

Hybrid Control and Switched Systems

Lecture #11 Stability of switched system: Arbitrary switching

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Summary

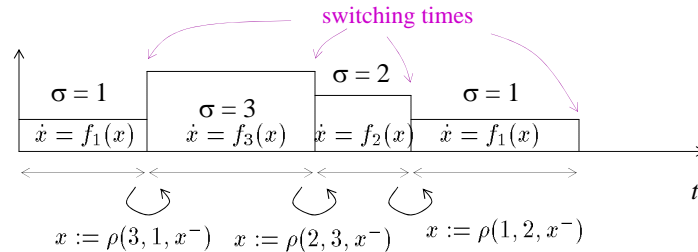
Stability under arbitrary switching

- Instability caused by switching
- Common Lyapunov function
- Converse results
- Algebraic conditions

Switched system

parameterized family of vector fields $\equiv f_p: \mathbb{R}^n \rightarrow \mathbb{R}^n \quad p \in Q$
 switching signal \equiv piecewise constant signal $\sigma: [0, \infty) \rightarrow Q$ parameter set
 $\mathcal{S} \equiv$ set of admissible pairs (σ, x) with σ a switching signal and x a signal in \mathbb{R}^n

$$\dot{x} = f_\sigma(x) \quad x = \rho(\sigma, \sigma^-, x^-) \quad (\sigma, x) \in \mathcal{S}$$



A **solution** to the switched system is a pair $(x, \sigma) \in \mathcal{S}$ for which

1. on every open interval on which σ is constant, x is a solution to

$$\dot{x} = f_{\sigma(t)}(x) \quad \text{time-varying ODE}$$
2. at every switching time t , $x(t) = \rho(\sigma(t), \sigma^-(t), x^-(t))$

Time-varying systems vs. Hybrid systems vs. Switched systems

Time-varying system \equiv for each initial condition $x(0)$ there is only one solution

$$\dot{x} = f_{\sigma(t)}(x) \quad (\text{all } f_p \text{ locally Lipschitz})$$

Hybrid system \equiv for each initial condition $q(0), x(0)$ there is only one solution

$$\dot{x} = f(q, x) \quad (q, x) = \Phi(q^-, x^-)$$

Switched system \equiv for each $x(0)$ there may be several solutions, one for each admissible σ

$$\dot{x} = f_\sigma(x) \quad x = \rho(\sigma, \sigma^-, x^-) \quad (\sigma, x) \in \mathcal{S}$$

the notions of stability, convergence, etc.
 must address "uniformity" over all solutions

Three notions of stability

Definition (class \mathcal{K} function definition):
 The equilibrium point x_{eq} is *stable* if $\exists \alpha \in \mathcal{K}$: α is independent of $x(t_0)$ and σ

$$\|x(t) - x_{eq}\| \leq \alpha(\|x(t_0) - x_{eq}\|) \quad \forall t \geq t_0 \geq 0, \|x(t_0) - x_{eq}\| \leq c$$

along any solution $(x, \sigma) \in \mathcal{S}$ to the switched system

Definition:
 The equilibrium point $x_{eq} \in \mathbb{R}^n$ is *asymptotically stable* if it is Lyapunov stable and for every solution that exists on $[0, \infty)$

$$x(t) \rightarrow x_{eq} \text{ as } t \rightarrow \infty.$$

Definition (class \mathcal{KL} function definition):
 The equilibrium point $x_{eq} \in \mathbb{R}^n$ is *uniformly asymptotically stable* if $\exists \beta \in \mathcal{KL}$:

$$\|x(t) - x_{eq}\| \leq \beta(\|x(t_0) - x_{eq}\|, t - t_0) \quad \forall t \geq t_0 \geq 0$$

along any solution $(x, \sigma) \in \mathcal{S}$ to the switched system β is independent of $x(t_0)$ and σ

exponential stability when $\beta(s, t) = c e^{-\lambda t} s$ with $c, \lambda > 0$

Stability under arbitrary switching

$$\dot{x} = f_\sigma(x) \quad x = \rho(\sigma, \sigma^-, x^-) \quad (\sigma, x) \in \mathcal{S}$$

$\mathcal{S}_{all} \equiv$ set of all pairs (σ, x) with σ piecewise constant and x piecewise continuous $\rho(p, q, x) = x \quad \forall p, q \in \mathcal{Q}, x \in \mathbb{R}^n$
no resets

any switching signal is admissible

If one of the vector fields $f_q, q \in \mathcal{Q}$ is unstable then the switched system is unstable

Why?

