

# Hybrid Control and Switched Systems

## Lecture #14 Computational methods to construct multiple Lyapunov functions & Applications

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### Summary

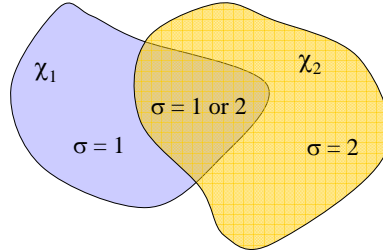
- Computational methods to construct multiple Lyapunov functions—Linear Matrix Inequalities (LMIs)
- Applications (vision-based control)

## Current-state dependent switching

$\chi := \{\chi_q \in \mathbb{R}^n: q \in \mathcal{Q}\} \equiv$  (not necessarily disjoint) covering of  $\mathbb{R}^n$ , i.e.,  $\cup_{q \in \mathcal{Q}} \chi_q = \mathbb{R}^n$

*Current-state dependent switching*

$\mathcal{S}[\chi] \equiv$  set of all pairs  $(\sigma, x)$  with  $\sigma$  piecewise constant and  $x$  piecewise continuous such that  $\forall t, \sigma(t) = q$  is allowed only if  $x(t) \in \chi_q$



Thus  $(\sigma, x) \in \mathcal{S}[\chi]$  if and only if  $x(t) \in \chi_{\sigma(t)} \forall t$

## Multiple Lyapunov functions

$$\dot{x} = f_\sigma(x) \quad x = \rho(\sigma, \sigma^-, x^-) \quad (\sigma, x) \in \mathcal{S} \subset \mathcal{S}[\chi]$$

**Theorem:**

Suppose there exists  $\alpha_1 \in \mathcal{K}_\infty$ ,  $\alpha_2 \in \mathcal{K}$  and a family of continuously differentiable, functions  $V_q: \mathbb{R}^n \rightarrow \mathbb{R}, q \in \mathcal{Q}$  such that

$$V_q(z) \geq \alpha_1(\|z\|), \quad \frac{\partial V_q}{\partial x}(z) f_q(z) \leq -\alpha_2(\|z\|) \quad \forall q \in \mathcal{Q}, z \in \chi_q$$

and at any  $z \in \mathbb{R}^n$  where a switching signal in  $\mathcal{S}$  can jump from  $p$  to  $q$

$$V_p(z) \geq V_q(\rho(q, p, z))$$

Then the origin is (glob) uniformly asymptotically stable.

## Multiple Lyapunov functions (linear)

$$\dot{x} = A_\sigma x \quad x = x^- \quad (\sigma, x) \in \mathcal{S} \subset \mathcal{S}[\chi]$$

no resets

**Theorem:**

Suppose there exist constants  $\delta, \epsilon \geq 0$  and symmetric matrices  $P_q \in \mathbb{R}^{n \times n}, q \in \mathcal{Q}$  such that

$$z' P_q z \geq \delta \|z\|^2 \quad z' (A_q' P_q + P_q A_q) z \leq -\epsilon \|z\|^2 \quad \forall q \in \mathcal{Q}, z \in \chi_q$$

and at any  $z \in \mathbb{R}^n$  where a switching signal in  $\mathcal{S}$  can jump from  $p$  to  $q$

$$z P_q z \leq z' P_p z \quad \text{check!}$$

Then the origin is uniformly asymptotically stable (actually exponentially).

Given the  $A_q, q \in \mathcal{Q}$

can we find  $P_q, q \in \mathcal{Q}$  and  $\chi := \{\chi_q; q \in \mathcal{Q}\}$  so that we have stability?

stabilization through choice  
of switching regions

## Multiple Lyapunov functions (linear)

$$\dot{x} = A_\sigma x \quad x = x^- \quad (\sigma, x) \in \mathcal{S} \subset \mathcal{S}[\chi]$$

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**Theorem:**

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and at any  $z \in \mathbb{R}^n$  where a switching signal in  $\mathcal{S}$  can jump from  $p$  to  $q$

$$z P_q z \leq z' P_p z \quad \text{check!}$$

Then the origin is uniformly asymptotically stable (actually exponentially).

Given  $P_q, q \in \mathcal{Q}$ , suppose we choose:

$$\begin{aligned} \chi_q &:= \{z \in \mathbb{R}^n : z' P_q z \geq z' P_p z \quad \forall p \in \mathcal{Q}\} \\ &= \{z \in \mathbb{R}^n : z' P_q z = \max_{p \in \mathcal{Q}} z' P_p z\} \end{aligned}$$

sigma must be equal to (one of) the largest  $V_q(z) := z' P_q z$   $\Rightarrow$  for switching to be possible at  $z \in \mathbb{R}^n$  from  $p$  to  $q$ , we must have  $V_q(z) = V_p(z)$   $\Rightarrow$  jump condition is always satisfied

## Multiple Lyapunov functions (linear)

$$\dot{x} = A_\sigma x \quad x = x^- \quad (\sigma, x) \in \mathcal{S} \subset \mathcal{S}[\chi]$$

no resets

**Theorem:**

Suppose there exist constants  $\delta, \epsilon \geq 0$  and symmetric matrices  $P_q \in \mathbb{R}^{n \times n}, q \in \mathcal{Q}$  such that

$$z' P_q z \geq \delta \|z\|^2 \quad z'(A'_q P_q + P_q A_q)z \leq -\epsilon \|z\|^2 \quad \forall q \in \mathcal{Q}, z \in \chi_q$$

and at any  $z \in \mathbb{R}^n$  where a switching signal in  $\mathcal{S}$  can jump from  $p$  to  $q$

$$z P_q z \leq z' P_p z \quad \text{check!}$$

Then the origin is uniformly asymptotically stable (actually exponentially).

