Hybrid Control and Switched Systems

Lecture #13
Stability under slow switching & state-dependent switching

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Summary

Stability under slow switching
- Dwell-time switching
- Average dwell-time
- Stability under brief instabilities

Stability under state-dependent switching
- State-dependent common Lyapunov function
- Stabilization through switching
- Multiple Lyapunov functions
- LaSalle’s invariance principle

Switched system

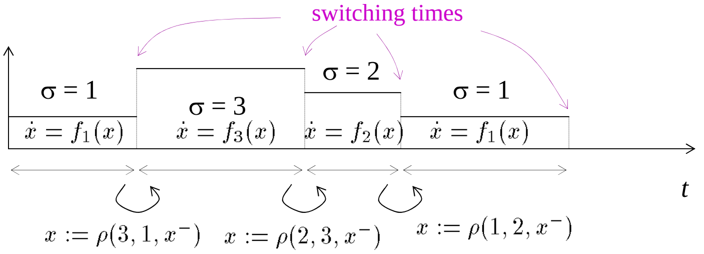
parameterized family of vector fields ≡ $f_p : \mathbb{R}^n \to \mathbb{R}^n$ $\quad p \in \mathcal{Q}$

parameter set

switching signal ≡ piecewise constant signal $\sigma : [0,\infty) \to \mathcal{Q}$

$\mathcal{S}$ ≡ set of admissible pairs (σ, x) with σ a switching signal and x a signal in $\mathbb{R}^n$

$$\dot{x} = f_\sigma(x) \qquad x = \rho(\sigma, \sigma^-, x^-) \qquad (\sigma, x) \in \mathcal{S}$$



A ***solution*** to the switched system is a pair $(\sigma, x) \in \mathcal{S}$ for which

1. on every open interval on which σ is constant, x is a solution to

$$\dot{x} = f_{\sigma(t)}(x)$$

time-varying ODE

2. at every switching time t, $x(t) = \rho(\sigma(t), \sigma^-(t), x^-(t))$

Three notions of stability

Definition (class $\mathcal{K}$ function definition):
The equilibrium point x_{eq} is ***stable*** if $\exists\, \alpha \in \mathcal{K}$:

α is independent of $x(t_0)$ and σ

$$\|x(t) - x_{eq}\| \leq \alpha(\|x(t_0) - x_{eq}\|) \;\; \forall\, t \geq t_0 \geq 0, \; \|x(t_0) - x_{eq}\| \leq c$$

along any solution $(\sigma, x) \in \mathcal{S}$ to the switched system

Definition:
The equilibrium point $x_{eq} \in \mathbb{R}^n$ is ***asymptotically stable*** if
it is Lyapunov stable and for every solution that exists on $[0,\infty)$

$$x(t) \to x_{eq} \text{ as } t \to \infty.$$

Definition (class $\mathcal{KL}$ function definition):
The equilibrium point $x_{eq} \in \mathbb{R}^n$ is ***uniformly asymptotically stable*** if $\exists\, \beta \in \mathcal{KL}$:

$$\|x(t) - x_{eq}\| \leq \beta(\|x(t_0) - x_{eq}\|, t - t_0) \;\; \forall\, t \geq t_0 \geq 0$$

along any solution $(\sigma, x) \in \mathcal{S}$ to the switched system

β is independent of $x(t_0)$ and σ

exponential stability when $\beta(s,t) = c\, e^{-\lambda t} s$ with $c, \lambda > 0$

Stability under slow switching

So far … $\dot{x} = f_\sigma(x) \qquad x = \rho(\sigma, \sigma^-, x^-) \qquad (\sigma, x) \in \mathcal{S}$

$\mathcal{S}_{\text{all}} \equiv$ set of all pairs (σ, x) with σ piecewise constant and x piecewise continuous

any switching signal is admissible

$\rho(p, q, x) = x \ \forall\, p,q \in \mathcal{Q},\ x \in \mathbb{R}^n$

no resets

Now… $\dot{x} = A_\sigma x \qquad x = R_{\sigma,\sigma^-} x^- \qquad (\sigma, x) \in \mathcal{S}$

switched linear systems

Slow switching:

$\mathcal{S}_{\text{dwell}}[\tau_D] \equiv$ switching signals with "dwell-time" $\tau_D > 0$, i.e., interval between consecutive discontinuities larger or equal to τ_D

Slow switching on the average:

$\mathcal{S}_{\text{ave}}[\tau_D, N_0] \equiv$ switching signals with "average dwell-time" $\tau_D > 0$ and "chatter-bound" $N_0 > 0$, i.e.,

$$N_\sigma(t, \tau) \leq N_0 + \frac{t - \tau}{\tau_D} \qquad \forall t > \tau \geq 0,\ \forall \sigma$$

# of discontinuities of σ in the open interval (τ, t)

$\mathcal{S}[\tau_D] = \mathcal{S}_{\text{ave}}[\tau_D, 1]$ *Why?*

Stability under slow switching

$\dot{x} = A_\sigma x \qquad x = R_{\sigma,\sigma^-} x^- \qquad (\sigma, x) \in \mathcal{S}_{\text{dwell}}[\tau_D]$

switched linear systems

$\mathcal{S}[\tau_D] \equiv$ switching signals with "dwell-time" $\tau_D > 0$, i.e., interval between consecutive discontinuities larger or equal to τ_D

Theorem: ($\mathcal{Q}$ finite)
If all A_q, $q \in \mathcal{Q}$ are asymptotically stable, there exists a dwell-time τ_D such that the switched system is uniformly (exponentially) asymptotically stable over $\mathcal{S}_{\text{dwell}}[\tau_D]$

Why?
1st For a switched linear system

$$x(t) = \Phi_\sigma(t, \tau) x(\tau)$$

state-transition matrix (σ-dependent)

$$\Phi_\sigma(t, \tau) := e^{A_{\sigma(t_k)}(t - t_k)} \left(\prod_{i=2}^{k} R_{\sigma(t_i),\sigma(t_{i-1})} e^{A_{\sigma(t_{i-1})}(t_i - t_{i-1})} \right) R_{\sigma(t_1),\sigma(\tau)} e^{A_{\sigma(\tau)}(t_1 - \tau}$$

