Transactions Briefs

Extension of the Cochrun–Grabel Method to Allow for Mutual Inductances

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***Abstract*—The Cochrun–Grabel (C–G) method, an algorithm for finding the characteristic polynomial of a circuit containing reactances, has so far been restricted to circuits not employing mutual inductances.**

In this paper we present an intuitive, yet rigorous, proof of the Cochrun-Grabel method for a general RLC circuit, and we extend the method to allow the analysis of an RLC circuit containing mutual inductances.

***Index Terms*—Circuit theory, Cochrun–Grabel method, method of time constants, mutual inductances.**

I. Introduction

The Cochrun–Grabel (C–G) approach [1] to the problem of finding the characteristic polynomial of a reactive circuit has two main features: first, it only requires the analysis of frequency-independent subcircuits derived from the circuit under study and second, it shows clearly how the polynomial coefficients depend on the circuit reactances.

The method may also be used for estimating the dominant pole of a multistage amplifier (see, e.g., [2]–[4]) where typically only the coefficient of the linear s term in the polynomial is calculated. The approximation so obtained is often very close to the actual pole value.

Extensions to the method have been presented in [5]–[7]. In this paper we give a more physical, yet rigorous, proof of the method,[1] and we extend it to handle general RLC circuits with mutual inductances.

II. The Cochrun–Grabel Method

It is well known that the characteristic polynomial of an RC circuit is a linear function of each capacitive admittance. The characteristic polynomial of an RLC circuit is also a linear function, but of each capacitive admittance and each inductive impedance. To see this it is sufficient to apply the modified nodal analysis [8] to a circuit, resulting in the equation

$$\begin{pmatrix} \mathbf{G}+\mathbf{C}s & \cdots \\ \cdots & \mathbf{R}+\mathbf{L}s \end{pmatrix}\begin{pmatrix} \mathbf{V} \\ \mathbf{I} \end{pmatrix} = \begin{pmatrix} \mathbf{i_i} \\ \mathbf{v_i} \end{pmatrix} \tag{1}$$

where s is the complex frequency as usual. It is clear that the role of the Cs's and the Ls's in the determinant expansion for the matrix in (1) is identical.

The characteristic polynomial [i.e., the determinant of the matrix in (1)] can be written in general as

$$p'(s) = a'_0 + a'_1 s + a'_2 s^2 + \cdots + a'_n s^n. \tag{2}$$

In the following, we will consider the normalized polynomial:

$$p(s) = \frac{p'(s)}{a'_0} = 1 + a_1 s + a_2 s^2 + \cdots + a_n s^n. \tag{3}$$

We have seen that $p(s)$ is a linear function of the circuit reactances. Therefore, we can write its coefficients as

$$a_i = \sum \alpha_{j_1 j_2 \ldots j_i} X_{j_1} X_{j_2} \ldots X_{j_i} \tag{4}$$

with X being either L or C. The summation is performed over the $\binom{n}{i}$ combinations of i reactive elements chosen from the n available reactive elements.

The main point upon which our construction of the C–G algorithm hinges is that $\alpha_{j_1 j_2 \ldots j_i}$ is independent of the X values (by the very definition of $\alpha_{j_1 j_2 \ldots j_i}$) so that $\alpha_{j_1 j_2 \ldots j_i}$ can be determined by assigning arbitrary values to each X. In particular, these values can be the limit values zero and infinity (for each $\alpha_{j_1 j_2 \ldots j_i}$ the value assigned to each X changes in general). In turn, the circuit becomes purely resistive when each X has the value zero or infinity. We will use this fact to show that the general expression for $\alpha_{j_1 j_2 \ldots j_i}$ is a product of resistances and conductances obtained from a resistive subset of the circuit under study

$$\alpha_{j_1 j_2 \ldots j_i} = \beta^0_{X_{j_1}} \beta^{X_{j_1}}_{X_{j_2}} \ldots \beta^{X_{j_1} X_{j_2} \ldots X_{j_{i-1}}}_{X_{j_i}} \tag{5}$$

where β_X is a resistance (conductance) if X is a capacitance (inductance). The superscripts $X_{j_1} \ldots X_{j_{h-1}}$ indicate that $\beta_{X_{j_h}}$ is to be calculated with $X_{j_1} \ldots X_{j_{h-1}}$ short circuited (open circuited) if they are capacitances (inductances), while the remaining capacitances (inductances) are open circuited (short circuited).