It is sufficient to find  $P_q, q \in \mathcal{Q}$  such that

$$z' P_q z \geq \delta \|z\|^2 \quad z'(A'_q P_q + P_q A_q)z \leq -\epsilon \|z\|^2 \quad \forall q \in \mathcal{Q}, z \in \chi_q$$

$$\chi_q := \{z \in \mathbb{R}^n : z' P_q z \geq z' P_p z \quad \forall p \in \mathcal{Q}\}$$

or equivalently  $\forall q \in \mathcal{Q}$ :

$$z' P_q z \geq z' P_p z \quad \forall p \in \mathcal{Q} \quad \Rightarrow \quad z' P_q z \geq \delta \|z\|^2 \quad z'(A'_q P_q + P_q A_q)z \leq -\epsilon \|z\|^2$$

## Finding multiple Lyapunov functions

We need to find  $P_q, q \in \mathcal{Q}$  such that  $\forall q \in \mathcal{Q}$

$$z' P_q z \geq z' P_p z \quad \forall p \in \mathcal{Q} \quad \Rightarrow \quad z' P_q z \geq \delta \|z\|^2 \quad z'(A'_q P_q + P_q A_q)z \leq -\epsilon \|z\|^2$$

Suppose we can find  $P_q, q \in \mathcal{Q}$  and constants  $\gamma_{p,q}, \mu_{p,q} \geq 0, p, q \in \mathcal{Q}$  such that

$$P_q \geq \delta I + \sum_{p \in \mathcal{Q} \setminus \{q\}} \mu_{p,q} (P_q - P_p)$$

$$A'_q P_q + P_q A_q \leq -\epsilon I - \sum_{p \in \mathcal{Q} \setminus \{q\}} \gamma_{p,q} (P_q - P_p)$$

Then

$$\begin{aligned} z' P_q z &\geq \delta \|z\|^2 + \sum_{p \in \mathcal{Q} \setminus \{q\}} \mu_{p,q} z'(P_q - P_p)z \\ z'(A'_q P_q + P_q A_q)z &\leq -\epsilon \|z\|^2 - \sum_{p \in \mathcal{Q} \setminus \{q\}} \gamma_{p,q} z'(P_q - P_p)z \\ &\geq 0 \text{ when } z' P_q z \geq z' P_p z \quad \forall p \in \mathcal{Q} \end{aligned}$$

## Constructing multiple Lyapunov functions

$$\dot{x} = A_\sigma x \quad x = x^- \quad (\sigma, x) \in \mathcal{S} \subset \mathcal{S}[\chi]$$

no resets

**Theorem:**

Suppose there exist constants  $\delta, \epsilon, \gamma_{p,q}, \mu_{p,q} \geq 0, p, q \in \mathcal{Q}$  and symmetric matrices  $P_q \in \mathbb{R}^{n \times n}, q \in \mathcal{Q}$  such that

$$P_q \geq \delta I + \sum_{p \in \mathcal{Q} \setminus \{q\}} \mu_{p,q} (P_q - P_p)$$

$$A'_q P_q + P_q A_q \leq -\epsilon I - \sum_{p \in \mathcal{Q} \setminus \{q\}} \gamma_{p,q} (P_q - P_p)$$

Then, for

$$\chi_q := \{z \in \mathbb{R}^n : z' P_q z \geq z' P_p z \quad \forall p \in \mathcal{Q}\}$$

the origin is uniformly asymptotically stable (actually exponentially).

## Linear Matrix Inequality (LMI)

$$F(x) \geq 0 \quad \text{where } F : \mathbb{R}^m \rightarrow \mathbb{R}^{n \times n} \equiv \text{affine function}$$

(i.e.,  $F(x) - F(0)$  is linear)

System of linear matrix inequalities:

$$\begin{cases} F_1(x) \geq 0 \\ F_2(x) \geq 0 \\ \vdots \\ F_k(x) \geq 0 \end{cases} \Leftrightarrow \bar{F}(x) \geq 0 \quad \bar{F}(x) := \begin{bmatrix} F_1(x) & 0 & \dots & 0 \\ 0 & F_2(x) & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & F_k(x) \end{bmatrix}$$

single LMI

System of linear matrix equalities and inequalities

$$\begin{cases} F(x) \geq 0 \\ G(x) = 0 \end{cases} \Leftrightarrow \begin{cases} F(x) \geq 0 \\ G(x) \geq 0 \\ -G(x) \geq 0 \end{cases} \Leftrightarrow \begin{bmatrix} F(x) & 0 & 0 \\ 0 & G(x) & 0 \\ 0 & 0 & -G(x) \end{bmatrix} \geq 0$$

single LMI

(however most numerical solvers handle equalities separately more efficiently)

*There are very efficient methods to solve LMIs (type 'help lmiLAB' in MATLAB)*

## Back to constructing multiple Lyapunov functions

$$\dot{x} = A_\sigma x \quad x = x^- \quad (\sigma, x) \in \mathcal{S} \subset \mathcal{S}[\chi]$$

no resets

**Theorem:**

Suppose there exist constants  $\delta, \varepsilon, \gamma_{p,q}, \mu_{p,q} \geq 0, p, q \in \mathcal{Q}$  and symmetric matrices  $P_q \in \mathbb{R}^{n \times n}, q \in \mathcal{Q}$  such that  $P_q - P_q' = 0$

$$P_q \geq \delta I + \sum_{p \in \mathcal{Q} \setminus \{q\}} \mu_{p,q} (P_q - P_p)$$

$$A_q' P_q + P_q A_q \leq -\varepsilon I - \sum_{p \in \mathcal{Q} \setminus \{q\}} \gamma_{p,q} (P_q - P_p)$$

Then, for

$$\chi_q := \{z \in \mathbb{R}^n : z' P_q z \geq z' P_p z \quad \forall p \in \mathcal{Q}\}$$

the origin is uniformly asymptotically stable (actually exponentially).

