Hybrid Control and Switched Systems

Lecture #10
Switched systems

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Summary

Switched systems
- Linear switched systems
- Lyapunov stability, asymptotic stability, exponential stability

Using switched systems to analyze complex hybrid systems

Switched system

parameterized family of vector fields $\equiv$ $f_p : \mathbb{R}^n \to \mathbb{R}^n$ $\quad p \in \mathcal{Q}$

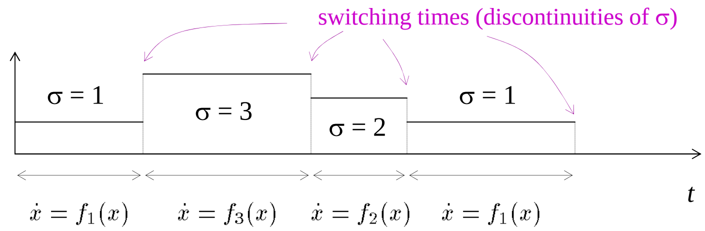
switching signal $\equiv$ piecewise constant signal $\sigma : [0,\infty) \to \mathcal{Q}$ $\quad$ parameter set

$\mathcal{S} \equiv$ set of admissible switching signals

E.g., $\mathcal{S} := \{\ \sigma : N_\sigma(\tau, t) \leq 1 + (t - \tau),\ \forall\, t > \tau \geq 0\ \}$

\# of discontinuities of σ in the interval (τ, t)

$$\dot{x} = f_\sigma(x) \qquad \sigma \in \mathcal{S}$$



A ***solution*** to the switched system is any pair (σ, x) with $\sigma \in \mathcal{S}$ and x a solution to

$$\dot{x} = f_{\sigma(t)}(x)$$

time-varying ODE

Switched system with state-dependent switching

parameterized family of vector fields $\equiv$ $f_p : \mathbb{R}^n \to \mathbb{R}^n$ $\quad p \in \mathcal{Q}$

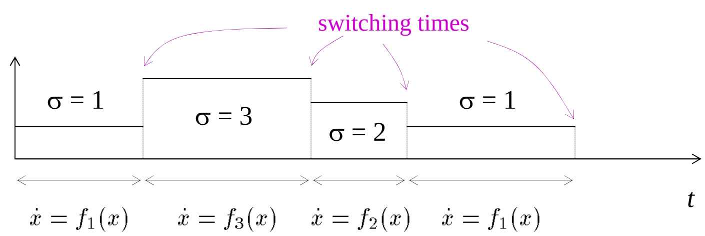
switching signal $\equiv$ piecewise constant signal $\sigma : [0,\infty) \to \mathcal{Q}$ $\quad$ parameter set

$\mathcal{S} \equiv$ set of admissible pairs (σ, x) with σ a switching signal and x a signal in $\mathbb{R}^n$

E.g., $\mathcal{S} := \{(\sigma, x) : N_\sigma(\tau, t) \leq 1 + \sup_{s \in (\tau,t)} \|x(s)\|\ (t - \tau),\ \forall\, t > \tau \geq 0\ \}$

for each x only some σ may be admissible

$$\dot{x} = f_\sigma(x) \qquad (\sigma, x) \in \mathcal{S}$$



A ***solution*** to the switched system is a pair $(\sigma, x) \in \mathcal{S}$ for which x is a solution to

$$\dot{x} = f_{\sigma(t)}(x)$$

time-varying ODE

Switched system with resets

parameterized family of vector fields $\equiv$ $f_p : \mathbb{R}^n \to \mathbb{R}^n$ $\quad p \in \mathcal{Q}$

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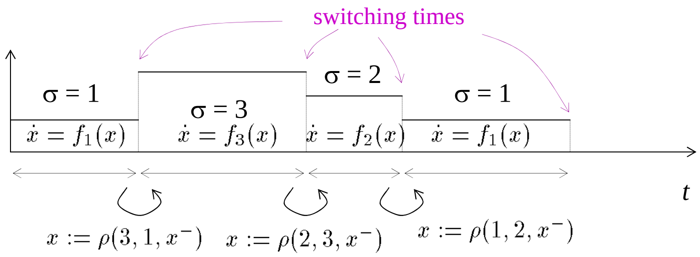
$\mathcal{S} \equiv$ set of admissible pairs (σ, x) with σ a switching signal and x a signal in $\mathbb{R}^n$

$$\dot{x} = f_\sigma(x) \qquad x = \rho(\sigma, \sigma^-, x^-) \qquad (\sigma, x) \in \mathcal{S}$$



