Hybrid Control and Switched Systems

Lecture #12
Controller realizations for stable switching

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Summary

Controller realization for stable switching

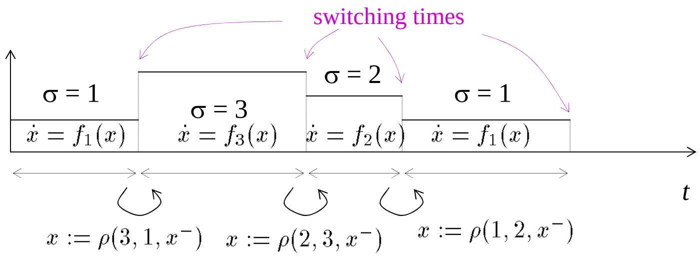
Switched system

parameterized family of vector fields $\equiv$ $f_p : \mathbb{R}^n \to \mathbb{R}^n$ $\quad p \in \mathcal{Q}$

switching signal $\equiv$ piecewise constant signal $\sigma : [0,\infty) \to \mathcal{Q}$ $\quad$ parameter set

$\mathcal{S} \equiv$ set of admissible pairs (σ, x) with σ a switching signal and x a signal in $\mathbb{R}^n$

$$\dot{x} = f_\sigma(x) \qquad x = \rho(\sigma, \sigma^-, x^-) \qquad (\sigma, x) \in \mathcal{S}$$



A ***solution*** to the switched system is a pair $(x, \sigma) \in \mathcal{S}$ for which

1. on every open interval on which σ is constant, x is a solution to
$$\dot{x} = f_{\sigma(t)}(x)$$
time-varying ODE
2. at every switching time t, $x(t) = \rho\big(\sigma(t), \sigma^-(t), x^-(t)\big)$

Three notions of stability

Definition (class $\mathcal{K}$ function definition):
The equilibrium point x_{eq} is ***stable*** if $\exists\, \alpha \in \mathcal{K}$: $\quad$ α is independent of $x(t_0)$ and σ

$$\|x(t) - x_{eq}\| \le \alpha(\|x(t_0) - x_{eq}\|) \;\; \forall\, t \ge t_0 \ge 0, \; \|x(t_0) - x_{eq}\| \le c$$

along any solution $(x, \sigma) \in \mathcal{S}$ to the switched system

Definition:
The equilibrium point $x_{eq} \in \mathbb{R}^n$ is ***asymptotically stable*** if
it is Lyapunov stable and for every solution that exists on $[0,\infty)$

$$x(t) \to x_{eq} \text{ as } t \to \infty.$$

Definition (class $\mathcal{KL}$ function definition):
The equilibrium point $x_{eq} \in \mathbb{R}^n$ is ***uniformly asymptotically stable*** if $\exists\, \beta \in \mathcal{KL}$:

$$\|x(t) - x_{eq}\| \le \beta(\|x(t_0) - x_{eq}\|, t - t_0) \;\; \forall\, t \ge t_0 \ge 0$$

along any solution $(x, \sigma) \in \mathcal{S}$ to the switched system $\quad$ β is independent of $x(t_0)$ and σ

exponential stability when $\beta(s,t) = c\, e^{-\lambda t} s$ with $c, \lambda > 0$

Stability under arbitrary switching

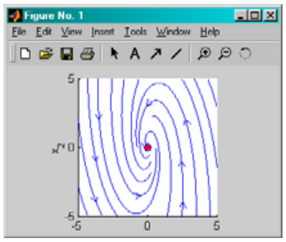
$$\dot{x} = f_\sigma(x) \qquad x = \rho(\sigma, \sigma^-, x^-) \qquad (\sigma, x) \in \mathcal{S}$$

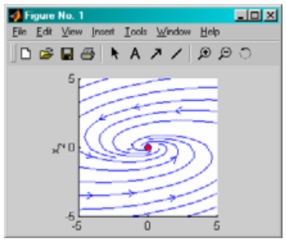
$\mathcal{S}_{\text{all}} \equiv$ set of all pairs (σ, x) with σ piecewise constant and x piecewise continuous

