Combinatorial Message Sharing for a Refined Multiple Descriptions Achievable Region

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***Abstract*—This paper presents a new achievable rate-distortion region for the L-channel multiple descriptions problem. Currently, the most popular region for this problem is due to Venkataramani, Kramer and Goyal [3]. Their encoding scheme is an extension of the Zhang-Berger scheme to the L-channel case and includes a combinatorial number of refinement codebooks, one for each subset of the descriptions. All the descriptions also share a single common codeword, which introduces redundancy, but assists in better coordination of the descriptions. This paper proposes a novel encoding technique involving 'Combinatorial Message Sharing', where every subset of the descriptions may share a distinct common message. This introduces a combinatorial number of shared codebooks along with the refinement codebooks of [3]. These shared codebooks provide a more flexible framework to trade off redundancy across the messages for resilience to descriptions loss. We derive an achievable rate-distortion region for the proposed technique, and show that it subsumes the achievable region of [3].**

***Index Terms*—Multiple descriptions coding, Source coding, Rate distortion theory**

I. Introduction

The Multiple Descriptions (MD) problem was proposed in the late seventies and has been a challenging problem since then. It has been studied extensively with results ranging from the derivation of asymptotic bounds [1], [2], [3], [4], [5], [6], [7] to practical approaches for multiple descriptions quantizer design [8]. It was originally viewed as a method to cope with channel failures, where multiple source descriptions are generated and sent over different paths. The encoder generates L descriptions for transmission over L available channels. It is assumed that the decoder receives a subset of the descriptions perfectly and the remaining are lost, as shown in Figure 1. The objective of the MD problem is to design the encoders (for each description) and decoders (for each possible received subset of the descriptions), with respect to an overall rate-distortion trade-off. The subtlety of the problem is due to the balance between the full reconstruction quality versus quality of individual descriptions; or noise free quality versus the amount of redundancy across descriptions needed to achieve resilience to descriptions loss.

The most well known achievable region currently known for the $L-$channel MD problem is due to Venkataramani, Kramer and Goyal (VGK) [3] for general sources and distortion measures. Their encoding scheme builds on the prior work for the 2-channel case by El-Gamal and Cover (EC) [1] and Zhang and Berger (ZB) [2] and introduces a combinatorial number of refinement codebooks, one for each subset of descriptions. A *single* common codeword is also shared among all the descriptions, which assists in controlling the redundancy across the messages, improving the rate-distortion trade-off. In this paper we present a new encoding scheme involving "Combinatorial Message Sharing" (CMS), where a unique common codeword is sent (shared) in each subset of the descriptions, thereby introducing a combinatorial number of *shared codebooks*, along with the refinement codebooks of [3]. The common codewords enable better coordination between descriptions, providing a refined over-all rate-distortion region. We derive an achievable rate-distortion region for the CMS scheme and show that it subsumes the achievable region due to VKG in [3]. We note that the underlying principle behind CMS has been shown in precursor work to be useful in related applicational contexts of routing for networks with correlated sources [10] and data storage for selective retrieval [11].

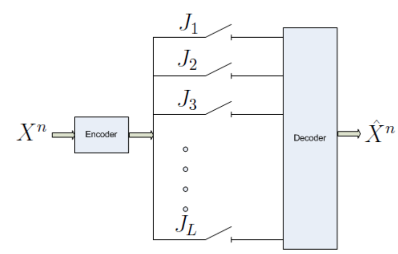

Figure 1. $L-$Channel Multiple Descriptions Setup : Each description is received error free or is completely lost at the decoder

The rest of the paper is organized as follows. In Section II, we formally state the $L-$channel MD setup and briefly describe the approaches and regions of EC [1], ZB [2] and VKG [3]. To keep the notation simple, we first describe in Section III-A, the CMS scheme for the 3 descriptions scenario and extend it to the general case in Section III-B.

II. Formal Definitions and Prior results

We first give a formal definition of the $L-$channel MD problem. We follow the notation in [3]. A source produces a sequence $X^n = \left(X^{(1)}, X^{(2)} \dots, X^{(n)}\right)$, which are n iid copies of a generic random variable X taking values in a finite alphabet $\mathcal{X}$. We denote $\mathcal{L} = \{1, \dots, L\}$. There are L encoding functions, $f_l(\cdot)\ l \in \mathcal{L}$, which map X^n to the descriptions $J_l = f_l(X^n)$, where J_l takes on values in the set $\{1, \dots B_l\}$. The rate of description l is defined as $R_l = \log_2(B_l)$. Description l is sent over channel l and is either received at