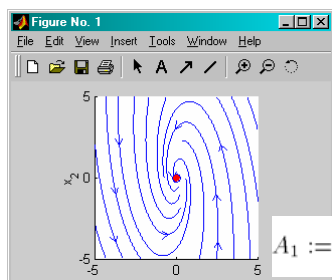
1. because the switching signals $\sigma(t) = q \quad \forall t$ is admissible
2. for this σ we cannot find $\alpha \in \mathcal{K}$ such that

$$\|x(t) - x_{eq}\| \leq \alpha(\|x(t_0) - x_{eq}\|) \quad \forall t \geq t_0 \geq 0, \|x(t_0) - x_{eq}\| \leq c$$

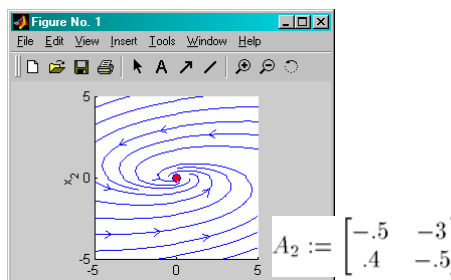
(must less for all σ)

But even if all $f_q, q \in \mathcal{Q}$ are stable the switched system may still be unstable ...

Stability under arbitrary switching



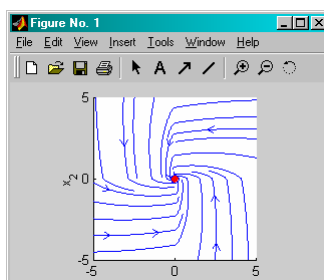
$\dot{x} = A_1 z$ asymptot. stable



$\dot{x} = A_2 z$ asymptot. stable

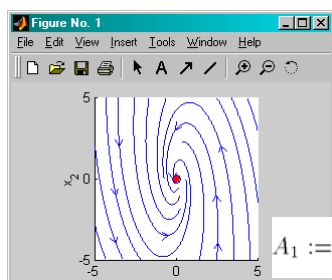
$$\sigma(t) := \begin{cases} 1 & x_1 x_2 \leq 0 \\ 2 & x_1 x_2 > 0 \end{cases}$$

$$\dot{x} = A_{\sigma} x$$

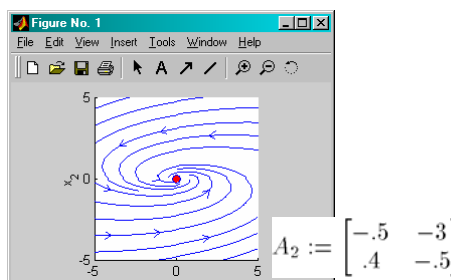


unstable.m

Stability under arbitrary switching



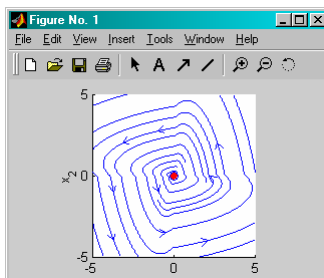
$\dot{x} = A_1 x$ asymptot. stable



$\dot{x} = A_2 x$ asymptot. stable

$$\sigma(t) := \begin{cases} 1 & x_1 x_2 \geq 0 \\ 2 & x_1 x_2 < 0 \end{cases}$$

$$\dot{x} = A_{\sigma} x$$



for some admissible switching signals the trajectories grow to infinity \Rightarrow switched system is unstable

unstable.m

Lyapunov's stability theorem (ODEs)

$$\dot{x} = f(x) \quad x \in \mathbb{R}^n$$

Definition (class \mathcal{K} function definition):

The equilibrium point $x_{\text{eq}} \in \mathbb{R}^n$ is **(Lyapunov) stable** if $\exists \alpha \in \mathcal{K}$:

$$\|x(t) - x_{\text{eq}}\| \leq \alpha(\|x(t_0) - x_{\text{eq}}\|) \quad \forall t \geq t_0 \geq 0, \|x(t_0) - x_{\text{eq}}\| \leq c$$

Theorem (Lyapunov):

Suppose there exists a continuously differentiable, positive definite, **radially unbounded** function $V: \mathbb{R}^n \rightarrow \mathbb{R}$ such that

$$\frac{\partial V}{\partial x}(z - x_{\text{eq}}) f(z) \leq 0 \quad \forall z \in \mathbb{R}^n$$

Then x_{eq} is a Lyapunov stable equilibrium and the **solution always exists globally**. Moreover, if $= 0$ only for $z = x_{\text{eq}}$ then x_{eq} is a (globally) **asymptotically stable equilibrium**.