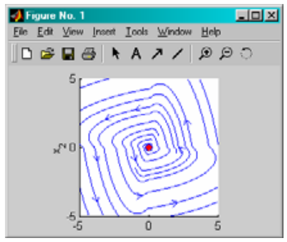
any switching signal is admissible

$\rho(p, q, x) = x \;\forall\, p,q \in \mathcal{Q}, x \in \mathbb{R}^n$

no resets





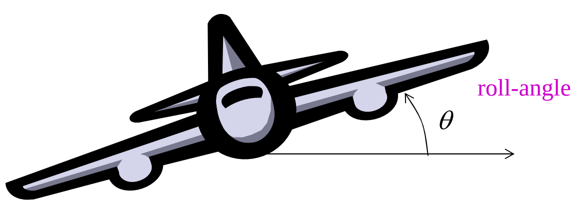


$$A_1 := \begin{bmatrix} -.5 & -.4 \\ 3 & -.5 \end{bmatrix} \qquad A_2 := \begin{bmatrix} -.5 & -3 \\ .4 & -.5 \end{bmatrix} \qquad \sigma(t) := \begin{cases} 1 & x_1 x_2 \geq 0 \\ 2 & x_1 x_2 < 0 \end{cases}$$

$\dot{x} = A_1 z$ asympot. stable $\qquad \dot{x} = A_2 z$ asympot. stable $\qquad \dot{x} = A_\sigma x$

Can we change the switching system to make it stable?

Example #11: Roll-angle control



$$\dddot{\theta} + 50.875\ddot{\theta} + 43.75\dot{\theta} = -1000u$$

θ is uniquely determined by u and the initial conditions

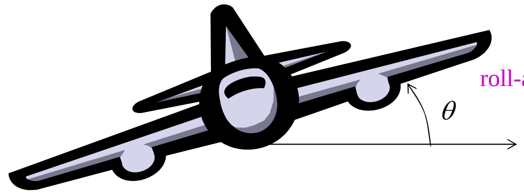
input-output model

state-space realization

$$x_P := \begin{bmatrix} \theta \\ \dot{\theta} \\ \ddot{\theta} \end{bmatrix} \quad \Rightarrow \quad \dot{x}_P = \underbrace{\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -50.875 & -43.75 & 0 \end{bmatrix}}_{A_P} x_P + \underbrace{\begin{bmatrix} 0 \\ 0 \\ -1000 \end{bmatrix}}_{b_P} u$$

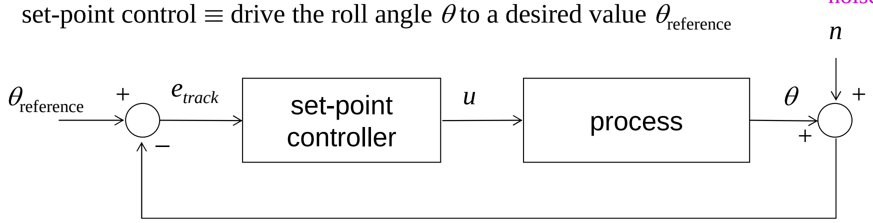
$$\theta = \underbrace{\begin{bmatrix} 1 & 0 & 0 \end{bmatrix}}_{c_P} x_P$$

Example #11: Roll-angle control



$$\omega = \dot{\theta}$$

set-point control ≡ drive the roll angle θ to a desired value $\theta_{\text{reference}}$



Example #11: Roll-angle control

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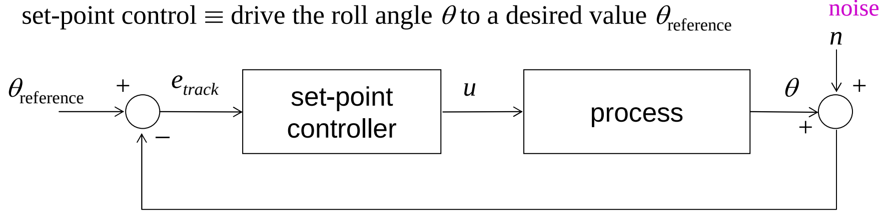