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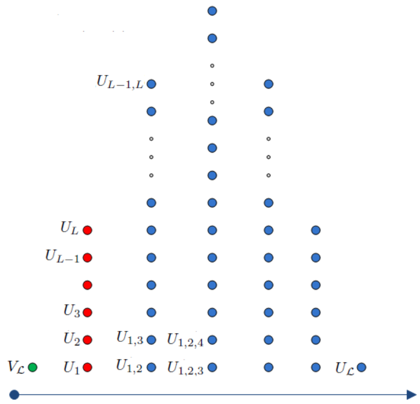

Figure 2. Codebook generation for VKG coding scheme. Green ($V_{\mathcal{L}}$) indicates 'common random variable'. Red (U_l $l \in \mathcal{L}$) indicates 'base layer random variables' and Blue ($U_{\mathcal{S}}$ $|\mathcal{S}| > 1$) indicates 'refinement random variables'.

the decoder error free or is completely lost. There are $2^L - 1$ decoding functions for each possible received combination of the descriptions $\hat{X}^n_{\mathcal{K}} = \left(\hat{X}^{(1)}_{\mathcal{K}}, \hat{X}^{(2)}_{\mathcal{K}} \ldots, \hat{X}^{(n)}_{\mathcal{K}}\right) = g_{\mathcal{K}}(J_l : l \in \mathcal{K})$, $\forall \mathcal{K} \subseteq \mathcal{L}$, $\mathcal{K} \neq \phi$, where $\hat{X}_{\mathcal{K}}$ takes on values on a finite set $\hat{\mathcal{X}}_{\mathcal{K}}$, and ϕ denotes the null set. The distortion at the decoder when a subset $\mathcal{K}$ of the descriptions is received is measured as $D_{\mathcal{K}} = E\left[\frac{1}{N}\sum_{t=1}^{n} d_{\mathcal{K}}(X^{(t)}, \hat{X}^{(t)}_{\mathcal{K}})\right]$, where the bounded distortion measures $d_{\mathcal{K}}(\cdot)$ are defined as $d_{\mathcal{K}} : \mathcal{X} \times \hat{\mathcal{X}}_{\mathcal{K}} \to \mathcal{R}$. We say that a rate-distortion tuple $(R_i, D_{\mathcal{K}} : i \in \mathcal{L}, \mathcal{K} \subseteq \mathcal{L}, \mathcal{K} \neq \phi)$ is achievable if there exit L encoding functions with rates $(R_1 \ldots, R_L)$ and $2^L - 1$ decoding functions yielding distortions $D_{\mathcal{K}}$. The closure of the set of all achievable rate-distortion tuples is defined as the '*L-channel multiple descriptions RD region*'. Note that, this region has $L + 2^L - 1$ dimensions.

In what follows, $2^{\mathcal{S}}$ denotes the set of all subsets (power set) of any set $\mathcal{S}$ and $|\mathcal{S}|$ denotes the set cardinality. Note that $|2^{\mathcal{S}}| = 2^{|\mathcal{S}|}$. $\mathcal{S}^c$ denotes the set complement. For two sets $\mathcal{S}_1$ and $\mathcal{S}_2$, we denote the set difference by $\mathcal{S}_1 - \mathcal{S}_2 = \{\mathcal{K} : \mathcal{K} \in \mathcal{S}_1, \mathcal{K} \notin \mathcal{S}_2\}$. We denote by $2^{\mathcal{S}} - \phi$, the set of all non-empty subsets of $\mathcal{S}$. We use the shorthand $\{U\}_{\mathcal{S}}$ for $\{U_{\mathcal{K}} : \mathcal{K} \in \mathcal{S}\}$[1].

A. VKG Encoding Scheme

The achievable region of [3] is denoted here $\mathcal{RD}_{VKG}$ and is described as follows. Let $(V_{\mathcal{L}}, \{U\}_{2^{\mathcal{L}}-\phi})$ be any set of 2^L random variables distributed jointly with X. Then, an RD tuple is said to be achievable if there exist functions $\psi_{\mathcal{S}}(\cdot)$ such that:

$$\sum_{l \in \mathcal{S}} R_l \geq |\mathcal{S}| I(X; V_{\mathcal{L}}) - H(\{U\}_{2^{\mathcal{S}}-\phi} | X, V_{\mathcal{L}}) + \sum_{\mathcal{K} \subseteq \mathcal{S}} H\left(U_{\mathcal{K}} | \{U\}_{2^{\mathcal{K}}-\phi-\mathcal{K}}\right) \quad (1)$$

$$D_{\mathcal{S}} \geq E\left[d_{\mathcal{S}}\left(X, \psi_{\mathcal{S}}(V_{\mathcal{L}}, \{U\}_{2^{\mathcal{S}}-\phi})\right)\right] \quad (2)$$

$\forall \mathcal{S} \subseteq \mathcal{L}$. The closure of the achievable tuples over all such 2^L random variables gives $\mathcal{RD}_{VKG}$.

