Abstract—For a wireless network in which every node is bounded in its energy supply, we define a new concept of network capacity called “bits-per-Joule capacity”, which is the maximum total number of bits that the network can deliver per Joule of energy deployed into the network. For a fixed network size, a finite number of information bits is delivered for each source-destination pair, under a fixed end-to-end probability of error constraint. We prove that under the one-to-one traffic model in which every node wants to send traffic to a randomly chosen destination node, the bits-per-Joule capacity of a stationary wireless network grows asymptotically as $\Omega\left(\frac{N}{\log N}\right)^{\frac{q-1}{2}}$, where $N$ is the number of nodes randomly deployed onto the surface of a sphere and $q$ is the path loss exponent. Further, the length of the block codes used grows only logarithmically in $N$, which indicates manageable decoder complexity as the network scales. The fact that the bits-per-Joule capacity grows with the number of nodes contrasts sharply with the scaling laws that have been derived for throughput capacity and implies that large-scale deployments for energy-limited sensor and ad hoc networks may be suitable for delay-tolerant data applications.

Index Terms—Capacity, wireless, ad hoc network, sensor network, energy conservation, scaling, networks, delay tolerant.

I. INTRODUCTION

W e develop a new metric called “bits-per-Joule capacity” to analyze the performance of energy-limited wireless sensor and ad hoc networks. Roughly speaking, we define the bits-per-Joule capacity of an energy-limited wireless network as the maximum number of bits that the network can deliver per Joule of energy deployed into the network. Much of the previous work [5][6][18][22][23] that analyzed the capacity of ad hoc and sensor networks has taken the throughput capacity (in bits-per-second) and the transport capacity (in bit-meters-per-second) as the main metrics with which to measure performance. The throughput and transport capacities are relevant for data-rate-intensive applications such as image and video transmission between devices with large energy supplies. However, many of the energy-limited networks in deployment today, such as underwater eco-sensing, habitat monitoring [24], animal-tracking [10], seismic exploration, and structural monitoring networks, transfer a relatively small, finite amount of data over tiny nodes that are severely limited in their energy supplies. By virtue of the recent low-cost, low-power implementations [11] of RF radios, hundreds to thousands of these energy-limited sensor nodes are expected to be deployed in the near future. New metrics are required to assess the capacity of these networks and to determine the potential for large-scale deployments that carry delay-tolerant data over energy-limited devices.

The bits-per-Joule capacity serves as a metric to measure the performance of such wireless networks made up of tiny nodes.1 The main difference between the throughput and bits-per-Joule measures is that the bits-per-Joule capacity measures the transmission of a finite number of bits across an energy-limited network. It is well-known that the transmission of a finite number of information bits reliably (in the Shannon sense) across an AWGN channel requires an infinite energy per bit. In contrast, it can be shown from [17] that the energy required for the transmission of a finite number of bits across a single link under a fixed, positive probability of error (used as a QoS metric) is finite, increases with the data rate, and decreases with the number of information bits per block (if block codes are used). Further, the energy per information bit decreases as the target probability of error increases.

In our context of nodes with severely limited energy supplies, we have to solve a number of problems that arise: (1) We must assure that as the number of nodes grows, the energy expenditure for a finite number of bits travelling across the network remains bounded. (2) We stipulate that an end-to-end probability of error of $\epsilon$ be guaranteed to all the transmissions. However, as the network scales and the number of hops to reach the destination increases, a probability of bit error that becomes tighter and tighter must be maintained on each of the links, driving up the transmit energy to send a finite number of bits. (In this paper, we use reductions in transmit energy

1In this paper, we focus on the energy-limited nature of both sensor and ad hoc networks and do not distinguish between sensor and ad hoc networks based on data-centricity [25].
due to shrinking internodal distances to counterbalance this effect.) (3) As the number of nodes increases, the relay traffic through a node increases, consuming the energy supply of that node for relays. (4) All the end-to-end transmissions must complete in a finite amount of time. (5) The buffer space used at each node must be finite. (6) The block lengths of all the channel codes must grow slowly with the number of nodes in the network to lead to realistic implementations. The main question is whether a strategy exists that can balance all these effects in favor of transmitting an increasing number of bits per Joule as the network scales.

The main result of this paper is the derivation of a strategy that satisfies the above conditions and establishes a lower bound for bits-per-Joule capacity that grows as a function of the number of energy-limited nodes randomly deployed over a fixed area. This growth is observed over a regime in which the transmit energy remains dominant over receiver and processor energies. Since novel low-energy transceiver architectures [11] continue to drive down the receiver and processing energies, the regime over which this growth holds will become more significant as the circuit technology advances. This main result stands in sharp contrast with the scaling laws derived [6][18] for throughput capacity, which indicate that the throughput capacity goes to zero with the number of nodes and that wireless networks cannot scale up in the number of users [6]. The main difference between the two scenarios is that for each network realization (with a fixed number of nodes), we transmit a finite number of bits as a function of the number of energy-limited nodes random deployed over a fixed area. This growth is observed over a regime in which the transmit energy remains dominant over receiver and processor energies. Since novel low-energy transceiver architectures [11] continue to drive down the receiver and processing energies, the regime over which this growth holds will become more significant as the circuit technology advances. This main result stands in sharp contrast with the scaling laws derived [6][18] for throughput capacity, which indicate that the throughput capacity goes to zero with the number of nodes and that wireless networks cannot scale up in the number of users [6]. The main difference between the two scenarios is that for each network realization (with a fixed number of nodes), we transmit a finite number of bits across a network of energy-limited nodes, whereas [6] seeks to transmit a constant throughput per node across a network of devices with no constraints on the energy supplies. The same finite bandwidth is used in both schemes and the difference in these scaling laws is not due to bandwidth expansion.

