

Hybrid Control and Switched Systems

Lecture #3 What can go wrong? Trajectories of hybrid systems

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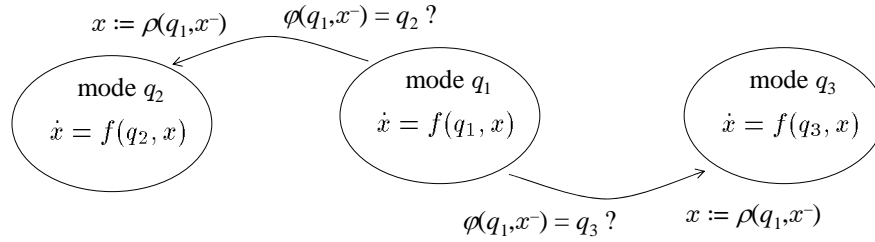


Summary

1. Trajectories of hybrid systems:
 - Solution to a hybrid system
 - Execution of a hybrid system
2. Degeneracies
 - Finite escape time
 - Chattering
 - Zeno trajectories
 - Non-continuous dependency on initial conditions

Hybrid Automaton (deterministic)

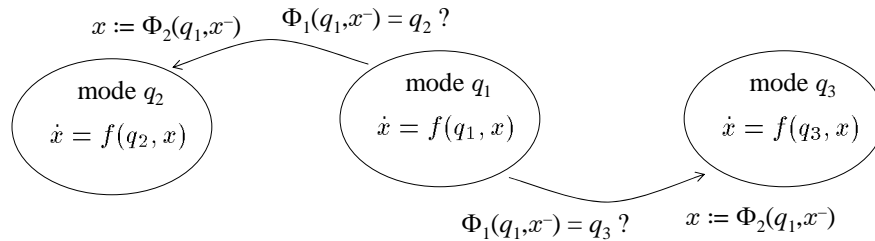
\mathcal{Q} \equiv set of discrete states
 \mathbb{R}^n \equiv continuous state-space
 $f: \mathcal{Q} \times \mathbb{R}^n \rightarrow \mathbb{R}^n$ \equiv vector field
 $\varphi: \mathcal{Q} \times \mathbb{R}^n \rightarrow \mathcal{Q}$ \equiv discrete transition
 $\rho: \mathcal{Q} \times \mathbb{R}^n \rightarrow \mathbb{R}^n$ \equiv reset map



Hybrid Automaton

\mathcal{Q} \equiv set of discrete states
 \mathbb{R}^n \equiv continuous state-space
 $f: \mathcal{Q} \times \mathbb{R}^n \rightarrow \mathbb{R}^n$ \equiv vector field
 $\Phi: \mathcal{Q} \times \mathbb{R}^n \rightarrow \mathcal{Q} \times \mathbb{R}^n$ \equiv discrete transition (& reset map)

$$\Phi(q, x) = \begin{bmatrix} \Phi_1(q, x) \\ \Phi_2(q, x) \end{bmatrix} = \begin{bmatrix} \varphi(q, x) \\ \rho(q, x) \end{bmatrix}$$

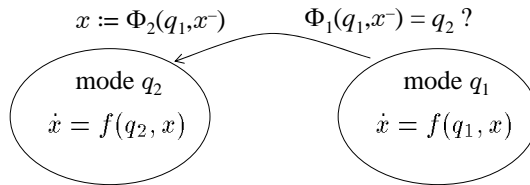


Compact representation of a hybrid automaton

$$\dot{x} = f(q, x) \quad (q, x) = \Phi(q^-, x^-) \quad q \in \mathcal{Q}, x \in \mathbb{R}^n$$

Solution to a hybrid automaton

$$\dot{x} = f(q, x) \quad (q, x) = \Phi(q^-, x^-) \quad q \in \mathcal{Q}, x \in \mathbb{R}^n$$



Definition: A **solution** to the hybrid automaton is a pair of right-continuous signals
 $x : [0, \infty) \rightarrow \mathbb{R}^n$ $q : [0, \infty) \rightarrow \mathcal{Q}$