The codebook generation is done using the following tree structure. First, $2^{nR''_{\mathcal{L}}}$ codewords of $V_{\mathcal{L}}$ are generated using the marginal distribution of $V_{\mathcal{L}}$. Conditioned on each codeword of $V_{\mathcal{L}}$, $2^{nR'_l}$ codewords of U_l are generated according to the conditional PMF $\prod_{t=1}^{n} P_{U_l|V_{\mathcal{L}}}(u_l^{(t)}|v_{\mathcal{L}}^{(t)})$ $\forall l \in \mathcal{L}$. Next, for each $j \in (1, \ldots, 2^{n(R''_{\mathcal{L}} + \sum_{l \in \mathcal{K}} R'_l)})$, a single codeword is generated for $U_{\mathcal{K}}(j)$ according to the conditional PMF $\prod_{t=1}^{n} P_{U_{\mathcal{K}}|V_{\mathcal{L}}, \{U\}_{2^{\mathcal{K}}-\phi-\mathcal{K}}}(u_{\mathcal{K}}^{(t)}|v_{\mathcal{L}}^{(t)}, \{u\}^{(t)}_{2^{\mathcal{K}}-\phi-\mathcal{K}})$ $\forall \mathcal{K} \subseteq \mathcal{L}$, $|\mathcal{K}| > 1$. Note that to generate the codebook for $U_{\mathcal{K}}$, we first need the codebooks for all $\{U\}_{2^{\mathcal{K}}-\phi-\mathcal{K}}$ and $V_{\mathcal{L}}$. Codewords of U_l (at rate R'_l) are sent in description l. Along with the '*private*' messages, each description also carries a '*shared message*' at rate $R''_{\mathcal{L}}$, which is the codeword of $V_{\mathcal{L}}$. Hence the rate of each description is given by $R_l = R'_l + R''_{\mathcal{L}}$. VKG showed that, to ensure finding a set of jointly typical codewords with the observed sequence, the rates must satisfy (1). It then follows from standard arguments (see for example "typical average lemma" [9]) that, if the random variables also satisfy (2), then the distortion constraints are met. The structure of encoding the auxiliary random variables is shown in Figure 2. We call $V_{\mathcal{L}}$ as the shared random variable, $U_l : l \in \mathcal{L}$ as the base layer random variables and all $U_{\mathcal{K}} : |\mathcal{K}| \geq 2$ as the refinement layers. Observe that the codebook generation follows the order: shared layer $\to$ base layer $\to$ refinement layer. Note that the refinement codewords are implicitly embedded in the private messages and assist in improving the distortion when multiple descriptions are received at the decoder.

The VKG scheme for the 2 descriptions scenario involves 4 auxiliary random variables V_{12}, U_1, U_2 and U_{12}. The VKG region was originally derived as an extension of the EC [1] and ZB [2] coding schemes for the 2-descriptions scenario. The first of the two regions was by EC and their rate region (denoted here by $\mathcal{RD}_{EC}$) is obtained by setting $V_{12} = \Phi$ in $\mathcal{RD}_{VKG}$, where Φ is a constant. ZB (their region is denoted here by $\mathcal{RD}_{ZB}$) later showed that, including the shared random variable can give strict improvement over $\mathcal{RD}_{EC}$. Their result, while perhaps counter-intuitive at first, clarifies the fact that, a shared message among the descriptions helps to better coordinate the messages, thereby providing a strictly improved RD region, even though it introduces redundancy. However, it has been shown that $\mathcal{RD}_{EC}$ is complete for some special cases of the setup (see for example [4], [5]).

We note that, it has recently been shown in [6] that the last layer of refinement random variables ($U_{\mathcal{L}}$) can be conveniently set to a constant (removed) without affecting the overall rate region. However, we continue to use this last layer for the time being as it is unclear if a similar result holds for the CMS coding scheme we describe in the following section. We note in passing that, other encoding schemes have been proposed in the literature for certain special cases of the $L-$channel MD setup [7], which improve on $\mathcal{RD}_{VKG}$. However, none of them have been shown to subsume $\mathcal{RD}_{VKG}$ for general sources and

[1]Note the difference between $\{U\}_{\mathcal{S}}$ and $U_{\mathcal{S}}$. $\{U\}_{\mathcal{S}}$ is a set of variables, whereas $U_{\mathcal{S}}$ is a single variable.

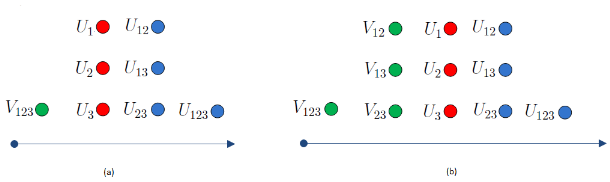

Figure 3. (a) Denotes the codebook generation order for VKG scheme. (b) represents the codebook generation for the proposed coding scheme.

distortion measure. Some of the new ideas introduced herein may have impact on these special cases, but such extensions are beyond the scope of this paper.