The rest of this paper is organized as follows: In Section II, we discuss the related work in this area and distinguish our work from previous results. In Section III, we state our basic assumptions and mathematical models. In Section IV, we develop the concept of the bits-per-Joule capacity of an energy-limited network. In Section V, we state an achievable asymptotic lower bound for bits-per-Joule capacity for a network that is randomly deployed onto the surface of a sphere, and we prove this result in Section VI. In Section VII, we discuss the results of this work.

II. RELATED WORK

The fundamental results for energy-efficient transmission on a point-to-point link can be traced back to [4]. A restatement of these results appears in [9], where capacity in bits-per-Joule is given for a single link on flat and frequency-selective additive white Gaussian noise (AWGN) channels. These concepts are re-investigated in [3] and [17]. In [2], the authors describe a joint source-channel coding technique that minimizes the total energy consumption subject to a maximum level of acceptable distortion. Even though transmit energy consumption is modeled in that paper, only the case of a point-to-point channel rather than a network is addressed. The design

III. BASIC ASSUMPTIONS AND MODELS

A. Basic Assumptions

We make the following basic assumptions in this paper: (1) Every node has a bounded amount of energy. (2) There is a finite number of nodes in the network. (3) Every node sees a wireless channel to every other node in the network. (4) Every node has the capability to encode and decode bits. (5) Every node has the capability to generate an unbounded number of information bits that it wishes to send to other nodes that it can select. (6) It takes a finite amount of time to generate a finite number of bits, to encode a finite number of bits and to decode a finite number of bits at each node. (7) Every node’s transmitter can vary its transmit power continuously. (8) There is a bounded bandwidth W (measured in Hz) within which all of the nodes in the network must operate. (9) Every node is allowed to relay its traffic via any other node in the network. (10) It is possible to start or stop the generation or transmission of a traffic stream at any node at any time, as decided by a global scheduler. (11) Receive and processing energy consumption is assumed to be much smaller than transmit energy consumption and is ignored. (12) It is possible to reduce the spectral efficiency continuously towards zero for any of the transmissions (via the use of channel coding).

B. Channel, Traffic and Network Models

In this subsection, we describe our channel, traffic and network models. We define a “node set” N to be a non-empty, finite collection of nodes and denote its cardinality by N. We denote the set of real numbers by R, the set of non-negative real numbers by R+, and the set of positive real numbers by R+. Similarly, we denote the set of integers by Z, the set of
non-negative integers by $\mathbb{Z}_+$, and the set of positive integers by $\mathbb{Z}^+$. Further, we let $\mathcal{B}$ denote the set \{0, 1\}.

We use the following multiuser channel model in this paper: a "memoryless additive white Gaussian noise (AWGN) channel model" $H$ on a node set $\mathcal{N}$ is a double $(\mathcal{G}, \tilde{\mathcal{N}})$ where: (1) $\mathcal{G} \in \mathbb{R}^{N \times N}$ is the "channel gain matrix", with $0 < G_{ij} < \infty \forall (i, j) \in \mathcal{N} \times \mathcal{N}$ with $i \neq j$, and (2) $\tilde{\mathcal{N}}(t) = (\mathcal{N}, \mathcal{R}^+) \rightarrow \mathbb{R}^N$ is a continuous-time vector Gaussian process such that the following hold: (a) $\Omega$ is a probability space, (b) $\text{Var}(\tilde{n}_j(t)) = \sigma_j^2 \delta(t) \forall t \in \mathbb{R}^+ \forall j \in \mathcal{N}$, where $\sigma_j^2 < \infty \forall j \in \mathcal{N}$, and (c) $\tilde{n}_j(t)$ is independent of $\tilde{n}_k(t')$ unless $j = k$ and $t = t'$. The received signal at node $j$ is given by:

$$Y_j(t) = \sum_{i \in \mathcal{N} \setminus \{j\}} \sqrt{G_{ij}} X_i(t) + \tilde{n}_j(t) \tag{1}$$

$\forall t \in \mathbb{R}^+$, where $X_i(t)$ is the signal emitted by node $i$ at time $t$.

We say that a node $l$ generates (or has) a traffic demand for node $m$ if $l$ wishes to send information to $m$. A "demand matrix" $\Gamma$ describes the set of source-destination pairs with positive demands: $\Gamma_{lm} = 1$ if node $l$ wishes to send traffic to node $m$, and $\Gamma_{lm} = 0$ otherwise. Since no node is allowed to have a traffic demand for itself, $\Gamma_{ll} = 0 \forall l$. An energy-limited network incorporates a channel model, node energies, and a demand matrix:

**Definition 1 (Energy-limited network):** An energy-limited network $G$ is a quadruple $(\mathcal{N}, H, \mathbf{E}, \mathbf{\Gamma})$ where: (1) $\mathcal{N}$ is a node set, (2) $H$ is a memoryless AWGN channel model, (3) $\mathbf{E} \in \mathbb{R}^N_{\mathbb{R}^+}$ is the "vector of node energies" with $0 < E_i < \infty \forall i \in \mathcal{N}$, and (4) $\mathbf{\Gamma} \in \mathbb{B}^{N \times N}$ is a demand matrix.

We will specify a node model for each node in the network. Each node is able to generate its own bits, decode the transmissions it receives, and encode the bits (for its own transmission as well as when acting as a relay). As stated in Section III-A, we will use a global scheduler that schedules these transmissions. A schedule for an energy-limited network specifies when each of the nodes in the network begins its GENERATE, ENCODE and DECODE operations. This will later allow us to state precisely how the bits will reach from the sources to the destinations. Further, each node maintains a FIFO queue in which the bits for its own transmissions and the bits to be relayed are queued. The following definitions formalize these arguments:

**Definition 2 (Schedule):** Let $G$ be an energy-limited network. A schedule $S$ on $G$ is a structure indexed by $j \in \mathcal{N}$ and $t \in \mathbb{R}^+$: $S_{jt} \leq t < t_0 + T$, where $t_0 \in \mathbb{R}$ is called the "starting time" and $T \in \mathbb{R}^+$ is called the "schedule length". Each element of $S$ is denoted by $S_{jt}$ and consists of the following: (1) The Boolean variables ENC ("Encode"), DEC ("Decode"), GEN ("Generate") and ENBL ("Enable"), and the values of these variables. (2) An "encoder index" $k \in \mathbb{N} \setminus \{j\}$ defined iff $ENC = 1$. (3) A "decoder index" $i \in \mathbb{N} \setminus \{j\}$ defined iff $DEC = 1$. If $ENC = 0$, then item (2) above is null, and if $DEC = 0$, then item (3) above is null.

