot p_1 / \tau_1$

for notational clarity we here write $q =$ electron charge

and not use q as a shorthand for $(1 - p)$.

Shot Noise (idealized)

Now consider small time increments Δt.

Time period Δt_i is the period $i \cdot \Delta t < t < (i+1) \cdot \Delta t$

Then

1) The # of electrons emitted in each Δt is independent of the # of electrons emitted in any other period Δt .

2) $p_{\Delta t_i}(k \text{ electrons}) = p_{\Delta t_i}(k) = p(k)$

$$= \binom{\Delta t / \tau}{k} \cdot p_1^k (1 - p_1)^{(\Delta t / \tau - 1)}$$

because there are $(\Delta t / \tau)$ trials each of probability p_1.

Shot Noise (idealized)

If we consider only cases where $p\cdot(\Delta t/\tau) >> 1,\ p << (1-p)$

the distribution will converge to a Gaussian, simplifying work :

$p_{\Delta t_i}(k) =$ Gaussian with mean $p\cdot(\Delta t/\tau)$ and variance $p\cdot(\Delta t/\tau)$

Consider separately the mean values and fluctuations :

$\Delta k = k - E[k] =$ fluctuation in flux,

where $E[k] = \bar{k} =$ mean electron flux in Δt.

Δk is Gaussian with variance $\sigma_k^2 = \bar{k}$

Shot Noise

For differening time steps Δt_i and Δt_j : $E\left[\Delta k_i \Delta k_j\right] = 0$ for $i \neq j$

But for the same time period : $E\left[\Delta k_i \Delta k_i\right] = \sigma_k^2$

So in general $E\left[\Delta k_i \Delta k_j\right] = \sigma_k^2 \cdot \delta_{i,j} = (r \cdot \Delta t) \cdot \delta_{i,j}$

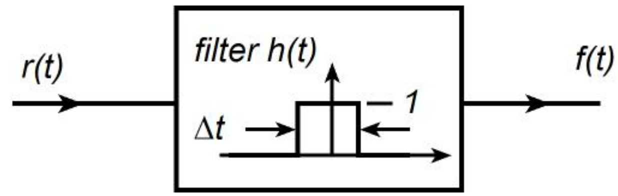
where $\delta_{i,j} = 1$ for $i = j$, $\delta_{i,j} = 0$ otherwise.

This is a sequence with discrete - time autocorrelation function

$R_{kk}(\Delta t_i \Delta t_j) = (r \cdot \Delta t) \cdot \delta_{i,j}$

We must work towards the continuous - time limit ($\Delta t \to 0$).

Shot Noise: Computing Power Spectrum



View as follows :

$r(t) =$ electron flux across boundary (# electrons/second)

...clearly a series of impulses at random times.

$f(t) =$# electrons crossing bondary between $t = -\Delta t$ and $t = 0$.

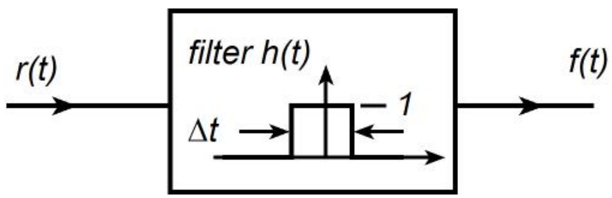
Then $f(t) = r(t) * h(t)$, where $h(t)$ is an impulse response :

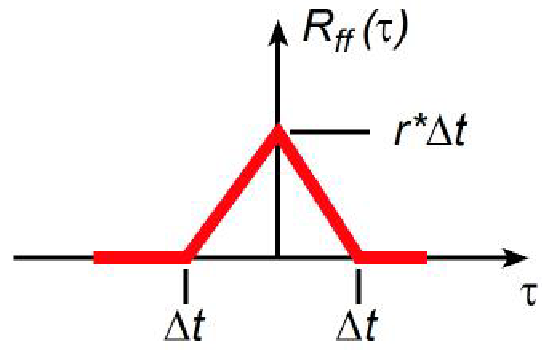
$h(t) = 1$ for $-\Delta t < t < 0$, $h(t) = 0$ otherwise.