III. Combinatorial Message Sharing

In this section, we describe the proposed encoding scheme which leads to a refined achievable rate region for the $L-$channel MD setup. To simplify notation and understanding, we first describe the scheme for the 3-descriptions case and offer intuitive arguments to show the achievability of the new region. We then extend the arguments to the $L-$channel case in Theorem 1.

A. 3-Descriptions scenario

The encoding order for the 3-descriptions VKG scheme is shown in Figure 3(a). Recall that the common codeword helps in coordinating the 3 descriptions. VKG employ *one* common codeword that is sent in all the 3 descriptions. However, when dealing with $L > 2$ descriptions, restricting to a single shared message might be suboptimal. The CMS scheme therefore will allow for 'combinatorial message sharing', i.e a common codeword may be sent in each (non-empty) subset of the descriptions.

The shared random variables are denoted by 'V'. The base and the refinement layer random variables are denoted by 'U'. For the 3 descriptions scenario, we have 11 auxiliary random variables which include 3 additional variables over the VKG scheme. These variables are denoted by V_{12}, V_{13} and V_{23}. The codeword corresponding to $V_\mathcal{S}$ is sent in *all* the descriptions $l \in \mathcal{S}$. For example, the codeword of V_{12} is sent in both the descriptions 1 and 2. This introduces a new layer in the encoding structure as shown in Figure 3 (b). This extra encoding layer can refine the control of redundancy across the messages, and hence lead to an improved rate-distortion region.

The codebook generation is done as follows. First, the codebook for V_{123} is generated containing $2^{nR''_{123}}$ independently generated codewords. Then codebooks for V_{12}, V_{13} and V_{23} (each containing $2^{nR''_{12}}$, $2^{nR''_{13}}$ and $2^{nR''_{23}}$ codewords respectively) are generated, conditioned on each codeword of V_{123}. Next, the base layer codebooks for U_l, $l \in \{1, 2, 3\}$ (each containing $2^{nR'_l}$ codewords) are generated conditioned on the codewords of all $V_\mathcal{S}$ such that $l \in \mathcal{S}$. For example, the codebooks for U_1 are generated conditioned on the codewords of V_{12}, V_{13} and V_{123}. Note that each codebook of U_1 contains $2^{nR'_1}$ codewords and there are such $2^{n(R''_{12}+R''_{13}+R''_{123})}$ codebooks.

The refinement layer codewords are generated similar to the VKG scheme. However, the codebook for $U_\mathcal{K}$, is now generated conditioned not only on the codewords of $\{U\}_{2^\mathcal{K}-\phi-\mathcal{K}}$ and $V_\mathcal{L}$, but also on the codewords of all $V_\mathcal{S}$ such that $|\mathcal{S} \cap \mathcal{K}| > 0$. For example, a single codeword for U_{12} is generated conditioned on each codeword tuple of $\{U_1, U_2, V_{12}, V_{13}, V_{23}, V_{123}\}$. Note that, there are $2^{n(R'_1+R'_2+R''_{12}+R''_{13}+R''_{123})}$ codewords in the codebook of U_{12}. Similarly codebooks for U_{13}, U_{23} and U_{123} are generated conditioned on codewords of $\{U_1, U_3, V_{12}, V_{13}, V_{23}, V_{123}\}$, $\{U_2, U_3, V_{12}, V_{13}, V_{23}, V_{123}\}$ and $\{U_1, U_2, U_3, V_{12}, V_{13}, V_{23}, V_{123}\}$ respectively.

The encoder, on observing X^n, tries to find a codeword from the codebook of V_{123} such that it is jointly typical with X^n. Using typicality arguments, it is easy to show that the probability of not finding such a codeword approaches zero if $R''_{123} \geq I(X; V_{123})$. Let us denote the selected codeword of V_{123} by v_{123}. The encoder next looks at the codebooks of V_{12}, V_{13} and V_{23}, which were generated conditioned on v_{123}, to find a triplet of codewords which are jointly typical with (X^n, v_{123}). It can be shown using arguments similar to [3], [9], that the probability of not finding such a triplet approaches zero if the following conditions are satisfied $\forall \mathcal{Q} \subseteq \{12, 13, 23\}$:

$$\sum_{\mathcal{K}\in\mathcal{Q}} R''_\mathcal{K} \geq \sum_{\mathcal{K}\in\mathcal{Q}} H\left(V_\mathcal{K}|V_{123}\right) - H(\{V\}_\mathcal{Q}|V_{123}, X) \qquad (3)$$