**Definition 3 (Node model):** Let $\mathcal{N}$ be a node set. A node model for node $j \in \mathcal{N}$ is a structure that consists of:

1) A family of $(N - 1)$ encoders $\tilde{E}_j$ such that $\tilde{E}_j : M_j \rightarrow X_{n_j}^{\mathcal{N} \setminus \{j\}} \forall k \in \mathcal{N} \setminus \{j\}$, where $M_j$ is the set of messages from which $j$ selects when it wants to send information to node $k$, $X_{n_j}^{\mathcal{N} \setminus \{j\}}$ is the output alphabet of the encoder $\tilde{E}_j$, and $n_j$ is the block length of encoder $\tilde{E}_j$. The "rate" of the encoder is defined as $R_j \equiv \frac{\log |M_j|}{n_j}$.

2) A family of $(N - 1)$ decoders $\tilde{D}_j$ such that $\tilde{D}_j : Y_{n_j}^{\mathcal{N} \setminus \{j\}} \rightarrow M_j \forall i \in \mathcal{N} \setminus \{j\}$, where $Y_{n_j}^{\mathcal{N} \setminus \{j\}}$ is the alphabet of the input to decoder $\tilde{D}_j$, and $M_j$ is the set of messages from which $i$ selects a message to send to node $j$.

3) A family of $(N - 1)$ bit-to-message mapping functions $\tilde{h}_j$ such that $\tilde{h}_j : B_{n_j}^{\mathcal{N} \setminus \{j\}} \rightarrow M_j \forall k \in \mathcal{N} \setminus \{j\}$, where $M_j$ is as in item 1 above, and $\tilde{l}_j \in \mathbb{Z}^+$ is the number of information bits encoded into a message in $M_j$.

4) A family of $(N - 1)$ message-to-bit mapping functions $\tilde{g}_j$ such that $\tilde{g}_j : M_j \rightarrow B_{n_j}^{\mathcal{N} \setminus \{j\}} \forall k \in \mathcal{N} \setminus \{j\}$, where $M_j$ as in item 2 above, and $l_j \in \mathbb{Z}^+$ is the number of information bits encoded into a message in $M_j$.

5) Boolean input variables ENC, DEC, GEN and ENBL (from a schedule $S$).

6) Input variables "decoding index" and "encoding index" (from a schedule $S$).

7) A "bit generator function" (or a "generator") $Q : B \rightarrow B$ that generates an information bit iff $ENBL \bullet GEN = 1$. Let $B_i[r]$ denote a bit generated by node $i$ at discrete-time $r$. We assume that each node can generate a possibly infinite number of bits and that $B_i[r]$ is statistically independent of $B_j[r']$ unless $i = j$ and $r = r'$.

8) A first-in-first-out ("FIFO") buffer that has the following capabilities: (a) Accepts information bits from decoder $\tilde{D}_i$ iff $ENBL \bullet DEC = 1$ and the decoding index is $i$. (b) Accepts information bits from the generator iff $ENBL \bullet GEN = 1$. (c) Puts out bits to encoder $\tilde{E}_j$ iff $ENBL \bullet ENC = 1$ and the encoding index is $j$.

**Definition 4 (Energy-limited network model):** An energy-limited network model $K_G$ consists of: (1) an energy-limited network $G = (\mathcal{N}, H, \mathbf{E}, \mathbf{\Gamma})$, (2) a node model $\forall i \in \mathcal{N}$, and (3) a schedule $S$ on $G$.

IV. Bits-per-Joule Capacity

We now proceed to a definition of achievability that is suitable for energy-limited wireless networks. This new notion of achievability is called $(\epsilon, \eta)$-achievability. One of our major goals in this paper is to assure a probability of information bit error from every source to its destination, of less than or equal to $\epsilon$, where $\epsilon > 0$ is chosen arbitrarily close to zero and then fixed. In wireless communications, achieving an arbitrarily low but fixed probability of information bit error from every source to every destination can serve as a measure of the Quality-of-Service (QoS) for that connection and is an acceptable criterion.

We denote the amount of traffic delivered end-to-end from node $l$ to node $m$ by $D_{lm}$, which is measured in bits (not bits per second). The $N \times N$ matrix $D$ is called the "traffic matrix". We define a function $f$ that maps the traffic on that network to a single number that incorporates the desirability criteria for a network designer. For example, this function may very well be the sum of all the traffic. In order to keep our framework general, we will parameterize our definition of achievable
by this function $f$. Further, we attribute the units of “bits” to $f(D)$.

Throughout the rest of this paper, we fix $G = (\mathcal{N}, H, E, \Gamma)$ to be an energy-limited network, and $K_G$ to be an energy-limited network model on $G$.

**Definition 5** ($(\epsilon, f)$-achievability): Let $b \in \mathbb{R}_+$, $\epsilon \in (0, 1]$ and $f : \mathbb{R}^{N \times N}_+ \rightarrow \mathbb{R}_+$. We say that $b$ bits-per-Joule is $(\epsilon, f)$-achievable on $K_G$ iff there exist (1) a family of $(N-1)$ encoders and a family of $(N-1)$ decoders in the node model of every node in $\mathcal{N}$, (2) a finite-length FIFO buffer in the node model of every node in $\mathcal{N}$, and (3) a finite-length schedule $S$ in $K_G$ such that

(a) A traffic matrix $D$ can be delivered with $\sum_{i \in \mathbb{E}} f(D) \geq b$, and

(b) $P_b \overset{\text{def}}{=} P\{\hat{B}_{lm} \neq B_{lm}\} \leq \epsilon$, where $B_{lm}$ denotes an information bit that $l$ wishes to send to $m$ and $\hat{B}_{lm}$ denotes the decision of node $m$ on information bit $B_{lm}$.