Why?

V can only stop decreasing when $x(t)$ reaches x_{eq}

but V must stop decreasing because it cannot become negative

Thus, $x(t)$ must converge to x_{eq}

Common Lyapunov function

$$\dot{x} = f_{\sigma}(x) \quad (\sigma, x) \in \mathcal{S}_{\text{all}}$$

Theorem:

Suppose there exists a continuously differentiable, positive definite, radially unbounded function $V: \mathbb{R}^n \rightarrow \mathbb{R}$ such that

$$\frac{\partial V}{\partial x}(z - x_{\text{eq}}) f_q(z) \leq W(z) \leq 0 \quad \forall z \in \mathbb{R}^n, q \in \mathcal{Q}$$

Then

1. the equilibrium point x_{eq} is Lyapunov stable
2. if $W(z) = 0$ only for $z = x_{\text{eq}}$ then x_{eq} is (glob) uniformly asymptotically stable.

The same V could be used to prove stability for all the unswitched systems

$$\dot{x} = f_q(x)$$

Common Lyapunov function

$$\dot{x} = f_\sigma(x) \quad (\sigma, x) \in \mathcal{S}_{\text{all}}$$

Theorem:

Suppose there exists a continuously differentiable, positive definite, radially unbounded function $V: \mathbb{R}^n \rightarrow \mathbb{R}$ such that

$$\frac{\partial V}{\partial x}(z - x_{\text{eq}})f_q(z) \leq W(z) \leq 0 \quad \forall z \in \mathbb{R}^n, q \in \mathcal{Q}$$

Then

1. the equilibrium point x_{eq} is Lyapunov stable
2. if $W(z) = 0$ only for $z = x_{\text{eq}}$ then x_{eq} is (glob) uniformly asymptotically stable.

Why? (for simplicity consider $x_{\text{eq}} = 0$)

1st Take an arbitrary solution (σ, x) and define $v(t) := V(x(t)) \forall t \geq 0$

$$\dot{v} = \frac{\partial V}{\partial x}(x)\dot{x} = \frac{\partial V}{\partial x}(x)f_\sigma(x) \leq W(x(t)) \leq 0$$

2nd Therefore

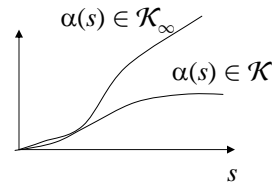
$$v(t) := V(x(t)) \leq v(0) := V(x(0)) \quad \forall t \geq 0$$

$V(x(t))$ is always bounded

Some facts about functions of class \mathcal{K} , \mathcal{K}_∞

class $\mathcal{K} \equiv$ set of functions $\alpha : [0, \infty) \rightarrow [0, \infty)$ that are

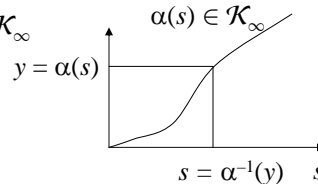
1. continuous
2. strictly increasing
3. $\alpha(0) = 0$



class $\mathcal{K}_\infty \equiv$ subset of \mathcal{K} containing those functions that are unbounded

Lemma 1: If $\alpha_1, \alpha_2 \in \mathcal{K}$ then $\alpha(s) := \alpha_1(\alpha_2(s)) \in \mathcal{K}$ (same for \mathcal{K}_∞)

Lemma 2: If $\alpha \in \mathcal{K}_\infty$ then α is invertible and $\alpha^{-1} \in \mathcal{K}_\infty$