Let us denote these codewords by v_{12}, v_{13} and v_{23}. The encoders next step is to find an index tuple (i_1, i_2, i_3) such that $(U_1(i_1), U_2(i_2), U_3(i_3), U_{12}(i_1, i_2), U_{13}(i_1, i_3), U_{23}(i_2, i_3), U_{123}(i_1, i_2, i_3))$ is jointly typical with $(X^n, v_{123}, v_{12}, v_{13}, v_{23})$. Here $U_1(i_1)$ denotes the i_1th codeword from the codebook of U_1 and $U_{12}(i_1, i_2)$ denotes the codeword of U_{12} generated conditioned on $(v_{123}, v_{12}, v_{13}, v_{23}, U_1(i_1), U_2(i_2))$. Similar notation is used for $U_2(i_2), U_3(i_3), U_{13}(i_1, i_3)$ etc. Again using similar arguments, we can show that the probability of not finding such an index tuple approaches zero if $\forall \mathcal{S} \subseteq \{1, 2, 3\}$:

$$\sum_{l\in\mathcal{S}} R'_l \geq \sum_{\mathcal{K}\in 2^\mathcal{S}-\phi} H\left(U_\mathcal{K}|\{U\}_{2^\mathcal{K}-\phi-\mathcal{K}}, \{V\}_{\mathcal{J}(\mathcal{K})}\right) - H\left(\{U\}_{2^\mathcal{S}-\phi}|V_{12}, V_{13}, V_{23}, V_{123}, X\right) \qquad (4)$$

where $\mathcal{J}(\mathcal{K}) = \{\mathcal{S} : \mathcal{S} \in \{12, 13, 23, 123\}, |\mathcal{K} \bigcap \mathcal{S}| > 0\}$. We denote the codewords corresponding to this index tuple by $(u_1, u_2, u_3, u_{12}, u_{13}, u_{23}, u_{123})$.

The encoder sends the index of the base layer codewords in the corresponding descriptions. It also sends the index of codewords corresponding to $V_\mathcal{S}$ in *all* the descriptions $l \in \mathcal{S}$. For example, in description 1, the encoder sends the indices corresponding to v_{123}, v_{12}, v_{13} and u_1. Similarly, descriptions 2 and 3 carry indices corresponding to $(v_{123}, v_{12}, v_{23}, u_2)$ and $(v_{123}, v_{13}, v_{23}, u_3)$ respectively. Therefore, rate for description l is $R_l = R'_l + \sum_{\mathcal{K}\in\mathcal{J}(l)} R''_\mathcal{K}$. Conditions on R_l can be obtained by substituting bounds from (3) and (4).

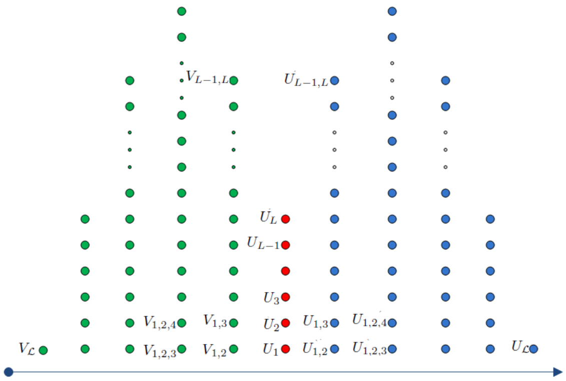

Figure 4. Codebook generation for the proposed coding scheme. Green ($V_{\mathcal{S}}$ $|\mathcal{S}| > 1$) indicates 'shared random variable'. Red (U_l $l \in \mathcal{L}$) indicates 'base layer random variables' and Blue ($U_{\mathcal{S}}$ $|\mathcal{S}| > 1$) indicates 'refinement random variables'.

The decoder, on receiving a subset of descriptions, estimates X^n based on all the messages embedded in either one of the descriptions. For example, if the decoder receives the descriptions 1 and 2, it can estimate X^n as $\psi_{12}(v_{123}, v_{12}, v_{13}, v_{23}, u_1, u_2, u_{12})$ for some function $\psi_{12}(\cdot)$. It follows from standard arguments [9], that if there exist functions $\psi_{\mathcal{S}}(\cdot)$ satisfying the following constraints, then distortion vector $\{D_{\mathcal{S}}\ \forall \mathcal{S} \in\ 2^{\mathcal{L}} - \phi\}$ is achievable.

$$D_{\mathcal{S}} \geq E\left[d_{\mathcal{S}}\left(X, \psi_{\mathcal{S}}\left(\{V\}_{\mathcal{J}(\mathcal{S})}, \{U\}_{2^{\mathcal{S}}-\phi}\right)\right)\right] \qquad (5)$$

An achievable RD region for the 3-descriptions MD setup is obtained by taking the closure of the achievable tuples over all such 11 auxiliary random variables. It will be shown in corollary 1 that, this rate-distortion region subsumes the VKG region.