We define the “bits-per-Joule capacity” of an energy-limited network model as the least upper bound of the number of bits-per-Joule $b$ that is $(\epsilon, f)$-achievable.

**Definition 6** (Bits-per-Joule capacity): Let $\epsilon \in (0, 1]$ and $f : \mathbb{R}^{N \times N}_+ \rightarrow \mathbb{R}_+$. The bits-per-Joule capacity of $K_G$ is defined as

$$C_J(K_G; \epsilon, f) \overset{\text{def}}{=} \sup\{b \mid b \text{ is } (\epsilon, f)\text{-achievable}\}$$

where the supremum is taken over the set of families of encoders and decoders in the node models and the set of schedules on $G$.

### V. ASYMPTOTIC BEHAVIOR

In this section, our main goal will be to prove an asymptotic $(\epsilon, f)$-achievable solution that will give a lower bound for bits-per-Joule capacity. We will prove this under the same deployment and traffic model as in [6]; however, we will also introduce a model for the nodes’ energies. The following is the statement of the traffic model:

**Definition 7** (One-to-one traffic model): For an energy-limited network model with more than 1 node, the one-to-one traffic model is defined as follows: Each node $i \in \mathcal{N}$ has a demand for exactly one other randomly picked node $s(i)$, and $s(i)$ and $s(j)$ are independent if $i \neq j$.

The path loss model describes the fall-off of transmit power with distance:

**Definition 8** (Path loss model): The path loss model for a memoryless AWGN channel model is the following relationship between the channel gain $G_{ij}$ and the distance $d_{ij}$ between nodes $i$ and $j$ with $i \neq j$: $G_{ij} = K^{-1}(d_{ij})dq_i^q \forall d_{ij} \geq d_0$ where $q \in \mathcal{Q}$ and $q \geq 2$, and $q$ is called the “path loss exponent”; $d_0 > 0$ and $d_0$ is called the “reference distance”; and $K(d_0, q) \in \mathbb{R}_+$ is a constant that depends on $d_0$ and $q$.

The following theorem is the main result of this paper:

**Theorem 1:** Assume that the nodes in $\mathcal{N}$ are deployed onto the surface $S^2$ of a fixed sphere. Let $T_i$ denote the position of node $i$, and let $\{T_i\}_{i \in \mathcal{N}}$ be i.i.d. uniform random variables on $S^2$. Assume that there exist deterministic scalars $E_{min}$ and $E_{max}$ that are functionally independent of $N$ such that $0 < E_{min} < E_{max} < \infty$, and $E_{min} \leq E_i \leq E_{max}$ \forall i \in \mathcal{N}$. Assume that the $\{\sigma_i\}_{i \in \mathcal{N}}$ in the channel model $H$ have an upper bound that is functionally independent of $N$.

Assume that the path loss model in Definition 8 holds. Define the function $f_{\text{sum}}(D) \overset{\text{def}}{=} \sum_{i \in \mathcal{N}} \sum_{m \in \mathcal{N}} D_{im}$. Then, under the one-to-one traffic model, there exists a deterministic positive constant $c_0$ that is functionally independent of $N$ (and dependent on $q$) such that $\forall \epsilon \in (0, 1]$ $\lim_{N \rightarrow \infty} P(C_J^{(N)}(K_G; \epsilon, f_{\text{sum}}) \geq c_0 \left(\frac{N}{\log N}\right)^{(q-1)/2}) = 1$

The above definition of $C_J$ on $N$ is explicitly noted via the superscript on $C_J$. This theorem says that the bits-per-Joule capacity of the network grows as $\Omega((N/ \log N)^{(q-1)/2})$. This result contrasts sharply with that of [6] which showed that the bits-per-second throughput capacity goes to zero with the number of nodes.

Next, we present the proof of this theorem.

### VI. PROOF OF THEOREM 1

**Proof Strategy:** The first part of the proof (which culminates in Lemma 3) constructs a transmission strategy to transfer a finite number of bits along a path $p$ from the source to the destination while satisfying the following four conditions: (1) Each link on this path operates at a point that is within a fixed factor of $\gamma$ of the minimum energy limit required to transmit reliably along that link. (2) The end-to-end probability of error (from the source and the destination) is maintained at $\epsilon$. (3) The transmission of every bit from the source to the destination completes in a finite amount of time. (4) The length of the FIFO buffer at each node on the path is finite. The proof strategy is as follows: First, a common rate $R$ is chosen (to be used on all of the links on the path) such that each link operates with the factor $\gamma$ of the minimum energy limit. Second, the length $n$ of a block code is chosen based on the data rate $R$, the number of nodes $N$ in the network (which gives an upper bound on the number of hops on the path) and the end-to-end probability of error $\epsilon$. An important result is that the length of the block codes used asymptotically grows as the logarithm of $N$, which shows that the block lengths scale well with network size under this strategy. Finally, we prove that the time required for the transmission to complete end-to-end is finite and that finite FIFO buffer lengths suffice at each node on the path.

The second part of the proof uses the Gupta-Kumar tessellation construction on a sphere. The end-to-end transmissions are scheduled by the global scheduler. The transmission along each of the paths (for each source-destination pair) can be done in any order, but we assume that one starts only after another path completes. The key point in this part of the proof is that the energy available per relay route increases as the distances shrink and the transmit powers are scaled down, and that this increase in the available transmit energy per route is faster than the rate at which the number of routes through a node grows. This leads to the rate of growth in the constructed lower bound for bits-per-Joule capacity.