Lemma 3: If $V: \mathbb{R}^n \rightarrow \mathbb{R}$ is positive definite and radially unbounded function

$$\text{then } \exists \alpha_1, \alpha_2 \in \mathcal{K}_\infty: \quad \alpha_1(\|x\|) \leq V(x) \leq \alpha_2(\|x\|) \quad \forall x \in \mathbb{R}^n$$

$$\|x\| \leq \alpha_1^{-1}(V(x))$$

$$\alpha_2^{-1}(V(x)) \leq \|x\|$$

Common Lyapunov function

$$\dot{x} = f_\sigma(x) \quad (\sigma, x) \in \mathcal{S}_{\text{all}}$$

Theorem:

Suppose there exists a continuously differentiable, positive definite, radially unbounded function $V: \mathbb{R}^n \rightarrow \mathbb{R}$ such that

$$\frac{\partial V}{\partial x}(z - x_{\text{eq}}) f_q(z) \leq W(z) \leq 0 \quad \forall z \in \mathbb{R}^n, q \in \mathcal{Q}$$

Then

1. the equilibrium point x_{eq} is Lyapunov stable
2. if $W(z) = 0$ only for $z = x_{\text{eq}}$ then x_{eq} is (glob) uniformly asymptotically stable.

Why? (for simplicity consider $x_{\text{eq}} = 0$)

2nd Therefore

$$v(t) := V(x(t)) \leq v(0) := V(x(0)) \quad \forall t \geq 0$$

3rd Since $\exists \alpha_1, \alpha_2 \in \mathcal{K}_\infty$: $\alpha_1(\|x\|) \leq V(x) \leq \alpha_2(\|x\|)$

$$\|x(t)\| \leq \alpha_1^{-1}(V(x(t))) \leq \alpha_1^{-1}(V(x(0))) \leq \alpha_1^{-1}(\alpha_2(\|x(0)\|)) \quad \forall t \geq 0$$

α_1^{-1} monotone

$V(x(t))$ is always bounded

Common Lyapunov function

$$\dot{x} = f_\sigma(x) \quad (\sigma, x) \in \mathcal{S}_{\text{all}}$$

Theorem:

Suppose there exists a continuously differentiable, positive definite, radially unbounded function $V: \mathbb{R}^n \rightarrow \mathbb{R}$ such that

$$\frac{\partial V}{\partial x}(z - x_{\text{eq}}) f_q(z) \leq W(z) \leq 0 \quad \forall z \in \mathbb{R}^n, q \in \mathcal{Q}$$

Then

1. the equilibrium point x_{eq} is Lyapunov stable
2. if $W(z) = 0$ only for $z = x_{\text{eq}}$ then x_{eq} is (glob) uniformly asymptotically stable.

Why? (for simplicity consider $x_{\text{eq}} = 0$)

3rd Since $\exists \alpha_1, \alpha_2 \in \mathcal{K}_\infty$: $\alpha_1(\|x\|) \leq V(x) \leq \alpha_2(\|x\|)$

$$\|x(t)\| \leq \alpha_1^{-1}(V(x(t))) \leq \alpha_1^{-1}(V(x(0))) \leq \alpha_1^{-1}(\alpha_2(\|x(0)\|)) \quad \forall t \geq 0$$

4th Defining $\alpha(s) := \alpha_1^{-1}(\alpha_2(s)) \in \mathcal{K}$ then

$$\|x(t)\| \leq \alpha(\|x(0)\|) \quad \forall t \geq 0$$

stability ! (1. is proved)

Common Lyapunov function

$$\dot{x} = f_\sigma(x) \quad (\sigma, x) \in \mathcal{S}_{\text{all}}$$

Theorem:

Suppose there exists a continuously differentiable, positive definite, radially unbounded function $V: \mathbb{R}^n \rightarrow \mathbb{R}$ such that

$$\frac{\partial V}{\partial x}(z - x_{\text{eq}}) f_q(z) \leq W(z) \leq 0 \quad \forall z \in \mathbb{R}^n, q \in \mathcal{Q}$$

Then

1. the equilibrium point x_{eq} is Lyapunov stable
2. if $W(z) = 0$ only for $z = x_{\text{eq}}$ then x_{eq} is (glob) uniformly asymptotically stable.