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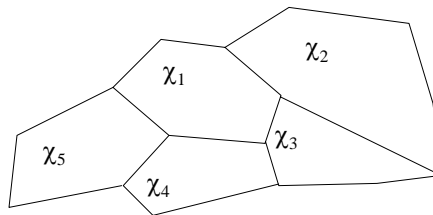
LMI on the unknowns  $\delta, \varepsilon > 0, \gamma_{p,q}, \mu_{p,q} \in \mathbb{R}, P_q \in \mathbb{R}^{n \times n}, p, q \in \mathcal{Q}$ ,

Only useful if we have the freedom to choose the switching region  $\chi_q$

## Convex polyhedral regions

$$\dot{x} = A_\sigma x \quad x = x^- \quad (\sigma, x) \in \mathcal{S} \subset \mathcal{S}[\chi]$$

$\chi_q, q \in \mathcal{Q} \equiv$  convex polyhedral with disjoint interiors



cell  $q \in \mathcal{Q}$ :

$$\chi_q := \left\{ z \in \mathbb{R}^n : E_q \begin{bmatrix} z \\ 1 \end{bmatrix} \geq 0 \right\} \quad \text{component-wise} \quad E_q \in \mathbb{R}^{m_q \times (n+1)}$$

boundary between regions  $p, q \in \mathcal{Q}$ :

$$\bar{\chi}_q \cap \bar{\chi}_p \subset \left\{ z \in \mathbb{R}^n : f_{pq} \begin{bmatrix} z \\ 1 \end{bmatrix} = 0 \right\} \quad f_{p,q} \in \mathbb{R}^{n+1}$$

## Constructing multiple Lyapunov functions

$$\dot{x} = A_\sigma x \quad x = x^- \quad (\sigma, x) \in \mathcal{S} \subset \mathcal{S}[\chi]$$

$$\chi_q := \left\{ z \in \mathbb{R}^n : E_q \begin{bmatrix} z \\ 1 \end{bmatrix} \geq 0 \right\} \quad \bar{\chi}_q \cap \bar{\chi}_p \subset \left\{ z \in \mathbb{R}^n : f'_{pq} \begin{bmatrix} z \\ 1 \end{bmatrix} = 0 \right\}$$

### Theorem:

Suppose there exist

1. symmetric matrices  $P_q \in \mathbb{R}^{(n+1) \times (n+1)}$ ,  $q \in \mathcal{Q}$
  2. vectors  $k_{pq} \in \mathbb{R}^{n+1}$ ,  $p, q \in \mathcal{Q}$
  3. constants  $\epsilon, \delta > 0$
  4. symmetric matrices with nonnegative entries  $U_q, W_q$ ,  $q \in \mathcal{Q}$  (not nec. pos. def)
- such that for every  $p, q \in \mathcal{Q}$  for which  $\chi_q$  and  $\chi_p$  have a common boundary

$$P_q - P_p = k_{pq} f'_{pq} + f_{pq} k'_{pq}$$

and  $\forall q \in \mathcal{Q}$

$$P_q - E_q^T W_q E_q \geq \delta \Pi^T \Pi \quad \Pi := \begin{bmatrix} I_{n \times n} & 0_{1 \times n} \end{bmatrix}$$

$$\Pi^T A_q^T \Pi P_q + P_q \Pi^T A_q \Pi + E_q^T U_q E_q \leq -\epsilon \Pi^T \Pi$$

the origin is uniformly asymptotically stable (actually exponentially).

LMI on the unknowns  $P_q, k_{pq}, \delta, \epsilon, U_q, W_q$

could be generalized  
to affine systems, i.e.,  
 $\dot{x} = A_\sigma x + b_\sigma$   
(homework!)

## Constructing multiple Lyapunov functions

$$\dot{x} = A_\sigma x \quad x = x^- \quad (\sigma, x) \in \mathcal{S} \subset \mathcal{S}[\chi]$$

$$\chi_q := \left\{ z \in \mathbb{R}^n : E_q \begin{bmatrix} z \\ 1 \end{bmatrix} \geq 0 \right\} \quad \bar{\chi}_q \cap \bar{\chi}_p \subset \left\{ z \in \mathbb{R}^n : f'_{pq} \begin{bmatrix} z \\ 1 \end{bmatrix} = 0 \right\}$$

$$P_q - P_p = k_{pq} f'_{pq} + f_{pq} k'_{pq} \quad *$$

$$P_q - E_q^T W_q E_q \geq \delta \Pi^T \Pi \quad \Pi := \begin{bmatrix} I_{n \times n} & 0_{1 \times n} \end{bmatrix}$$

$$\Pi^T A_q^T \Pi P_q + P_q \Pi^T A_q \Pi + E_q^T U_q E_q \leq -\epsilon \Pi^T \Pi$$

Why?

Consider multiple Lyapunov functions:  $V_q(z) := \begin{bmatrix} z^T & 1 \end{bmatrix} P_q \begin{bmatrix} z \\ 1 \end{bmatrix} \quad q \in \mathcal{Q}$

1<sup>st</sup> On the boundary between  $\chi_q$  and  $\chi_p$ :

$$f'_{pq} \begin{bmatrix} z \\ 1 \end{bmatrix} = 0 \quad \Rightarrow \quad V_q(z) = V_p(z)$$

## Constructing multiple Lyapunov functions

$$\dot{x} = A_\sigma x \quad x = x^- \quad (\sigma, x) \in \mathcal{S} \subset \mathcal{S}[\chi]$$

$$\chi_q := \left\{ z \in \mathbb{R}^n : E_q \begin{bmatrix} z \\ 1 \end{bmatrix} \geq 0 \right\} \quad \bar{\chi}_q \cap \bar{\chi}_p \subset \left\{ z \in \mathbb{R}^n : f'_{pq} \begin{bmatrix} z \\ 1 \end{bmatrix} = 0 \right\}$$

$$\begin{aligned} P_q - P_p &= k_{pq} f'_{pq} + f_{pq} k'_{pq} \\ P_q - E_q^T W_q E_q &\geq \delta \Pi^T \Pi \quad \Pi := \begin{bmatrix} I_{n \times n} & 0_{1 \times n} \end{bmatrix} \\ \Pi^T A_q^T \Pi P_q + P_q \Pi^T A_q \Pi + E_q^T U_q E_q &\leq -\epsilon \Pi^T \Pi \end{aligned}$$

Why?

