Mixed Signal IC Design
Notes set 6:
Mathematics of Electrical Noise

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Strategy

There is not time in 145c/218c to develop this subject in detail.

Strategy :
give backround sufficient for correct calculation of
SNR, spectral densities, correlation functions,
signal correlations, error rates.

More detail can be found in my noise class notes (on the web),
or in the literature. Van der Zeil's book is comprehensive.

Topics

Math:
distributions, random variables, expectations,
pairs of RV, joint distributions, covariance and correlations.
Random processes, stationarity, ergodicity, correlation functions,
autocorrelation function, power spectral density.

Noise models of devices:
thermal and shot noise.
Models of resistors, diodes, transitors, antennas.

Circuit noise analysis:
network representation. Solution.
Total output noise. Total input noise. 2 generator model.
En/In model. Noise figure, noise temperature.
Signal / noise ratio.

random variables

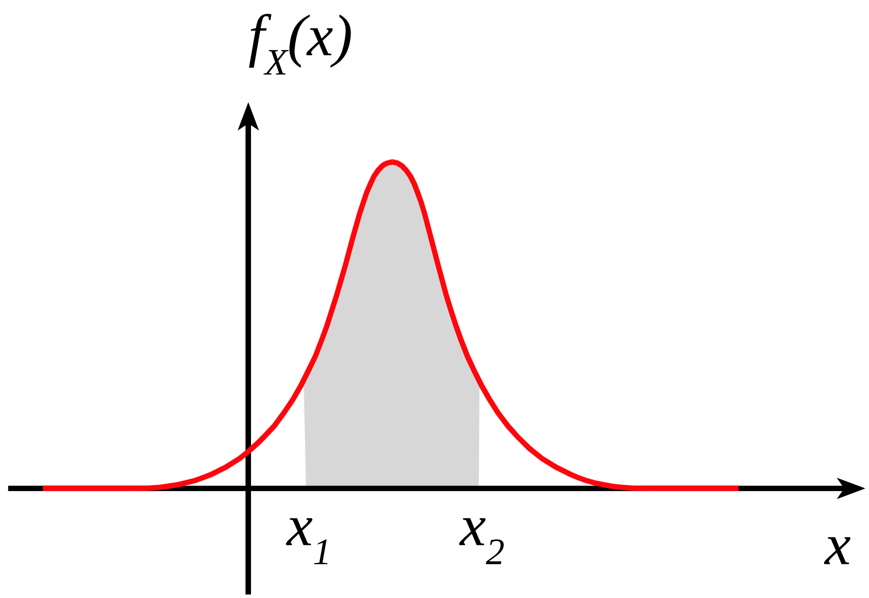
The first step: random Variables

During an experiment, a random variable X takes on a particular value x.

The probability that x lies between x_1 and x_2 is

$$P\{x_1 < x < x_2\} = \int_{x_1}^{x_2} f_X(x)dx$$

$f_X(x)$ is the probability distribution function.



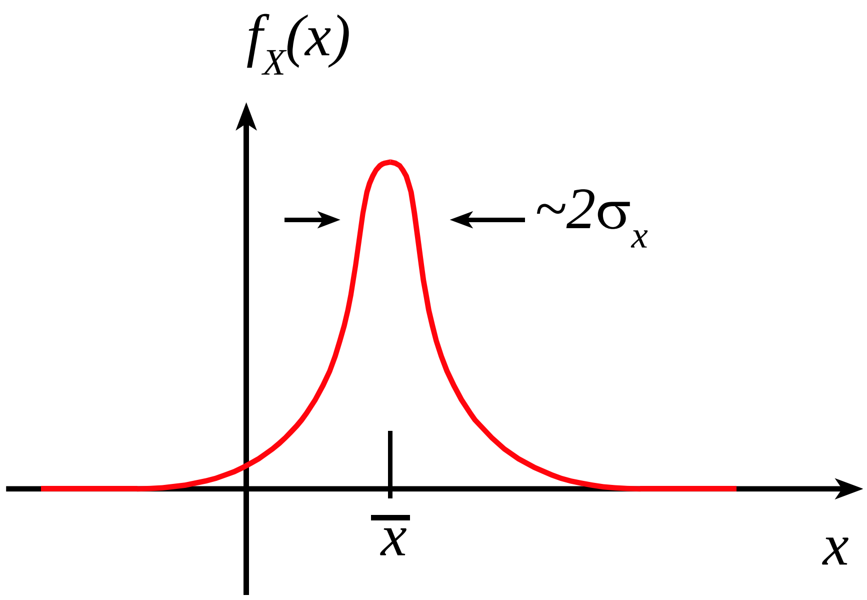
Example: The Gaussian Distribution

The Gaussian distribution :

$$f_X(x) = \frac{1}{\sqrt{2\pi\sigma_x^2}} \exp\left(\frac{-\left(x-\overline{x}\right)^2}{2\sigma_x^2} \right)$$

We will define shortly the mean ($\overline{x}$) and the standard deviation (σ_x^2).

Because of the *central limit theorem*, physical random processes arising from the sum of many small effects have probability distributions close to that of the Gaussian.



Mean values and expectations

Expectation of a function g(X) of the random variable X

$$E[g(x)] = \int_{-\infty}^{+\infty} g(x) f_X(x) dx$$

Mean Value of X

$$\langle X \rangle = \overline{X} = E[X] = \int_{-\infty}^{+\infty} x f_X(x) dx$$

Expected value of X^2

$$\langle X^2 \rangle = E[X^2] = \int_{-\infty}^{+\infty} x^2 f_X(x) dx$$

Variance

The variance σ_x^2 of X is its root-mean-square deviation from its average value

$$\sigma_X^2 = \left\langle \left(X - \overline{x} \right)^2 \right\rangle = E\left[\left(X - \overline{x} \right)^2 \right] = \int_{-\infty}^{+\infty} \left(x - \overline{x} \right)^2 f_X(x)\,dx$$