$t_1, t_2, t_3, \ldots, t_k \equiv$ switching times of σ in the interval $[t, \tau)$

Stability under slow switching

$$\dot{x} = A_\sigma x \qquad x = R_{\sigma,\sigma^-} x^- \qquad (\sigma, x) \in \mathcal{S}_{\text{dwell}}[\tau_D]$$

switched linear systems

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If all A_q, $q \in \mathcal{Q}$ are asymptotically stable, there exists a dwell-time τ_D such that the switched system is uniformly (exponentially) asymptotically stable over $\mathcal{S}_{\text{dwell}}[\tau_D]$

Why?
2st Since all the A_q, $q \in \mathcal{Q}$ are asymptotically stable: $\exists\, c, \lambda_0 > 0 \ \|e^{A_q t}\| \le c\, e^{-\lambda_0 t}$

3nd Taking norms of the state-transition matrix…

$$\|\Phi_\sigma(t,\tau)\| \le \|e^{A_{\sigma(t_k)}(t-t_k)}\| \Big(\prod_{i=2}^{k} \|R_{\sigma(t_i),\sigma(t_{i-1})}\| \, \|e^{A_{\sigma(t_{i-1})}(t_i-t_{i-1})}\|\Big)$$
$$\|R_{\sigma(t_1),\sigma(\tau)}\| \, \|e^{A_{\sigma(\tau)}(t_1-\tau)}\|$$
$$\le c e^{-\lambda_0(t-t_k)} \Big(\prod_{i=2}^{k} rce^{-\lambda_0(t_i-t_{i-1})}\Big) rce^{-\lambda_0(t_1-\tau)}$$
$$r := \max_{p,q\in\mathcal{Q}} \|R_{p,q}\|$$

Stability under slow switching

$$\dot{x} = A_\sigma x \qquad x = R_{\sigma,\sigma^-} x^- \qquad (\sigma, x) \in \mathcal{S}_{\text{dwell}}[\tau_D]$$

switched linear systems

$\mathcal{S}[\tau_D] \equiv$ switching signals with "dwell-time" $\tau_D > 0$, i.e., interval between consecutive discontinuities larger or equal to τ_D

Theorem: ($\mathcal{Q}$ finite)
If all A_q, $q \in \mathcal{Q}$ are asymptotically stable, there exists a dwell-time τ_D such that the switched system is uniformly (exponentially) asymptotically stable over $\mathcal{S}_{\text{dwell}}[\tau_D]$

Why?
3nd $\|\Phi_\sigma(t,\tau)\| \le ce^{-\lambda_0(t-t_k)} \Big(\prod_{i=2}^{k} rce^{-\lambda_0(t_i-t_{i-1})}\Big) rce^{-\lambda_0(t_1-\tau)}$

4th Pick $\tau_D > 0$, $\lambda \in (0, \lambda_0)$ such that

$$rce^{-\lambda_0 \Delta} \le e^{-\lambda\Delta} \quad \forall \Delta \ge \tau_D$$

Always possible? yes:

$$\left.e^{(\lambda_0-\lambda)\Delta} \ge rc \quad \Leftrightarrow \quad \Delta \ge \frac{\log rc}{\lambda_0 - \lambda}\right\} \text{can pick} \qquad \tau_D := \frac{\log rc}{\lambda_0 - \lambda}$$

Stability under slow switching

$$\dot{x} = A_\sigma x \qquad x = R_{\sigma,\sigma^-} x^- \qquad (\sigma, x) \in \mathcal{S}_{\text{dwell}}[\tau_D]$$

switched linear systems

$\mathcal{S}[\tau_D] \equiv$ switching signals with “dwell-time” $\tau_D > 0$, i.e., interval between consecutive discontinuities larger or equal to τ_D

Theorem: (Q finite)
If all A_q, $q \in Q$ are asymptotically stable, there exists a dwell-time τ_D such that the switched system is uniformly (exponentially) asymptotically stable over $\mathcal{S}_{\text{dwell}}[\tau_D]$

Why?

3nd $\|\Phi_\sigma(t,\tau)\| \leq c e^{-\lambda_0 (t-t_k)} \left(\prod_{i=2}^{k} r c e^{-\lambda_0 (t_i - t_{i-1})} \right) r c e^{-\lambda_0 (t_1 - \tau)}$

4th $\lambda \in (0, \lambda_0)$ & $r c e^{-\lambda_0 \Delta} \leq e^{-\lambda \Delta} \quad \forall \Delta \geq \tau_D$

5th Then

$$\|\Phi_\sigma(t,\tau)\| \leq c e^{-\lambda (t-t_k)} \left(\prod_{i=2}^{k} e^{-\lambda (t_i - t_{i-1})} \right) r c e^{-\lambda (t_1 - \tau)} = r c^2 e^{-\lambda (t-\tau)}$$

exponential convergence to zero
(with rate independent of σ)

Stability under slow switching

$$\dot{x} = A_\sigma x \qquad x = R_{\sigma,\sigma^-} x^- \qquad (\sigma, x) \in \mathcal{S}_{\text{dwell}}[\tau_D]$$

switched linear systems

$\mathcal{S}[\tau_D] \equiv$ switching signals with “dwell-time” $\tau_D > 0$, i.e., interval between consecutive discontinuities larger or equal to τ_D

Theorem: (Q infinite)
Assuming the sets $\{A_q : q \in Q\}$ & $\{R_{p,q} : p, q \in Q\}$ are compact.
If all A_q, $q \in Q$ are asymptotically stable, there exists a dwell-time τ_D such that the switched system is uniformly (exponentially) asymptotically stable over $\mathcal{S}_{\text{dwell}}[\tau_D]$

Stability under slow switching on the average

$$\dot{x} = A_\sigma x \qquad x = R_{\sigma,\sigma^-} x^- \qquad (\sigma, x) \in \mathcal{S}_{\text{ave}}[\tau_D, N_0]$$

switched linear systems

$\mathcal{S}_{\text{ave}}[\tau_D, N_0] \equiv$ switching signals with "average dwell-time" $\tau_D > 0$ and "chatter-bound" $N_0 > 0$, i.e., $N_\sigma(t,\tau) \leq N_0 + \frac{t-\tau}{\tau_D}$

\# of switchings in (τ, t)