Consider multiple Lyapunov functions  $V_q(z) := [z^T \ 1] P_q \begin{bmatrix} z \\ 1 \end{bmatrix} \quad q \in \mathcal{Q}$ ,

$$\begin{aligned} \text{2nd On } \chi_q: \quad V_q(z) &= [z^T \ 1] P_q \begin{bmatrix} z \\ 1 \end{bmatrix} \\ &= [z^T \ 1] (P_q - E_q^T W_q E_q) \begin{bmatrix} z \\ 1 \end{bmatrix} + [z^T \ 1] \underbrace{E_q^T W_q E_q}_{\geq 0} \begin{bmatrix} z \\ 1 \end{bmatrix} \\ &\geq [z^T \ 1] (P_q - E_q^T W_q E_q) \begin{bmatrix} z \\ 1 \end{bmatrix} \\ &\geq \delta [z^T \ 1] \Pi^T \Pi \begin{bmatrix} z \\ 1 \end{bmatrix} = \delta \|z\|^2 \end{aligned}$$

## Constructing multiple Lyapunov functions

$$\dot{x} = A_\sigma x \quad x = x^- \quad (\sigma, x) \in \mathcal{S} \subset \mathcal{S}[\chi]$$

$$\chi_q := \left\{ z \in \mathbb{R}^n : E_q \begin{bmatrix} z \\ 1 \end{bmatrix} \geq 0 \right\} \quad \bar{\chi}_q \cap \bar{\chi}_p \subset \left\{ z \in \mathbb{R}^n : f'_{pq} \begin{bmatrix} z \\ 1 \end{bmatrix} = 0 \right\}$$

$$\begin{aligned} P_q - P_p &= k_{pq} f'_{pq} + f_{pq} k'_{pq} \\ P_q - E_q^T W_q E_q &\geq \delta \Pi^T \Pi \quad \Pi := \begin{bmatrix} I_{n \times n} & 0_{1 \times n} \end{bmatrix} \\ \Pi^T A_q^T \Pi P_q + P_q \Pi^T A_q \Pi + E_q^T U_q E_q &\leq -\epsilon \Pi^T \Pi \quad * \end{aligned}$$

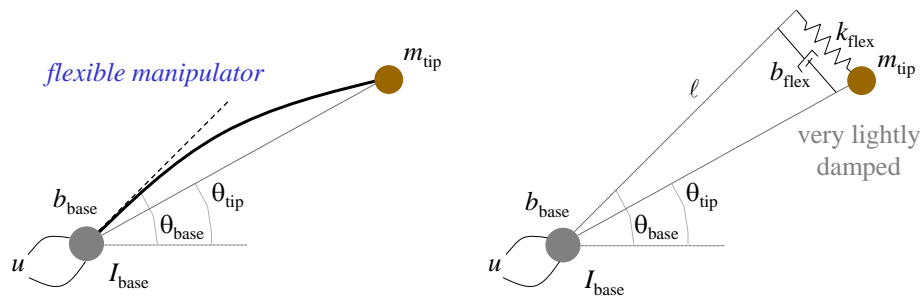
Why?

Consider multiple Lyapunov functions  $V_q(z) := [z^T \ 1] P_q \begin{bmatrix} z \\ 1 \end{bmatrix} \quad q \in \mathcal{Q}$ ,

3rd On  $\chi_q$ :

$$\begin{aligned} \frac{\partial V_q}{\partial z}(z) A_q z &= [z^T \ 1] (\Pi^T A_q^T \Pi P_q + P_q \Pi^T A_q \Pi) \begin{bmatrix} z \\ 1 \end{bmatrix} \\ &= [z^T \ 1] (\Pi^T A_q^T \Pi P_q + P_q \Pi^T A_q \Pi + E_q^T U_q E_q) \begin{bmatrix} z \\ 1 \end{bmatrix} - [z^T \ 1] E_q^T U_q E_q \begin{bmatrix} z \\ 1 \end{bmatrix} \\ &\leq -\epsilon [z^T \ 1] \Pi^T \Pi \begin{bmatrix} z \\ 1 \end{bmatrix} = -\epsilon \|z\|^2. \end{aligned}$$

## Example: Vision-based control of a flexible manipulator



*4<sup>th</sup> dimensional small bending approximation*

$$m_{\text{tip}} \ell^2 \ddot{\theta}_{\text{tip}} = \ell k_{\text{flex}} (\theta_{\text{base}} - \theta_{\text{tip}}) + \ell b_{\text{flex}} (\dot{\theta}_{\text{base}} - \dot{\theta}_{\text{tip}})$$

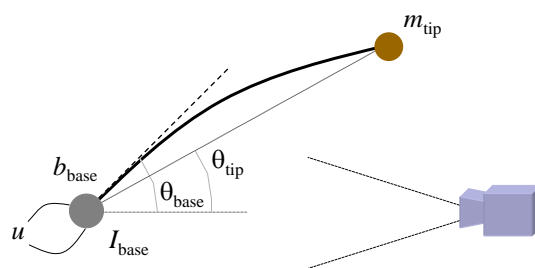
$$I_{\text{base}} \ddot{\theta}_{\text{base}} = -b_{\text{base}} \dot{\theta}_{\text{base}} + \ell k_{\text{flex}} (\theta_{\text{tip}} - \theta_{\text{base}}) + \ell b_{\text{flex}} (\dot{\theta}_{\text{tip}} - \dot{\theta}_{\text{base}}) + k_{\text{motor}} u$$

**Control objective:** drive  $\theta_{\text{tip}}$  to zero, using feedback from

$\theta_{\text{base}} \rightarrow$  encoder at the base

$\theta_{\text{tip}} \rightarrow$  machine vision (essential to increase the damping of the flexible modes in the presence of noise)