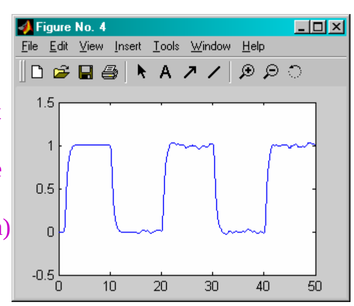
controller 1

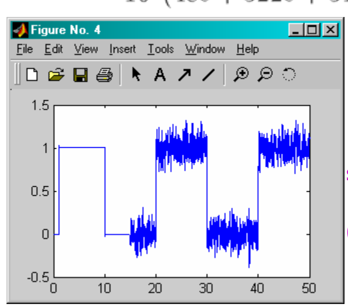
$$\dddot{u} + 63\ddot{u} + 751\dot{u} + 4471u = 6.7\ddot{e} + 340\dot{e} + 316e$$

controller 2

$$\dddot{u} + 974\ddot{u} + 4.7 \times 10^5\dot{u} + 1.2 \times 10^8 u = 10^6(48\ddot{e} + 322\dot{e} + 316e)$$

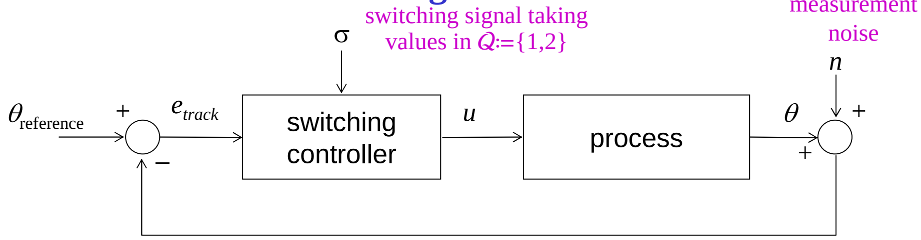
slow but not very sensitive to noise (low-gain)

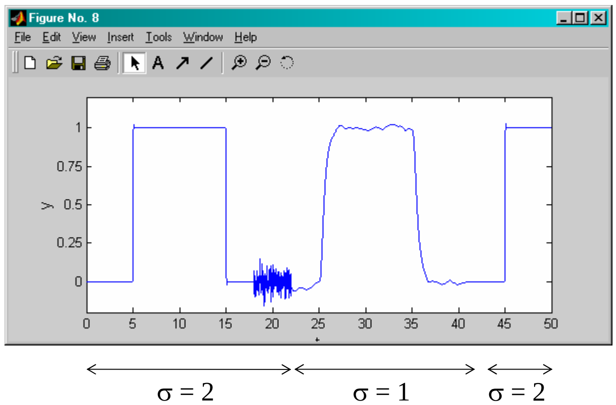




fast but very sensitive to noise (high-gain)

Switching controller





How to build the switching controller to avoid instability ?

Realization theory (SISO)

*n*th order input-output model $\qquad u, y \in \mathbb{R}$

$$y^{(n)} + \alpha_1 y^{(n-1)} + \cdots + \alpha_{n-1}\dot{y} + \alpha_n y = \beta_1 u^{(n-1)} + \cdots + \beta_{n-1}\dot{u} + \beta_n u$$

state-space model $\qquad$ for short $\alpha(y) = \beta(u)$

$$\dot{x} = Ax + bu \qquad x \in \mathbb{R}^m \equiv \text{state}$$
$$y = cx$$

Definition:
(A, b, c) is called a *realization* of the input-output model if the two models have the same solution y for every given u and zero initial conditions.

Theorem:

1. (A, b, c) is a realization of the IO model if and only if

$$c(sI - A)^{-1}b = \frac{\beta(s)}{\alpha(s)} \qquad \alpha(s) := s^n + \alpha_1 s^{n-1} + \cdots + \alpha_{n-1}s + \alpha_n$$
$$\beta(s) := \beta_1 s^{n-1} + \cdots + \beta_{n-1}s + \beta_n$$

2. Any nth order IO model has a realization with $x \in \mathbb{R}^n$
3. If all roots of $\alpha(s)$ have negative real part, A can be chosen asymptotically stable
4. For any nonsingular matrix $T \in \mathbb{R}^{m\times m}$, if ($A$, b, c) is a realization of an IO model then (TAT^{-1}, Tb, cT^{-1}) is also a realization of the same model