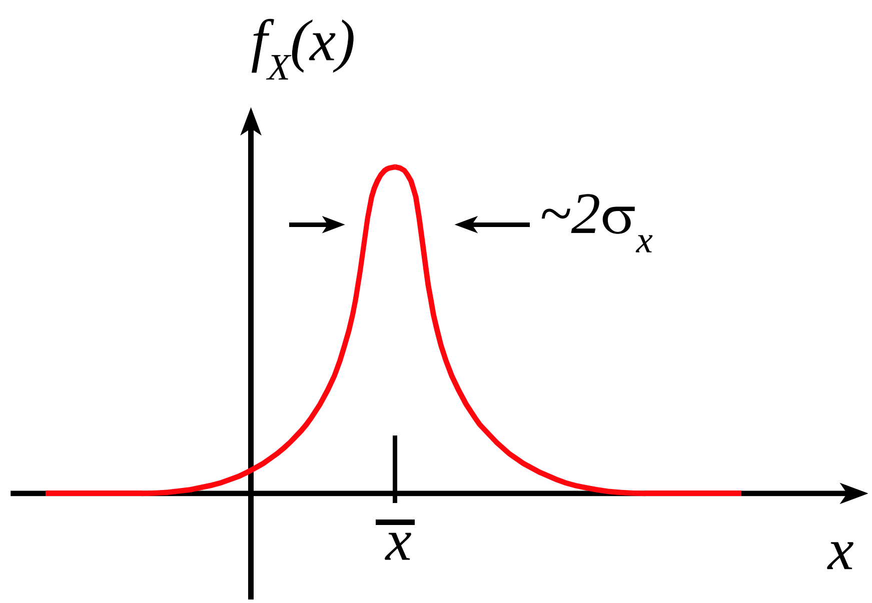
The standard deviation σ_x of X is simply the square root of the variance

Returning to the Gaussian Distribution

The notation describing the Gaussian distribution :

$$f_X(x) = \frac{1}{\sqrt{2\pi\sigma_x^2}} \exp\left(\frac{-(x-\bar{x})^2}{2\sigma_x^2} \right)$$

should now be clear.



Variance vs Expectation of the Square

$$\sigma_x^2 = \left\langle \left(X - \bar{x}\right)^2 \right\rangle = \left\langle \left(X - \bar{x}\right)\left(X - \bar{x}\right) \right\rangle$$

$$= \left\langle X^2 - 2X \cdot \bar{x} + \left(\bar{x}\right)^2 \right\rangle$$

$$= \left\langle X^2 \right\rangle - 2 \cdot \bar{x}\left\langle X \right\rangle + \left\langle \left(\bar{x}\right)^2 \right\rangle$$

$$= \left\langle X^2 \right\rangle - 2 \cdot \bar{x} \cdot \bar{x} + \left(\bar{x}\right)^2$$

$$\sigma_x^2 = \left\langle X^2 \right\rangle - \left(\bar{x}\right)^2$$

The variance is the expectation of the square minus the square of the expectation.

Pairs of Random Variables

To understand random processes,
we must first understand pairs of random variables.

In an experiment, a pair of random variables X and Y
takes on specific particular values x and y.

Their joint behavior is described by the joint distribution $f_{XY}(x,y)$

$$P\{A < x < B \text{ and } C < y < D\} = \int_C^D \int_A^B f_{XY}(x,y)\,dx\,dy$$

Pairs of Random Variables

Marginal distributions must also be defined

$$P\{A < x < B\} = \int_{-\infty}^{+\infty}\int_{A}^{B} f_{XY}(x,y)\,dxdy$$

$$= \int_{A}^{B} f_X(x)\,dx$$

and similarly for Y :

$$P\{C < y < D\} = \int_{C}^{D}\int_{-\infty}^{+\infty} f_{XY}(x,y)\,dxdy$$

$$= \int_{C}^{D} f_Y(y)\,dy$$

Statistical Independence

In the case where

$$f_{XY}(x,y) = f_X(x)\,f_Y(y),$$

the variables are said to be statistically independent.

This is not generally expected.

Expectations of a pair of random variables

The expectation of a function g(X, Y) of the random variables Y and Y is

$$E[g(x,y)] = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} g(x,y) f_{XY}(x,y)\,dx\,dy$$

Expectation of X :

$$E[X] = \bar{x} = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} x f_{XY}(x,y)\,dx\,dy = \int_{-\infty}^{+\infty} x f_X(x)\,dx$$

Expectation of X^2

$$E[X^2] = \langle X^2 \rangle = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} x^2 f_{XY}(x,y)\,dx\,dy = \int_{-\infty}^{+\infty} x^2 f_X(x)\,dx$$

...and similarly for Y and Y^2.

Correlation between random variables

The correlation of X and Y is

$$R_{XY} = E[XY] = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} xy \bullet f_{XY}(x,y)dxdy$$

The covariance of X and Y is

$$C_{XY} = E[(X-\overline{x})(Y-\overline{y})] = E[XY - \overline{x}Y - X\overline{y} + \overline{xy}]$$
$$= R_{XY} - \overline{x}\cdot\overline{y}$$

Note that correlation and covariance are the same if either X or Y have zero mean values.

Correlation versus Covariance

When we are working with voltages and currents, we usually separate the mean value (DC bias) from the time - varying component.

The random variables then have zero mean.

Correlation is then equal to covariance.

It is therefore common in circuit noise analysis to use the two terms interchangably.

But, nonzero mean values can return when we e.g. calculate conditional distributions.

Be careful.

Correlation Coefficient

The correlation coefficient of X and Y is

$$\rho_{XY} = C_{XY} / \sigma_X \sigma_Y$$

Note the (standard) confusion in terminology between correlation and covariance.

Sum of TWO Random Variables

Sum of two random variables : $Z = X + Y$

$$E\left[Z^2\right] = E\left[(X+Y)^2\right] = E\left[X^2 + 2XY + Y^2\right]$$
$$= E\left[X^2\right] + E\left[Y^2\right] + 2R_{XY}$$

If X and Y both have zero means

$$E\left[Z^2\right] = E\left[X^2\right] + E\left[Y^2\right] + 2C_{XY}$$

This emphasizes the role of correlation.

Pairs of Jointly Gaussian Random Variables

If X and Y are Jointly Gaussian :

$$f_{XY}(x,y) = \frac{1}{2\pi\sigma_X\sigma_Y\sqrt{1-\rho_{XY}^2}}$$

$$\times \exp\left[-\frac{1}{2(1-\rho_{XY}^2)}\cdot\left(\frac{(x-\bar{x})^2}{\sigma_X^2}+\frac{(x-\bar{x})(y-\bar{y})}{\sigma_X\sigma_Y}+\frac{(y-\bar{y})^2}{\sigma_Y^2}\right)\right]$$

This definition can be extended to a larger # of variables.

In general, we can have a Jointly Gaussian random vector

$(X_1, X_2, \cdots, X_n)$

which is specified by a set of

means $\bar{x}_i$, variances $E\left[x_i x_i\right]$, and covariances $E\left[x_i x_j\right]$

Linear Operations on JGRV's

If X and Y are Jointly Gaussian, and if we define

$V = aX + bY$ and $W = cX + dY$

Then V and W are also Jointly Gaussian.

This is stated without proof; the result arises because convolution of 2 Gaussian functions produces a Gaussian function.

The result holds for JGRVs of any number.