Why? (for simplicity consider $x_{\text{eq}} = 0, W(z) \rightarrow 0$ as $z \rightarrow \infty$) $\alpha_2^{-1}(V(x)) \leq \|x\|$

1st As long as $W \rightarrow 0$ as $z \rightarrow \infty, \exists \alpha_3 \in \mathcal{K}$

$$W(x) \leq -\alpha_3(\|x\|) \leq -\alpha_3(\alpha_2^{-1}(V(x)))$$

2nd Take an arbitrary solution (σ, x) and define $v(t) := V(x(t)) \forall t \geq 0$

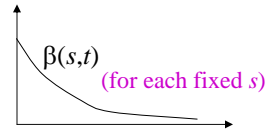
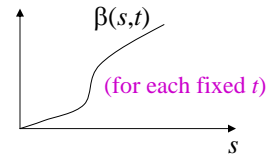
$$\dot{v} = \frac{\partial V}{\partial x}(x) \dot{x} = \frac{\partial V}{\partial x}(x) f_\sigma(x) \leq W(x(t)) \leq -\alpha(v)$$

$$\alpha(s) := \alpha_3(\alpha_2^{-1}(s)) \in \mathcal{K}$$

Some more facts about functions of class $\mathcal{K}, \mathcal{KL}$

class $\mathcal{KL} \equiv$ set of functions $\beta: [0, \infty) \times [0, \infty) \rightarrow [0, \infty)$ s.t.

1. for each fixed $t, \beta(\cdot, t) \in \mathcal{K}$
2. for each fixed $s, \beta(s, \cdot)$ is monotone decreasing and $\beta(s, t) \rightarrow 0$ as $t \rightarrow \infty$



Lemma 3: Given $\alpha \in \mathcal{K}$, in case

$$\dot{v} \leq -\alpha(v)$$

then $\exists \beta \in \mathcal{KL}$ such that

$$v(t) \leq \beta(v(t_0), t - t_0) \quad t \geq t_0$$

After some work ...
$$\beta(s, t) = \begin{cases} \eta^{-1}(\eta(s) + t) & s > 0 \\ 0 & s = 0 \end{cases} \quad \eta(s) := - \int_1^s \frac{dr}{\alpha(r)}$$

Common Lyapunov function

$$\dot{x} = f_\sigma(x) \quad (\sigma, x) \in \mathcal{S}_{\text{all}}$$

Theorem:

Suppose there exists a continuously differentiable, positive definite, radially unbounded function $V: \mathbb{R}^n \rightarrow \mathbb{R}$ such that

$$\frac{\partial V}{\partial x}(z - x_{\text{eq}}) f_q(z) \leq W(z) \leq 0 \quad \forall z \in \mathbb{R}^n, q \in \mathcal{Q}$$

Then

1. the equilibrium point x_{eq} is Lyapunov stable
2. if $W(z) = 0$ only for $z = x_{\text{eq}}$ then x_{eq} is (glob) uniformly asymptotically stable.

Why? (for simplicity consider $x_{\text{eq}} = 0, W(z) \rightarrow 0$ as $z \rightarrow \infty$)

2nd Take an arbitrary solution (σ, x) and define $v(t) := V(x(t)) \forall t \geq 0$

$$\dot{v} = \frac{\partial V}{\partial x}(x) \dot{x} = \frac{\partial V}{\partial x}(x) f_\sigma(x) \leq W(x(t)) \leq -\alpha(v)$$

3rd Then

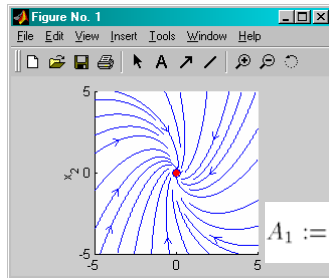
$$v(t) \leq \beta(v(0), t) \quad t \geq 0$$