Realization theory (SISO)

nth order input-output model $u, y \in \mathbb{R}$

$$y^{(n)} + \alpha_1 y^{(n-1)} + \cdots + \alpha_{n-1}\dot{y} + \alpha_n y = \beta_1 u^{(n-1)} + \cdots + \beta_{n-1}\dot{u} + \beta_n u$$

state-space realization of the IO model for short $\alpha(y) = \beta(u)$

$$\dot{x} = Ax + bu \qquad x \in \mathbb{R}^m \equiv \text{state}$$
$$y = cx$$

Suppose A is asymptotically stable: $\exists\, P > 0,\ P A + A' P = -I$

$(\underbrace{P^{1/2} A P^{-1/2}}_{\bar{A}}, \underbrace{P^{1/2} b}_{\bar{b}}, \underbrace{c P^{-1/2}}_{\bar{c}})$ is also a realization of the IO model

$$\underbrace{P^{-1/2}(A'P + PA)P^{-1/2}}_{\bar{A} + \bar{A}'} = -P^{-1} < 0$$

Theorem:
Given any nth order input-output model for which all roots of $\alpha(s)$ have negative real parts, it is always possible to find a realization for it, for which

$$A + A' = Q < 0$$

Switching between input-output models

$\mathcal{M} := \{\ \alpha_q(y) = \beta_q(u) : q \in \mathcal{Q}\ \} \equiv$ finite family of nth order input-output models

Theorem:
There exists a family of realizations for $\mathcal{M}$

$$\mathcal{R} := \{\ (A_q, b_q, c_q) : q \in \mathcal{Q}\ \}$$

such that the switched system

$$\dot{x} = A_\sigma x$$

is exponentially stable for arbitrary switching

Why?
1st Choose realizations such that $A_q + A_q' = -Q_q < 0\ \forall\, q \in \mathcal{Q}$
2nd The function $V(z) = z' z$ is a common Lyapunov function for the switched system: continuously differentiable, positive definite, radially unbounded,

$$\frac{\partial V}{\partial x} A_q x = x'(A_q + A_q')x = -x' Q_q x < 0 \quad \forall x \neq 0$$

system is uniformly asymptotically stable $\Rightarrow$ exponentially stable

Back to switching controllers…

controller 1

$$\dddot{u} + 63\ddot{u} + 751\dot{u} + 4471u = 6.7\ddot{e} + 340\dot{e} + 316e$$

realization: $\dot{z} = F_1 z + g_1 e$, $u = h_1 z$

$$F_1 = \begin{bmatrix} -63 & -23 & -17 \\ 32 & 0 & 0 \\ 0 & 8 & 0 \end{bmatrix} \quad g_1 = \begin{bmatrix} 4 \\ 0 \\ 0 \end{bmatrix}$$

$$h_1 = \begin{bmatrix} 1.7 & 2.7 & .31 \end{bmatrix}$$

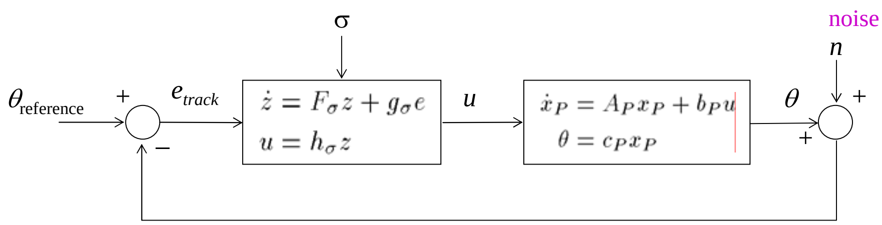
controller 2

$$\dddot{u} + 974\ddot{u} + 4.7 \times 10^5 \dot{u} + 1.2 \times 10^8 u = 10^6(48\ddot{e} + 322\dot{e} + 316e)$$