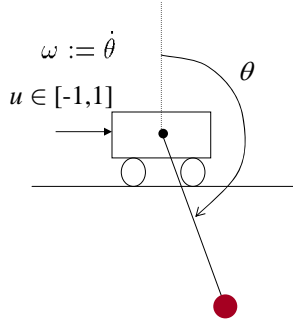
such that

1. x is piecewise differentiable & q is piecewise constant
2. on any interval (t_1, t_2) on which q is constant and x continuous

$$x(t) = x(t_1) + \int_{t_1}^t f(q(t_1), x(\tau)) d\tau \quad \forall t \in [t_1, t_2) \quad \text{continuous evolution}$$

3. $(q(t), x(t)) = \Phi(q^-(t), x^-(t)) \quad \forall t \geq 0$ discrete transitions

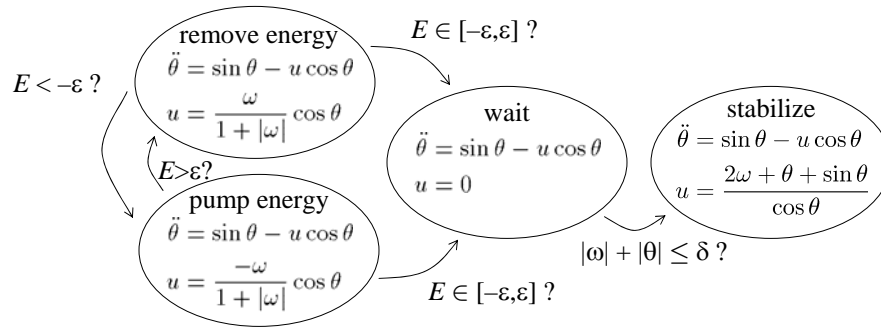
Example #4: Inverted pendulum swing-up



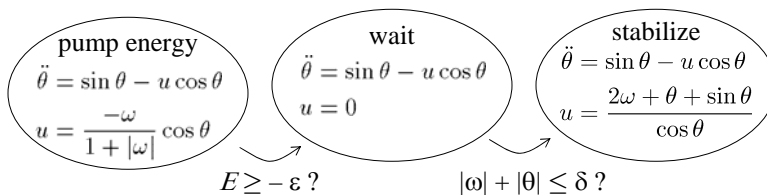
$$\ddot{\theta} = \sin \theta - u \cos \theta \quad E := \frac{1}{2} \omega^2 + (\cos \theta - 1)$$

Hybrid controller:

- 1st pump/remove energy into/from the system by applying maximum force, until $E \approx 0$
- 2nd wait until pendulum is close to the upright position
- 3th next to upright position use feedback linearization controller



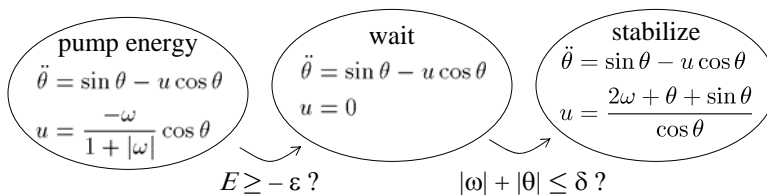
Example #4: Inverted pendulum swing-up



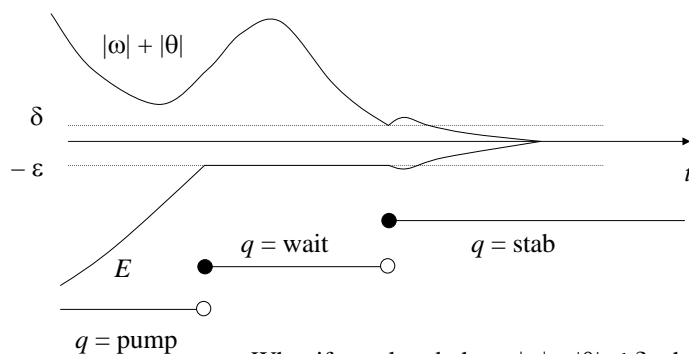
$\mathcal{Q} = \{ \text{pump, wait, stab} \}$ $\mathbb{R}^2 = \text{continuous state-space}$

$$f(q, x) = \dots \quad \Phi(q, x) = \begin{cases} (\text{pump}, x) & q = \text{pump}, E < -\epsilon \\ (\text{wait}, x) & q = \text{pump}, E \geq -\epsilon \\ (\text{wait}, x) & q = \text{wait}, |\omega| + |\theta| > \delta \\ (\text{stab}, x) & q = \text{wait}, |\omega| + |\theta| \leq \delta \\ (\text{stab}, x) & q = \text{stab} \end{cases}$$