Work with fluctuations $\Delta r = r - \bar{r}, \Delta f = f - \bar{f}$

Δf is Gaussian with variance $\sigma_f^2 = r\Delta t$

Shot Noise: Computing Power Spectrum



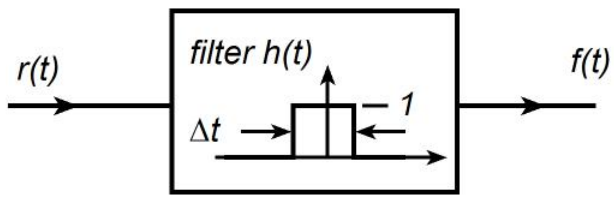


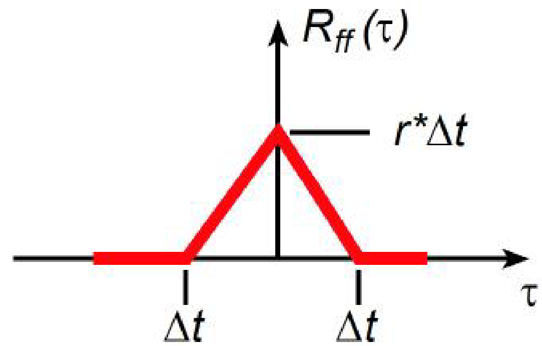
The autocorrelation function of Δf must be as shown because

(1) the variance of Δf is $r\Delta t$ and

(2) $\Delta f(t)$ and $\Delta f(t+\tau)$ represent time-averages of $r(t)$ whose averaging time-windows overlap only if τ is less than Δt

Shot Noise: Computing Power Spectrum





Power spectral density of Δf : $S_{\Delta f \Delta f}(j\omega) = \bar{r} \cdot (\Delta t)^2 \cdot \left[\frac{\sin(\omega \Delta t / 2)}{(\omega \Delta t / 2)} \right]^2$

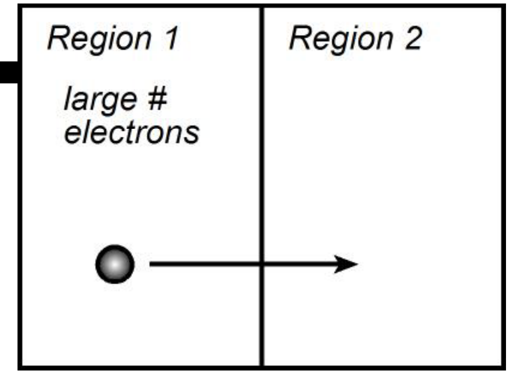
Filter Transfer function : $\| H(j\omega) \| = \left[\frac{\sin(\omega \Delta t / 2)}{(\omega \Delta t / 2)} \cdot (\Delta t) \right]$

But $S_{\Delta f \Delta f}(j\omega) = \| H(j\omega) \|^2 \cdot S_{\Delta r \Delta r}(j\omega)$

Power spectral density of Δr :

$$S_{\Delta r \Delta r}(j\omega) = S_{\Delta f \Delta f}(j\omega) \| H(j\omega) \|^{-2} = \bar{r}$$

Shot Noise: Computing Power Spectrum



This simple calculation has shown 2 things :

1) The shot noise process has a power spectrum, constant over all frequencies, equal to the flux. The spectrum *does not* decrease at frequencies higher than the inverse of the average emission time.

2) If we filter shot noise with some impulse response Δt, such that $\bar{r}\Delta t >> 1$, then the filtered process is Gaussian.

If we filter instead with $\bar{r}\Delta t$ not $>> 1$, the process is no longer Gaussian, but it remains constant - power with frequency (white).

DC current and Noise Current

To convert from electron fluxes to currents :

$$I = q \cdot r \quad \rightarrow \quad \bar{I} = I_{DC} = q\bar{r} \quad \rightarrow \quad \sigma_I^2 = q^2 \sigma_r^2$$

$$S_I(j\omega) = q^2 S_r(j\omega) = q^2 \bar{r} = q(q\bar{r}) = q\bar{I}$$

Using instead single-sided Hz-based spectra :