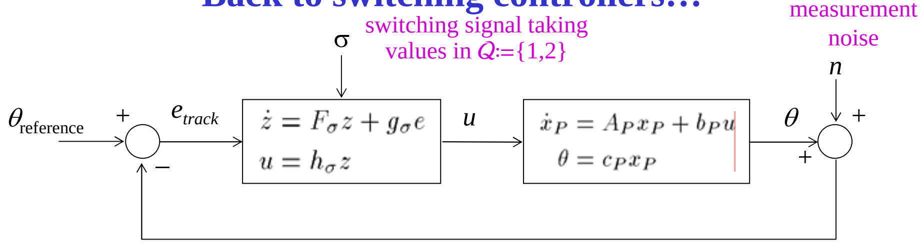
realization: $\dot{z} = F_2 z + g_2 e$, $u = h_2 z$

$$F_2 = \begin{bmatrix} -974 & -459 & -229 \\ 1024 & 0 & 0 \\ 0 & 512 & 0 \end{bmatrix} \quad g_2 = \begin{bmatrix} 8192 \\ 0 \\ 0 \end{bmatrix}$$

$$h_2 = \begin{bmatrix} 5859 & 38 & 0.074 \end{bmatrix}$$



Back to switching controllers…



overall system:

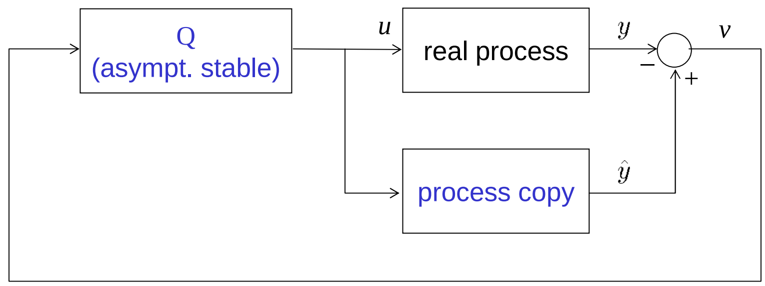
$$\begin{bmatrix} \dot{x}_P \\ \dot{z} \end{bmatrix} = \begin{bmatrix} A_P & b_P h_\sigma \\ -g_\sigma c_P & F_\sigma \end{bmatrix} \begin{bmatrix} x_P \\ z \end{bmatrix} + \begin{bmatrix} 0 \\ g_\sigma \end{bmatrix} (\theta_{\text{ref}} - n)$$

$$\dot{x} = A_\sigma x + b_\sigma(\theta_{\text{ref}} - n) \qquad A_q := \begin{bmatrix} A_P & b_P h_q \\ -g_q c_P & F_q \end{bmatrix} \; q \in \mathcal{Q} := \{1, 2\}$$

Assuming each controller was properly designed, each A_q is asymptotically stable but the overall switched systems could still be unstable
This can be avoided by proper choice of the controller realizations

Youla parameterization (non-switched systems)

Assume process is asymptotically stable

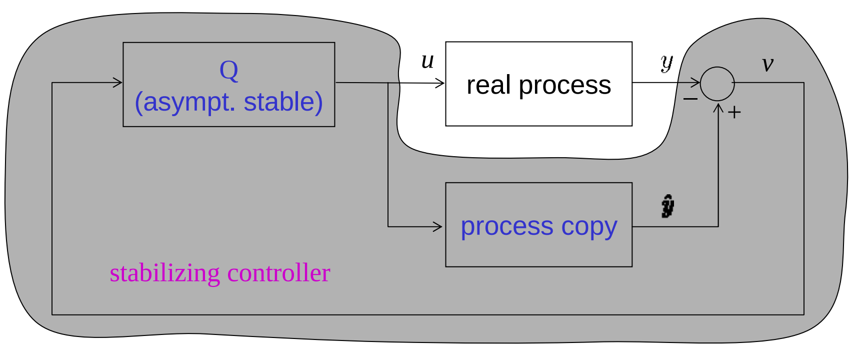


1. If the real process and its copy have the same initial conditions ⇒ $v = 0 \ \forall\, t$ otherwise v converges to zero exponentially fast
2. Since the Q system is asymptotically stable, u converges to zero exponentially fast

No matter what we choose for Q, as long as it is asymptotically stable, the overall system is asymptotically stable

Youla parameterization (non-switched systems)

Assume process is asymptotically stable



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Youla parameterization (non-switched)