Example #4: Inverted pendulum swing-up

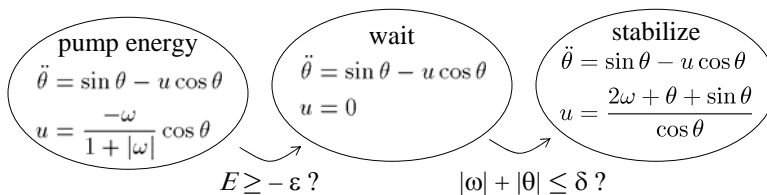


$\mathcal{Q} = \{ \text{pump, wait, stab} \}$ $\mathbb{R}^2 = \text{continuous state-space}$

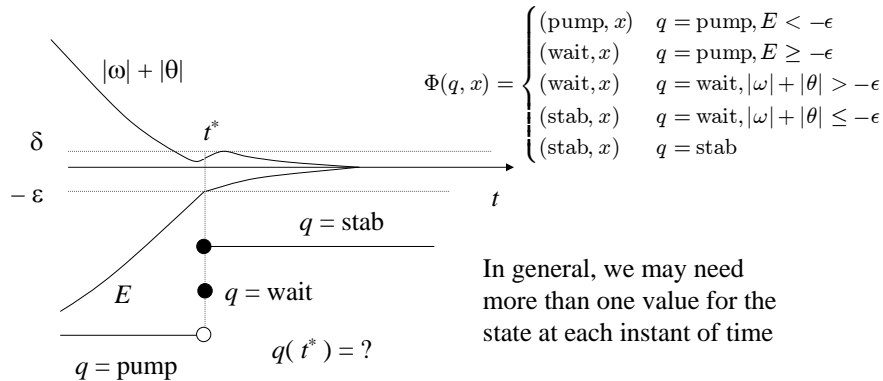


What if we already have $|\omega| + |\theta| \leq \delta$ when E reaches $-\epsilon$?

Example #4: Inverted pendulum swing-up



$Q = \{ \text{pump, wait, stab} \}$ $\mathbb{R}^2 = \text{continuous state-space}$



In general, we may need more than one value for the state at each instant of time

Hybrid signals

Definition: A **hybrid time trajectory** is a (finite or infinite) sequence of closed intervals

$$\tau = \{ [\tau_i, \tau'_i] : \tau_i \leq \tau'_i, \tau'_i = \tau_{i+1}, i=1,2, \dots \}$$

(if τ is finite the last interval may be open on the right)

$\mathcal{T} \equiv$ set of hybrid time trajectories

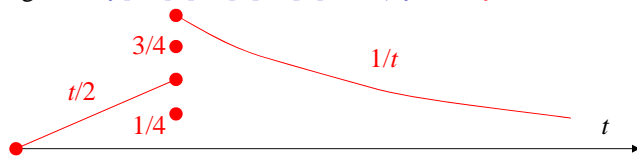
Definition: For a given $\tau = \{ [\tau_i, \tau'_i] : \tau_i \leq \tau'_i, \tau_{i+1} = \tau'_i, i=1,2, \dots \} \in \mathcal{T}$

a **hybrid signal defined on τ** with values on \mathcal{X} is a sequence of functions

$$x = \{ x_i : [\tau_i, \tau'_i] \rightarrow \mathcal{X} \quad i=1,2, \dots \}$$

$x : \tau \rightarrow \mathcal{X} \equiv$ hybrid signal defined on τ with values on \mathcal{X}

E.g., $\tau := \{ [0,1], [1,1], [1,1], [1,+\infty) \}$, $x := \{ t/2, 1/4, 3/4, 1/t \}$

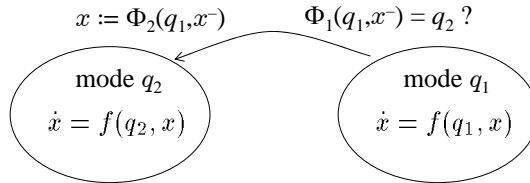


$[0,1]$ $[1,1]$
 $[1,1]$ $[1,+\infty)$

A hybrid signal can take multiple values for the same time-instant

Execution of a hybrid automaton

$$\dot{x} = f(q, x) \quad (q, x) = \Phi(q, x^-) \quad q \in \mathcal{Q}, x \in \mathbb{R}^n$$

