Zero-error source-channel coding with source side information at the decoder

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Let $C$ be a discrete memoryless channel with transition probability distribution $p_{Y|X}(y|x)$, $x \in \mathcal{X}$, $y \in \mathcal{Y}$, where $\mathcal{X}$ and $\mathcal{Y}$ are finite sets. Let $(\mathcal{S}_U, \mathcal{S}_V)$ be a pair of memoryless correlated sources producing a pair of random variables $(U, V)$ from a finite set $\mathcal{U} \times \mathcal{V}$ at each instant. Alice, the sender, has access to $U$ while Bob, the receiver, has access to $V$. Alice and Bob are connected by the channel $C$. We wish to decide if Alice can convey $n$ realizations of $U$ in $n$ channel uses for some $n$. The asymptotically vanishing probability of error case offers the following possibility: By Slopeian-Wolf, we can encode $U$ at the rate $H(U|V)$. Channel coding makes reliable transmission possible if $H(U|V) < C(G_U)$.

Consider the pentagon graph $G = (\{1, 2, 3, 4, 5\}, E)$, where $(i, j) \in E$ if $|i - j| = 1 \pmod{5}$. Let $G_U = G_X = G$. $R^*(G_U) = \log \frac{5}{4}$ and $C(G_X) = \frac{1}{5} \log 5$. So, we cannot transmit the source through the channel by separate coding. On the other hand, a scalar joint source-channel code exists since the pentagon graph is its own complement. Therefore, separate source and channel coding is suboptimal.

**Unrestricted Input case**: Our first result is that separate coding is suboptimal in the UI case. Source and channel coding can be done separately only if $R^*(G_U) \leq C(G_X)$. Consider the pentagon graph $G = (\{0, 1, 2, 3, 4\}, E)$, where $(i, j) \in E$ if $i \neq j$. Let $G_U = G_X = G$. $R^*(G_U) = \log \frac{5}{2}$ and $C(G_X) = \frac{1}{2} \log 5$. So, we cannot transmit the source through the channel by separate coding. On the other hand, a scalar joint source-channel code exists since the pentagon graph is its own complement. Therefore, separate source and channel coding is suboptimal.

**Restricted Input case**: In this case separate source and channel coding is possible only if $R_m(G_U) \leq C(G_X)$. Our proof of the suboptimality of separate coding employs the theta function, $\theta(G)$, defined by Lovász. Lovász proved that $\log \theta(G) \geq C(G)$ for any graph $G$. One of our key results is:

**Theorem 1.** For any graph $G = (V, E)$, $\log \theta(G) \leq R_m(G)$.

We now have the string of inequalities $C(G) \leq \log \theta(G) \leq R_m(G)$. If any of these inequalities is strict for some graph, then separate coding is suboptimal in the RI case as well. Indeed, such graphs do exist. For example, if $G_X$ is the Schlafli graph, $C(G_X) \leq \log 7 < \theta(G_X) = \log 9$.

**How large are the gains?** Given a source-channel pair $(G_U, G_X)$, let us rephrase the problem as: how many channel uses are required per source symbol to enable zero-error transmission? With separate coding, the channel uses per symbol is $\frac{R_m(G_U)}{C(G)}$. Using a recent result by Alon [3], we show that:

**Theorem 2.** Given any $l$, we can find a graph $G$ such that

$$\frac{R^*(G)}{C(G)} \geq \frac{R_m(G)}{C(G)} \geq l.$$ 

This means that both $\frac{R^*(G_U)}{C(G_U)}$ and $\frac{R_m(G_U)}{C(G_U)}$ can be arbitrarily large even when $G_U = G_X$, the case where a zero-error (scalar) joint source-channel code exists with one channel use per source symbol.

**References**