4th Going back to $x \dots$

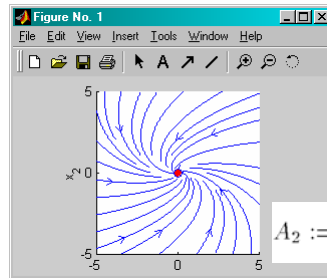
$$\|x(t)\| \leq \alpha_1^{-1}(\beta(\alpha_2(\|x(0)\|), t)) \quad t \geq 0$$

class \mathcal{KL} function
(check !!!)
independent of σ

Example



$$A_1 := \begin{bmatrix} -1 & .25 \\ -1 & -1 \end{bmatrix}$$



$$A_2 := \begin{bmatrix} -1 & -1 \\ .25 & -1 \end{bmatrix}$$

$$\dot{x} = A_\sigma x$$

Defining $V(x_1, x_2) := x_1^2 + x_2^2$

common Lyapunov function

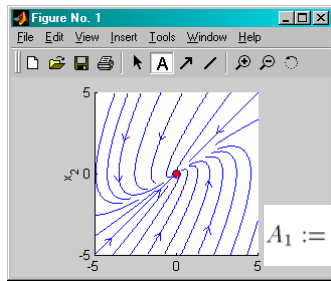
$$\frac{\partial V}{\partial x} A_1 x = -x_1^2 - 1.4375x_2^2 - (x_1 + .75x_2)^2 < 0$$

$$\frac{\partial V}{\partial x} A_2 x = -x_1^2 - 1.4375x_2^2 - (x_1 + .75x_2)^2 < 0$$

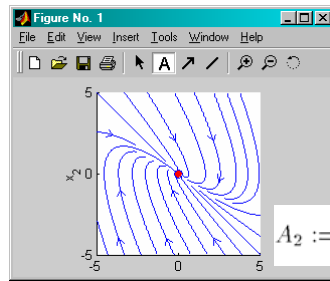
uniform asymptotic stability

stable.m

Example



$$A_1 := \begin{bmatrix} 0 & -1 \\ 1 & -2 \end{bmatrix}$$



$$A_2 := \begin{bmatrix} 0 & 1 \\ -1 & -2 \end{bmatrix}$$

$$\dot{x} = A_\sigma x$$

Defining $V(x_1, x_2) := x_1^2 + x_2^2$

common Lyapunov function

$$\frac{\partial V}{\partial x} A_1 x = -4x_2^2 \leq 0$$

$$\frac{\partial V}{\partial x} A_2 x = -4x_2^2 \leq 0$$

stability (not asymptotic)
(problems, e.g., close to the $x_2=0$ axis)

stable.m

Converse result

$$\dot{x} = f_\sigma(x) \quad (\sigma, x) \in \mathcal{S}_{\text{all}}$$

Theorem:

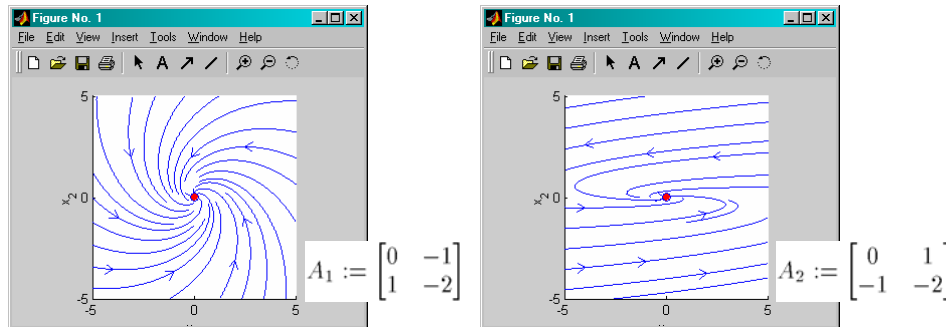
Assume Q is finite. The switched system is uniformly asymptotically stable (on \mathcal{S}_{all}) if and only if there exists a common Lyapunov function, i.e., continuously differentiable, positive definite, radially unbounded function $V: \mathbb{R}^n \rightarrow \mathbb{R}$ such that

$$\frac{\partial V}{\partial x} (z - x_{\text{eq}}) f_q(z) \leq W(z) < 0 \quad \forall z \in \mathbb{R}^n \setminus \{0\}, q \in Q$$

Note that...