Equation (5) enables (4) to be expressed as a sum of products of time constants, which is the form most usually taken by the C–G theorem

$$a_i = \sum \left(\beta^0_{X_{j_1}} X_{j_1}\right) \cdot \left(\beta^{X_{j_1}}_{X_{j_2}} X_{j_2}\right) \cdots \left(\beta^{X_{j_1} X_{j_2} \ldots X_{j_{i-1}}}_{X_{j_i}} X_{j_i}\right). \tag{6}$$

A. Proof of the Method

In the following, a network with only one capacitor and one inductor is considered, but it will be clear that the proof is easily extended to the general case. The characteristic polynomial for such an LC network is, according to (3) and (4)

$$p(s) = 1 + a_1 s + a_2 s^2 = 1 + (\alpha_L L + \alpha_C C)s + \alpha_{LC} LCs^2. \tag{7}$$

We must now supply a procedure for finding α_L, α_C, and α_{LC}.

We start by letting $C \to 0$. In this case $p(s) \to 1 + \alpha_L Ls$ and α_L is readily found through the well-known relation $\alpha_L = G^0_L$, G^0_L being the conductance seen by L when C is open circuited. Similarly, α_C is equal to R^0_C the resistance seen by C when L is short circuited (since in this case $L \to 0$).

It remains to derive α_{LC}. We start by noticing that if we let $L = kC$, k in the appropriate units, then for $k \to \infty$ the value $-1/\alpha_L L$ is a root of $p(s)$

$$\lim_{k\to\infty} p\left(\frac{-1}{\alpha_L L}\right) = \lim_{k\to\infty}\left(1 - \frac{\alpha_L L + \alpha_C C}{\alpha_L L} + \frac{\alpha_{LC} LC}{\alpha_L^2 L^2}\right) = \lim_{k\to\infty} \frac{1}{k}\left(\frac{-\alpha_C}{\alpha_L} + \frac{\alpha_{LC}}{\alpha_L^2}\right) = 0.$$

We rewrite now $p(s)$ as an explicit function of its roots

$$p(s) = (1 + \tau_1 s)(1 + \tau_2 s) = 1 + (\tau_1 + \tau_2)s + \tau_1 \tau_2 s^2. \tag{8}$$

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[1] The original proof was limited to RC and RL circuits only.

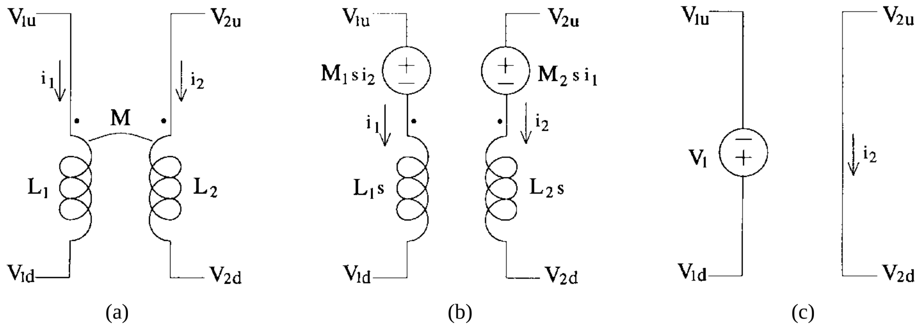

Fig. 1. (a) Circuit containing the mutual inductance M. (b) The mutual inductance modeled as a pair of current-controlled voltage sources, with $M_1 = M_2 = M$. (c) Transformed circuit for the calculation of $R^0_{M_1} = V_1/i_2$.

Since $-1/\alpha_L L$ is a root of $p(s)$ we can set $\tau_1 = \alpha_L L$. From (7) and (8) we obtain then

$$\tau_2 = \frac{\tau_1 \tau_2}{\tau_1} = \frac{\alpha_{LC} LC}{\alpha_L L} = \frac{\alpha_{LC} C}{\alpha_L}. \tag{9}$$

Thus, τ_2 depends on C, but not on L. Furthermore, since the condition $k \to \infty$ implies $L \to \infty$ for a finite value of C, C sees a circuit where L can be replaced by an open circuit, so that we can write τ_2 as

$$\tau_2 = R^L_C C \tag{10}$$

with R^L_C defined as the resistance seen by C with L open circuited. From (9) and (10) we arrive at the final expression for α_{LC}

$$\alpha_{LC} = \alpha_L R^L_C = G^0_L R^L_C.$$

By interchanging the roles of L and C we obtain the alternative relation $\alpha_{LC} = R^0_C G^C_L$.

The above procedure is straightforwardly generalized to a network with n reactive elements, for which (5) holds.