A ***solution*** to the switched system is a pair (σ, x) $\in \mathcal{S}$ for which

1. on every open interval on which σ is constant, x is a solution to
$$\dot{x} = f_{\sigma(t)}(x)$$
time-varying ODE
2. at every switching time t, $x(t) = \rho(\sigma(t), \sigma^-(t), x^-(t))$

Time-varying systems vs. Hybrid systems vs. Switched systems

Time-varying system $\equiv$ for each initial condition $x(0)$ there is only one solution

$$\dot{x} = f_{\sigma(t)}(x)$$

(all f_p locally Lipschitz)

Hybrid system $\equiv$ for each initial condition $q(0)$, $x(0)$ there is only one solution

$$\dot{x} = f(q, x) \qquad (q, x) = \Phi(q^-, x^-)$$

Switched system $\equiv$ for each $x(0)$ there may be several solutions, one for each admissible σ

$$\dot{x} = f_\sigma(x) \qquad x = \rho(\sigma, \sigma^-, x^-) \qquad (\sigma, x) \in \mathcal{S}$$

the notions of stability, convergence, etc.
must address "uniformity" over all solutions

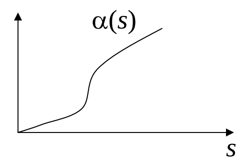
Stability of ODEs

$$\dot{x} = f(x) \qquad x \in \mathbb{R}^n$$

equilibrium point $\equiv x_{eq} \in \mathbb{R}^n$ for which $f(x_{eq}) = 0$

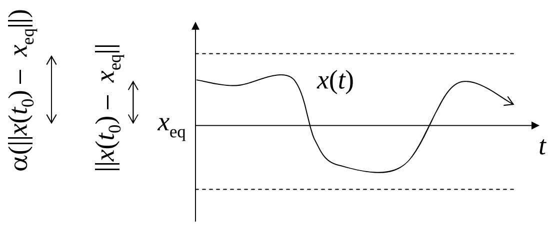
class $\mathcal{K} \equiv$ set of functions $\alpha : [0,\infty) \to [0,\infty)$ that are

1. continuous
2. strictly increasing
3. $\alpha(0)=0$



Definition (class $\mathcal{K}$ function definition):
The equilibrium point x_{eq} is (***Lyapunov) stable*** if $\exists\ \alpha \in \mathcal{K}$:

$$\|x(t) - x_{eq}\| \leq \alpha(\|x(t_0) - x_{eq}\|) \ \forall\, t \geq t_0 \geq 0,\ \|x(t_0) - x_{eq}\| \leq c$$



Stability of switched systems

$$\dot{x} = f_\sigma(x) \qquad (\sigma, x) \in \mathcal{S}$$

equilibrium point $\equiv x_{eq} \in \mathbb{R}^n$ for which $f_q(x_{eq}) = 0\ \forall\, q \in \mathcal{Q}$

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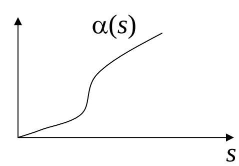

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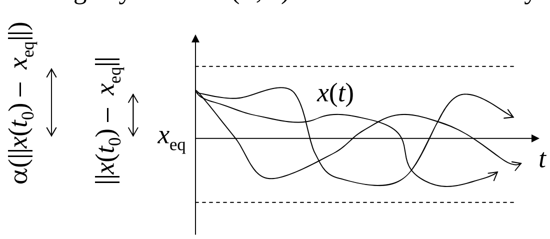
along any solution $(\sigma, x) \in \mathcal{S}$ to the switched system

α is independent of $x(t_0)$ and σ



in switched systems one is only concerned about boundedness or convergence of the continuous state

Asymptotic stability of ODEs

$$\dot{x} = f(x) \qquad x \in \mathbb{R}^n$$

equilibrium point $\equiv x_{\text{eq}} \in \mathbb{R}^n$ for which $f(x_{\text{eq}}) = 0$