Probability distribution after a Linear Operation on JGRV's

$$\overline{V} = E[V] = E[aX + bY] = a\overline{X} + b\overline{Y} \quad \text{and} \quad \overline{W} = c\overline{X} + d\overline{Y}$$

$$\sigma_V^2 = E[V^2] - \overline{V}^2 = a^2 E[X^2] + b^2 E[Y^2] + 2ab \cdot E[XY] - (a\overline{X} + b\overline{Y})^2$$

$$\sigma_W^2 = E[W^2] - \overline{W}^2 = c^2 E[X^2] + d^2 E[Y^2] + 2cd \cdot E[XY] - (c\overline{X} + d\overline{Y})^2$$

$$C_{VW} = E[VW] - \overline{V}\,\overline{W} = E[(aX + bY)(cX + dY)] - \overline{V}\,\overline{W}$$

$$= E[acX^2 + (ad + bc)XY + bdY^2] - \overline{V}\,\overline{W}$$

$$= acE[X^2] + (ad + bc)E[XY] + bd \cdot E[Y^2] - (a\overline{X} + b\overline{Y})(c\overline{X} + d\overline{Y})$$

tedious details

We can now calculate the joint distribution of V and W.

$$f_{VW}(v, w) = \frac{1}{2\pi\sigma_V\sigma_W\sqrt{1-\rho_{VW}^2}}$$

$$\times \exp\left[-\frac{1}{2(1-\rho_{VW}^2)} \cdot \left(\frac{(v-\overline{v})^2}{\sigma_V^2} + \frac{(v-\overline{v})(w-\overline{w})}{\sigma_V\sigma_W} + \frac{(w-\overline{w})^2}{\sigma_W^2} \right) \right]$$

Why are JGRV's Important ?

The math on the last slide was tedious but there is a clear conclusion :

With JGRV's subjected to linear operations, it is sufficient to keep track of means, correlations, and variances.

With this information, distribution functions can always be simply found.

This vastly simplifies calculations of noise propagation in linear systems (linear circuits).

Uncorrelated Variables.

Uncorrelated :

$$C_{XY} = 0$$

Statistically independent :

$$f_{XY}(x,y) = f_X(x)\, f_Y(y)$$

Independence implies zero correlation.

Zero correlation does not imply independence.

For JGRV's, uncorrelated does imply independence

Summing of Noise (Random) Voltages

Two voltages are applied to the resistor R

The power dissipated in the resistor is a random variable P

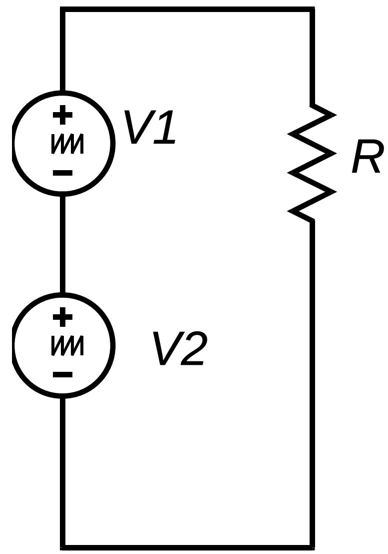
$$E[P]=\langle P\rangle=\frac{1}{R}\left\langle (V_1+V_2)^2\right\rangle=\frac{1}{R}\left\langle V_1^{\,2}+2V_1V_2+V_2^{\,2}\right\rangle$$

$$=\frac{1}{R}\left\langle V_1^{\,2}\right\rangle+\frac{1}{R}2C_{V_1V_2}+\frac{1}{R}\left\langle V_2^{\,2}\right\rangle$$

$$=\frac{1}{R}\left\langle V_1^{\,2}\right\rangle+\frac{1}{R}2\sigma_{V_1V_2}+\frac{1}{R}\left\langle V_2^{\,2}\right\rangle$$

$$=\frac{1}{R}\left\langle V_1^{\,2}\right\rangle+\frac{1}{R}\left\langle V_2^{\,2}\right\rangle+\frac{1}{R}2\rho_{V_1V_2}\sigma_{V_1}\sigma_{V_2}$$

$$=\frac{1}{R}\left\langle V_1^{\,2}\right\rangle+\frac{1}{R}2\left\langle V_1V_2\right\rangle+\frac{1}{R}\left\langle V_2^{\,2}\right\rangle$$



The noise powers of the two random generators do not add -

- a correllation term must be included.

The instantaneous time values of the random noise voltages do add.

Shot Noise as a Random Variable

The fiber has transmission probability p.

Send one photon, and call the # of received photons N_1.

$$E[N_1] = \overline{N}_1 = p \quad \text{and} \quad E[N_1^2] = p \quad \text{so} \quad \sigma_{N_1}^2 = E[N_1^2] - \overline{N}_1^2 = p - p^2$$

If we now send many photons (M of them), transmission of each is statistically independent, so --- calling the # of received photons N,

$$E[N] = M \cdot E[N_1] = Mp \quad \text{and} \quad \sigma_N^2 = M \cdot \sigma_{N_1}^2 = M(p - p^2)$$

Now suppose $M >> 1$, $p << 1$, and $Mp >> 1$,

$$\rightarrow \sigma_N^2 = \overline{N}$$

The variance of the count approaches the mean value of the count.

Thermal Noise as a Random Variable

A capacitor C is connected to a resistor R .

The resistor is in equilibrium with a "reservoir" (a warm room) at temperature T

R can exchange energy with the room in the form of heat.

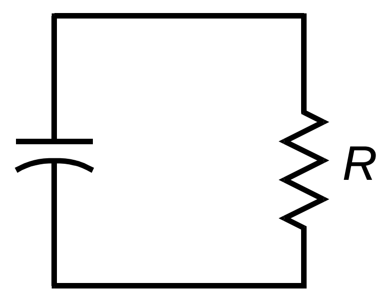
C can dissipate no power : it establishes thermal equilibrium with the room via the resistor.