III. Mutual Inductances

The C–G method cannot yet handle mutual inductances, and we want to extend the algorithm to allow for the presence of such components in the network (Fig. 1). Let the mutual inductance M be modeled as two current-controlled voltage sources $M_1 i_2$ and $M_2 i_1$. The characteristic polynomial is then linear in M_1 and M_2 as (11) clearly shows

$$\begin{pmatrix} L_1 s & M_1 s & \cdot & \cdot & -1 & 1 & 0 & 0 \\ M_2 s & L_2 s & \cdot & \cdot & 0 & 0 & -1 & 1 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{pmatrix} \begin{pmatrix} i_1 \\ i_2 \\ \cdot \\ \cdot \\ v_{1u} \\ v_{1d} \\ v_{2u} \\ v_{2d} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \cdot \end{pmatrix}. \tag{11}$$

Letting every X except M_1 approach zero, we obtain $p(s) = 1 + \alpha_{M_1} M_1 s$. On the other hand, (2) yields

$$p'(s) = \Delta_{12} M_1 s + \Delta \tag{12}$$

where Δ is the determinant of the matrix in (11) and Δ_{12} is the determinant of the matrix obtained by removing the first row and the second column from the matrix in (11). Both determinants are calculated with every X set to zero. Equation (12) yields $\alpha_{M_1} = \Delta_{12}/\Delta$. Since $\Delta/\Delta_{12} = V_1/i_2$ for the circuit in Fig. 1(c), we obtain the relation $\alpha_{M_1} = 1/R^0_{M_1}$, $R^0_{M_1}$ being the transresistance V_1/i_2. The transresistances $R^{X_{i_1} \cdots X_{i_j}}_{M_1}$ are determined in the same way; however, if one or more mutual inductance values approach infinity (e.g., one or more $X_{i_j} = M_{i_j}$) we cannot proceed so easily as for $X_{i_j} = L_{i_j}, C_{i_j}$, where the branch containing L_{i_j} (C_{i_j}) simply becomes an open circuit (short circuit). Referring to Fig. 2(a)–(b), if $M_i \to \infty$, then $M_i s i_j \to \infty$ as well unless $i_j \to 0$. Since no voltage in the circuit can grow indefinitely, i_j does tend to zero, and the branch connecting V_{ju} and V_{jd} can be modeled as a nullator, with constitutive equations $i_j = 0$, $V_{ju} - V_{jd} = 0$. The branch connecting V_{iu} and V_{id} can, in turn, be modeled as a norator (i_j = *unknown*, $V_{ju} - V_{jd}$ = *unknown*) [9], Fig. 2(c). Having so transformed the circuit, the transresistance $R^{M_i}_{M_1}$ can be found by using standard circuit analysis techniques. In general, if

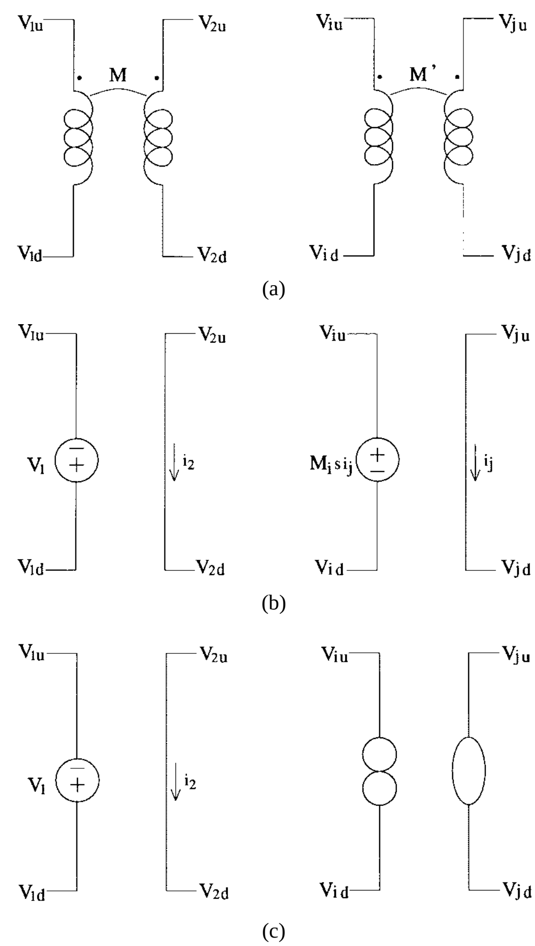

Fig. 2. (a) Circuit containing two mutual inductances M and M'. As shown in Fig. 1(b), each mutual inductance can be modeled as a pair of current-controlled voltage sources, with $M_1 = M_2 = M$, $M_i = M_j = M'$. (b) Transformed circuit for the calculation of $R^{M_i}_{M_1} = V_1/i_2|_{M_i \to \infty}$. (c) The nullator–norator pair makes the analysis of (b) easier.