B. $L-$Channel case

The fundamental idea here is same as the 3-descriptions scenario, albeit involving more complex notation. The codebook generation is done in an order as shown in Figure 4. First, the codebook for $V_{\mathcal{L}}$ is generated. Then, the codebooks for $V_{\mathcal{S}}$, $|S| = W$ are generated in the order $W = L-1, L-2\ldots 2$. This is followed by the generation of the base layer codebooks, i.e. U_S, $|S| = 1$. Then, the refinement layer codebooks corresponding to U_S, $|S| = W$ are generated in the order $W = 2, 3\ldots, L$. Each codebook is generated conditioned on a subset of the previously generated codewords. The specifics of codebook generation will be described as part of the proof of Theorem 1.

Before stating the theorem, we define the following subsets of $2^{\mathcal{L}}$:

$$\begin{aligned} \mathcal{I}_W &= \{\mathcal{S} : \mathcal{S} \in 2^{\mathcal{L}},\ |\mathcal{S}| = W\} \\ \mathcal{I}_{W+} &= \{\mathcal{S} : \mathcal{S} \in 2^{\mathcal{L}},\ |\mathcal{S}| > W\} \end{aligned} \qquad (6)$$

Let $\mathcal{B}$ be any non-empty subset of $\mathcal{L}$ with $|\mathcal{B}| \leq\ W$. We define the following subsets of $\mathcal{I}_W$ and $\mathcal{I}_{W+}$:

$$\begin{aligned} \mathcal{I}_W(\mathcal{B}) &= \{\mathcal{S} : \mathcal{S} \in \mathcal{I}_W,\ \mathcal{B} \subseteq \mathcal{S}\} \\ \mathcal{I}_{W+}(\mathcal{B}) &= \{\mathcal{S} : \mathcal{S} \in \mathcal{I}_{W+},\ \mathcal{B} \subseteq \mathcal{S}\} \end{aligned} \qquad (7)$$

We also define:

$$\mathcal{J}(\mathcal{K}) = \bigcup_{l \in \mathcal{K}} \mathcal{I}_{1+}(l) = \{\mathcal{S} : \mathcal{S} \subseteq \mathcal{L}, |\mathcal{S}| > 1, |\mathcal{K} \cap \mathcal{S}| > 0\} \qquad (8)$$

Let $(\{V\}_{\mathcal{J}(\mathcal{L})}, \{U\}_{2^{\mathcal{L}}-\phi})$ be any set of $2^{L+1} - L - 2$ random variables jointly distributed with X. We define the quantities $\alpha_W(\mathcal{Q})$ and $\beta(\mathcal{S})$ as follows:

$$\begin{aligned} \alpha_W(\mathcal{Q}) = & \sum_{\mathcal{K} \in \mathcal{Q}} H\left(V_{\mathcal{K}} | \{V\}_{\mathcal{I}_{W+}(\mathcal{K})}\right) \\ & -H\left(\{V\}_{\mathcal{Q}} | \{V\}_{\mathcal{I}_{W+}}, X\right)\ \ \forall \mathcal{Q} \subseteq \mathcal{I}_W \qquad (9) \\ \beta(\mathcal{S}) = & \sum_{\mathcal{K} \subseteq \mathcal{S}} H\left(U_{\mathcal{K}} | \{U\}_{2^{\mathcal{K}}-\phi-\{\mathcal{K}\}}, \{V\}_{\mathcal{J}(\mathcal{K})}\right) \\ & -H\left(\{U\}_{2^{\mathcal{S}}-\phi} | \{V\}_{\mathcal{I}_{1+}}, X\right)\ \ \forall \mathcal{S} \subseteq \mathcal{L} \qquad (10) \end{aligned}$$

We follow the convention $\alpha_W(\phi) = \beta(\phi) = 0$. We next state the achievable region using the CMS scheme in the following theorem.

Theorem 1. *Let $(\{V\}_{\mathcal{J}(\mathcal{L})}, \{U\}_{2^{\mathcal{L}}-\phi})$ be any set of $2^{L+1} - L - 2$ random variables jointly distributed with X defined on arbitrary finite alphabets. Let $\alpha_W(\mathcal{Q})$ and $\beta(\mathcal{S})$ be defined as in (9) and (10). Let $R_{\mathcal{K}}^{''}$ $\forall \mathcal{K} \in \mathcal{J}\ (\mathcal{L})$ and $R_l^{'}$ $\forall l \in \mathcal{L}$ be any set of rate tuples satisfying:*

$$\begin{aligned} \sum_{\mathcal{K} \in \mathcal{Q}} R_{\mathcal{K}}^{''} &\geq \alpha_W(\mathcal{Q})\ \forall \mathcal{Q} \subseteq \mathcal{I}_W, W \in \mathcal{L} \\ \sum_{l \in \mathcal{S}} R_l^{'} &\geq \beta(\mathcal{S})\ \forall \mathcal{S} \subseteq \mathcal{L} \end{aligned}$$

then, the RD region for the $L-$channel MD problem contains the rates and distortions for which there exist functions $\psi_{\mathcal{S}}(\cdot)$, such that

$$R_l \geq R_l^{'} + \sum_{\mathcal{K} \in \mathcal{J}(l)} R_{\mathcal{K}}^{''} \qquad (11)$$

$$D_{\mathcal{S}} \geq E\left[d_{\mathcal{S}}\left(X, \psi_{\mathcal{S}}\left(\{V\}_{\mathcal{J}(\mathcal{S})}, \{U\}_{2^{\mathcal{S}}-\phi}\right)\right)\right] \qquad (12)$$