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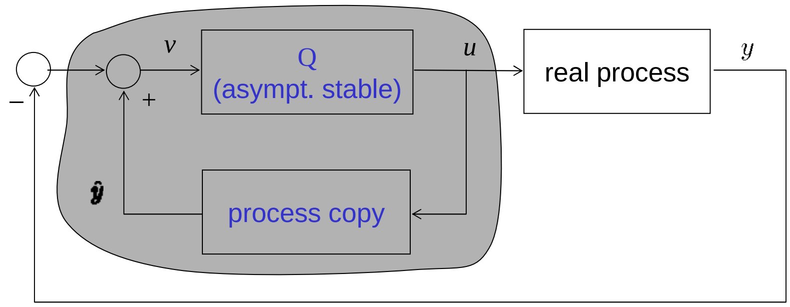
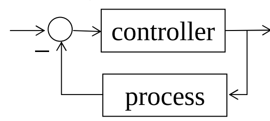


Theorem [Youla-Bongiorno]:

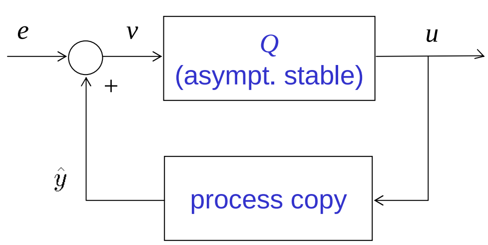
1. For any asymptotically stable *Q*, this controller asymptotically stabilizes the overall system
2. Any controller that asymptotically stabilizes the overall system is of this form, for an asymptotically stable *Q* with the same IO model as:



a similar parameterization also exists when the process is not asymptotically stable…

Why?

"Youla" realizations



realization for *Q*

$$\dot x_Q = \bar A x_Q + \bar b v$$
$$u = \bar c x_Q$$

realization for the process copy

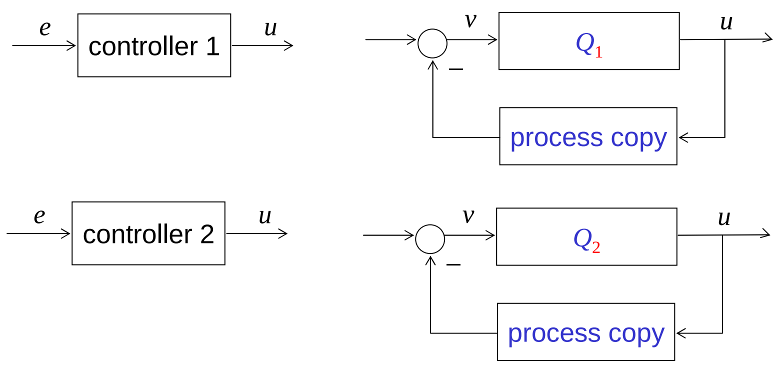
$$\dot x_C = A_P x_C + b_P u$$
$$\hat y = c_P x_C$$

realization for the controller

$$\begin{bmatrix} \dot x_C \\ \dot x_Q \end{bmatrix} = \begin{bmatrix} A_P & b_P \bar c \\ \bar b c_P & \bar A \end{bmatrix} \begin{bmatrix} x_C \\ x_Q \end{bmatrix} + \begin{bmatrix} 0 \\ \bar b \end{bmatrix} e$$
$$y = \begin{bmatrix} 0 & \bar c \end{bmatrix} \begin{bmatrix} x_C \\ x_Q \end{bmatrix}$$

In general these realizations are not minimal

and back again to multiple controllers…

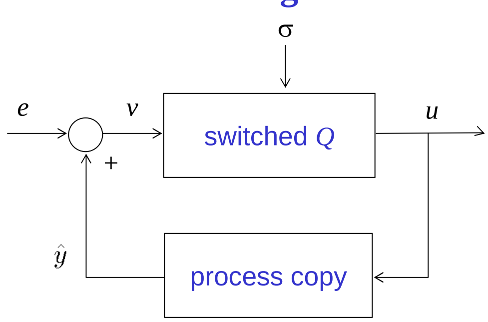


$$\dot{z} = \begin{bmatrix} A_P & b_P \bar{c}_q \\ \bar{b}_q c_P & \bar{A}_q \end{bmatrix} z + \begin{bmatrix} 0 \\ \bar{b}_q \end{bmatrix} e$$