Theorem: (Q finite)
If all the A_q, $q \in Q$ are asymptotically stable, there exists an average dwell-time τ_D such that for every chatter-bound N_0 the switched system is uniformly (exponentially) asymptotically stable over $\mathcal{S}_{\text{ave}}[\tau_D, N_0]$

Why?
1st As before …

$$\|\Phi_\sigma(t,\tau)\| \leq c e^{-\lambda_0(t-t_k)} \left(\prod_{i=2}^{k} r c e^{-\lambda_0(t_i - t_{i-1})} \right) r c e^{-\lambda_0(t_1-\tau)}$$
$$= c(rc)^k e^{-\lambda_0(t-\tau)}$$

(w.l.g we assume $r\,c > 1$)

2nd But k is the number of switchings in $[t,\tau)$ so $k \leq 1 + N_\sigma(t,\tau) \leq 1 + N_0 + \frac{t-\tau}{\tau_D}$

$$\|\Phi_\sigma(t,\tau)\| \leq c(rc)^{1+N_0+\frac{t-\tau}{\tau_D}} e^{-\lambda_0(t-\tau)} = c e^{\log(rc)(1+N_0) - (\lambda_0 - \frac{\log(rc)}{\tau_D})(t-\tau)}$$

exponential decrease as long as $\lambda_0 > \frac{\log rc}{\tau_D} \quad \Leftrightarrow \quad \tau_D > \frac{\log rc}{\lambda_0}$

Stability under slow switching on the average

$$\dot{x} = A_\sigma x \qquad x = R_{\sigma,\sigma^-} x^- \qquad (\sigma, x) \in \mathcal{S}_{\text{ave}}[\tau_D, N_0]$$

switched linear systems

$\mathcal{S}_{\text{ave}}[\tau_D, N_0] \equiv$ switching signals with "average dwell-time" $\tau_D > 0$ and "chatter-bound" $N_0 > 0$, i.e., $N_\sigma(t,\tau) \leq N_0 + \frac{t-\tau}{\tau_D}$

\# of switchings in (τ, t)

Theorem: (Q infinite)
Assuming the sets $\{A_q : q \in Q\}$ & $\{R_{p,q} : p, q \in Q\}$ are compact.
If all the A_q, $q \in Q$ are asymptotically stable, there exists an average dwell-time τ_D such that for every chatter-bound N_0 the switched system is uniformly (exponentially) asymptotically stable over $\mathcal{S}_{\text{ave}}[\tau_D, N_0]$

1. Same results would hold for any subset of $\mathcal{S}_{\text{ave}}[\tau_D, N_0]$
2. Some versions of these results also exist for nonlinear systems
3. One may still have stability if some of the A_q are unstable, provided that σ does not "dwell" on these values for a long time (switching under brief instabilities)

So far… state-independent switching

$$\dot{x} = f_\sigma(x) \qquad x = x^- \qquad (\sigma, x) \in \mathcal{S}$$

no resets

Arbitrary switching:

$\mathcal{S}_{\text{all}} \equiv$ set of all pairs (σ, x) with σ piecewise constant and x piecewise continuous

$$\dot{x} = A_\sigma x \qquad x = R_{\sigma,\sigma^-} x^- \qquad (\sigma, x) \in \mathcal{S}$$

switched linear systems

Slow switching:

$\mathcal{S}[\tau_D] \equiv$ switching signals with "dwell-time" $\tau_D > 0$, i.e., interval between consecutive discontinuities larger or equal to τ_D

Slow switching on the average:

$\mathcal{S}_{\text{ave}}[\tau_D, N_0] \equiv$ switching signals with "average dwell-time" $\tau_D > 0$ and "chatter-bound" $N_0 > 0$, i.e.,

$$N_\sigma(t,\tau) \leq N_0 + \frac{t-\tau}{\tau_D} \qquad \forall t > \tau \geq 0,\ \forall \sigma$$

# of discontinuities of σ in the open interval (τ, t)

Current-state dependent switching

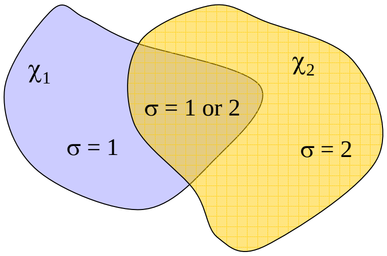
$$\dot{x} = f_\sigma(x) \qquad x = x^- \qquad (\sigma, x) \in \mathcal{S}$$

no resets

$\chi := \{\chi_q \in \mathbb{R}^n : q \in \mathcal{Q}\} \equiv$ (not necessarily disjoint) covering of $\mathbb{R}^n$, i.e., $\cup_{q\in\mathcal{Q}} \chi_q = \mathbb{R}^n$

Current-state dependent switching

$\mathcal{S}[\chi] \equiv$ set of all pairs (σ, x) with σ piecewise constant and x piecewise continuous such that $\forall\, t$, $\sigma(t) = q$ is allowed only if $x(t) \in \chi_q$



Thus $(\sigma, x) \in \mathcal{S}[\chi]$ if and only if $x(t) \in \chi_{\sigma(t)}\ \forall\, t$

Common Lyapunov function for arbitrary switching

$$\dot{x} = f_\sigma(x) \qquad x = x^- \qquad (\sigma, x) \in \mathcal{S}_{\text{all}}$$

Theorem:
Suppose there exists a continuously differentiable, positive definite, radially unbounded function $V: \mathbb{R}^n \to \mathbb{R}$ such that

$$\frac{\partial V}{\partial x}(z - x_{\text{eq}}) f_q(z) \leq W(z) \leq 0 \qquad \forall q \in \mathcal{Q},\ z \in \mathbb{R}^n$$

Then for arbitrary switching $\mathcal{S}_{\text{all}}$

1. the equilibrium point x_{eq} is Lyapunov stable
2. if $W(z) = 0$ only for $z = x_{\text{eq}}$ then x_{eq} is (glob) uniformly asymptotically stable.