## Example: Vision-based control of a flexible manipulator



To achieve high accuracy in the measurement of  $\theta_{\text{tip}}$  the camera must have a small field of view

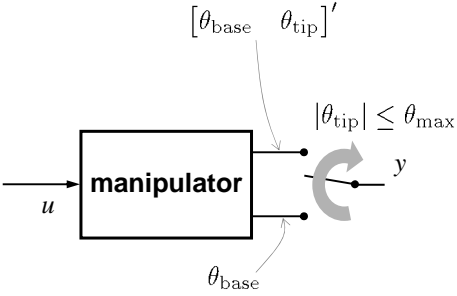
feedback output: 
$$y := \begin{cases} \begin{bmatrix} \theta_{\text{base}} & \theta_{\text{tip}} \end{bmatrix}' & |\theta_{\text{tip}}| \leq \theta_{\text{max}} \\ \theta_{\text{base}} & |\theta_{\text{tip}}| > \theta_{\text{max}} \end{cases}$$

**Control objective:** drive  $\theta_{\text{tip}}$  to zero, using feedback from

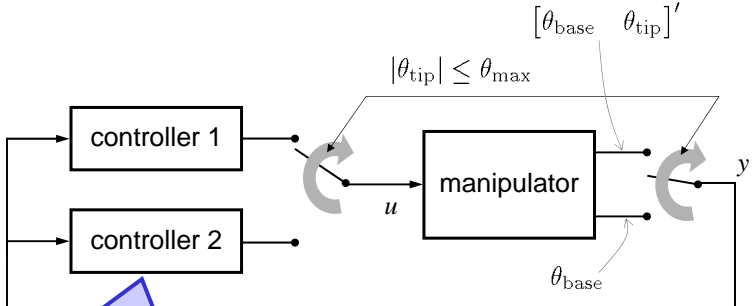
$\theta_{\text{base}} \rightarrow$  encoder at the base

$\theta_{\text{tip}} \rightarrow$  machine vision (essential to increase the damping of the flexible modes in the presence of noise)

### Switched process

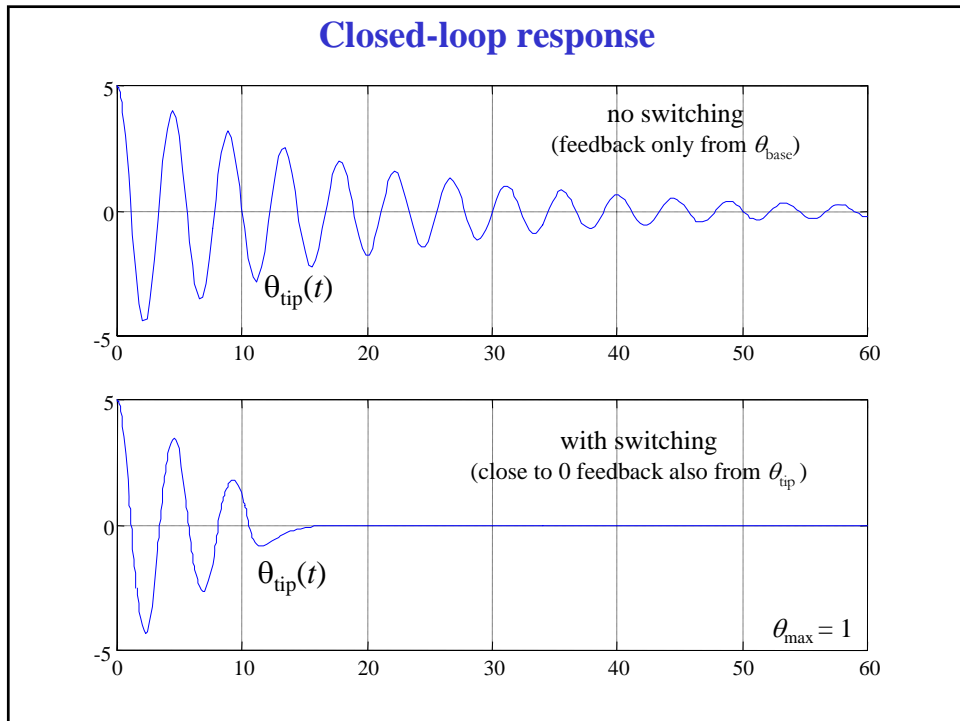
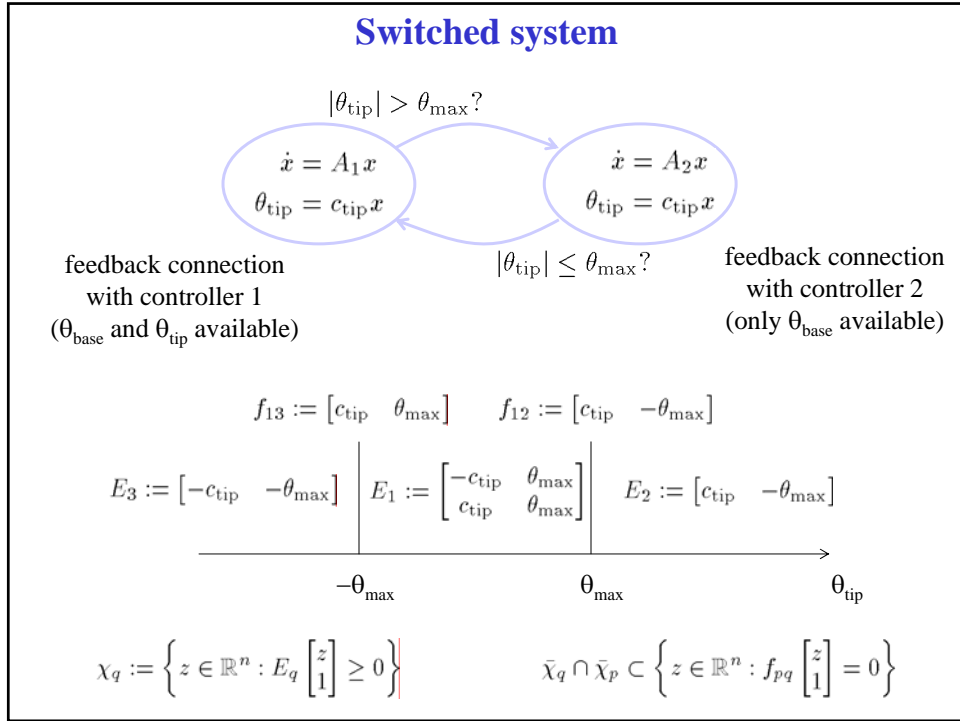