The closure of the achievable tuples over all such $2^{L+1} - L - 2$ random variables is denoted by $\mathcal{RD}_{CMS}$.

Proof: Due to space constraints, we omit the detailed proof here. We only briefly describe the codebook generation scheme and error analysis.

Codebook Generation : Suppose we are given $P\left(\{v\}_{\mathcal{J}(\mathcal{L})}, \{u\}_{2^{\mathcal{L}}} | x\right)$ and $\psi_{\mathcal{S}}(\cdot)$ satisfying (12). The codebook generation begins with $V_{\mathcal{L}}$. We independently generate $2^{nR_{\mathcal{L}}^{''}}$ codewords of $V_{\mathcal{L}}$, $v_{\mathcal{L}}^n(j_{\mathcal{L}})$ $j_{\mathcal{L}} \in \{1 \ldots 2^{nR_{\mathcal{L}}^{''}}\}$, according to the PMF $\prod_{t=1}^n P_{V_{\mathcal{L}}}(v_{\mathcal{L}}^{(t)})$. For each codeword $v_{\mathcal{L}}^n(j_{\mathcal{L}})$, we independently generate $2^{nR_{\mathcal{S}}^{''}}$ codewords of $V_{\mathcal{S}}$ $\forall \mathcal{S} \in \mathcal{I}_{L-1}$, according to $\prod_{t=1}^n P_{V_{\mathcal{S}}|V_{\mathcal{L}}}(v_{\mathcal{S}}^{(t)} | v_{\mathcal{L}}^{(t)})$. We denote these codewords by $v_{\mathcal{S}}^n(j_{\mathcal{L}}, j_{\mathcal{S}})$ $j_{\mathcal{S}} \in \{1 \ldots 2^{nR_{\mathcal{S}}^{''}}\}$. This procedure for generating the codebooks of the shared random variables continues. $2^{nR_{\mathcal{S}}^{''}}$ codewords of $V_{\mathcal{S}}$ are independently generated for each codeword tuple of $\{V\}_{\mathcal{I}_{W+}(\mathcal{S})}$ according to $\prod_{t=1}^n P_{V_{\mathcal{S}}|\{V\}_{\mathcal{I}_{W+}(\mathcal{S})}}(v_{\mathcal{S}}^{(t)} | \{v\}_{\mathcal{I}_{W+}(\mathcal{S})}^{(t)})$. These codewords are denoted by $v^n_{\mathcal{S}}(\{j\}_{\mathcal{I}_{W+}(\mathcal{S})}, j_{\mathcal{S}})\ j_{\mathcal{S}} \in \{1 \dots 2^{nR''_{\mathcal{S}}}\}$. Note that to generate the codebooks for $V_{\mathcal{S}}\ \forall \mathcal{S} \in \mathcal{I}_W$, we need the codebooks of $V_{\mathcal{S}}\ \forall \mathcal{S} \in \mathcal{I}_{W+}(\mathcal{Q})$. The codebook generation follows the order indicated in figure 4.

Once all the codebooks of shared random variables are generated, the codebooks for the base layer random variables are generated. For each codeword tuple of $\{V\}_{I_{1+}(l)}$, $2^{nR'_l}$ codewords of U_l are generated independently according to $\prod_{t=1}^n P_{U_l|(\{V\}_{\mathcal{I}_{1+}(l)})}(u_l^{(t)}|\{v\}^{(t)}_{\mathcal{I}_{1+}(l)})$ and are denoted by $u_l^n\left(\{j\}_{I_{1+}(l)}, i_l\right)\ i_l \in \{1 \dots 2^{nR'_l}\}$. Then the codebooks for the refinement layers are formed by assigning a codeword $u^n_{\mathcal{S}}(\{j\}_{\mathcal{J}(\mathcal{S})}, \{i\}_{\mathcal{S}})$ to each $\mathcal{S} \in 2^{\mathcal{L}} - \{(1),(2)\dots(L)\}$ and $\forall \{j\}_{\mathcal{J}(\mathcal{S})}, \{i\}_{\mathcal{S}}$. These codewords are generated according to $\prod_{t=1}^n P_{U_{\mathcal{S}}|\{V\}_{\mathcal{J}(\mathcal{S})},\{U\}_{2^{\mathcal{S}}-\mathcal{S}}}(u^{(t)}_{\mathcal{S}}|\{v\}^{(t)}_{\mathcal{J}(\mathcal{S})}, \{u\}^{(t)}_{2^{\mathcal{S}}-\mathcal{S}})$.