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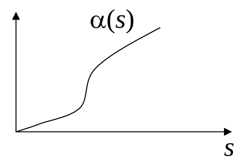
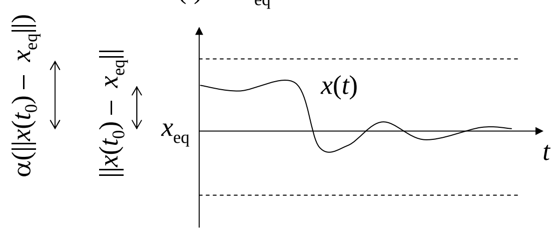


Definition:
The equilibrium point x_{eq} is ***(globally) asymptotically stable*** if it is Lyapunov stable and for every initial state the solution exists on $[0,\infty)$ and

$$x(t) \to x_{\text{eq}} \text{ as } t \to \infty.$$



Asymptotic stability of switched systems

$$\dot{x} = f_\sigma(x) \qquad (\sigma, x) \in \mathcal{S}$$

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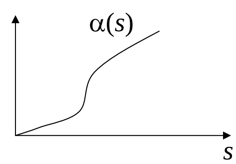

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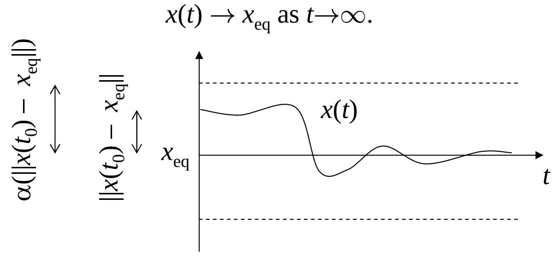
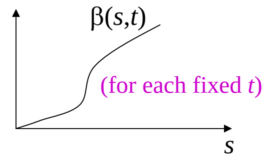


Asymptotic stability of ODEs

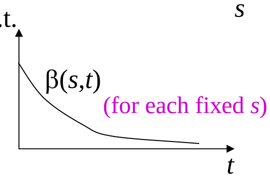
$$\dot{x} = f(x) \qquad x \in \mathbb{R}^n$$



equilibrium point ≡ $x_{eq} \in \mathbb{R}^n$ for which $f(x_{eq}) = 0$

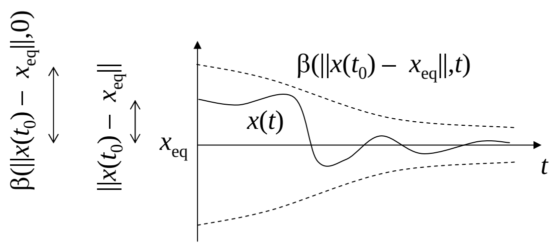
class $\mathcal{KL}$ ≡ set of functions $\beta : [0,\infty)\times[0,\infty)\to[0,\infty)$ s.t.

1. for each fixed t, $\beta(\cdot,t) \in \mathcal{K}$
2. for each fixed s, $\beta(s,\cdot)$ is monotone decreasing and $\beta(s,t) \to 0$ as $t\to\infty$



Definition (class $\mathcal{KL}$ function definition):
The equilibrium point x_{eq} is ***(globally) asymptotically stable*** if $\exists\, \beta\in\mathcal{KL}$:

$$\|x(t) - x_{eq}\| \leq \beta(\|x(t_0) - x_{eq}\|, t - t_0) \quad \forall\, t \geq t_0 \geq 0$$



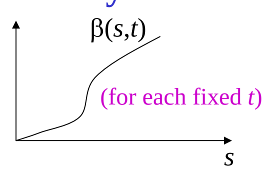
We have ***exponential stability*** when

$$\beta(s,t) = c\, e^{-\lambda t}\, s$$

with $c,\lambda > 0$

Uniform asymptotic stability of switched systems

$$\dot{x} = f_\sigma(x) \qquad (\sigma, x) \in \mathcal{S}$$



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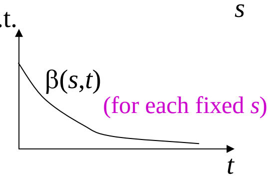
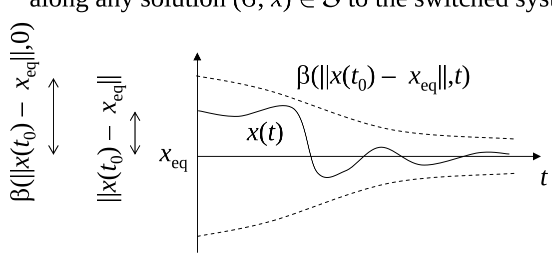