### Switched process



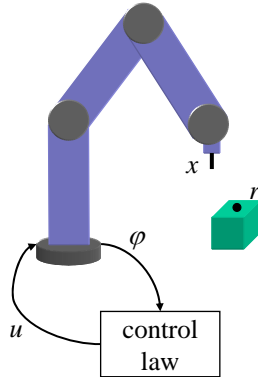
controller 1 optimized for feedback from  $\theta_{base}$  and  $\theta_{tip}$   
and  
controller 2 optimized for feedback only from  $\theta_{base}$

E.g., LQG controllers that minimize  $\lim_{T \rightarrow \infty} \frac{1}{T} E \left[ \int_0^T \theta_{tip}^2 + \theta_{base}^2 + \rho u^2 dt \right]$



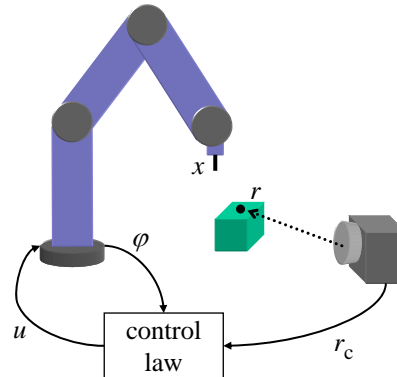
## Visual servoing

**Goal:** Drive the position  $x$  of the end-effector to a reference point  $r$



### encoder-based control

- reference  $r$  in pre-specified position
- end-effector position  $x$  determined from the joint angles

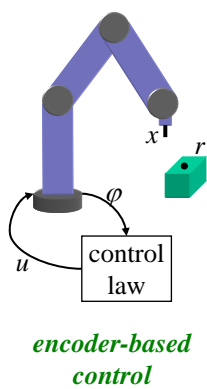


### mixed vision/encoder-based control

- reference  $r$  determined from the camera measurement  $r_c$
- end-effector position  $x$  determined from the joint angles

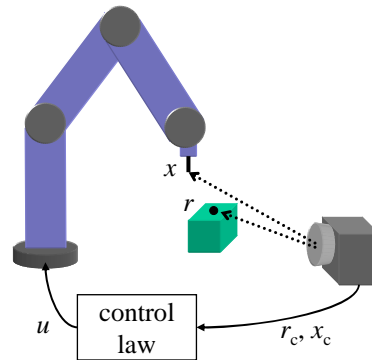
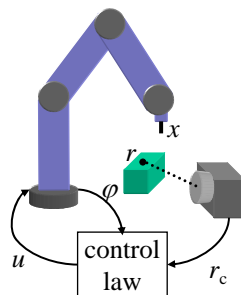
## Visual servoing

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### encoder-based control

### mixed vision/encoder-based control



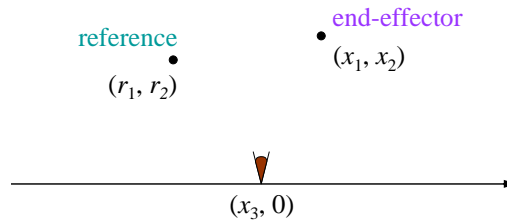
### vision-based control

- reference  $r$  determined from the camera measurement  $r_c$
- end-effector position  $x$  determined from the camera measurement  $x_c$

Typically guarantees zero error, even in the presence of manipulator and camera modeling errors

## Prototype problem

**Goal:** Drive the position  $(x_1, x_2)$  of the end-effector to the reference point  $(r_1, r_2)$ , using visual feedback



For simplicity:

1. Normalized pin-hole camera moves along a straight line perpendicular to its optical axis

$$\left. \begin{aligned} y_1 &= \frac{x_1 - x_3}{x_2} & y_2 &= \frac{r_1 - x_3}{r_2} \end{aligned} \right\} \text{camera measurements} \\ \text{(robot \& reference)}$$

2. Feedback linearized kinematic model for the motions of robots and camera

$$\dot{x}_1 = u_1 \quad \dot{x}_2 = u_2 \quad \dot{x}_3 = u_3$$

## Stabilizability by output-feedback

$$\dot{x}_1 = u_1 \quad \dot{x}_2 = u_2 \quad \dot{x}_3 = u_3 \quad y_1 = \frac{x_1 - x_3}{x_2} \quad y_2 = \frac{r_1 - x_3}{r_2}$$

Can we design an output-feedback controller that asymptotically stabilizes the equilibrium point

$$x_1 = r_1 \quad x_2 = r_2 \quad x_3 = \alpha$$

for some  $\alpha \in \mathbb{R}$  ?

Local linearization around equilibrium yields:

$$\begin{bmatrix} \dot{\tilde{x}}_1 \\ \dot{\tilde{x}}_2 \\ \dot{\tilde{x}}_3 \end{bmatrix} = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} \quad \begin{bmatrix} \tilde{y}_1 \\ \tilde{y}_2 \end{bmatrix} = \frac{1}{r_2} \begin{bmatrix} 1 & \alpha - r_1 & -1 \\ 0 & r_2 & -1 \end{bmatrix} \begin{bmatrix} \tilde{x}_1 \\ \tilde{x}_2 \\ \tilde{x}_3 \end{bmatrix} \quad \text{not detectable}$$

*Manifestation of a fundamental limitation on the class of controllers that asymptotically stabilize the system...*

## Stabilizability by output-feedback

$$\dot{x}_1 = u_1 \quad \dot{x}_2 = u_2 \quad \dot{x}_3 = u_3 \quad y_1 = \frac{x_1 - x_3}{x_2} \quad y_2 = \frac{r_1 - x_3}{r_2} \quad (1)$$

Can we design an output-feedback controller that asymptotically stabilizes the equilibrium point

$$x_1 = r_1 \quad x_2 = r_2 \quad x_3 = \alpha$$

for some  $\alpha \in \mathbb{R}$  ?