$$\tilde{S}_I(jf) = 2q\bar{I}$$

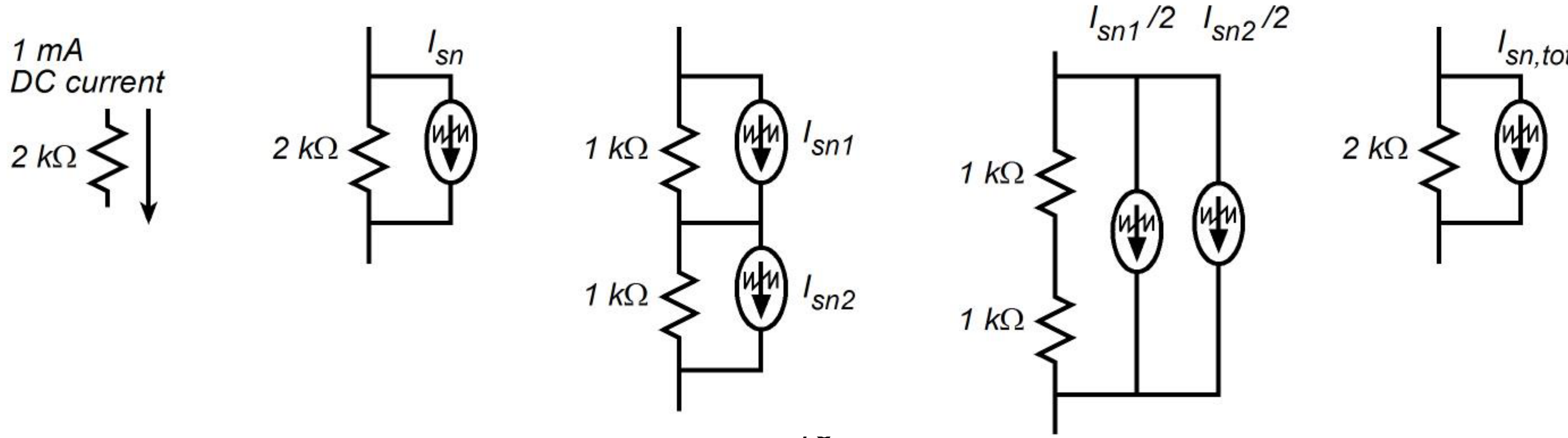
Resistors don't have shot noise

Note we have assumed independent emissions

As electrons move in resistors, electric fields are produced. These fields produce forces on other nearby electrons, thereby modifying their motions.

Electron passages through resistors during DC current flow is therefore not a sequence of independent emission events.

Resistor shot noise ? Reductio ad absurdum



Assume : resistors have shot noise : $\tilde{S}_{I_{sn}I_{sn}} = 2q(1\,\text{mA})$

Following the illustrated manipulations,

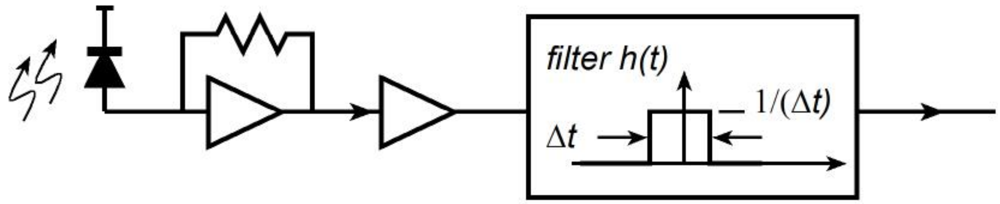
$$I_{SN,tot} = (I_{sn1} + I_{sn2})/2 \quad \text{where } \tilde{S}_{I_{sn1}I_{sn1}} = \tilde{S}_{I_{sn2}I_{sn2}} = 2q(1\,\text{mA})$$

$$\text{So } \tilde{S}_{I_{sn,tot}I_{sn,tot}} = \left(\tilde{S}_{I_{sn1}I_{sn1}} + \tilde{S}_{I_{sn2}I_{sn2}}\right)/4 = q(1\,\text{mA})$$

$$\tilde{S}_{I_{sn}I_{sn}} \neq \tilde{S}_{I_{sn,tot}I_{sn,tot}}$$

Breaking a resistor into a vast # of small resistors, a similar argument shows *how* shot noise is suppressed in resistors.

Shot Noise Example: Optical Receiver



Photocurrent is amplified and filtered over a time $\Delta t = T_{bit}$.
Assume a receiver limited by detector leakage shot noise; not a typical situation.

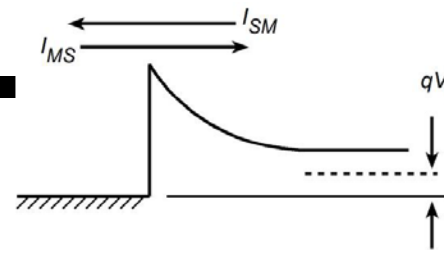
$I_{\text{leak}} = 100$ nA, 40 Gb/s data rate.