From thermodynamics, any independent degree of freedom of a system at temperature T has mean energy kT/2, hence

$$\langle E \rangle = kT/2$$

$$\langle CV^2/2 \rangle = kT/2$$

$$\langle V^2 \rangle = kT/C$$



The noise voltage has variance kT/C.

random processes

Random Processes

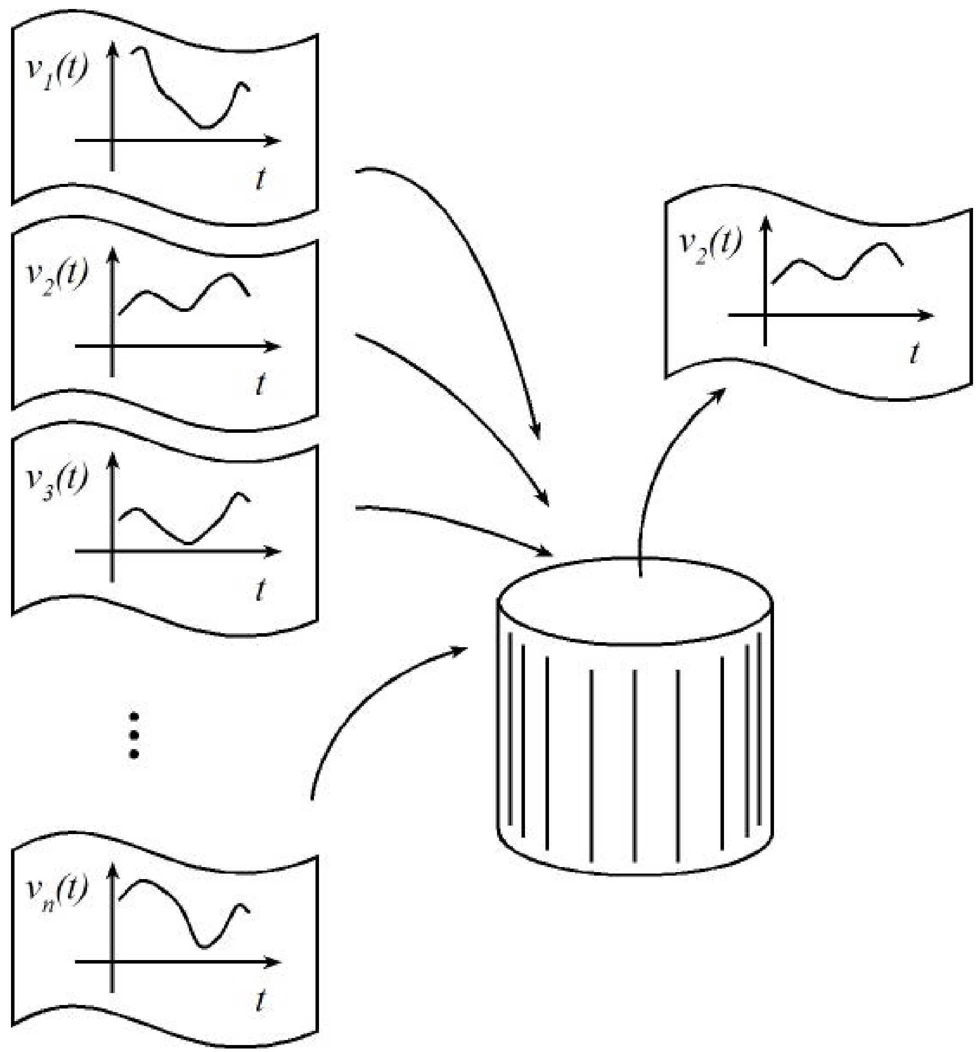
Draw a set of graphs, on separate sheets of paper, of functions of voltage vs. time.

Put them into a garbage can.

This garbage can is called the probability sample space.

Pick out one sheet at random. This is our random function of time.

The random process is $V(t)$. The particular outcome is v(t)

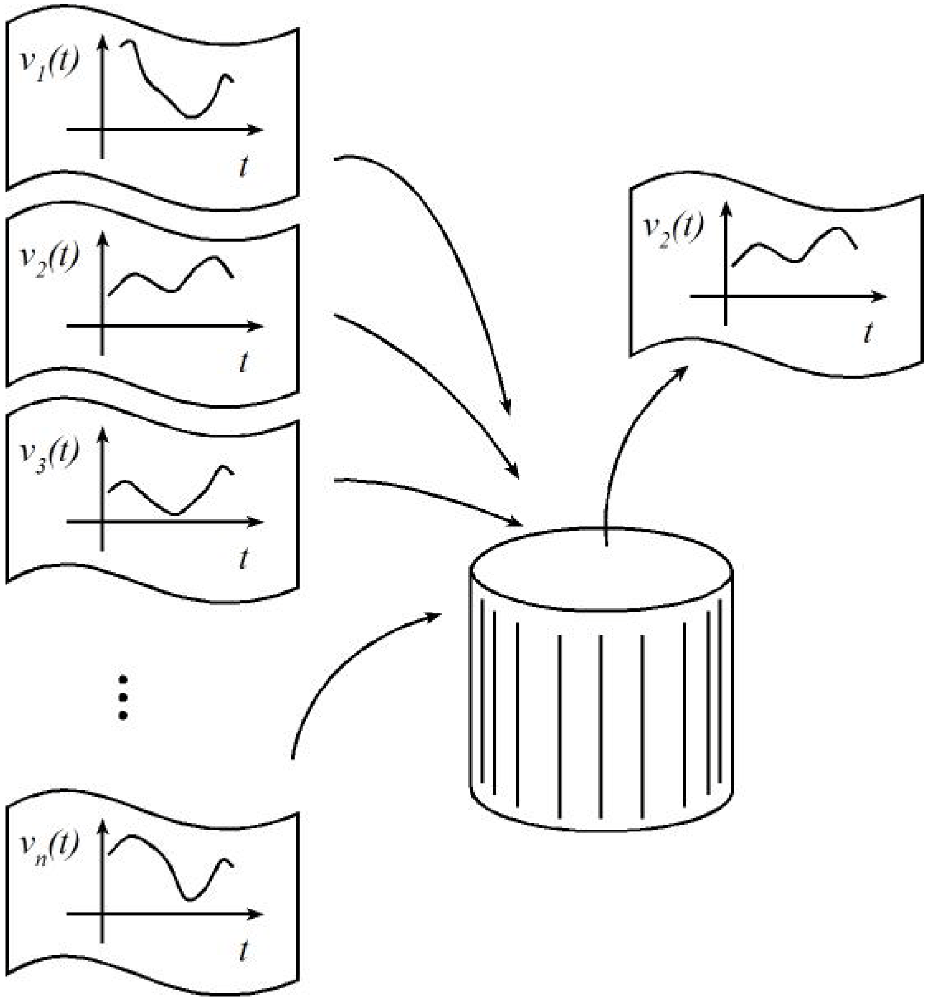


Time Averages vs. Sample Space Averages

Recall the definion of the expectation of a function g(X) of a random variable X

$$E[g(x)] = \int_{-\infty}^{+\infty} g(x)\, f_X(x)\, dx = \overline{g}$$

$\overline{g}$ is the *average value* of g, where the average is over the sample space.