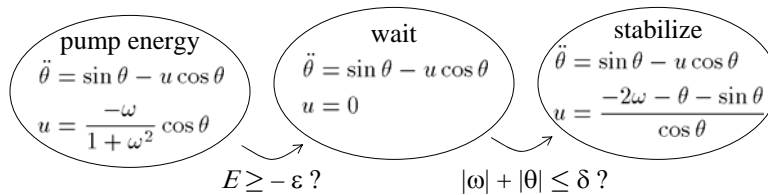


Definition: An **execution** of the hybrid automaton is a pair of hybrid signals
 $x : \tau \rightarrow \mathbb{R}^n \quad q : \tau \rightarrow \mathcal{Q} \quad \tau = \{ [\tau_i, \tau'_i] : i=1, 2, \dots \} \in \mathcal{T}$
 such that

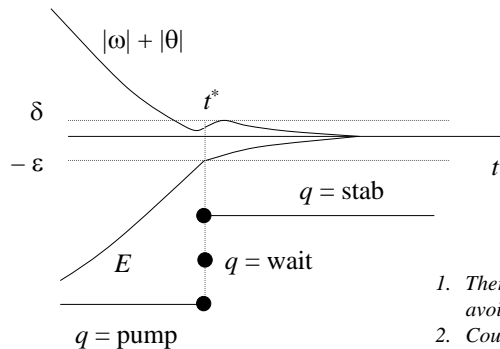
1. on any $[\tau_i, \tau'_i] \in \tau$, q_i is constant and continuous evolution

$$x_i(t) = x_i(\tau_i) + \int_{\tau_i}^t f(q_i(\tau_i), x_i(\tau)) d\tau \quad \forall t \in [\tau_i, \tau'_i]$$
2. $(q(\tau_{i+1}), x(\tau_{i+1})) = \Phi(q(\tau'_i), x(\tau'_i))$ discrete transitions

Example #4: Inverted pendulum swing-up



$\mathcal{Q} = \{ \text{pump, wait, stab} \}$ $\mathbb{R}^2 = \text{continuous state-space}$



$$\tau := \{ [0, t^*], [t^*, t^*], [t^*, +\infty) \}$$

$$q := \{ \text{pump, wait, stab} \}$$

$$x := \{ \dots, \dots, \dots \}$$

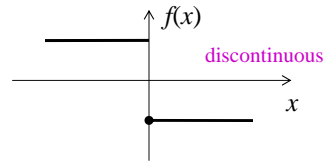
1. There are other concepts of solution that also avoid this problem [Teel 2005]
2. Could one "fix" this hybrid system to still work with the usual notion of solution?

What can go wrong?

1. Problems in the continuous evolution :
 - existence
 - uniqueness
 - finite escape
2. Problems in the hybrid execution:
 - Chattering
 - Zeno
3. Non-continuous dependency on initial conditions

Existence

$$\dot{x} = f(x) = \begin{cases} -1 & x \geq 0 \\ 1 & x < 0 \end{cases}$$



There is no solution to this differential equation that starts with $x(0) = 0$

$$x(t) = \int_0^t f(x(\tau)) d\tau \quad \forall t \geq 0$$

Why? on any interval $[0, \varepsilon)$ x cannot: remain zero, become positive, or become negative.

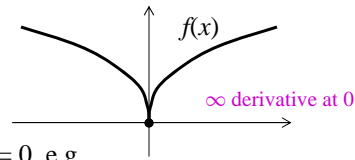
($x = 0$ would make some sense)

Theorem [Existence of solution]

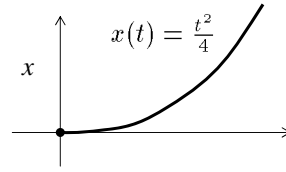
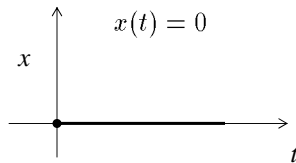
If $f: \mathbb{R}^n \rightarrow \mathbb{R}^n$ is **continuous**, then $\forall x_0 \in \mathbb{R}^n$ there exists at least one solution with $x(0) = x_0$, defined on some interval $[0, \varepsilon)$