The encoder, on observing a sequence X^n attempts to find a set of codewords, one for each variable, such that they are all jointly typical. If the encoder succeeds in finding such a set, from the typical average lemma [9], it follows that the average distortions are less than $D_{\mathcal{S}},\ \forall \mathcal{S} \in\ 2^{\mathcal{L}} - \phi$. However, if the encoder fails to find such a set, the average distortions are upper bounded by d_{max} (as the distortion measures are assumed to be bounded). Hence, if the probability of finding a set of jointly typical codewords approaches 1 the distortion conditions, (12), are met. We next show that, this probability approaches 1 if the rates satisfy (11).

Probability of error analysis : Let E_{VU} be the event of not finding a set of jointly typical codewords. We have,

$$P(E_{VU}) = P(E_V) + P(E_{VU}|E^c_V)P(E^c_V) \qquad (13)$$

where E_V denotes the event of not finding a set of 'shared codewords' which are jointly typical with X^n.

Conditions on $R''_{\mathcal{S}}$: We can rewrite $P(E_V)$ as,

$$\begin{aligned} P(E_V) \ &= \ P(E_{V,L}) + P(E_{V,L-1}|E^c_{V,L})P(E^c_{V,L}) + \\ &\quad \dots + P(E_{V,2}|E^c_{V,3})P(E^c_{V,3}) \end{aligned} \qquad (14)$$

where $E_{V,W}$ denotes the event of not finding a set of codewords of $\{V\}_{(I_W, I_{W+})}$ which are jointly typical with X^n. It can be shown that, $P(E_{V,W}|E^c_{V,W+1})$ can be made arbitrarily small if $\sum_{\mathcal{K}\in\mathcal{Q}} R''_{\mathcal{K}} > \alpha_W(\mathcal{Q})\ \forall \mathcal{Q} \subseteq \mathcal{I}_W$.

Conditions on $R'_{\mathcal{S}}$: Similar arguments as in [3] can be used to show that, $P(E_{VU}|E^c_V)$ can be made arbitrarily small if $\sum_{l\in\mathcal{S}} R'_l > \beta(\mathcal{S})\ \forall \mathcal{S} \subseteq \mathcal{L}$.

Conditions on $R_{\mathcal{S}}$: Recall that the encoder sends the codewords of $V_{\mathcal{S}}$ (at rate $R''_{\mathcal{S}}$) in all the descriptions $l \in \mathcal{S}$. It also sends the codewords of U_l (at rate R'_l) in description l. Therefore the rate of description l can be bounded by:

$$R_l \geq R'_l + \sum_{\mathcal{K}\in\mathcal{J}(l)} R''_{\mathcal{K}} \qquad (15)$$

Combining (13), (14) and (15), proves the theorem. ■

Note that, the bounds on R_l in (15) are implicit in terms of the bounds on R'_l and $R''_{\mathcal{K}}$. Expressing them directly in terms information theoretic quantities is considerably harder and will be considered as part of future work.

Corollary 1. *The achievable rate-distortion region of Theorem 1 subsumes the VKG region, i.e.* $\mathcal{RD}_{VKG} \subseteq \mathcal{RD}_{CMS}$

Proof: The proof follows directly by setting $V_{\mathcal{S}} = \Phi\ \forall \mathcal{S}$ such that $|\mathcal{S}| < L$. We then have $\alpha_W(\mathcal{Q}) = 0\ \forall \mathcal{Q}, W < L$. Substituting in (11), we get

$$R_{(\mathcal{S})} \geq \beta(\mathcal{S}) + |\mathcal{S}|\alpha_L(\mathcal{L}) \qquad (16)$$

which is same as (2). ■

IV. Comment on strict enlargement

At the time of the submission of this manuscript, we did not know whether the CMS scheme leads to a strictly larger rate-distortion region compared to the VKG scheme, i.e., $\mathcal{RD}_{VKG} \subset \mathcal{RD}_{CMS}$. We recently proved that, this is indeed true, i.e., a combinatorial number of shared codewords, one for every subset of descriptions, can be used to achieve points outside the VKG region. The proof is available on arXiv [12] for interested readers.

V. Conclusion

In this paper, we proposed a new encoding scheme for the $L-$channel multiple descriptions problem. This results in a new achievable rate-distortion region which subsumes the most famous achievable region for this problem. The proposed encoding adds controlled redundancy by including a common codeword in every subset of the descriptions. The possible impact of the new ideas presented here, on practical multiple descriptions encoder design and on certain special cases of the setting will be studied as part of future work.

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