With our random process definition, we can define an average over the sample space at some particular time t_1 :

$$E[g(v(t_1))] = \int_{-\infty}^{+\infty} g(v(t_1))\, f_V(v(t_1))\, d(v(t_1))$$

We can also define an average of the function over time:

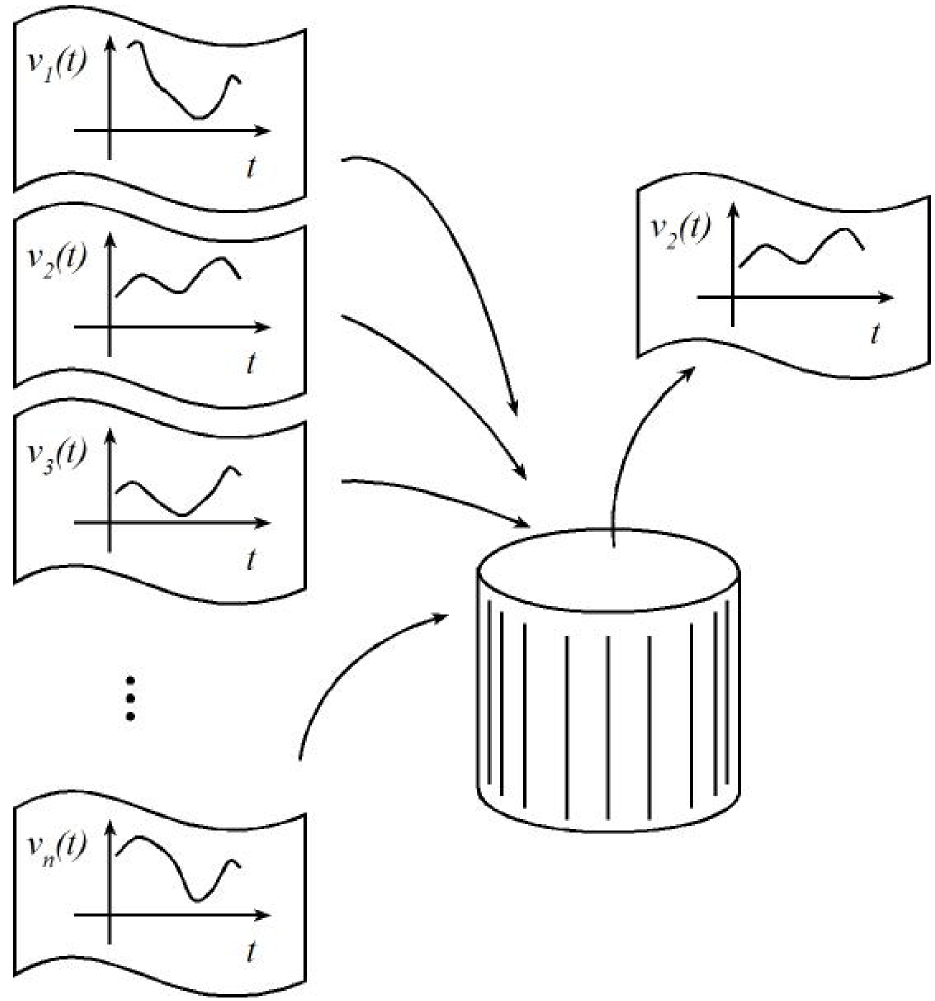
$$A[g(v(t))] = \int_{-\infty}^{+\infty} g(v(t))\, dt$$

Ergodic Random Processes

An Ergodic random process has averages over time equal to averages over the statistical sample space

$$E\left[g(v(t_1))\right] = A\left[g(v(t))\right]$$

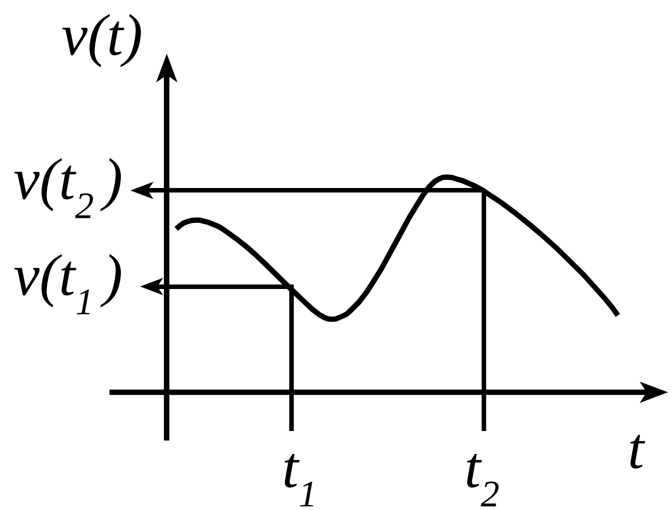
In some sense, we have made "random variation with time" equivalent to "random variation over the sample space"



Time Samples of Random Processes

With time samples at times t_1 and t_2
the random process $V(t)$ has values $V(t_1)$ and $V(t_2)$.

$V(t_1)$ and $V(t_2)$ have some joint probability distribution.
They might (or might not) be jointly Gaussian.



Random Waveforms are Random Vectors

Using Nyquist's sampling theorem,

if a random signal is bandlimited,

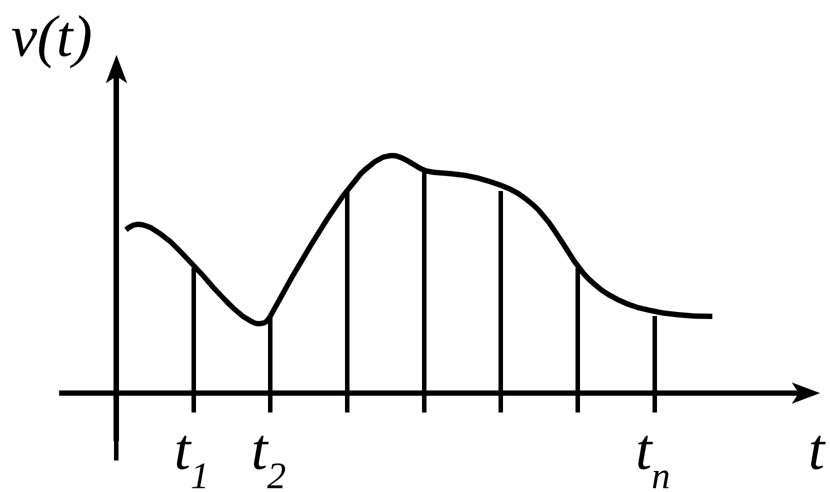
and if we pick regularly - spaced time samples $t_1 ... t_n$,

we convert our random process into a random vector.

We can thus analyze random signals using

vector analysis and geometry.

This is mostly beyond the scope of this class.



Stationary Random Processes

The statistics of a stationary process do not vary with time.