Uniqueness

$$\dot{x} = f(x) = \sqrt{|x|}$$



There are multiple solutions that start at $x(0) = 0$, e.g.,



Definitions: A function $f: \mathbb{R}^n \rightarrow \mathbb{R}^n$ is **Lipschitz continuous** if in any bounded subset of \mathcal{S} of \mathbb{R}^n there exists a constant c such that

$$\|f(x) - f(y)\| \leq c\|x - y\| \quad \forall x, y \in \mathcal{S}$$

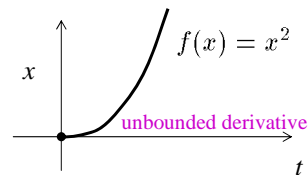
(f is differentiable almost everywhere and the derivative is bounded on any bounded set)

Theorem [Uniqueness of solution]

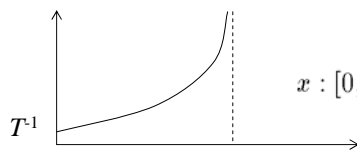
If $f: \mathbb{R}^n \rightarrow \mathbb{R}^n$ is **Lipschitz continuous**, then $\forall x_0 \in \mathbb{R}^n$ there a single solution with $x(0) = x_0$, defined on some interval $[0, \varepsilon)$

Finite escape time

$$\dot{x} = x^2$$



Any solution that does not start at $x(0) = 0$ is of the form



$$x: [0, T) \rightarrow \mathbb{R} \quad x(t) = \frac{1}{T-t} \quad t \in [0, T)$$

T finite escape \equiv solution x tends to ∞ in finite time

Definitions: A function $f: \mathbb{R}^n \rightarrow \mathbb{R}^n$ is **globally Lipschitz continuous** if there exists a constant c such that

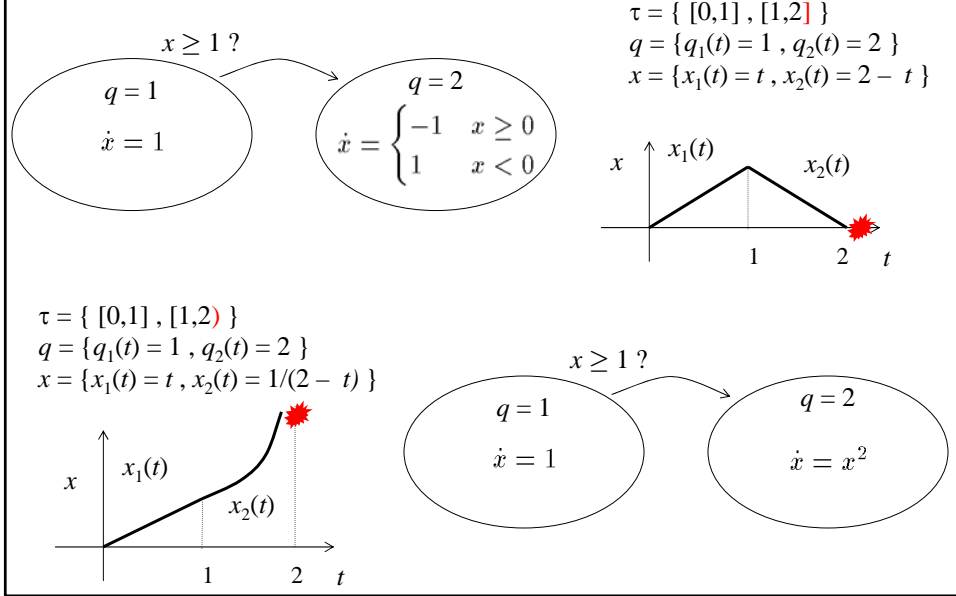
$$\|f(x) - f(y)\| \leq c\|x - y\| \quad \forall x, y \in \mathbb{R}^n$$

f grows no faster than linearly

Theorem [Uniqueness of solution]

If $f: \mathbb{R}^n \rightarrow \mathbb{R}^n$ is **globally Lipschitz continuous**, then $\forall x_0 \in \mathbb{R}^n$ there a single solution with $x(0) = x_0$, defined on $[0, \infty)$