Definition (class $\mathcal{KL}$ function definition):
The equilibrium point x_{eq} is ***uniformly asymptotically stable*** if $\exists\, \beta\in\mathcal{KL}$:

$$\|x(t) - x_{eq}\| \leq \beta(\|x(t_0) - x_{eq}\|, t - t_0) \quad \forall\, t \geq t_0 \geq 0$$

along any solution $(\sigma, x) \in \mathcal{S}$ to the switched system

β is independent of $x(t_0)$ and σ



We have ***exponential stability*** when

$$\beta(s,t) = c\, e^{-\lambda t}\, s$$

with $c,\lambda > 0$

Three notions of stability

Definition (class $\mathcal{K}$ function definition):
The equilibrium point x_{eq} is ***stable*** if $\exists\, \alpha \in \mathcal{K}$:

$$\|x(t) - x_{eq}\| \le \alpha(\|x(t_0) - x_{eq}\|) \;\; \forall\, t \ge t_0 \ge 0,\; \|x(t_0) - x_{eq}\| \le c$$

along any solution $(x, \sigma) \in \mathcal{S}$ to the switched system

α is independent of $x(t_0)$ and σ

Definition:
The equilibrium point $x_{eq} \in \mathbb{R}^n$ is ***asymptotically stable*** if
it is Lyapunov stable and for every solution that exists on $[0,\infty)$

$$x(t) \to x_{eq} \text{ as } t \to \infty.$$

Definition (class $\mathcal{KL}$ function definition):
The equilibrium point $x_{eq} \in \mathbb{R}^n$ is ***uniformly asymptotically stable*** if $\exists\, \beta \in \mathcal{KL}$:

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along any solution $(\sigma, x) \in \mathcal{S}$ to the switched system

β is independent of $x(t_0)$ and σ

exponential stability when $\beta(s,t) = c\, e^{-\lambda t} s$ with $c,\lambda > 0$

Example

$$\dot{x} = \sigma x$$

$\mathcal{S} \equiv$ set of piecewise constant switching signals taking values in $\mathcal{Q} := \{-1, +1\}$

$\mathcal{S} \equiv$ set of piecewise constant switching signals taking values in $\mathcal{Q} := \{-1, 0\}$

$\mathcal{S} \equiv$ set of piecewise constant switching signals taking values in $\mathcal{Q} := \{-1, 0\}$
with infinitely many switches

$\mathcal{S} \equiv$ set of piecewise constant switching signals taking values in $\mathcal{Q} := \{-1, 0\}$
with infinitely many switches and interval between consecutive discontinuities bounded below by 1

$\mathcal{S} \equiv$ set of piecewise constant switching signals taking values in $\mathcal{Q} := \{-1, 0\}$
with infinitely many switches and interval between consecutive discontinuities below by 1 and above by 2

Example

$$\dot{x} = \sigma x$$

$\mathcal{S} \equiv$ set of piecewise constant switching signals taking values in $\mathcal{Q} := \{-1, +1\}$
unstable

$\mathcal{S} \equiv$ set of piecewise constant switching signals taking values in $\mathcal{Q} := \{-1, 0\}$
stable but not asympt.

$\mathcal{S} \equiv$ set of piecewise constant switching signals taking values in $\mathcal{Q} := \{-1, 0\}$
with infinitely many switches
stable but not asympt.