Why? (for simplicity consider $x_{\text{eq}} = 0$)
1st Take an arbitrary solution (σ, x) and define $v(t) := V(\,x(t)\,)\ \forall\, t \geq 0$

$$\dot{v} = \frac{\partial V}{\partial x}(x)\dot{x} = \frac{\partial V}{\partial x}(x) f_\sigma(x) \leq W\big(x(t)\big) \leq 0$$

2nd Therefore

$$v(t) := V\big(x(t)\big) \leq v(0) := V\big(x(0)\big) \qquad \forall t \geq 0$$

$V(\,x(t)\,)$ is always bounded…

Common Lyapunov function for current-state dep. switching

$$\dot{x} = f_\sigma(x) \qquad x = x^- \qquad (\sigma, x) \in \mathcal{S}[\chi]$$

Theorem:
Suppose there exists a continuously differentiable, positive definite, radially unbounded function $V: \mathbb{R}^n \to \mathbb{R}$ such that

$$\frac{\partial V}{\partial x}(z - x_{\text{eq}}) f_q(z) \leq W(z) \leq 0 \qquad \forall q \in \mathcal{Q},\ \boxed{z \in \chi_q}$$

Then for current-state dependent switching $\mathcal{S}[\chi]$

1. the equilibrium point x_{eq} is Lyapunov stable
2. if $W(z) = 0$ only for $z = x_{\text{eq}}$ then x_{eq} is (glob) uniformly asymptotically stable.

Why? (for simplicity consider $x_{\text{eq}} = 0$)
1st Take an arbitrary solution (σ, x) and define $v(t) := V(\,x(t)\,)\ \forall\, t \geq 0$

$$\dot{v} = \frac{\partial V}{\partial x}(x)\dot{x} = \frac{\partial V}{\partial x}(x) f_\sigma(x) \leq W\big(x(t)\big) \leq 0$$

still holds because $x(t) \in \chi_{\sigma(t)}$

2nd Therefore

$$v(t) := V\big(x(t)\big) \leq v(0) := V\big(x(0)\big) \qquad \forall t \geq 0$$

Same conclusions as before …

Common Lyapunov function for current-state dep. switching

$$\dot{x} = f_\sigma(x) \qquad x = x^- \qquad (\sigma, x) \in \mathcal{S}[\chi]$$

Theorem:
Suppose there exists a continuously differentiable, positive definite, radially unbounded function V: $\mathbb{R}^n \to \mathbb{R}$ such that

$$\frac{\partial V}{\partial x}(z - x_{\text{eq}}) f_q(z) \le W(z) \le 0 \qquad \forall q \in \mathcal{Q}, \boxed{z \in \chi_q}$$

Then for current-state dependent switching $\mathcal{S}[\chi]$

1. the equilibrium point x_{eq} is Lyapunov stable
2. if $W(z) = 0$ only for $z = x_{eq}$ then x_{eq} is (glob) uniformly asymptotically stable.

Note that:

- Same conclusion would hold for any subset of $\mathcal{S}[\chi]$
- Some (or all) the unswitched systems may not be stable
$$\dot{x} = f_q(x)$$
- This theorem does not guarantee existence of solutions (as opposed to the usual Lyapunov Theorem and the ones for state independent switching)…

Common Lyapunov function for current-state dep. switching

$$\dot{x} = f_\sigma(x) \qquad x = x^- \qquad (\sigma, x) \in \mathcal{S}[\chi]$$

Theorem:
Suppose there exists a continuously differentiable, positive definite, radially unbounded function V: $\mathbb{R}^n \to \mathbb{R}$ such that

$$\frac{\partial V}{\partial x}(z - x_{\text{eq}}) f_q(z) \le W(z) \le 0 \qquad \forall q \in \mathcal{Q}, \boxed{z \in \chi_q}$$

Then for current-state dependent switching $\mathcal{S}[\chi]$

1. the equilibrium point x_{eq} is Lyapunov stable
2. if $W(z) = 0$ only for $z = x_{eq}$ then x_{eq} is (glob) uniformly asymptotically stable.

E.g., $\mathcal{Q} := \{-1, +1\}$, $\chi_{-1} := [0,\infty)$, $\chi_{+1} := (-\infty,0)$

$$\dot{x} = \sigma = \begin{cases} -1 & x \ge 0 \\ +1 & x < 0 \end{cases} \qquad f_{-1}(x) := -1 \quad f_{+1}(x) := +1$$

no solutions exists

For $x_{eq} = 0$ is an equilibrium point and for $V(z) := z^2$

$$\frac{\partial V}{\partial x}(z - x_{\text{eq}}) f_q(z) = \begin{cases} -2z & q = -1,\ z \ge 0 \\ 2z & q = +1,\ z < 0 \end{cases} \le 0$$

Stabilization through switching

Given a family of unstable vector fields f_q, $q \in \mathcal{Q}$

Is there a covering χ *for which the current-state dependent set of switching signals* $\mathcal{S}[\chi]$ *results in stability?*

Theorem:
If there exists a set of constants $\lambda_q \geq 0$, $q \in \mathcal{Q}$ such that $\sum_q \lambda_q = 1$ and x_{eq} is an (asymptotically) stable equilibrium point of the ODE

$$\dot{x} = \sum_{q \in \mathcal{Q}} \lambda_q f_q(x)$$

convex combination of the f_q

then there is a current-state dependent set of switching signals $\mathcal{S}[\chi]$ for which x_{eq} is an (asymptotically) stable equilibrium point of the switched system.

Why?

1st Since the convex combination is asymptotically stable, it has a Lyapunov function *V*:

$$\frac{\partial V}{\partial x}(z - x_{\text{eq}}) \sum_q \lambda_q f_q(z) \leq W(z) \leq 0 \qquad \forall z \in \mathbb{R}^n$$

$$\Rightarrow \quad \sum_q \lambda_q \Big(\frac{\partial V}{\partial x}(z - x_{\text{eq}}) f_q(z) - W(z) \Big) \leq 0$$

since all the $\lambda_q \geq 0$, for every *z*, at least one of the terms must be ≤ 0

Stabilization through switching

Given a family of unstable vector fields f_q, $q \in \mathcal{Q}$

Is there a covering χ *for which the current-state dependent set of switching signals* $\mathcal{S}[\chi]$ *results in stability?*

Theorem:
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$$\dot{x} = \sum_{q \in \mathcal{Q}} \lambda_q f_q(x)$$

convex combination of the f_q

then there is a current-state dependent set of switching signals $\mathcal{S}[\chi]$ for which x_{eq} is an (asymptotically) stable equilibrium point of the switched system.