**Lemma:** There exist no locally Lipschitz functions

$$F: \mathbb{R}^n \times \mathbb{R}^2 \times [0, \infty) \rightarrow \mathbb{R}^n \quad G: \mathbb{R}^n \times \mathbb{R}^2 \times [0, \infty) \rightarrow \mathbb{R}^3$$

for which the closed-loop system defined by (1) and

$$\dot{z} = F(z, y_1, y_2, t) \quad [u_1 \quad u_2 \quad u_3] = G(z, y_1, y_2, t)$$

has an asymptotically stable equilibrium point with

$$x_1 = r_1 \quad x_2 = r_2$$

## Stabilizability by output-feedback

$$\dot{x}_1 = u_1 \quad \dot{x}_2 = u_2 \quad \dot{x}_3 = u_3 \quad y_1 = \frac{x_1 - x_3}{x_2} \quad y_2 = \frac{r_1 - x_3}{r_2} \quad (1)$$

This limitation is not caused by lack of controllability/observability

**Lemma 2:** For every  $r_1, r_2 \in \mathbb{R}$ , the system (1) is observable. Moreover, there exists a constant input which allows one to uniquely determine the initial state by observing the output over any finite interval  $(t_0, t_0+T)$ . Indeed, for

$$\begin{aligned} u_1 = u_2 = 0 & \quad \rightarrow \text{robot stopped} \\ u_3 = 1 & \quad \rightarrow \text{camera moving at constant speed} \end{aligned}$$

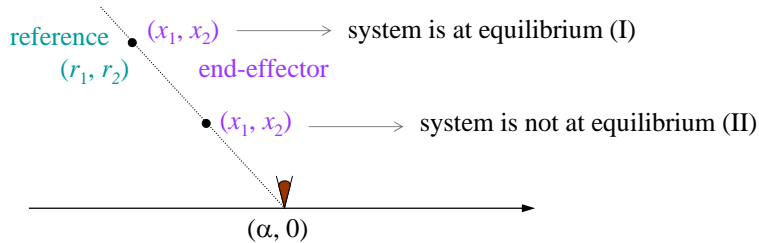
there exists a state reconstruction function  $h_T$  for which

$$\begin{matrix} \text{final state} \\ \rightarrow \end{matrix} \begin{bmatrix} x_1(t_0+T) \\ x_2(t_0+T) \\ x_3(t_0+T) \end{bmatrix} = \begin{bmatrix} x_1(t_0) \\ x_2(t_0) \\ x_3(t_0) + T \end{bmatrix} = h_T \left( \underbrace{\int_{t_0}^{t_0+T} [y_1 \quad ty_1 \quad y_2 \quad ty_2] dt}_{\text{measured outputs}} \right) + \begin{bmatrix} r_1 \\ r_2 \\ r_1 \end{bmatrix}$$

relative position with respect to reference

## Limitation on asymptotic stabilization

**Goal:** Drive the position  $(x_1, x_2)$  of the end-effector to the reference point  $(r_1, r_2)$ , using visual feedback



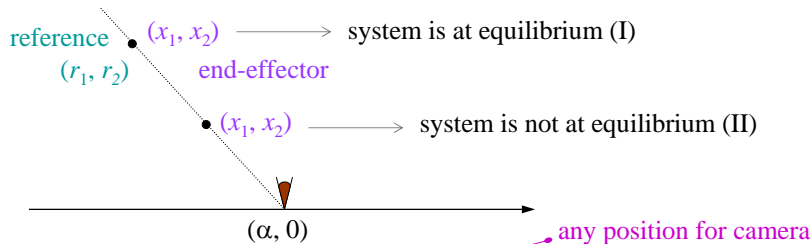
To disambiguate between cases I and II,  
the camera must move away from  $(\alpha, 0)$



This would violate the stability of the equilibrium point in case I

## Limitation on asymptotic stabilization

**Goal:** Drive the position  $(x_1, x_2)$  of the end-effector to the reference point  $(r_1, r_2)$ , using visual feedback



There is no need to stabilize the camera and the state of the controller to a fixed equilibrium. It is sufficient to make the set

$$\mathcal{G} := \{ [r_1 \ r_2 \ \alpha \ z]' \in \mathbb{R}^{n+3} : \alpha \in \mathbb{R}, z \in \mathbb{R}^n \}$$