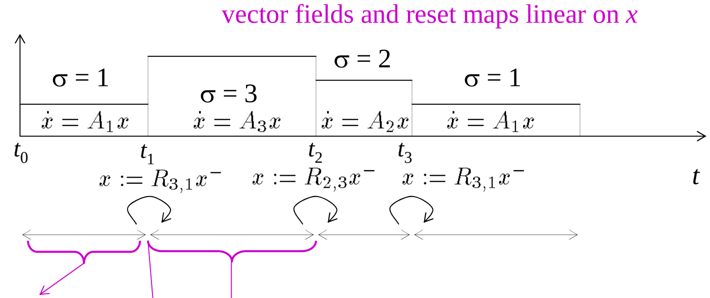
$\mathcal{S} \equiv$ set of piecewise constant switching signals taking values in $\mathcal{Q} := \{-1, 0\}$
with infinitely many switches and interval between consecutive discontinuities bounded below by 1
asympt. stable

$\mathcal{S} \equiv$ set of piecewise constant switching signals taking values in $\mathcal{Q} := \{-1, 0\}$
with infinitely many switches and interval between consecutive discontinuities below by 1 and above by 2
uniformly asympt. stable

Linear switched systems

$$\dot{x} = A_\sigma x \qquad x = R_{\sigma,\sigma^-} x^- \qquad (\sigma, x) \in \mathcal{S} \qquad A_q, R_{q,q'} \in \mathbb{R}^{n\times n} \quad q,q' \in \mathcal{Q}$$

vector fields and reset maps linear on x



$$x(t) = e^{A_1(t-t_0)} x(t_0)$$

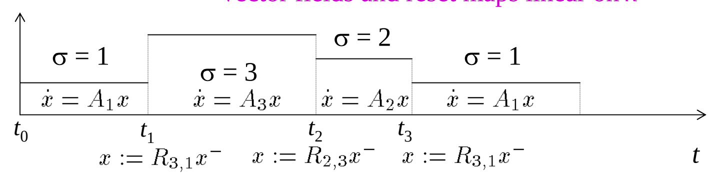
$$x(t_1) = R_{3,1} x^-(t_1) = R_{3,1} e^{A_1(t_1-t_0)} x(t_0)$$

$$x(t) = e^{A_3(t-t_1)} x(t_1) = e^{A_3(t-t_1)} R_{3,1} e^{A_1(t_1-t_0)} x(t_0)$$

Linear switched systems

$\dot{x} = A_\sigma x \qquad x = R_{\sigma,\sigma^-} x^- \qquad (\sigma, x) \in \mathcal{S} \qquad A_q, R_{q,q'} \in \mathbb{R}^{n\times n} \quad q,q' \in Q$

vector fields and reset maps linear on x



$$x(t) = \Phi_\sigma(t,\tau) x(\tau)$$

state-transition matrix for the switched system (σ-dependent)

$$\Phi_\sigma(t,\tau) := e^{A_{\sigma(t_k)}(t-t_k)} R_{\sigma(t_k),\sigma(t_{k-1})} e^{A_{\sigma(t_{k-1})}(t_k - t_{k-1})} \dots$$
$$\dots R_{\sigma(t_2),\sigma(t_1)} e^{A_{\sigma(\tau)}(t_1-\tau)} \qquad t \geq \tau$$

$t_1, t_2, t_3, \dots, t_k \equiv$ switching times of σ in the interval $[t,\tau)$

Linear switched systems

$\dot{x} = A_\sigma x \qquad x = R_{\sigma,\sigma^-} x^- \qquad (\sigma, x) \in \mathcal{S} \qquad A_q, R_{q,q'} \in \mathbb{R}^{n\times n} \quad q,q' \in Q$

$x(t) = \Phi_\sigma(t,\tau) x(\tau)$ state-transition matrix (σ-dependent)

$$\Phi_\sigma(t,\tau) := e^{A_{\sigma(t_k)}(t-t_k)} R_{\sigma(t_k),\sigma(t_{k-1})} e^{A_{\sigma(t_{k-1})}(t_k - t_{k-1})} \dots$$
$$\dots R_{\sigma(t_1),\sigma(\tau)} e^{A_{\sigma(\tau)}(t_1-\tau)} \qquad t \geq \tau$$

$t_1, t_2, t_3, \dots, t_k \equiv$ switching times of σ in the interval $[t,\tau)$

Analogous to what happens for (unswitched) linear systems:

1. $\Phi_\sigma(\tau,\tau) = I \quad \forall \tau$
2. $\Phi_\sigma(t,s)\, \Phi_\sigma(s,\tau) = \Phi_\sigma(t,\tau) \quad \forall\, t \geq s \geq \tau$ (semi-group property)
3. if t is not a switching time, $\Phi_\sigma(t,\tau)$ is differentiable at t and
$$\frac{\mathrm{d}}{\mathrm{d}t}\Phi_\sigma(t,\tau) = A_{\sigma(t)} \Phi_\sigma(t,\tau)$$
4. if t is a switching time,
$$\Phi_\sigma(t,\tau) = R_{\sigma(t),\sigma^-(t)} \Phi_\sigma^-(t,\tau)$$

for a given σ, Φ_σ is a "solution" to the switched system with resets

5. variation of constants formula holds for systems with inputs

but now Φ_σ may not be nonsingular (will be singular if one of the $R_{q,q'}$ are)

Uniform vs. exponential stability

$\dot{x} = A_\sigma x \qquad x = R_{\sigma,\sigma^-} x^- \qquad (\sigma, x) \in \mathcal{S} \qquad A_q, R_{q,q'} \in \mathbb{R}^{n\times n} \quad q,q' \in Q$

state-independent switching $\equiv$ $\mathcal{S}$ is such that

$(\sigma, x) \in \mathcal{S} \Rightarrow (\sigma, z) \in \mathcal{S}$

for any other piecewise continuous z

only σ determines whether or not (σ, x) is admissible

Theorem:
For switched linear systems with state-independent switching, uniform asymptotic stability implies exponential stability (two notions are equivalent)

Outline…

1st By uniform asymptotic stability $\exists\, \beta \in \mathcal{KL}$: $\|x(t)\| \le \beta(\|x(t_0)\|, t - t_0)\ \ \forall\, t \ge t_0 \ge 0$

2nd Choose T sufficiently large so that $\beta(1,T) = \gamma = e^{-\lambda} < 1$ ($\lambda > 0$)

3rd Pick arbitrary solution $(\sigma, x) \in \mathcal{S}$

4th Consider another solution (σ, x^*) starting at $x^*(\tau_1) = z := x(\tau_1)/\|x(\tau_1)\|$. Then

$x(\tau_2) = \Phi_\sigma(\tau_2,\tau_1)\, x(\tau_1) = \|x(\tau_1)\|\, \Phi_\sigma(\tau_2,\tau_1)\, z = \|x(\tau_1)\|\, x^*(\tau_2)$

$\|x^*(\tau_2)\| \le \beta(\|z\|, \tau_2 - \tau_1) = \beta(1, \tau_2 - \tau_1)$

$\Rightarrow \quad \|x(\tau_2)\| \le \beta(1, \tau_2 - \tau_1)\, \|x(\tau_1)\|$

exponential decrease of γ^k any interval of length $\ge k\,T$

Uniform vs. exponential stability

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For switched linear systems with state-independent switching, uniform asymptotic stability implies exponential stability (two notions are equivalent)

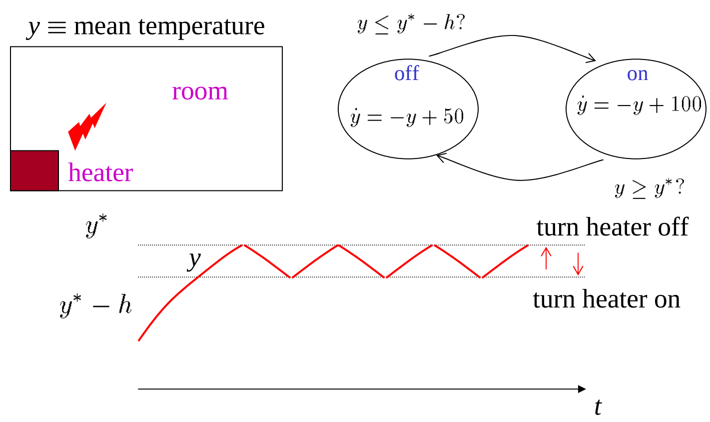
Outline…

4th … $\|x(\tau_2)\| = \beta(1, \tau_2 - \tau_1)\, \|x(\tau_1)\|$

5th Given an arbitrary interval $[t_0,t]$, break it into $k :=$ floor$\big((t - t_0)/T\big)$ intervals of length T plus one interval of length smaller than T …