Why?

2nd Define $\chi_q := \underbrace{\Big\{ z \in \mathbb{R}^n : \frac{\partial V}{\partial x}(z - x_{\text{eq}}) f_q(z) - W(z) \leq 0 \Big\}} \quad q \in \mathcal{Q}$

1. every point in $\mathbb{R}^n$ belongs to one of the χ_q
   $\Rightarrow \chi := \{ \chi_q : q \in \mathcal{Q} \}$ form a covering
2. $\frac{\partial V}{\partial x}(z - x_{\text{eq}}) f_q(z) \leq W(z) \leq 0 \qquad \forall q \in \mathcal{Q},\ z \in \chi_q$

V is a common Lyapunov function for current-state dep. switching

Stabilization through switching

Given a family of unstable vector fields f_q, $q \in \mathcal{Q}$

Is there a covering χ for which the current-state dependent set of switching signals $\mathcal{S}[\chi]$ results in stability?

Theorem:
If there exists a set of constants $\lambda_q \geq 0$, $q \in \mathcal{Q}$ such that $\sum_q \lambda_q = 1$ and x_{eq} is an (asymptotically) stable equilibrium point of the ODE

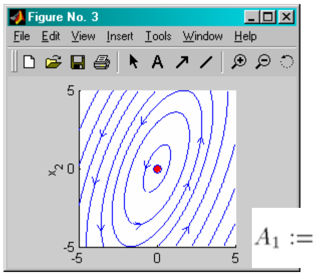
$$\dot{x} = \sum_{q \in \mathcal{Q}} \lambda_q f_q(x)$$

convex combination of the f_q

then there is a current-state dependent set of switching signals $\mathcal{S}[\chi]$ for which x_{eq} is an (asymptotically) stable equilibrium point of the switched system.

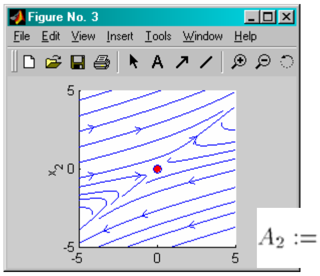
But these covers may lead to non-existence of solution (Zeno)

Example



$$A_1 := \begin{bmatrix} 1 & -1 \\ 3 & -1 \end{bmatrix}$$

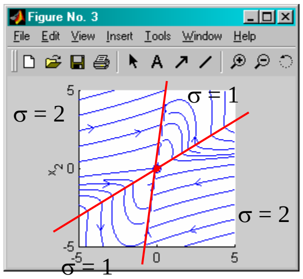
$\dot{x} = A_1 z$ stable but not asympt.



$$A_2 := \begin{bmatrix} -2 & 3 \\ -.5 & 1 \end{bmatrix}$$

$\dot{x} = A_2 z$ unstable

$$\dot{x} = A_\sigma x$$



The two regions actually intersect. One can use this to prevent Zeno (e.g., through hysteresis)…

Multiple Lyapunov functions

$$\dot{x} = f_\sigma(x) \qquad x = \rho(\sigma, \sigma^-, x^-) \qquad (\sigma, x) \in \mathcal{S} \subset \mathcal{S}[\chi]$$

$V_q : \mathbb{R}^n \to \mathbb{R}$, $q \in \mathcal{Q} \equiv$ family of Lyapunov functions (cont. dif., pos. def., rad. unb.)

$$\frac{\partial V_q}{\partial x}(z - x_{\text{eq}}) f_q(z) \le W(z) \le 0 \qquad \forall q \in \mathcal{Q},\ z \in \chi_q$$

Given a solution (σ, x) and defining $v(t) := V_{\sigma(t)}(\,x(t)\,)\ \forall\, t \ge 0$

1. On an interval $[\tau, t)$ where $\sigma = q$ (constant)

v decreases

$$\dot{v} = \frac{\partial V_q}{\partial x}(x)\dot{x} = \frac{\partial V_q}{\partial x}(x) f_\sigma(x) = \frac{\partial V_q}{\partial x}(x) f_q(x) \le W\big(x(t)\big) \le 0$$

2. But at a switching time t, where $\sigma^-(t) = p \ne \sigma(t) = q$,

$$v^-(t) = V_p\big(x^-(t)\big) \qquad v(t) = V_q\big(x(t)\big)$$

v may be discontinuous

(even without reset)

Multiple Lyapunov functions

$$\dot{x} = f_\sigma(x) \qquad x = \rho(\sigma, \sigma^-, x^-) \qquad (\sigma, x) \in \mathcal{S} \subset \mathcal{S}[\chi]$$

$V_q : \mathbb{R}^n \to \mathbb{R}$, $q \in \mathcal{Q} \equiv$ family of Lyapunov functions (cont. dif., pos. def., rad. unb.)

$$\frac{\partial V_q}{\partial x}(z - x_{\text{eq}}) f_q(z) \le W(z) \le 0 \qquad \forall q \in \mathcal{Q},\ z \in \chi_q$$

Given a solution (σ, x) and defining $v(t) := V_{\sigma(t)}(\,x(t)\,)\ \forall\, t \ge 0$

1. On an interval $[\tau, t)$ where $\sigma = q$ (constant)

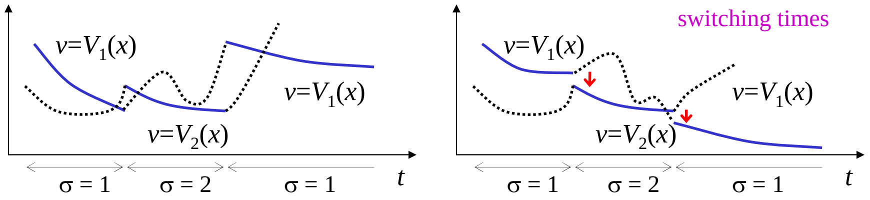
v decreases

$$\dot{v} = \frac{\partial V_q}{\partial x}(x)\dot{x} = \frac{\partial V_q}{\partial x}(x) f_\sigma(x) = \frac{\partial V_q}{\partial x}(x) f_q(x) \le W\big(x(t)\big) \le 0$$