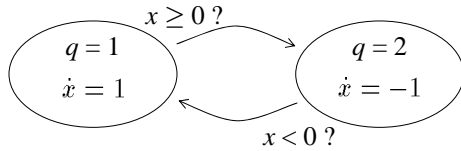
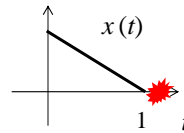
Degenerate executions due to problems in the continuous evolution



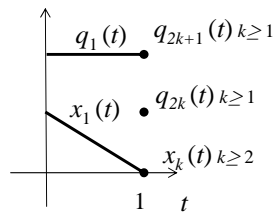
Chattering

$$\dot{x} = \begin{cases} -1 & x \geq 0 \\ 1 & x < 0 \end{cases}$$

There is no solution past $x = 0$



but there is an execution



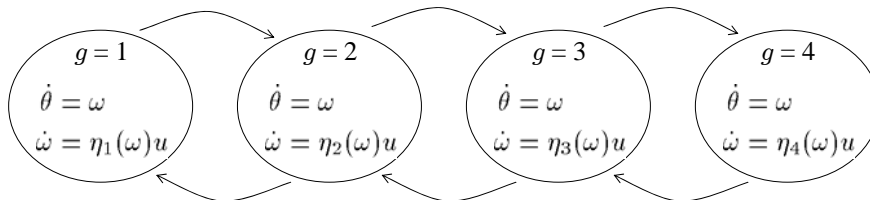
$\tau = \{ [0,1], [1], [1], [1], \dots \}$
 $q = \{ q_1(t) = 2, q_2(t) = 1, q_1(t) = 2, \dots \}$
 $x = \{ x_1(t) = 1 - t, x_2(t) = 0, x_2(t) = 0, \dots \}$

Chattering execution $\equiv \tau$ is infinite but after some time, all intervals are singletons

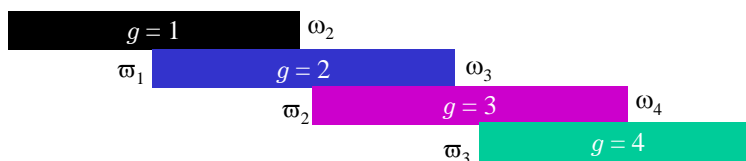
Example #3: Semi-automatic transmission

$v(t) \in \{ \text{up, down, keep} \} \equiv$ drivers input (discrete)

$v = \text{up}$ or $\omega \geq \omega_2$? $v = \text{up}$ or $\omega \geq \omega_3$? $v = \text{up}$ or $\omega \geq \omega_4$?

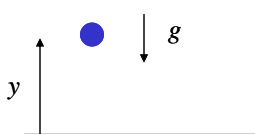


$v = \text{down}$ or $\omega \leq \varpi_1$? $v = \text{down}$ or $\omega \leq \varpi_2$? $v = \text{down}$ or $\omega \leq \varpi_3$?



If the driver sets $v(t) = \text{up} \forall t \geq t^*$ and $\omega(t^*) \leq \varpi_1$ one gets chattering. *For ever?*

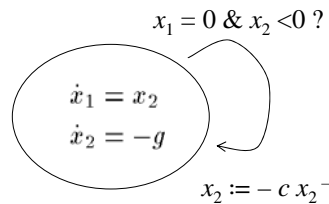
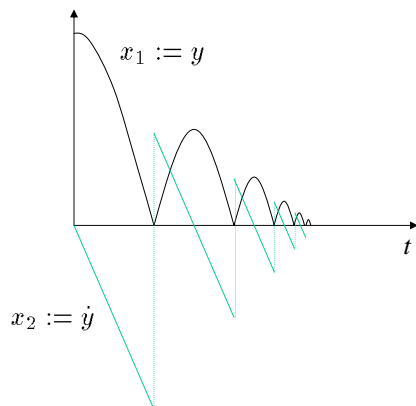
Example #1: Bouncing ball (Zeno execution)