$$\|x(t)\| \le \beta(1,0)\, e^{\frac{\lambda(t-t_0)}{T}}\, \|x(t_0)\|$$

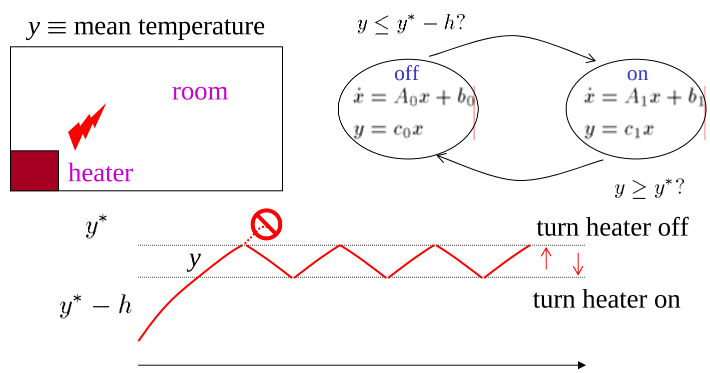
Example #2: Thermostat



The state of the system remains bounded as $t \to \infty$:

$$\min\{y(0), y^* - h\} \leq y(t) \leq \max\{y(0), y^*\} \qquad \forall t \geq 0$$

Example #2: Thermostat



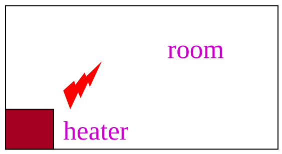
A_0, A_1 asymptotically stable (all eigenvalues with negative real part)

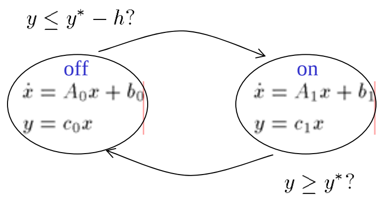
1. if system would stay in off mode forever then
   eq. state $x_{eq} = A_0^{-1} b_0$ is asymptotically stable & $y \to y_{off} := c_0 A_0^{-1} b_0 \leq y^*-h$
2. if system would stay in on mode forever then
   eq. state $x_{eq} = A_1^{-1} b_1$ is asymptotically stable & $y \to y_{on} := c_1 A_1^{-1} b_1 \geq y^*$

With switching, does the overall state x of the system remains bounded as $t \to \infty$?

Example #2: Thermostat

$y \equiv$ mean temperature





One option to prove that the state remains bounded:

1st Establish a bound of how fast switching can occur:
on an interval (τ, t) the maximum number of switchings $N(\tau, t)$ is bounded by

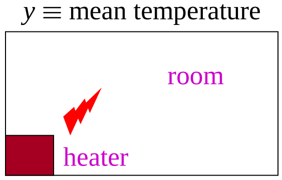
$$N(\tau, t) \leq 1 + \frac{c \sup_{s \in (\tau, t)} \|x(s)\|}{h}(t - \tau)$$

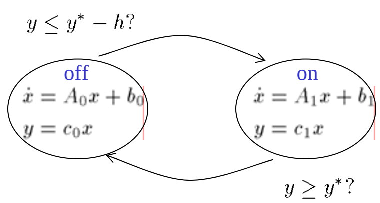
Why? maximum derivative of y is proportional to ||x|| and between two consecutive switchings y must have a variation of h

a (sequence) property of the discrete-component of the state

Example #2: Thermostat

$y \equiv$ mean temperature





One option to prove that the state remains bounded:

1st Establish a bound of how fast switching can occur:
on an interval (τ, t) the maximum number of switchings $N(\tau, t)$ is bounded by

x is a solution to the following (state-dependent) switching system:

$$\dot{x} = A_\sigma x + b_\sigma$$

with

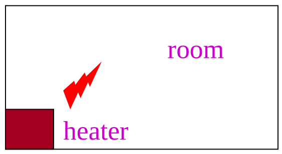
$$\mathcal{S} := \left\{(\sigma, x) : N_\sigma(\tau, t) \leq 1 + \frac{c \sup_{s \in (\tau, t)} \|x(s)\|}{h}(t - \tau),\ \forall t \geq \tau \geq 0\right\}$$

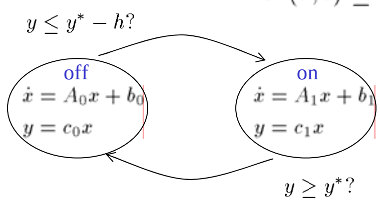
(tough to analyze directly…)