2. But at a switching time t, where $\sigma^-(t) = p \ne \sigma(t) = q$,

$$v^-(t) = V_p\big(x^-(t)\big) \qquad v(t) = V_q\big(x(t)\big)$$

we would be okay if *v* would not increase at switching times



Multiple Lyapunov functions

$$\dot{x} = f_\sigma(x) \qquad x = \rho(\sigma, \sigma^-, x^-) \qquad (\sigma, x) \in \mathcal{S} \subset \mathcal{S}[\chi]$$

Theorem: ($\mathcal{Q}$ finite)

Suppose there exists a family of continuously differentiable, positive definite, radially unbounded functions $V_q: \mathbb{R}^n \to \mathbb{R}$, $q \in \mathcal{Q}$ such that

$$\frac{\partial V_q}{\partial x}(z - x_{\text{eq}}) f_q(z) \leq W(z) \leq 0 \qquad \forall q \in \mathcal{Q},\ z \in \chi_q$$

and at any $z \in \mathbb{R}^n$ where a switching signal in $\mathcal{S}$ can jump from p to q

$$V_p(z) \geq V_q\big(\rho(q, p, z)\big)$$

Then

1. the equilibrium point x_{eq} is Lyapunov stable
2. if $W(z) = 0$ only for $z = x_{\text{eq}}$ then x_{eq} is (glob) uniformly asymptotically stable.

Why? (for simplicity consider $x_{\text{eq}} = 0$)

1st Take an arbitrary solution (σ, x) and define $v(t) := V_\sigma(\,x(t)\,)\ \forall\, t \geq 0$

while σ is constant: $\dot{v} = \dfrac{\partial V_\sigma}{\partial x}(x)\dot{x} = \dfrac{\partial V_\sigma}{\partial x}(x) f_\sigma(x) \leq W\big(x(t)\big) \leq 0$

and, at points of discontinuity of σ: $v^-(t) \geq v(t)$ does not increase

from now on same as before …

Multiple Lyapunov functions

$$\dot{x} = f_\sigma(x) \qquad x = \rho(\sigma, \sigma^-, x^-) \qquad (\sigma, x) \in \mathcal{S} \subset \mathcal{S}[\chi]$$

Theorem: ($\mathcal{Q}$ finite)

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Why? (for simplicity consider $x_{\text{eq}} = 0$)

2nd Since $\exists\ \alpha_1, \alpha_2 \in \mathcal{K}_\infty$: $\quad \alpha_1(\|x\|) \leq V_q(x) \leq \alpha_2(\|x\|)$

$$\|x(t)\| \leq \alpha_1^{-1}\big(\alpha_2(\|x(0)\|)\big) \qquad \forall t \geq 0$$

3rd If $\exists\ \alpha_3$: $\quad W(x) \leq -\alpha_3(\|x\|)$

class $\mathcal{KL}$ function independent of σ

$$\|x(t)\| \leq \alpha_1^{-1}\big(\beta(\alpha_2(\|x(0)\|), t)\big) \qquad t \geq 0$$

Multiple Lyapunov functions

$$\dot{x} = f_\sigma(x) \qquad x = \rho(\sigma, \sigma^-, x^-) \qquad (\sigma, x) \in \mathcal{S} \subset \mathcal{S}[\chi]$$

Theorem: ($\mathcal{Q}$ finite)

Suppose there exists a family of continuously differentiable, positive definite, radially unbounded functions $V_q: \mathbb{R}^n \to \mathbb{R}$, $q \in \mathcal{Q}$ such that

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$$V_p(z) \geq V_q\big(\rho(q, p, z)\big)$$

Then

1. the equilibrium point x_{eq} is Lyapunov stable
2. if $W(z) = 0$ only for $z = x_{\mathrm{eq}}$ then x_{eq} is (glob) uniformly asymptotically stable.

The V_q's need not be positive definite and radially unbounded "everywhere"

It is enough that $\exists\ \alpha_1, \alpha_2 \in \mathcal{K}_\infty$: $\alpha_1(\|z\|) \leq V_q(z) \leq \alpha_2(\|z\|)$ $\quad \forall\ q \in \mathcal{Q},\ z \in \chi_q$

LaSalle's Invariance Principle (ODE)

$$\dot{x} = f(x) \qquad x \in \mathbb{R}^n$$

$M \in \mathbb{R}^n$ is an invariant set $\equiv x(t_0) \in M \Rightarrow x(t) \in M\ \forall\ t \geq t_0$

Theorem (LaSalle Invariance Principle):

Suppose there exists a continuously differentiable, positive definite, radially unbounded function $V: \mathbb{R}^n \to \mathbb{R}$ such that

$$\frac{\partial V}{\partial x}(z - x_{\mathrm{eq}}) f(z) \leq W(z) \leq 0 \qquad \forall z \in \mathbb{R}^n$$

Then x_{eq} is a Lyapunov stable equilibrium and the solution always exists globally. Moreover, $x(t)$ converges to the largest invariant set M contained in

$$E := \{\ z \in \mathbb{R}^n : W(z) = 0\ \}$$

Note that:

1. When $W(z) = 0$ only for $z = x_{\mathrm{eq}}$ then $E = \{x_{\mathrm{eq}}\}$.
   Since $M \subset E$, $M = \{x_{\mathrm{eq}}\}$ and therefore $x(t) \to x_{eq} \Rightarrow$ asympt. stability
2. Even when E is larger then $\{x_{\mathrm{eq}}\}$ we often have $M = \{x_{\mathrm{eq}}\}$ and can conclude asymptotic stability.