asymptotically stable

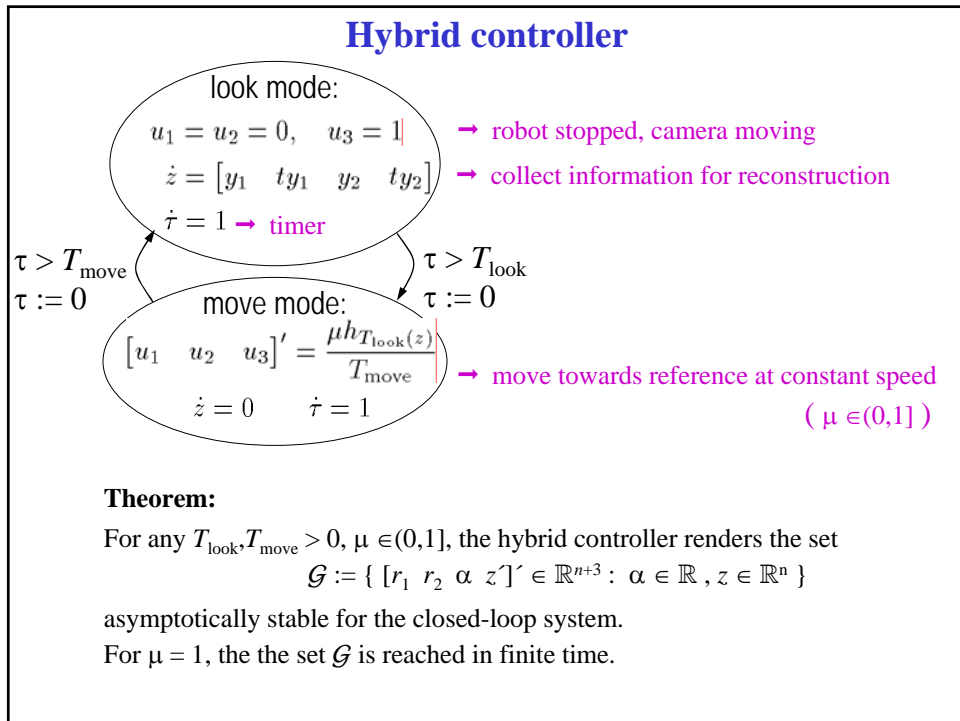
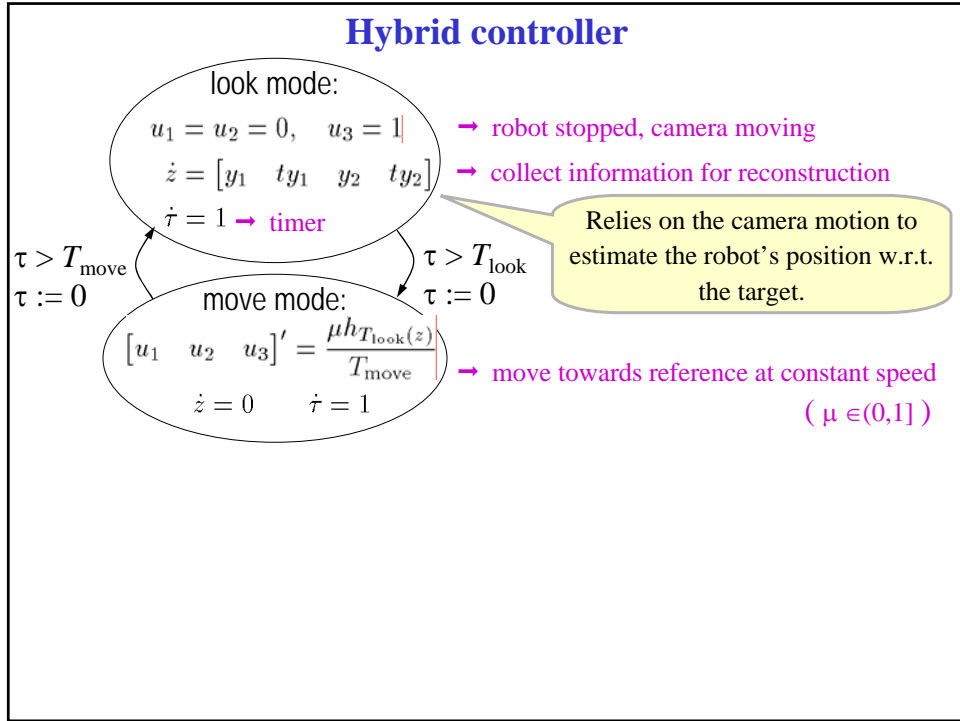
any state for controller

The set  $\mathcal{G}$  is asymptotically stable if  $\forall \varepsilon > 0, \exists \delta > 0$  such that  $\forall t_0 \geq 0$

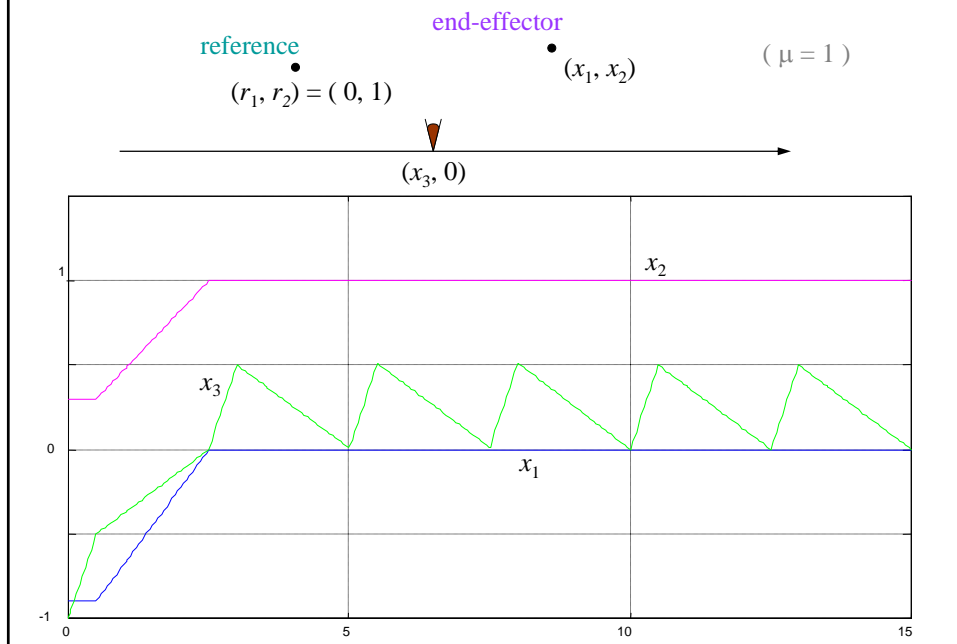
$$d(x_0, \mathcal{G}) < \delta \Rightarrow d(x(t), \mathcal{G}) < \varepsilon$$

and  $d(x(t), \mathcal{G}) \rightarrow 0$  as  $t \rightarrow \infty$

solution to closed-loop  
(required to exist globally)



## Stateflow simulation



## Next lecture...

Applications: Supervisory control