$$y = \begin{bmatrix} 0 & \bar{c}_q \end{bmatrix} z$$

$(\bar{A}_q, \bar{b}_q, \bar{c}_q)$ realization for Q_q

Switching controller



Switched Q

$$\dot{x}_Q = \bar{A}_\sigma x_Q + \bar{b}_\sigma v$$
$$u = \bar{c}_\sigma x_Q$$

realization for the process copy

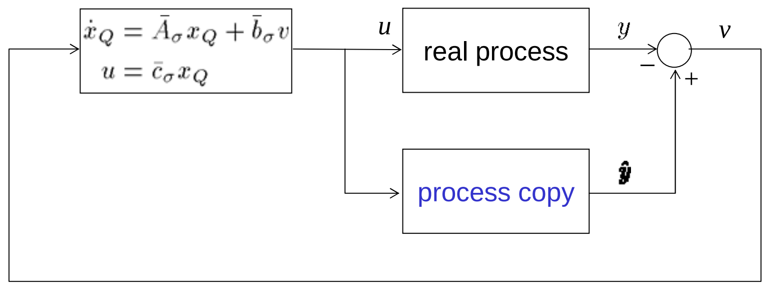
switched controller

$$\dot{z} = F_\sigma z + g_\sigma e$$
$$u = h_\sigma z$$

$$A_q := \begin{bmatrix} A_P & b_P \bar{c}_q \\ \bar{b}_q c_P & \bar{A}_q \end{bmatrix} \quad b_q := \begin{bmatrix} 0 \\ \bar{b}_q \end{bmatrix}$$

$$c_q := \begin{bmatrix} 0 & \bar{c}_q \end{bmatrix}$$

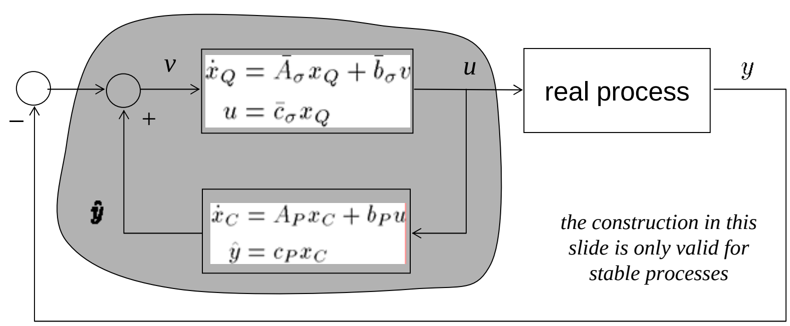
Switched closed-loop



1. If the real process and its copy have the same initial conditions ⇒ $v = 0\ \forall\ t$ otherwise v converges to zero exponentially fast
2. If the switched Q system is asymptotically stable, u converges to zero exponentially fast and the overall system is asymptotically stable

Always possible by appropriate choice of realizations for each Q_q
(e.g., by choosing realizations so that $V(z) = z' z$ is a common Lyapunov function)

Switched closed-loop



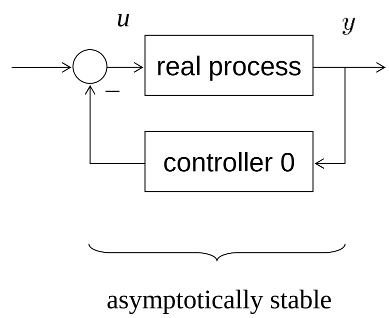
Theorem:
For every family of input-output controller models, there always exist a family a controller realizations such that the switched closed-loop systems is exponentially stable for arbitrary switching.

One can actually show that there exists a common quadratic Lyapunov function for the closed-loop.