**Part 1:**

A “path” on $\mathcal{N}$ is a sequence of nodes in $\mathcal{N}$ that has at least 2 nodes. The path model may be conceived of as the implementation of a path:
Definition 9 (Path model): Let $p$ be a path on $\mathcal{N}$. A path model $O^p$ on $(p, K_G)$ consists of (1) a concatenation of the node models of the nodes on path $p$ in the order that the nodes appear in path $p$, and (2) a schedule $S^p$ on $G$.

A path that has no repetition of any node in $\mathcal{N}$ is called “acyclic”. The “length” of an acyclic path is the number of nodes on the path minus 1. Throughout the rest of this proof, for any acyclic path $p$ defined, the sequence of nodes on $p$ will be given by $l \rightarrow k_1 \rightarrow k_2 \rightarrow k_3 \rightarrow \ldots \rightarrow k_{L-1} \rightarrow m$ with $k_0 = l$ (the first node of $p$) and $k_L = m$ (the last node of $p$).

Definition 10 (Simple path codec): Let $R \in \mathbb{R}^+$, $n \in \mathbb{Z}^+$, and let $M$ be a set of $2^n R$ messages. Let $p$ be an acyclic path. Assume that a finite number $B$ of information bits is to be sent from $l$ to $m$ via $p$. A simple path codec is a path model on $(p, K_G)$ such that:

1. $R_{k_{i-1}, k_i} = R$ and $n_{k_{i-1}, k_i} = n \forall i : 1 \leq i \leq L$, and
2. $M_{k_{i-1}, k_i} = M \forall i : 1 \leq i \leq L$, and
3. $W_{k_{i-1}, k_i} = \hat{W}_{k_{i-1}, k_i} \forall i : 1 \leq i \leq L - 1$; that is, $\hat{h}_{k_{i-1}, k_i}(g_{k_{i-1}, k_i}(w)) = w \forall w \in M \forall i : 1 \leq i \leq L - 1$, and
4. The schedule $S^p$ is given in Table I. Each subtable in this table is read as follows: The Boolean variables index the columns and continuous-time intervals index the row, $t_0$ is the time at which schedule $S^p$ starts and $T_0 \in \mathbb{R}^+$ is a length of time that is at least as long as the maximum decoder, encoder and generator delays for $B$ bits across all the nodes in $O^p$. A “1” in row $r$ and column $c$ below indicates that the Boolean variable that appears on that column $c$ is 1 during the time interval described by the the right hand side of row $r$. A “0” indicates that this variable is 0 during that time interval. An “X” indicates that we “don’t care” about the value of this variable during that time interval. An “=” indicates that this variable is 0 during that time interval. An “=” indicates that this variable is 1 during that time interval. An “N” indicates that the Boolean variable bit error of at most $\epsilon$ in row $r$ and column $c$ below indicates that the Boolean variable that appears on that column $c$ is 1 during the time interval described by the the right hand side of row $r$. A “0” indicates that this variable is 0 during that time interval. An “X” indicates that we “don’t care” about the value of this variable during that time interval. An “=” indicates that this variable is 0 during that time interval. An “=” indicates that this variable is 1 during that time interval.

Remark: Since all of the transmissions on schedule $S^p$ above start at the earliest at $t_0$ and finish by time $t_0 + (L + 1)T_0$, the length (in seconds) of schedule $S^p$ is $(L + 1)T_0$.

Definition 11 (Probability of information bit error): Let $O^p$ be a path model on $K_G$. The probability of information bit error on the path model $O^p$ is denoted by $P_b(O^p)$ and is defined as $P_b(O^p) \equiv P\{\hat{B}_{lm}(O^p) \neq B_{lm}(O^p)\}$ where (1) $B_{lm}$ is an information bit that wishes to send to $m$ via path $p$, and (2) $\hat{B}_{lm}(O^p)$ is the decision of node $m$ on information bit $B_{lm}$, when the path model $O^p$ is used to deliver the bit from $l$ to $m$.

Definition 12 (Probability of message error on a link): The probability of message error on the link from node $j$ to node $k$ is defined as $P_{e-j,k}^{1-k} \equiv P\{\hat{W}_{jk} \neq W_{jk}\}$ where: (1) $W_{jk}$ is the output of $h_{jk}$ in node $j$, (2) $W_{jk}$ is the output of $D_{jk}$ in node $k$.

Our main idea in achieving an end-to-end probability of message bit error of at most $\epsilon$ is to achieve a probability of message error of $\frac{1}{L+1}$ at each link on a path:

Lemma 1: Let $p$ be an acyclic path on $\mathcal{N}$, of length $L$. Let $\epsilon \in (0,1]$, $N \geq 2$, and $N' \in \mathbb{Z}^+$ with $N' \geq L + 1$. Let $O^p$ be a simple path code on $(p, K_G)$. If $P_{e-j,k}^{1-k} \leq \frac{\epsilon}{L+1} \forall i : 0 \leq i \leq L - 1$, then $P_b(O^p) \leq \epsilon$.

Proof: Let $P_b(O^p) \equiv 1 - P_b(O^p)$. Then, $P_b(O^p) \geq \prod_{i=0}^{L-1} P\{W_{k_i,k_{i+1}} = W_{k_i,k_{i+1}} \forall i : 0 \leq i \leq L - 1\} = \prod_{i=0}^{L-1} (1 - \frac{\epsilon}{L+1}) \geq (1 - \epsilon) \geq (1 - \epsilon)$, where we have used the fact that $W_{k_0,k_1} \rightarrow W_{k_1,k_2} \rightarrow \ldots \rightarrow W_{k_{L-1},k_L}$ forms a Markov chain.

Recall [15] that the minimum energy per information bit required for reliable reception (in the Shannon sense) at rate $R$ on an AWGN channel with noise variance $\sigma^2$ is $E_\sigma(R) \equiv \frac{2^R - 1}{\sigma^2 2^R}$. Let $\epsilon / R$ this to equation exists. Now, since $\frac{1}{R}E_\sigma(R)$ depends only on $R$, the solution does not depend on $\sigma$.