N^{th} – order stationarity :

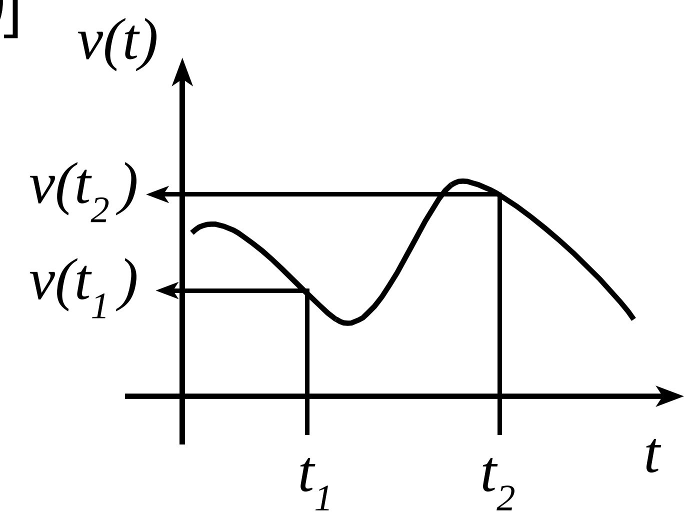
$$E\left[f\left(V(t_1),V(t_2),\ldots,V(t_n)\right)\right]=E\left[f\left(V(t_1+\tau),V(t_2+\tau),\ldots,V(t_n+\tau)\right)\right]$$

..and lower orders

2^{nd} – order stationarity :

$$E\left[f\left(V(t_1),V(t_2)\right)\right]=E\left[f\left(V(t_1+\tau),V(t_2+\tau)\right)\right]$$

lower orders $\rightarrow E\left[f\left(V(t_1)\right)\right]=E\left[f\left(V(t_1+\tau)\right)\right]$



Restrictions on the random processes we consider

We will make following restrictions to make analysis tractable:

The process will be **Ergodic**.

The process will be **stationary** to any order: all statistical properties are independent of time. Many common processes are not stationary, including integrated white noise and 1/f noise.

The process will be **Jointly Gaussian**. This means that if the values of a random process X(t) are sampled at times t_1, t_2, etc, to form random variables $X_1=X(t_1)$, etc, then X_1,X_2, etc. are a jointly Gaussian random variable.

In nature, many random processes result from the sum of a vast number of small underlying random processses. From the central limit theorem, such processes can frequently be expected to be Jointly Gaussian.

Variation of a random process with time

For the random process X(t), look at $X_1=X(t_1)$ and $X_2=X(t_2)$.

$$R_{X_1X_2}=E[X_1X_2]=\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}x_1x_2\bullet f_{X_1X_2}(x_1,x_2)dx_1dx_2$$

To compute this we need to know the joint probability distribution. We have assumed a Gaussian process. The above is called the Autocorrellation function. IF the process is stationary, it is a function only of (t_1-t_2)=tau, and hence

$$R_{XX}(\tau)=E[X(t)X(t+\tau)]$$

this is the autocorrellation function. It describes how rapidly a random voltage varies with time….

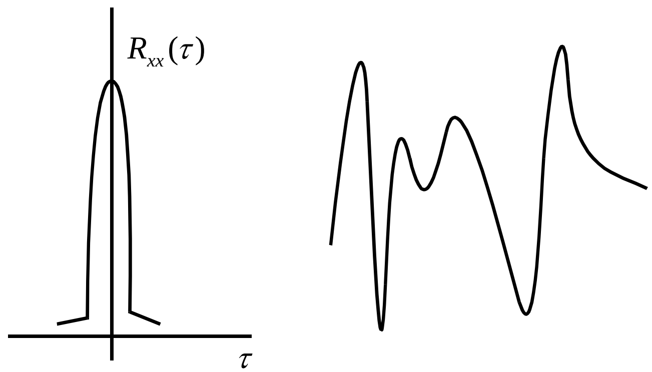
PLEASE recall we are assuming zero-mean random processes (DC bias subtracted). Thus the autocorrellation and the auto-covariance are the same

Variation of a random process with time

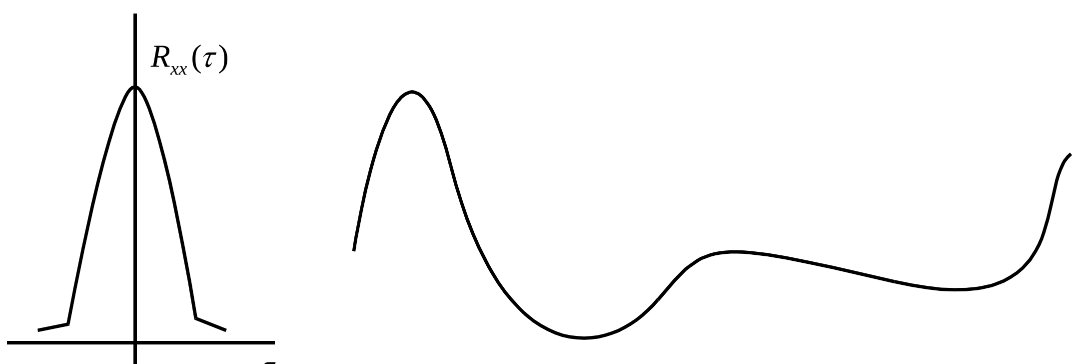
Note that $R_{XX}(0) = E\left[X(t)X(t)\right] = \sigma_X^2$ gives the variance of the random process.

The autocorrelation function gives us variance of the random process and the correlation between its values for two moments in time. If the process is Gaussian, this is enough to completely describe the process.

Narrow autocorrelation:
Fast variation



Broad autocorrelation
Slow variation



Autocorrelation is an Estimate of the Variation with Time

If random variables X and Y are **Jointly Gaussian**, and have **zero mean**,
then knowledge of the value y of the outome of Y results in a best estimate of X as follows:

$$E\left[X \middle| Y = y\right] = \left\langle X \middle| Y = y\right\rangle = \frac{R_{XY}}{\sigma_Y^2} y$$

"The expected value of the random variable X, given that the random variable Y has value y is ..."

Hence, the autocorrellation function tells us the degree to which the signal at time t **is related to the signal at time** $t+\tau$

A narrow autocorrelation is indicative of a quickly-varying random process

Power spectral densities

The autocorrellation function describes how a random process evolves with time.