In general the realizations are not minimal

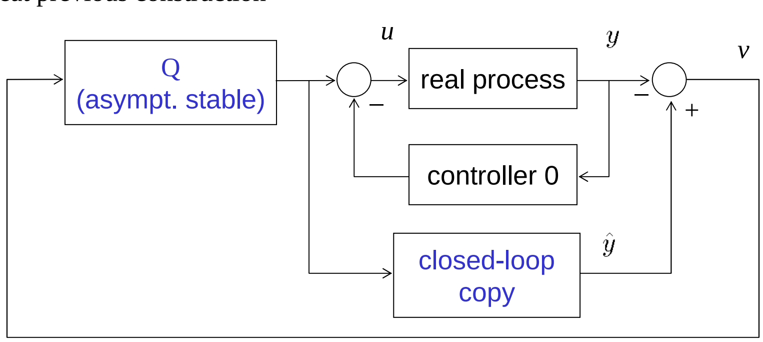
Non-asymptotically stable processes

1^{st} Pick one stabilizing “nominal” controller



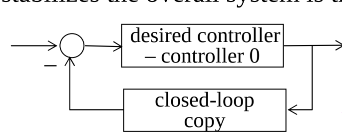
Non-asymptotically stable processes

2^{nd} repeat previous construction



Theorem [Youla-Bongiorno]:

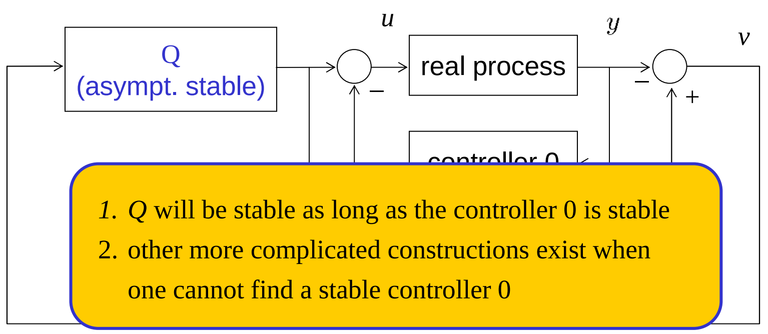
1. For any asymptotically stable Q, this controller asymptotically stabilizes the overall system
2. Any controller that asymptotically stabilizes the overall system is this form, for an appropriately chosen Q:



Why?

Non-asymptotically stable processes

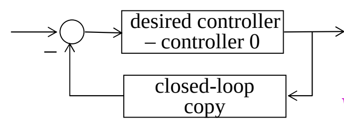
2^{nd} repeat previous construction



1. Q will be stable as long as the controller 0 is stable
2. other more complicated constructions exist when one cannot find a stable controller 0

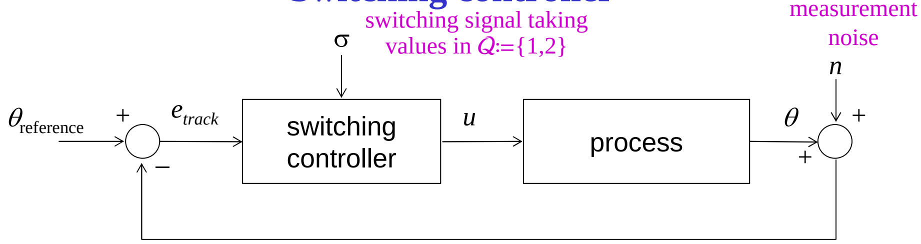
Theorem [Youla-Bongiorno]:

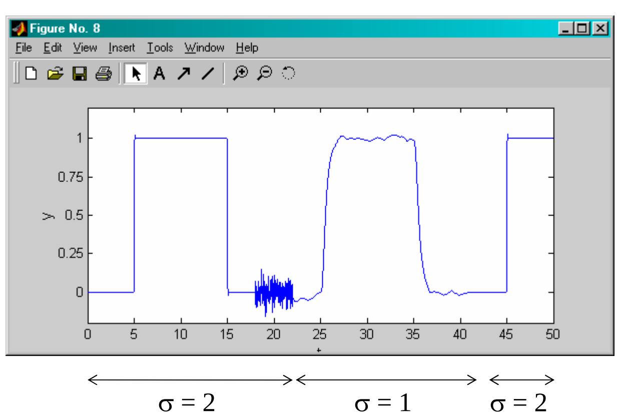
1. For any asymptotically stable Q, this controller asymptotically stabilizes the overall system
2. Any controller that asymptotically stabilizes the overall system is this form, for an appropriately chosen Q:



Why?

Switching controller





By proper choice of the controllers realization we can have stability for arbitrary switching.

Next lecture…

Stability under slow switching

- Dwell-time switching
- Average dwell-time
- Stability under brief instabilities