LaSalle's Invariance Principle (linear system)

$$\dot{x} = Ax \qquad x \in \mathbb{R}^n$$

$M \in \mathbb{R}^n$ is an invariant set if $x(0) \in M \Rightarrow x(t) \in M \ \forall\, t \geq 0$

Theorem (LaSalle Invariance Principle–linear system, quadratic V):
Suppose there exists a positive definite matrix P

$$A' P + P A \leq -Q \leq 0$$

Then the system is stable.
Moreover, $x(t)$ converges to the largest invariant set M contained in

$$E := \{ z \in \mathbb{R}^n : \ Q z = 0 \}$$

Note that:

1. Since $Q \geq 0$ we can always write $Q = C' C$ …

LaSalle's Invariance Principle (linear system)

$$\dot{x} = Ax \qquad x \in \mathbb{R}^n$$

$M \in \mathbb{R}^n$ is an invariant set if $x(0) \in M \Rightarrow x(t) \in M \ \forall\, t \geq 0$

Theorem (LaSalle Invariance Principle–linear system, quadratic V):
Suppose there exists a positive definite matrix P

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Then the system is stable.
Moreover, $x(t)$ converges to the largest invariant set M contained in

$$E := \{ z \in \mathbb{R}^n : C z = 0 \}$$

Why? show that $C'Cz = 0 \Rightarrow Cz = 0$

Note that:

2. When $Q > 0$ then $E = \{0\}$.
   Since $M \subset E$, $M = \{0\}$ and therefore $x(t) \to 0 \Rightarrow$ asympt. stability
3. Even when E is larger then $\{0\}$ we often have $M = \{0\}$ and can conclude asymptotic stability.

When does this happen ?

Asymptotic stability from LaSalle's IP

$$\dot{x} = Ax \qquad x \in \mathbb{R}^n$$

$M \in \mathbb{R}^n$ is an invariant set if $x(0) \in M \Rightarrow x(t) \in M \;\forall\, t \geq 0$

$M \equiv$ largest invariant set contained in $E := \{\, z \in \mathbb{R}^n : C\,z = 0 \,\}$

$x_0 \in M$ if and only if $x(t) := e^{A\,t}\,x_0 \in M \subset E \quad \forall\, t \geq 0$

$$\left.\begin{array}{ccl} Ce^{At}x_0 = 0 & \overset{t=0}{\Rightarrow} & Cx_0 = 0 \\ \Downarrow \tfrac{d}{dt} & & \\ CAe^{At}x_0 = 0 & \overset{t=0}{\Rightarrow} & CAx_0 = 0 \\ \Downarrow \tfrac{d}{dt} & & \\ CA^2e^{At}x_0 = 0 & \overset{t=0}{\Rightarrow} & CA^2x_0 = 0 \\ \vdots & & \\ CA^ke^{At}x_0 = 0 & \overset{t=0}{\Rightarrow} & CA^kx_0 = 0 \quad \forall k \geq 0 \end{array}\right\} \quad \begin{bmatrix} C \\ CA \\ CA^2 \\ \vdots \\ CA^k \\ \vdots \end{bmatrix} x_0 = 0$$

$\Updownarrow$ *(Why?)*

$$\begin{bmatrix} C \\ CA \\ CA^2 \\ \vdots \\ CA^{n-1} \end{bmatrix} x_0 = 0$$

Asymptotic stability from LaSalle's IP

$$\dot{x} = Ax \qquad x \in \mathbb{R}^n$$

$M \in \mathbb{R}^n$ is an invariant set if $x(0) \in M \Rightarrow x(t) \in M \;\forall\, t \geq 0$

$M \equiv$ largest invariant set contained in $E := \{\, z \in \mathbb{R}^n : C\,z = 0 \,\}$

$x_0 \in M$ if and only if $x(t) := e^{A\,t}\,x_0 \in M \subset E \quad \forall\, t \geq 0$

$$M := \left\{ z \in \mathbb{R}^n : \begin{bmatrix} C \\ CA \\ CA^2 \\ \vdots \\ CA^{n-1} \end{bmatrix} z = 0 \right\}$$

(check that this is indeed an invariant set …)

LaSalle’s Invariance Principle (linear system)

$$\dot{x} = Ax \qquad x \in \mathbb{R}^n$$

$M \in \mathbb{R}^n$ is an invariant set if $x(0) \in M \Rightarrow x(t) \in M \ \forall\, t \geq 0$

Theorem (LaSalle Invariance Principle–linear system, quadratic V):
Suppose there exists a positive definite matrix P

$$A'P + PA \leq -C'C \leq 0$$

Then the system is stable. Moreover, $x(t)$ converges to

observability matrix
of the pair (C,A)

$$M := \{z \in \mathbb{R}^n : Oz = 0\} \qquad O := \begin{bmatrix} C \\ CA \\ CA^2 \\ \vdots \\ CA^{n-1} \end{bmatrix}$$

When O is nonsingular, we have asymptotic stability
(pair (C,A) is said to be observable)

Back to switched linear systems…

$$\dot{x} = A_\sigma x \qquad x = R_{\sigma,\sigma^-} x^- \qquad (\sigma, x) \in \mathcal{S}$$

Theorem: ($\mathcal{Q}$ finite)

Suppose there exist positive definite matrices $P_q \in \mathbb{R}^{n\times n}$, $q \in \mathcal{Q}$ such that

$$A_q' P_q + P_q A_q \leq -C_q' C_q \leq 0 \qquad \forall\, q \in \mathcal{Q}$$

and at any $z \in \mathbb{R}^n$ where a switching signal in $\mathcal{S}[\chi]$ can jump from p to q

$$z' P_p z \geq z' R'_{q\,p} P_q R_{q\,p} z$$

from general theorem

Then the switched system is stable.
Moreover, if every pair (C_q, A_q), $q \in \mathcal{Q}$ is observable then

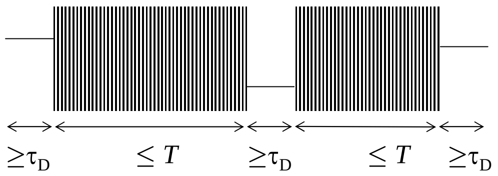
1. if $\mathcal{S} \subset \mathcal{S}_{\text{weak-dwell}}$ then it is asymptotically stable
2. if $\mathcal{S} \subset \mathcal{S}_{\text{p-dwell}}[\tau_D, T]$ then it is uniformly asymptotically stable.