1. This result generalized for infinite Q but one needs extra technical assumptions
2. The sufficiency was already established. It turns out that the existence of a common Lyapunov function is also necessary.
3. Finding a common Lyapunov function may be difficult.
E.g., even for linear systems V may not be quadratic

Example



$$\dot{x} = A_\sigma x$$

The switched system is uniformly exponentially stable for arbitrary switching but there is no common quadratic Lyapunov function

stable.m

Algebraic conditions for stability under arbitrary switching

$$\dot{x} = A_\sigma x \quad (\sigma, x) \in \mathcal{S}_{\text{all}} \quad \text{linear switched system}$$

Suppose $\exists m \geq n, M \in \mathbb{R}^{m \times n}$ full rank & $\{ B_q \in \mathbb{R}^{m \times m} : q \in \mathcal{Q} \}$:

$$M A_q = B_q M \quad \forall q \in \mathcal{Q}$$

Defining $z := Mx$

$$\dot{z} = M\dot{x} = M A_\sigma x = B_\sigma Mx = B_\sigma z$$

Theorem: If $V(z) = z' z$ is a common Lyapunov function for

$$\dot{z} = B_\sigma z \quad \left(\frac{\partial V}{\partial z} B_q z = z'(B_q + B_q')z \right)$$

i.e.,

$$B_q' + B_q < 0 \quad \forall q \in \mathcal{Q}$$

then the original switched system is uniformly (exponentially) asymptotically stable

Why?

1. If $V(z) = z' z$ is a common Lyapunov function then z converges to zero exponentially fast
2. $z := Mx \Rightarrow M' z = M' Mx \Rightarrow (M' M)^{-1} M' z = x \Rightarrow x$ converges to zero exponentially fast

Algebraic conditions for stability under arbitrary switching

$$\dot{x} = A_\sigma x \quad (\sigma, x) \in \mathcal{S}_{\text{all}} \quad \text{linear switched system}$$

Suppose $\exists m \geq n, M \in \mathbb{R}^{m \times n}$ full rank & $\{ B_q \in \mathbb{R}^{m \times m} : q \in \mathcal{Q} \}$:

$$M A_q = B_q M \quad \forall q \in \mathcal{Q}$$

Defining $z := Mx$

$$\dot{z} = M \dot{x} = M A_\sigma x = B_\sigma M x = B_\sigma z$$

Theorem: If $V(z) = z' z$ is a common Lyapunov function for

$$\dot{z} = B_\sigma z \quad \left(\frac{\partial V}{\partial z} B_q z = z'(B_q + B_q') z \right)$$

i.e., $B_q' + B_q < 0 \quad \forall q \in \mathcal{Q}$

then the original switched system is uniformly (exponentially) asymptotically stable

It turns out that ...

If the original switched system is uniformly asymptotically stable then such an M always exists (for some $m \geq n$) but may be difficult to find...

Commuting matrices

$$\dot{x} = A_\sigma x \quad (\sigma, x) \in \mathcal{S}_{\text{all}} \quad \text{linear switched system}$$

$$x(t) = \Phi_\sigma(t, \tau)x(\tau) \quad \text{state-transition matrix } (\sigma\text{-dependent})$$

$$\Phi_\sigma(t, \tau) := e^{A_{\sigma(t_k)}(t-t_k)} e^{A_{\sigma(t_{k-1})}(t_k-t_{k-1})} \dots e^{A_{\sigma(\tau)}(t_1-\tau)} \quad t \geq \tau$$

$t_1, t_2, t_3, \dots, t_k \equiv$ switching times of σ in the interval $[t, \tau]$

Recall: in general $e^M e^N \neq e^{M+N} \neq e^N e^M$ unless $MN = NM$

Suppose that for all $p, q \in \mathcal{Q}, A_p A_q = A_q A_p$

$$\begin{aligned} \Phi_\sigma(t, \tau) &= e^{A_{\sigma(t_k)}(t-t_k) + A_{\sigma(t_{k-1})}(t_k-t_{k-1}) + \dots + A_{\sigma(\tau)}(t_1-\tau)} \\ &= e^{\sum_{q \in \mathcal{Q}} A_q T_q} = \prod_{q \in \mathcal{Q}} e^{A_q T_q} \end{aligned}$$

$T_q \equiv$ total time $\sigma = q$ on (t, τ)

Assuming all A_q are asymptotically stable: $\exists c, \lambda_0 > 0 \|e^{A_q t}\| \leq c e^{-\lambda_0 t}$

$$\|\Phi_\sigma(t, \tau)\| \leq c^{|\mathcal{Q}|} e^{-\lambda \sum_q T_q} = c^{|\mathcal{Q}|} e^{-\lambda(t-\tau)}$$