Find its Fourier transform:

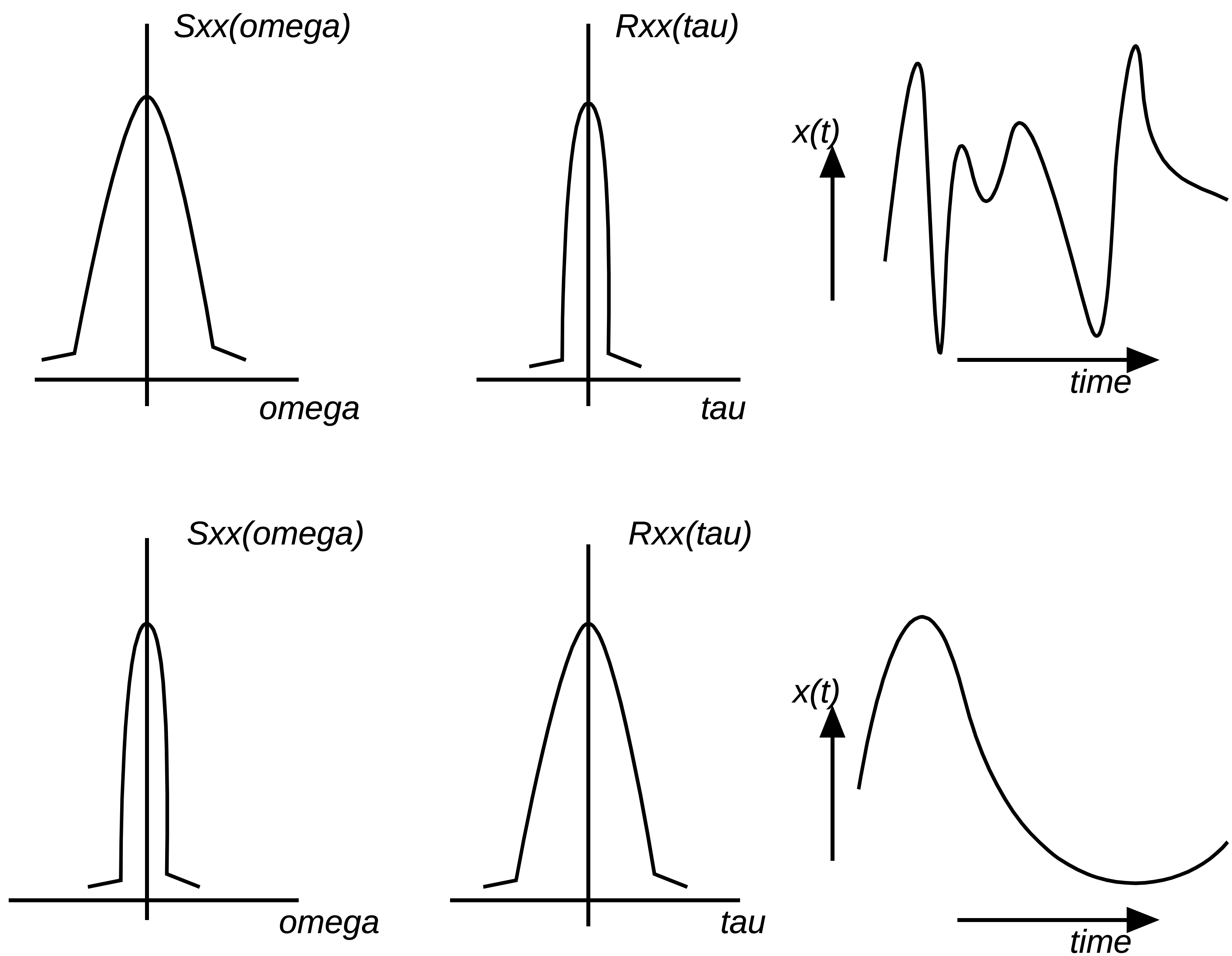
$$S_{XX}(\omega)=\int_{-\infty}^{+\infty} R_{XX}(\tau)\exp(-j\omega\tau)d\tau$$

This is called the power spectral density of the signal.

Remembering the usual Fourier transform relationships, if the power spectrum is broad, the autocorrellation function is narrow, and the signal varies rapidly--it has content at high frequencies, and the voltages of any two points are strongly related only if the two points are close together in time.

If the power spectrum is narrow, the autocorrellation function is broad, and the signal varies slowly--it has content only at low frequencies and the voltages of any two points are strongly related unless if the two points are broadly separated in time.

Power spectral densities



Power Spectral Densities

Recall that the power spectral density is the Fourier transform of the autocorrelation function

$$S_{XX}(\omega) = \int_{-\infty}^{+\infty} R_{XX}(\tau)\exp(-j\omega\tau)d\tau$$

The inverse transform holds, so that

$$R_{XX}(\tau) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} S_{XX}(\omega)\exp(j\omega\tau)d\omega$$

specifically,

$$R_{XX}(0) = \sigma_X^2 = \frac{1}{2\pi}\int_{-\infty}^{+\infty} S_{XX}(\omega)d\omega$$

So, if σ_X^2 is called the power in the process, then integrating the power spectral density will give us the power.

This is the justification for the term, "power spectral density"

Correlated Random Processes

Two processes can be statistically related.

Consider two random processes X(t) and Y(t).

Define the cross-correllation function of the processes

$$R_{XY}(\tau) = E\left[X(t)Y(t+\tau)\right]$$

They will have a cross-spectral density as follows:

$$S_{XY}(\omega) = \int_{-\infty}^{+\infty} R_{XY}(\tau)\exp(-j\omega\tau)d\tau$$

and therefore $R_{XY}(\tau) = \dfrac{1}{2\pi}\displaystyle\int_{-\infty}^{+\infty} S_{XY}(\omega)\exp(j\omega\tau)d\omega$

Single-Sided Hz-based Spectral Densities

Double - Sided Spectral Densities

$$R_{XX}(\tau) = E\left[X(t)X(t+\tau)\right] = \frac{1}{2\pi}\int_{-\infty}^{+\infty} S_{XX}(j\omega)\exp(j\omega\tau)\,d\omega$$

$$S_{XX}(j\omega) = \int_{-\infty}^{+\infty} R_{XX}(\tau)\exp(-j\omega\tau)\,d\tau$$

Single - Sided Hz - based Spectral Densities

$$R_{XX}(\tau) = E\left[X(t)X(t+\tau)\right] = \frac{1}{2}\int_{-\infty}^{+\infty} \tilde{S}_{XX}(jf)\exp(j2\pi f\tau)\,df$$

$$\tilde{S}_{XX}(jf) = 2\int_{-\infty}^{+\infty} R_{XX}(\tau)\exp(-j2\pi f\tau)\,d\tau$$

Single-Sided Hz-based Spectral Densities- Why ?

Why this notation ?