Input referred noise $\tilde{S}_{I_{leak}} = 2q(100\text{ nA}) = 3.2(10^{-16})\text{A}^2/\text{Hz}$.

$\Delta t \cdot \bar{r} = (100\text{ nA})/q \cdot (1/40\text{ Gb/s}) = 16$ electrons

We expect 16 ± 4 (mean $\pm$ one std. deviation) leakage electrons per bit period, but the distribution will be Poisson, not Gaussian.

Shot noise in a thin Schottky diode



Current in thermionic limit, approximate analysis.

$$I_{ms} = I_0 \qquad I_{sm} = I_0 \cdot \exp(qV_f / kT)$$

We postulate that these are independent currents, each of which arises from electron being thermally excited above the barrier energy with probability $p(E_i) \propto \exp(-E_i / kT)$

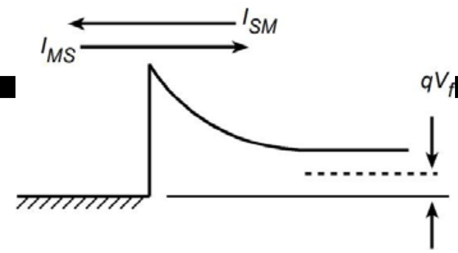
Total current in thermionic limit, approximate analysis.

$$I = I_{sm} - I_{ms} = I_0 \cdot \left[\exp(qV_f / kT) - 1\right]$$

If we treat both I_{sm} and I_{ms} as independent processes driven from independent electron emissions, then

$$\tilde{S}_{I_n I_n} = 2qI_{ms} + 2qI_{sm}$$

Shot noise in a thin Schottky diode



First consider zero bias :

$$g \equiv dI / dV = qI_0 / kT \text{ at } V_f = 0 \text{ Volts, while } I_{ms} = I_{sm} = I_0$$

So :

$$\tilde{S}_{I_n I_n} = 2qI_{ms} + 2qI_{sm} = 4qI_0 = 4kTg$$

Hence the available power :

$$\partial P_{avail} / \partial f = (1/4g) \cdot \tilde{S}_{I_n I_n} = kT$$

This is required from thermodynamics; otherwise we would have a net flow of noise power if we were to connect the diode in parallel with a resistor.

Yet, something is wrong : what happens for $hf > kT$?

High-frequency diode shot noise : UV crisis again

The available noise power at zero bias must be

$$\frac{dP_{avail}}{df} = \left[\frac{hf}{2} + \frac{hf}{\exp(hf/kT)-1}\right] \xrightarrow{hf<<kT} kT$$

Yet we have derived

$$\frac{dP_{avail}}{df} = kT \text{ at all frequencies}$$

This suggests that either the shot noise spectral density should be

$$\tilde{S}_{II}(jf) = 2qI_0\left[\frac{hf}{2kT} + \frac{hf/kT}{\exp(hf/kT)-1}\right]$$

...or that the diode zero - bias conductance increases for $hf > kT$

High-frequency diode shot noise : UV crisis again

This discrepancy is not widely noted in the literature, yet must be important for THz devices operated with cooling.

We have earlier shown that S_{II} does not decrease for frequencies higher than the inverse of the mean flux; this is not our resolution.

Think : if an electron occupies a state 200 meV above the Fermi energy, the equilibrium probabilty of this is $\sim e^{-7.7}$. If this electron then passes over the barrier and is lost, how long does it take to re-fill the state ?

High-frequency diode shot noise : UV crisis again

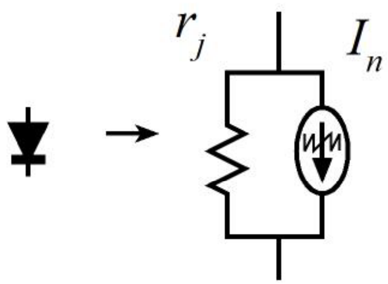
We had assumed equilibrium thermodynamics in order to obtain the probability of state occupancy. We have further assumed independent electron emissions. Yet, we have not calculated from equilibrium thermodynamics the rate at which such states would be filled and emptied, hence we have no basis for claiming independent events and a white spectral density.

We would expect, from $\Delta E \Delta t \sim \hbar,$ that states be repopulated at time delays $\Delta t \sim \hbar / \Delta E.$
We would therefore expect that our derivation would fail for frequencies $f > \Delta E / \hbar \approx kT / \hbar.$

There must surely be a way to calculate the statistics of time variation of state occupancy, and from this calculate the power spectrum of (thermal emission) shot noise.