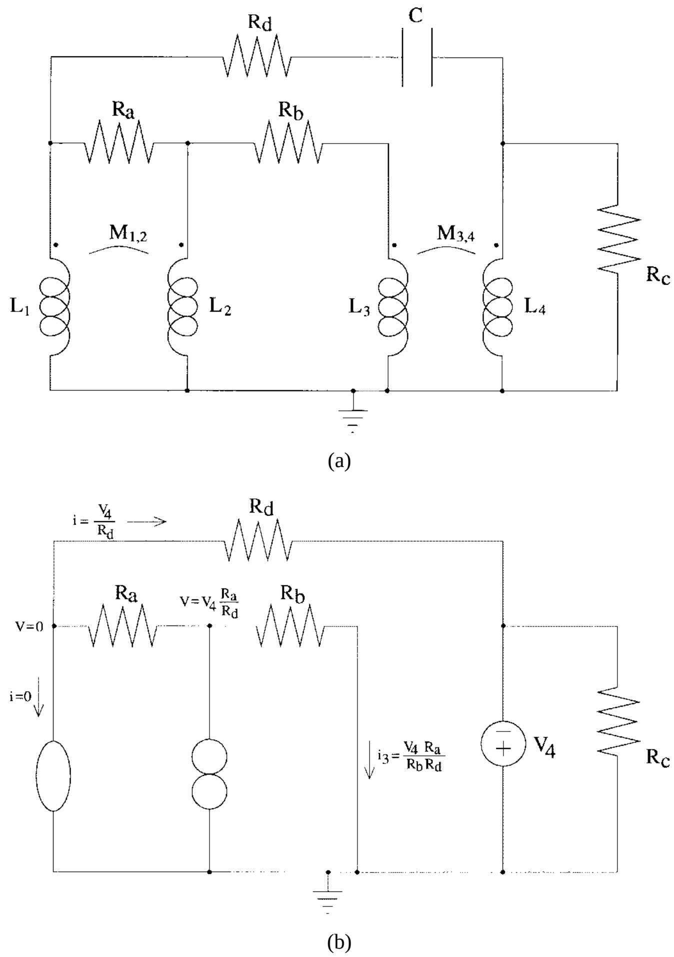

Fig. 3. (a) Example of a circuit containing two mutual inductances. (b) Transformed circuit for the calculation of $R_{M_4}^{M_2C}$.

n mutual inductances approach an infinite value, n nullator-norator pairs appear in the circuit. It is worth noting that we have to calculate $R_{M_1}^{M_i}$ only if $i \neq 2$, that is, M_i and M_1 are not the same physical mutual inductance. If $i = 2$, then $R_{M_2}^0 R_{M_1}^{M_2} = -R_{L_2}^0 R_{L_1}^{L_2}$, which is obvious if we expand the determinant of the matrix in (11) as $\Delta = (L_1 L_2 s^2 - M_1 M_2 s^2)\Delta' + g$, where Δ' is the determinant of the matrix obtained removing the first two rows and columns from the matrix in (11), and g does not contain the products $L_1 L_2$ and $M_1 M_2$. The same determinant expansion shows that the equality $R_{M_2}^{X_1 \ldots X_j} R_{M_1}^{X_1 \ldots X_j M_2} = -R_{L_2}^{X_1 \ldots X_j} R_{L_1}^{X_1 \ldots X_j L_2}$ holds in general. Equation (11) also shows that the product $L_i M_j,\ i,j = 1,2$ cannot appear in $p(s)$. We have furthermore $R_X^{X_i \ldots X_j M_1 M_2} = R_X^{X_i \ldots X_j L_1 L_2}$, since if $M_1, M_2 \to \infty$, then $i_1, i_2 \to 0$, while $V_{1u} - V_{1d} =$ *unknown* and $V_{2u} - V_{2d} =$ *unknown*. These are in turn the conditions imposed by the constraints $L_1, L_2 \to \infty$. Thus, only few transresistances have to be calculated explicitly.

Finally, we notice that $\alpha_{j_1 j_2 \ldots j_i}$ is much easier to derive if we build it as

$$\alpha_{j_1 j_2 \ldots j_i} = \beta_{X_{j_1}}^0 \ldots \beta_{X_{j_h}}^{X_{j_1} \ldots X_{j_{h-1}}} \beta_{M_{j_{h+1}}}^{X_{j_1} \ldots X_{j_h}} \ldots \beta_{M_{j_i}}^{X_{j_1} \ldots X_{j_h} M_{j_{h+1}} \ldots M_{j_{i-1}}}$$

where $X_{j_1} \ldots X_{j_h}$ are not mutual inductances. In this way we avoid the use of nullators and norators in the calculation of the resistances seen by capacitors and inductors.