Example #2: Thermostat

$$N(\tau,t) \le N_0 + \frac{t-\tau}{\tau_D}$$

$y \equiv$ mean temperature





One option to prove the state remains bounded:

a (sequence) property of the continuous-dynamics

2nd Estimate how large x can be from y:

For the following (state independent) switching systems

$$\begin{cases} \dot{x} = A_\sigma x + b_\sigma \\ y = c_\sigma x \end{cases} \qquad \mathcal{S} := \left\{ \sigma : N_\sigma(\tau,t) \le N_0 + \frac{t-\tau}{\tau_D},\ \forall t \ge \tau \ge 0 \right\}$$

there exist constants $\alpha \ge 1$, $\beta, \gamma > 0$ such that

$$\|x(t)\| \le \alpha \|x(\tau)\| + \beta + \gamma \underbrace{\sup_{s \in (\tau,t)} \|y(s)\|}_{\le y^*}$$

- constants α, β, γ depend on N_0 & τ_D
- to prove this one needs the system to be observable from y

Example #2: Thermostat

1st On an interval (τ,t) the maximum number of switchings $N(\tau,t)$ is bounded by

$$N(\tau,t) \le 1 + \frac{c \sup_{s \in (\tau,t)} \|x(s)\|}{h}(t-\tau)$$

2nd Assuming that the max. number of switchings $N(\tau,t)$ on (τ,t) is bounded by

$$N(\tau,t) \le N_0 + \frac{t-\tau}{\tau_D}$$

Then there exist constants $\alpha \ge 1$, $\beta, \gamma > 0$ such that

$$\|x(t)\| \le \alpha \|x(\tau)\| + \beta + \gamma y^*$$

3rd For any choice of τ_D and h such that

$$\alpha \|x(0)\| + \beta + \gamma y^* < \frac{h}{c\,\tau_D}$$

x must be bounded for any solution compatible with 1 & 2 above.

Hint: prove by contradiction that

$$\frac{c\|x(s)\|}{h} \le \frac{1}{\tau_D} \quad \forall s \ge 0$$

Proof…

We will show that

$$\frac{c\|x(t)\|}{h} < \frac{1}{\tau_D} \quad \forall t \geq 0 \qquad (*)$$

1nd For $s = 0$, (*) holds because … $\alpha \geq 1$

$$\alpha\|x(0)\| + \beta + \gamma y^* < \frac{h}{c\,\tau_D} \quad \Rightarrow \quad \frac{c}{h}\|x(0)\| < \frac{1}{\alpha\,\tau_D} \leq \frac{1}{\tau_D}$$

2nd By contradiction suppose that (*) holds strictly for $t \in [0,t^*)$ and with equality at $t = t^*$. Then

$$N(0,t^*) \leq 1 + \frac{c \sup_{s\in(0,t^*)} \|x(s)\|}{h} t \leq 1 + \frac{1}{\tau_D}$$

Therefore, we conclude that

$$\|x(t^*)\| \leq \alpha\|x(0)\| + \beta + \gamma y^* < \frac{h}{c\,\tau_D}$$

$\|x(t^*)\|$ can never reach $h / (c\,\tau_D)$!

Discrete/continuous decoupling

1st x is a solution to the following (state-dependent) switching system:

$$\dot{x} = A_\sigma x + b_\sigma$$

$$\mathcal{S} := \left\{(\sigma,x) : N_\sigma(\tau,t) \leq 1 + \frac{c \sup_{s\in(\tau,t)} \|x(s)\|}{h}(t-\tau)\right\}$$

property of the discrete evolution

2nd For the following (state-independent) switching system:

$$\begin{cases} \dot{x} = A_\sigma x + b_\sigma \\ y = c_\sigma x \end{cases}$$

$$\mathcal{S} := \left\{\sigma : N_\sigma(\tau,t) \leq N_0 + \frac{t-\tau}{\tau_D}\right\}$$

There exist constants α, β, γ such that

$$\|x(t)\| \leq \alpha\|x(\tau)\| + \beta + \gamma y^*$$

property of a (state-independent) switching systems

property of the interconnection

Next lecture…

Stability under arbitrary switching

- Instability caused by switching
- Common Lyapunov function
- Converse results
- Algebraic conditions