$|\mathcal{Q}| \equiv$ # elements in \mathcal{Q}

Commuting matrices

$$\dot{x} = A_\sigma x \quad (\sigma, x) \in \mathcal{S}_{\text{all}} \quad \text{linear switched system}$$

$$x(t) = \Phi_\sigma(t, \tau)x(\tau) \quad \text{state-transition matrix } (\sigma\text{-dependent})$$

$$\Phi_\sigma(t, \tau) := e^{A_{\sigma(t_k)}(t-t_k)} e^{A_{\sigma(t_{k-1})}(t_k-t_{k-1})} \dots e^{A_{\sigma(\tau)}(t_1-\tau)} \quad t \geq \tau$$

$t_1, t_2, t_3, \dots, t_k \equiv$ switching times of σ in the interval $[t, \tau]$

Recall: in general $e^M e^N \neq e^{M+N} \neq e^N e^M$ unless $MN = NM$

Theorem:

If \mathcal{Q} is finite all $A_q, q \in \mathcal{Q}$ are asymptotically stable and

$$A_p A_q = A_q A_p \quad \forall p, q \in \mathcal{Q}$$

then the switched system is uniformly (exponentially) asymptotically stable

Triangular structures

$$\dot{x} = A_\sigma x \quad (\sigma, x) \in \mathcal{S}_{\text{all}} \quad \text{linear switched system}$$

Theorem:

If all the matrices $A_q, q \in \mathcal{Q}$ are asymptotically stable and upper triangular or all lower triangular then the switched system is uniformly (exponentially) asymptotically stable

Why?

One can find a common quadratic Lyapunov function of the form $V(x) = x' P x$ with P diagonal... check!

Triangular structures

$$\dot{x} = A_\sigma x \quad (\sigma, x) \in \mathcal{S}_{\text{all}} \quad \text{linear switched system}$$

Theorem:

If all the matrices $A_q, q \in Q$ are asymptotically stable and upper triangular or all lower triangular then the switched system is uniformly (exponentially) asymptotically stable

Theorem:

If there is a nonsingular matrix $T \in \mathbb{R}^{n \times n}$ such that all the matrices $B_q = T A_q T^{-1}$ ($T^{-1} B_q T = A_q$) are upper triangular or all lower triangular then the switched system is uniformly (exponentially) asymptotically stable

Why?

1st The B_q have a common quadratic Lyapunov function $V(x) = x' P x$, i.e.,

$$P B_q + B_q' P < 0$$

2nd Therefore

$$P T T^{-1} B_q + B_q' T^{-1} T' P < 0 \Rightarrow \underbrace{T' P T}_{Q} T^{-1} B_q T + \underbrace{T' B_q' T^{-1}}_{A_q'} \underbrace{T' P T}_{Q} < 0$$

$Q := T' P T$ is a common Lyapunov function

Commuting matrices

$$\dot{x} = A_\sigma x \quad (\sigma, x) \in \mathcal{S}_{\text{all}} \quad \text{linear switched system}$$

Theorem:

If Q is finite all $A_q, q \in Q$ are asymptotically stable and

$$A_p A_q = A_q A_p \quad \forall p, q \in Q$$

then the switched system is uniformly (exponentially) asymptotically stable

Another way of proving this result...

From Lie Theorem if a set of matrices commute then there exists a common similarity transformation that upper triangularizes all of them

There are weaker conditions for simultaneous triangularization
(Lie Theorem actually provides the necessary and sufficient condition
 \equiv Lie algebra generated by the matrices must be solvable)

Commuting vector fields

$$\dot{x} = f_\sigma(x) \quad (\sigma, x) \in \mathcal{S}_{\text{all}} \quad \text{nonlinear switched system}$$

Theorem: If all unswitched systems

$$\dot{x} = f_q(x) \quad q \in \mathcal{Q}$$

are asymptotically stable and

$$\frac{\partial f_p}{\partial x} f_q = \frac{\partial f_q}{\partial x} f_p \quad \forall p, q \in \mathcal{Q}$$

then the switched system is uniformly asymptotically stable

For linear vector fields becomes: $A_p A_q x = A_q A_p x \quad \forall x \in \mathbb{R}^n, p, q \in \mathcal{Q}$

Next lecture...

Controller realization for stable switching

Stability under slow switching

- Dwell-time switching
- Average dwell-time
- Stability under brief instabilities