IV. Example

As an example of the method previously described, the transresistance $R_{M_4}^{M_2C}$ for the network in Fig. 3(a) will be calculated. We begin by transforming the circuit as shown in Fig. 3(b) where inductances L_1 and L_2 have been replaced by a nullator and a norator, respectively. The presence of the nullator $(v = 0)$ forces the current through R_d to assume value V_4/PR_d. Since no current can flow in the nullator, the voltage at the norator output has value $V_4 R_a/PR_d$ from which the relations $i_3 = V_4 R_a / P R_b R_d$ and $R_{M_4}^{M_2C} = V_4/Pi_3 = R_b R_d / P R_a$ immediately follow. Although the calculations above are rather simple, it is clear that the presence of multiple mutual inductances in the circuit can make the application of the C–G method cumbersome.

With the exception of $R_{M_4}^{M_2C}$ and $R_{M_3}^{M_1C}$, all other transresistances are directly found by inspection. Some values are listed below.

$$\begin{aligned}
&R_{M_1}^0 = R_{M_2}^0 = -R_a\\
&R_{M_3}^0 = R_{M_4}^0 = -\infty\\
&R_{M_1}^C = R_{M_2}^C = -R_a\\
&R_{M_3}^C = R_{M_4}^C = -\infty\\
&R_{M_1}^{L_3} = R_{M_1}^{L_4} = R_{M_2}^{L_3} = R_{M_2}^{L_4} = -R_a\\
&R_{M_3}^{L_1} = R_{M_3}^{L_2} = R_{M_4}^{L_1} = R_{M_4}^{L_2} = -\infty\\
&R_{M_1}^0 R_{M_2}^{M_1} = -R_{L_1}^0 R_{L_2}^{L_1} = -R_a R_b\\
&R_{M_3}^0 R_{M_4}^{M_3} = -R_{L_3}^0 R_{L_4}^{L_3} = -R_b R_c\\
&R_{M_4}^{M_2C} = R_{M_3}^{M_1C} = \frac{R_b R_d}{R_a}\\
&R_{M_1}^{L_3C} = R_{M_2}^{L_3C} = R_{M_1}^{L_4C} = R_{M_2}^{L_4C} = -R_a\\
&R_{M_1}^{M_3M_4} = -R_{M_1}^{L_3L_4} = -R_a\\
&R_{M_2}^{M_3M_4} = -R_{M_2}^{L_3L_4} = -R_a.
\end{aligned}$$

V. Conclusion

The work has shown how the Cochrun-Grabel method may be extended to RLC circuits containing mutual inductances. The time constants associated with the mutual inductances depend on various transresistances seen by the mutual inductances. Transresistances are no harder to find than resistances unless more mutual inductances are present. In this case, calculations may become less straightforward because of the appearance of nullator–norator pairs in the circuit.

References

[1] B. L. Cochrun and A. Grabel, "A method for the determination of the transfer function of electronic circuits," *IEEE Trans. Circuit Theory*, vol. CT-20, pp. 16–20, Jan. 1973.

[2] P. R. Gray and R. G. Mayer, *Analog Integrated Circuits*, 2nd ed. New York: Wiley, 1984, ch. 7.

[3] S. Rosenstark, *Feedback Amplifier Principles* New York: Macmillan, 1986, ch. 5.

[4] J. Millman and A. Grabel, *Microelectronics*, 2nd ed. New York: McGraw-Hill, 1987, ch. 11.

[5] A. M. Davis and E. A. Moustakas, "Decomposition analysis of active networks," *Int. J. Electron.*, Vol. 46, no. 5, pp. 449–456, 1979.

[6] R. M. Fox and S. G. Glee, "Extension of the open-circuit time-constant method to allow for transcapacitances," *IEEE Trans. Circuits Syst.*, vol. 37, pp. 1167–1171, Sept. 1990.

[7] P. Andreani and S. Mattisson, "Characteristic polynomial and zero polynomial with the Cochrun-Grabel method," *Int. J. Circ. Theor. Appl.*, vol. 26, pp. 287–292, May–June 1998.

[8] R. K. Ruehli, C. W. Ho, and P. A. Brennan, "The modified nodal approach to network analysis," *IEEE Trans. Circuits Syst.*, Vol. CAS-22, pp. 504–509, June 1975.

[9] J. Vlach and K. Singhal, *Computer Methods for Circuit Analysis and Design*. New York: Van Nostrand, 1983, ch. 4.