The signal power in the bandwidth $\{f_{\text{low}}, f_{\text{high}}\}$

$$\text{Power} = \frac{1}{2}\int_{-f_{\text{high}}}^{-f_{\text{low}}} \tilde{S}_{XX}(jf)\,df + \frac{1}{2}\int_{f_{\text{low}}}^{f_{\text{high}}} \tilde{S}_{XX}(jf)\,df = \int_{f_{\text{low}}}^{f_{\text{high}}} \tilde{S}_{XX}(jf)\,df$$

$\rightarrow \tilde{S}_{XX}(jf)$ is directly the Watts of signal power

per Hz of signal bandwidth

at frequencies lying close to the frequency f.

Single-Sided Hz-based Cross Spectral Densities

Double - Sided Cross Spectral Densities

$$R_{XY}(\tau) = E[X(t)Y(t+\tau)] = \frac{1}{2\pi}\int_{-\infty}^{+\infty} S_{XY}(j\omega)\exp(j\omega\tau)d\omega$$

$$S_{XY}(j\omega) = \int_{-\infty}^{+\infty} R_{XY}(\tau)\exp(-j\omega\tau)d\tau$$

Single - Sided Hz - based Cross Spectral Densities

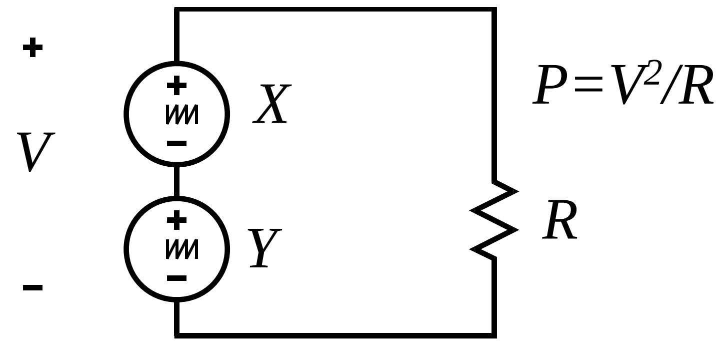
$$R_{XY}(\tau) = E[X(t)Y(t+\tau)] = \frac{1}{2}\int_{-\infty}^{+\infty} \tilde{S}_{XY}(jf)\exp(j2\pi f\tau)df$$

$$\tilde{S}_{XY}(jf) = 2\int_{-\infty}^{+\infty} R_{XY}(\tau)\exp(-j2\pi f\tau)d\tau$$

$\tilde{S}_{XY}(jf)$ is also often written as $\dfrac{d}{df}\langle XY\rangle$

Example: Cross Spectral Densities

$$V(t) = X(t) + Y(t)$$



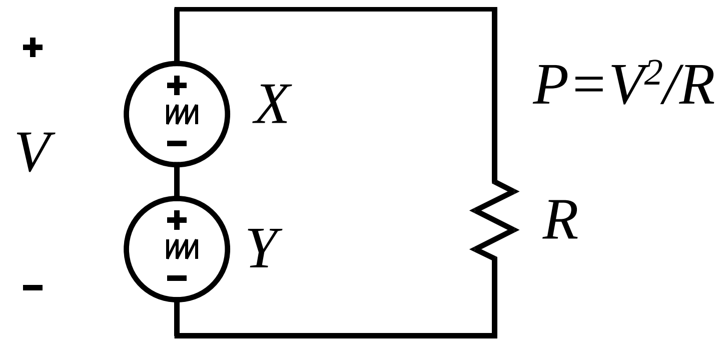
$$R_{VV}(\tau) = E\left[\left(X(t)+Y(t)\right)\left(X(t+\tau)+Y(t+\tau)\right)\right]$$
$$= R_{XX}(\tau) + R_{YY}(\tau) + R_{XY}(\tau) + R_{YX}(\tau)$$

$$S_{VV}(j\omega) = S_{XX}(j\omega) + S_{YY}(j\omega) + S_{XY}(j\omega) + S^*_{XY}(j\omega)$$
$$= S_{XX}(j\omega) + S_{YY}(j\omega) + 2\cdot \mathrm{Re}\left\{S_{XY}(j\omega)\right\}$$

Or, in single-sided spectral densities

$$\tilde{S}_{VV}(jf) = \tilde{S}_{XX}(jf) + \tilde{S}_{YY}(jf) + 2\cdot \mathrm{Re}\left\{\tilde{S}_{XY}(jf)\right\}$$

Example: Cross Spectral Densities



The Power $P = V^2(t)/R$ has expected value

$E[V(t)V(t)/R] = R_{VV}(0)/R$

And in the bandwidth between f_{low} and f_{high},

$$\overline{P} = \int_{f_{low}}^{f_{high}} \tilde{S}_{VV}(jf)\,df \ldots$$

Integrating with respect to frequency (over whatever bandwidth is relevant) gives the total (expected) power dissipated in R.

Note that the cross - spectral density is relevant.

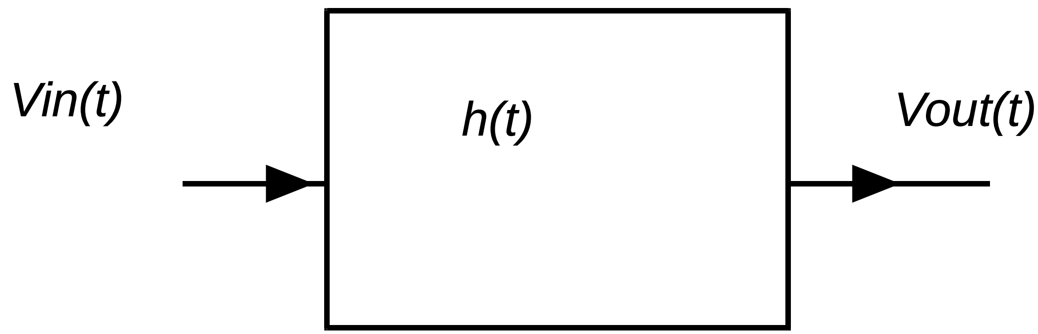
Noise passing through filters & linear electrical networks

If the filter has impulse response $h(t)$ and transfer function $H(j\omega)$,
then for any $v_{in}(t) \rightarrow v_{out}(t)$, $V_{out}(j\omega) = H(j\omega)V_{in}(j\omega)$

We can show

$$S_{V_{out}V_{in}}(j\omega) = H(j\omega)S_{V_{in}V_{in}}(j\omega)$$

$$S_{V_{in}V_{out}}(j\omega) = S_{V_{in}V_{in}}(j\omega)H^*(j\omega)$$



and

$$S_{V_{out}V_{out}}(j\omega) = \left\| H(j\omega) \right\|^2 S_{V_{in}V_{in}}(j\omega)$$