Lemma 2: Let $\gamma \in (0,1)$. Then, there exists an $R^* \in \mathbb{R}^+$ that is independent of $\sigma$ such that $E_\sigma(R^*) = 2\sigma^2 \ln(2)$. Hence, a solution $R^*$ to this equation exists. Now, since $\frac{1}{R}E_\sigma(R)$ depends only on $R$, the solution does not depend on $\sigma$.

Lemma 3: Let $p$ be an acyclic path on $\mathcal{N}$. Let $\gamma \in (0,1)$, and $\epsilon \in (0,1]$. For every $i : 1 \leq i \leq L$, let $C_{k_{i-1}, k_i}$ denote the transmit energy per bit consumed by node $k_{i-1}$ to transmit to node $k_i$. Then, there exists a $N_0 \geq 2$ such that for every $N \geq N_0$, there exists a path model $O^p$ with a finite-length FIFO buffer for every node, and a finite-length schedule $S^p$ in $K_G$ such that (a) $C_{k_{i-1}, k_i} \leq \frac{2 \sigma^2 \ln(2)}{R_{k_{i-1}, k_i} \rightarrow k_i}$ Joules of transmit energy per information bit is consumed by node $k_{i-1}$ on link $k_{i-1} \rightarrow k_i \forall i : 1 \leq i \leq L$, (b) $P_b(O^p) \leq \epsilon$, and (c) Every bit from $l$ to $m$ through path $p$ is delivered in a finite amount of time.

Proof of Lemma 3: We use a simple path code as the path model. We divide the proof into three parts: In Part A of the proof, we describe the choice of the rate $R$ and the block length $n$ which will be common to all of the links on the path. In Part B, we describe the code for each link and perform the analysis of the probability of message error. In Part C, we describe the probability of information bit error from source $l$ to destination $m$ and show that the transmission completes in a finite amount of time.

Part A: Choice of $(R, n)$: By Lemma 2, there exists an $R^* > 0$ such that $E_\sigma(R^*) = \frac{2\sigma^2 \ln(2)}{R}$ for every $i : 1 \leq i \leq L$. By the lemma, this $R^*$ depends only on $\gamma$, and not on any of the $\sigma_i$ or $N$. Let $Q^+$ denote the set of positive rational numbers. Since $Q^+$ is dense in $\mathbb{R}^+$, there exists an $R \in Q^+$ such that $0 < R < R^*$. Fix this $R$. Let $N' = \max\left\{N, \left\lceil \frac{\epsilon}{R^* - R} \right\rceil + 1 \right\}$.

Let $n = \max\left\{1, \frac{\log_2(3N') - \log_2(\epsilon)}{R^* - R - \frac{\gamma}{2}} \right\}$ and let $n$ be the minimum integer such that $n \geq n$ and $nR$ is an integer. Hence we have fixed $R$ and $n$. (Remark: By the above choice, $R$ is functionally independent of $N$, whereas $n$ asymptotically grows as $\sim \log(N)$.)

Part B: Link analysis: Let $i \rightarrow j$ be any link on path $p$.

1. Generation of the codebook: Let $P_j \equiv \frac{R^* 2\sigma^2 \ln(2)}{2^R}$. Let $w$ denote the index of a codeword such that $w \in \{1, 2, \ldots, 2^R\}$.  

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Let $X^{(i,j)}(w)$ denote the codeword that corresponds to index $w$. This codeword is an n-dimensional vector with each entry denoted by $X_r^{(i,j)}(w)$ and generated i.i.d. from a Gaussian distribution (denoted by $\mathcal{N}$) as $\mathcal{N}(0, P_r \frac{\sigma_r^2}{G_i})$.

2) Encoding: The encoder $\bar{E}_{ij}$ at each link $ij$ on path $p$ is given as follows: After the generation of the codebook, the codebook is revealed to both sender $i$ and receiver $j$. In order to send the message index $w$, $i$ sends the $w$th codeword $X^{(i,j)}(w)$ in the codebook.

3) Decoding: The decoder $\bar{D}_{ij}$ at each link $ij$ on path $p$ is given as follows: Each decoder uses typical set decoding: A codeword is declared as received if it is jointly typical with the received vector. If the power of the received signal is greater than $P_0$ or if there is more than one codeword that is jointly typical with the received vector, an error is declared [1].

4) Bit-to-message and message-to-bit mappings: The bit-to-message function $h_{k_0,k_1}$ is given as follows: $nR$ successive bits (in a block) are mapped to the set of messages $M_i$ via any bijective mapping. (This bijective mapping is chosen and then fixed.) Then, $h_{k_0,k_1}$ is revealed to node $k_L$. The message-to-bit function $g_{k_L+1,k_2}$, at node $k_L$, is the inverse of $h_{k_0,k_1}$.

5) Analysis of the probability of message error: Under schedule $S_p$, the received signal at each node on path $p$ reduces from (1) to $Y_{k_i}(t) = \sqrt{G_{k_{i-1},k_i}}X_{k_{i-1},i}(t) + \bar{n}_k(t)$ for every $i : 1 \leq i \leq L$. Since we use typical set decoding [1] at each decoder, the probability of message error satisfies $P_e^{k_{i-1},k_i} \leq 2^{-n(R - \frac{\epsilon}{G_{k_{i-1},k_i}} - \frac{\ln(\gamma_{k_{i-1},k_i})}{\ln(2)})}$ for every $i : 1 \leq i \leq L$. In this expression, $X_{k_{i-1},k_i}$ denotes the output of node $k_{i-1}$ and $Y_{k_{i-1},k_i}$ is the input to the decoder of node $k_i$. Now,

$$I(X_{k_{i-1},k_i}, Y_{k_{i-1},k_i}) = \frac{1}{2} \log_2(1 + \frac{P_{k_i}}{\sigma_{k_i}^2}) \leq R^*$$

$$= \frac{1}{2} \log_2(1 + R^* \frac{2\sigma_k^2 \ln(2)}{\gamma_{k_i}^2}) \leq R^* \quad (6)$$

$$= R^*$$

$$= R^*$$

Above, (6) follows from Shannon’s capacity formula and the fact that scaling by the (constant, deterministic) channel gain $G_{ij}$ does not change the channel capacity, and (8) follows from the definition of $R^*$. Now, (4) and (5) imply that $2^{-n(R - R - \frac{\epsilon}{G_{k_{i-1},k_i}})} \leq \frac{\epsilon}{G_{k_{i-1},k_i}}$. Hence, for every $i$ with $1 \leq i \leq L$: $P_{e^{k_{i-1},k_i}} \leq 2^{-n(R - R - \frac{\epsilon}{G_{k_{i-1},k_i}})}$. Note that by (5), $n \geq \frac{1}{\gamma_{k_{i-1},k_i}}$; hence, there is at least one information bit to transmit per message.