I have not seen this in the literature, and am presently uncertain how to derive it myself.

Shot noise in a forward biased diode



Now consider strong forward bias :

$$\rightarrow I \cong I_0 \exp(qV_f / kT)$$

and $\tilde{S}_{I_n I_n} = 2qI_{ms} + 2qI_{sm} \cong 2qI_{sm} = 2qI$

while $g \equiv dI / dV = qI / kT$

So, the diode is modelled as shown, with

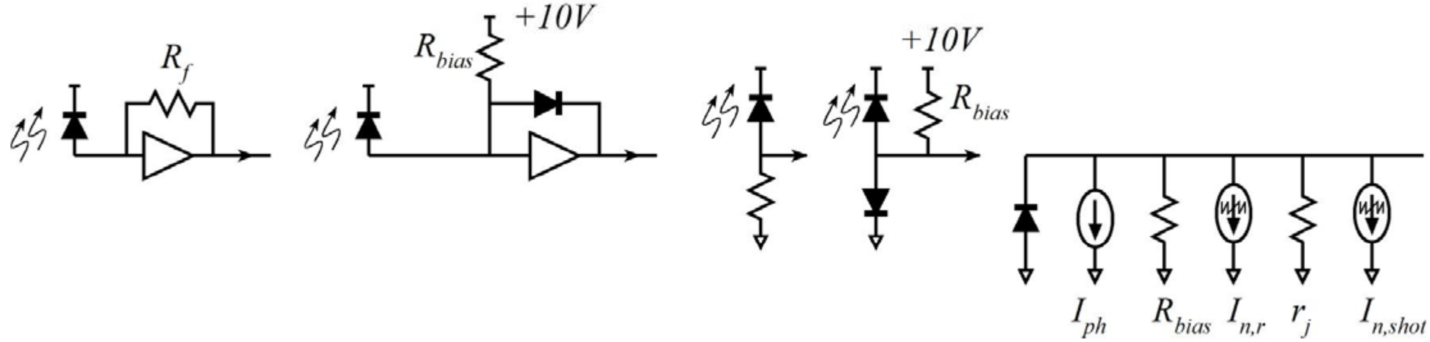
$\tilde{S}_{I_n I_n} = 2qI$ and $g = 1/r_j = qI / kT$

Hence the available power :

$$\partial P_{avail} / \partial f = (1/4g) \cdot \tilde{S}_{I_n I_n} = kT / 2$$

The biased diode can be modelled as a resistor at half ambient temperature.

Forward biased diode as a low-noise resistor



A forward - biased diode produces less noise than a resistor of equal conductance.

Left 2 images : real case of transimpedance optical amplifier.

Next 2 images : simplified case of photodiode with simple load.

Final image : equivalent circuit

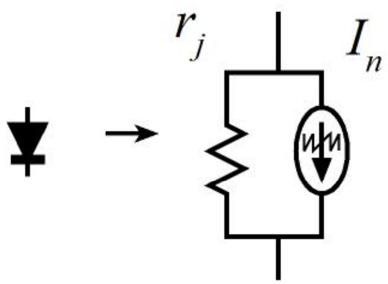
$R_{bias} = 100\,\text{k}\Omega$ so that $I_{bias} = 100\,\mu\text{A}$ giving $r_j = 26\text{ mV}/100\,\mu\text{A} = 260\,\Omega$

$\tilde{S}_{I_{n,r}I_{n,r}} = 4kT/(100\,\text{k}\Omega)$ $\quad \tilde{S}_{I_{n,shot}I_{n,shot}} = 2q(100\,\mu\text{A}) = 2kT/260\,\Omega$

Total : $\tilde{S}_{I_{n,tot}I_{n,tot}} \cong 2kT/260\,\Omega$

This is half that produced by instead loading the photodiode in $260\,\Omega$.

Shot noise in a reverse biased diode



Now consider strong reverse bias :

$$\rightarrow I \cong I_0$$

$$\text{and } \tilde{S}_{I_n I_n} = 2qI_{ms} + 2qI_{sm} \cong 2qI_0 = 2qI_0$$

$$\text{while } g \equiv dI/dV = \frac{d}{dV}\left[I_0 \cdot (\exp(qV_f/kT) - 1)\right]$$

Under strong reverse bias, g becomes very small, and the available noise power far exceeds kT.