Sets of switching signals

$\mathcal{S}_{\text{dwell}}[\tau_D]$ ≡ switching signals with “dwell-time” $\tau_D > 0$, i.e., interval between consecutive discontinuities larger or equal to τ_D

$\mathcal{S}_{\text{ave}}[\tau_D, N_0]$ ≡ switching signals with “average dwell-time” $\tau_D > 0$ and “chatter-bound” $N_0 > 0$, i.e., $N_\sigma(t,\tau) \leq N_0 + \frac{t-\tau}{\tau_D}$

$\mathcal{S}_{\text{p-dwell}}[\tau_D, T]$ ≡ switching signals with “persistent dwell-time” $\tau_D > 0$ and “period of persistency” $T > 0$, i.e., ∃ infinitely many intervals of length $\geq \tau_D$ on which sigma is constant & consecutive intervals with this property are separated by no more than T



$\mathcal{S}_{\text{weak-dwell}} := \cup_{\tau_D > 0} \mathcal{S}_{\text{p-dwell}}[\tau_D, +\infty] \equiv$ each σ has persistent dwell-time > 0

$$\mathcal{S}_{\text{dwell}}[\tau_D] \subset \mathcal{S}_{\text{ave}}[\tau_D, N_0] \subset \mathcal{S}_{\text{p-dwell}}[\gamma\,\tau_D, T] \subset \mathcal{S}_{\text{weak-dwell}} \subset \mathcal{S}_{\text{all}}$$

$$\gamma \in (0,1), \quad T := \frac{N_0 - \gamma}{1-\gamma}\gamma\tau_D$$

LaSalle’s IP for switched systems

$$\dot{x} = A_\sigma x \qquad x = R_{\sigma,\sigma^-} x^- \qquad (\sigma, x) \in \mathcal{S}$$

Theorem: (Q finite)

Suppose there exist positive definite matrices $P_q \in \mathbb{R}^{n\times n}$, $q \in Q$ such that

$$A_q' P_q + P_q A_q \leq -C_q' C_q \leq 0 \qquad \forall q \in Q$$

and at any $z \in \mathbb{R}^n$ where a switching signal in $\mathcal{S}[\chi]$ can jump from p to q

$$V_p(z) \geq V_q(R_{qp}\, z)$$

Then the switched system is stable.

from general theorem

Moreover, if every pair (C_q, A_q), $q \in Q$ is observable then

1. if $\mathcal{S} \subset \mathcal{S}_{\text{weak-dwell}}$ then it is asymptotically stable
2. if $\mathcal{S} \subset \mathcal{S}_{\text{p-dwell}}[\tau_D, T]$ then it is uniformly asymptotically stable.

$\mathcal{S}_{\text{p-dwell}}[\tau_D, T]$ ≡ switching signals with “persistent dwell-time” $\tau_D > 0$ and “period of persistency” $T > 0$, i.e., ∃ infinitely many intervals of length $\geq \tau_D$ on which sigma is constant & consecutive intervals with this property are separated by no more than T

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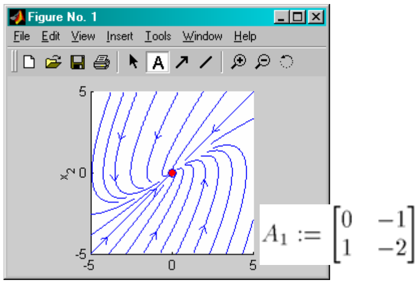
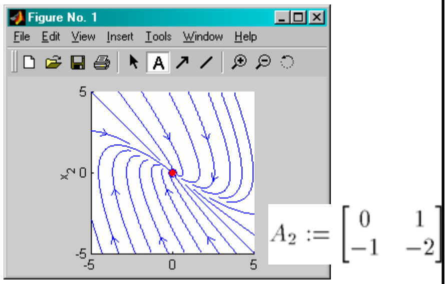
$$\mathcal{S}_{\text{dwell}}[\tau_D] \subset \mathcal{S}_{\text{ave}}[\tau_D, N_0] \subset \mathcal{S}_{\text{p-dwell}}[\gamma\,\tau_D, T] \subset \mathcal{S}_{\text{weak-dwell}} \subset \mathcal{S}_{\text{all}}$$

Example



$$A_1 := \begin{bmatrix} 0 & -1 \\ 1 & -2 \end{bmatrix}$$



$$A_2 := \begin{bmatrix} 0 & 1 \\ -1 & -2 \end{bmatrix}$$

$$\dot{x} = A_\sigma x$$

Choosing $P_1 = P_2 = I$ *common Lyapunov function*

$$A_q' P_q + P_q A_q = -c_q' c_q \leq 0 \qquad c_q := \begin{bmatrix} 0 & 2 \end{bmatrix} \qquad \forall q \in \{1,2\}$$

$$O_q := \begin{bmatrix} c_q \\ c_q A_q \end{bmatrix} = \begin{bmatrix} 0 & 2 \\ \pm 2 & -4 \end{bmatrix} \quad q \in \{1,2\}$$ *nonsingular (observable)*

1. One can find $\sigma \notin \mathcal{S}_{\text{weak-dwell}}$ for which we do not have asymptotic stability
2. Stability is not uniform on $\mathcal{S}_{\text{weak-dwell}}$, because one can find $\sigma \in \mathcal{S}_{\text{weak-dwell}}$ for which convergence is "arbitrarily slow" *(problems, e.g., close to the x_2=0 axis)*

LaSalle's IP for switched systems

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and at any $z \in \mathbb{R}^n$ where a switching signal in $\mathcal{S}[\chi]$ can jump from p to q

$$V_p(z) \geq V_q(R_{q\,p}\, z)$$

Then the switched system is stable. from general theorem

Moreover, if every pair (C_q, A_q), $q \in \mathcal{Q}$ is observable then

1. if $\mathcal{S} \subset \mathcal{S}_{\text{weak-dwell}}$ then it is asymptotically stable
2. if $\mathcal{S} \subset \mathcal{S}_{\text{p-dwell}}[\tau_D, T]$ then it is uniformly asymptotically stable.

a) Finiteness of $\mathcal{Q}$ could be replaced by compactness
b) In some cases it is sufficient for all pairs (C_q, A_q), $q \in \mathcal{Q}$ to be detectable (e.g., when $A_q = A + B F_q$)
c) When the pairs (C_q, A_q), $q \in \mathcal{Q}$ are not observable x converges to the smallest subspace $\mathcal{M}$ that is invariant for all unswitched system and contains the kernels of all O_q
d) There are nonlinear versions of this result (no uniformity?)

Next lecture…

- Computational methods to construct multiple Lyapunov functions—Linear Matrix Inequalities (LMIs)
- Applications (vision-based control)