6) Energy-per-bit: It follows from Part A of the proof that $C_{k_{i-1},k_i} \leq \frac{2\gamma_{k_{i-1},k_i}}{G_{k_{i-1},k_i}}$ for every $i : 1 \leq i \leq L$.

7) Finite link transmission time: Since $R > 0$ and the encoding and decoding delays (in seconds) are finite for a finite number of bits, the transmission of a finite number of bits on this link completes in finite time.

8) Finite FIFO buffer length: We fix the buffer length in the node model of every node to $\min_{i,j} \frac{E_t}{\gamma_{i,j}} G_{ij}$. Since $0 < E_t < \infty \forall i \in \mathcal{N}$ and $0 < C_{ij} < \infty \forall (i,j) \in \mathcal{N} \times \mathcal{N}$ with $i \neq j$, this length is finite.

Part C: Source-to-destination analysis: By Lemma 1, $P_b(O^p) \leq \epsilon$. Further, since the delay through each node is finite for a finite number of bits and there is a finite number of nodes on $p$, it follows that the transmission of a finite number of bits from $l$ to $m$ finishes in finite time. This completes the proof of Lemma 3.

Part 2: In this part, we prove Theorem 1 using the results of Part 1. We divide the proof of this theorem into seven sections. In the first section, we describe the Gupta-Kumar tessellation construction [6] that we employ. In the second section, we specify the connectivity of the network. In the third section, we describe how to route the traffic. In the fourth section, we derive an upper bound on the transmit energy cost of a link as a function of $N$. In the fifth section, we derive a lower bound on the energy in each cell. In the sixth section, we derive an upper bound on the energy to transmit a codeword. In the seventh section, we derive the asymptotic result.

1) Tessellation construction: Construct the Voronoi tessellation of Gupta-Kumar on $S^2$. In this tessellation, each cell contains a disk of radius $\rho(N)$ and is contained in a disk of radius $\frac{100 \log N}{N}$ on $S^2$. We design a scheme that delivers $\delta(N)$ bits from every node to its destination under the one-to-one traffic model. By Lemma 4.14 in [6], there exists a sequence $\eta N$ such that $P_{\sup_{\gamma \in V_N}}(100 \log N N^{-1}) \leq \epsilon' \delta(N) \sqrt{N \log N} \geq 1 - \eta N$, where $V_N$ is the entire set of cells in the tessellation and $\epsilon'$ is a deterministic positive constant.

2) Connectivity: Let $i \in \mathcal{N}$, and let $V(i) = \{v(i)\}$ be the cell in which $i$ is located. For each $i$, we assume that $i$ has links to all of the nodes in $V(i)$ and to all of the nodes in the cells that are adjacent to $V(i)$.

Note that in [6], $\lambda(N)$ bits-per-second is delivered end-to-end, while $\delta(N)$ bits is delivered end-to-end above.
3) Routing traffic: There are $N$ traffic streams to be delivered under the one-to-one traffic model. Each stream $(l, s(l))$ will be delivered (as in [6]) along a straight line from its source $l$ to its destination $s(l)$. Each hop is from a node in one cell to a node in the adjacent cell, unless $l$ and $s(l)$ are in the same cell. Explicitly, for every cell $V$: (a) Any traffic that originates from a node $i \in V$ and is destined for $j \in V$, $i \neq j$, will be sent on the direct link from $i$ to $j$. (b) In the last hop of any traffic that traverses more than 1 cell, the traffic is directly transmitted to the destination node. (c) For any transit traffic through cell $V$, the traffic is relayed to a node in the next cell along the straight line from the source to the destination.

In order to route the traffic from $l$ to $m$, we use a simple path codec on the path formed. We concatenate the schedules along the straight line from the source to the destination.

4) Lower bound on the number of bits per node: The maximum distance over which any node needs to transmit is $8 \rho(N)$, which is on the order of the size of a cell, that is, $\sqrt{\log N}$. Therefore, a lower bound $b_{ij}$ on the number of bits that a node can transmit before it runs out of energy is given as follows. Let $L(i)$ denote the set of nodes to which node $i$ has links in the network. (Recall that $L(i)$ consists of the nodes in $V(i)$ and the nodes in the cells adjacent to $V(i)$.)

$$b_{ij} \geq \max_{j \in L(i)} \frac{E_{\text{min}}}{\gamma_{ij}} \frac{2 \sigma^2 q}{\gamma_{ij} \log N} = \frac{2 \log (2) K(d_i, q)}{\gamma_{ij} \log N} \max_{j \in L(i)} \sigma^2 d_{ij}^2$$

$$\geq \frac{2 \log (2) K(d_i, q)}{\gamma_{ij} \log N} \max_{j \in L(i)} \sigma^2 d_{ij}^2 \frac{N}{\gamma_{ij}}$$

5) Energy in a cell: Fix a cell $V$ in the set of cells $V_N$ on the sphere. Let $K_V$ denote the set of nodes in $V$. By the proof of Lemma 4.8 in [6], $P(|K_V| \geq 50 \log N \forall V \in V_N) > 1 - \eta_N$ where $\eta_N$ is a deterministic sequence with $\eta_N \to 0$. Let $M_k \subseteq K_V$ denote the $(50 \log N)$ nodes that are in the cell “with high probability (whp)”, where we use this phrase to signify the above, as in [6]. Since every node has at least $E_{\text{min}} > 0$ Joules of energy, there are at least $(E_{\text{min}}/50 \log N)$ Joules of energy in cell $V$, whp.

6) Energy to transmit a codeword: Since there is a finite number of nodes on path $p$ and $d_{k(i)}^N \in O(\sqrt{\log N})$, there exists an $\alpha' \in \mathbb{R}^+$ such that $d_{k(i)}^N \leq \alpha' \sqrt{\log N}$ for all $i = 1, \ldots, L$. Note that $n \sim \Theta(\log N)$ by the definition of $n$. Since $R$ is functionally independent of $N$, the number of information bits per codeword, $nR$, is $\Theta(\log N)$. Then, an upper bound on the transmit energy consumed by $k_i$ to transmit a codeword to $k_i$ is given by

$$nRC_{k_{i-1}, k_i} \leq nR \frac{2 \sigma^2 q}{\gamma_{k_{i-1}, k_i}} \log (2)$$

7) Asymptotics:

(a) The number of bits available in a cell will be partitioned among the traffic streams above such that each traffic stream receives roughly the same amount. Since there are $c' \sqrt{N \log N}$ routes through each cell whp [6], each traffic stream can be allocated at least the following number of bits whp:

$$\frac{(50 \log N)}{c' \sqrt{N \log N}} = \gamma c' \left( \frac{N}{\log N} \right)^{(q-1)/2}$$

(b) Each route is served by exactly one node in each cell; hence, there are $\frac{c'}{50} \sqrt{N \log N}$ routes per node. Each node allocates at least $(50 \log N)$ Joules to each of its routes, which decreases more slowly in $N$ than the upper bound on the energy per codeword in (16); hence, the number of codewords that a node transfers per route asymptotically grows with $N$.

(c) There are $N$ traffic streams under the one-to-one traffic model, and the total energy in the network is bounded above by $E_{\text{max}} N$. Therefore, the total number of bits delivered divided by the total number of Joules in the network is at least $\frac{N \gamma c' \left( \frac{N}{\log N} \right)^{(q-1)/2}}{E_{\text{max}} N}$. Since we used a simple path codec on each source-to-destination transmission, the probability of information bit error on each source-to-destination delivery is less than or equal to $\epsilon$. Each transmission is guaranteed to finish in finite time, and since there is a finite number $N$ of traffic streams, the transmission of all of the traffic streams finishes in finite time. Hence we have shown that $c_q \left( \frac{N}{\log N} \right)^{(q-1)/2}$ bits-per-Joule is $(\epsilon, \text{fsum})$-achievable, where $c_q \left( \frac{N}{\log N} \right)^{(q-1)/2}$.

Then, $C_J(K_G; \epsilon, \text{fsum}) \geq \sup_{0 < \gamma < c_q} \gamma c_q \left( \frac{N}{\log N} \right)^{(q-1)/2} \gamma$.

This proves the theorem.

VII. CONCLUSIONS

The asymptotic growth of bits-per-Joule capacity implies that while large-scale wireless sensor and ad hoc networks might not be suitable for data-rate-intensive applications (due to the scaling laws for throughput and transport capacities [6]), they appear suitable for delay-tolerant data applications over energy-limited devices, such as in sensor networks [10][24].

An important part of the constructive proof in this paper is the decoupling of rate and transmit power adaptations. A single spectral efficiency $R$ is fixed based only on $\gamma$ (an
independent parameter controlled by the network designer, and the proof shows that \( \gamma_{eq} \left( \frac{N}{\log N} \right) ^{(q-1)/2} \) bits-per-Joule is \((\epsilon, f_{sum})\)-achievable for the network. Transmit power adaptation (in response to shrinking internodal distances) controls the asymptotic form of this growth, whereas rate adaptation controls only its coefficient \( 0 < \gamma < 1 \). Hence, contrary to the common wisdom about energy-efficient networks, a wireless network that operates at a fixed, high spectral efficiency (hence low \( \gamma \)) on each of its links sees the same form of asymptotic growth in bits-per-Joule, albeit with a low coefficient. (In this respect, this analysis differs significantly from [8] and [20] that require that the links operate in the power-limited regime.)

The underlying reason for the growth of bits-per-Joule capacity is that the transmit energy available in a cell (namely \( \log N \) nodes per cell times \( \left( \frac{N}{\log N} \right)^{q/2} \)) grows faster in \( N \) than the number of relay routes \( N \log N \) that a cell has to carry. The decrease in transmit energy consumption occurs due to shrinking internodal distances, and the increase in the number of routes occurs due to the growing number of geographically separated sources and destinations under the one-to-one traffic model. In contrast with [6], we are able to use the transmit power adaptation to our advantage because unlike throughput, bits-per-Joule capacity is not decreased when transmissions are scheduled in non-overlapping time slots.

Since we were not concerned with end-to-end delay in this paper, we used a simple path codec that sent all the bits in one shot from each source to each destination, and scheduled all the transmissions free of interference. Since each node has to wait for all the other streams to finish, the delay grows at least linearly with \( N \) in this simple scheme. This delay can be reduced by allowing time slot reuse in distant parts of the network, and by sending first the delay-critical data on a route that uses a high spectral efficiency \( R \) on each of its links, and a priority queuing policy at each of its nodes. However, further work is necessary to completely understand the relationship between delay and bits-per-Joule capacity.

Cooperative listen-and-amplify schemes [8] have shown an increase in power efficiency by a factor of at least \( \sqrt{N} \) for scenarios in which the distances are so short that the path loss model no longer holds. The bits-per-Joule capacity growth in this paper relies on the path loss model and hence operates in a different regime. Further work is necessary to understand the relationship between the bits-per-Joule capacities in these different regimes.

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