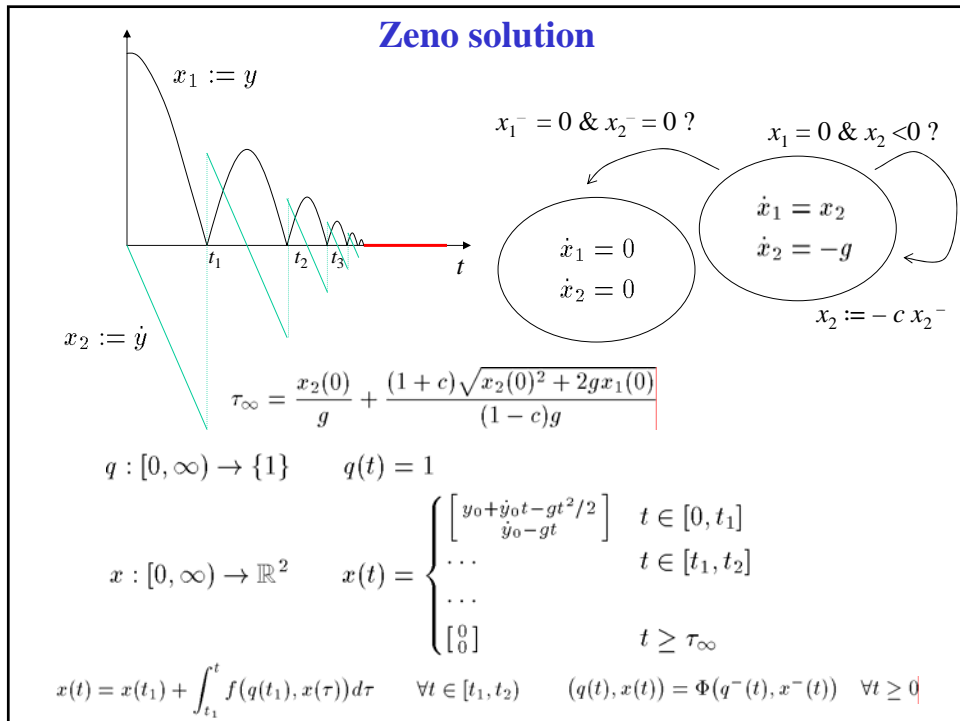
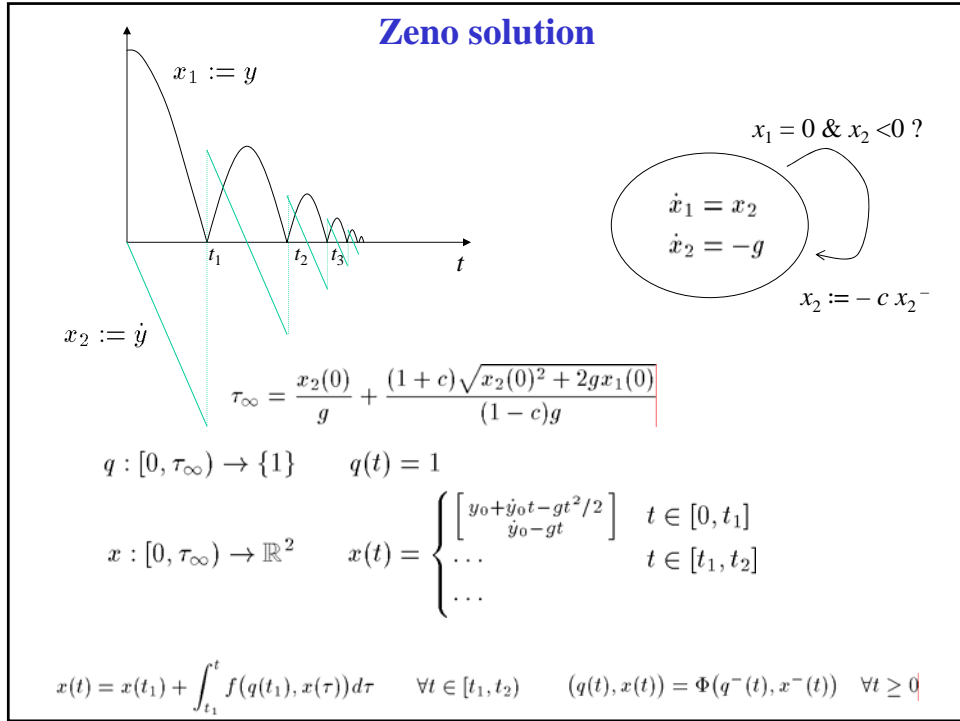


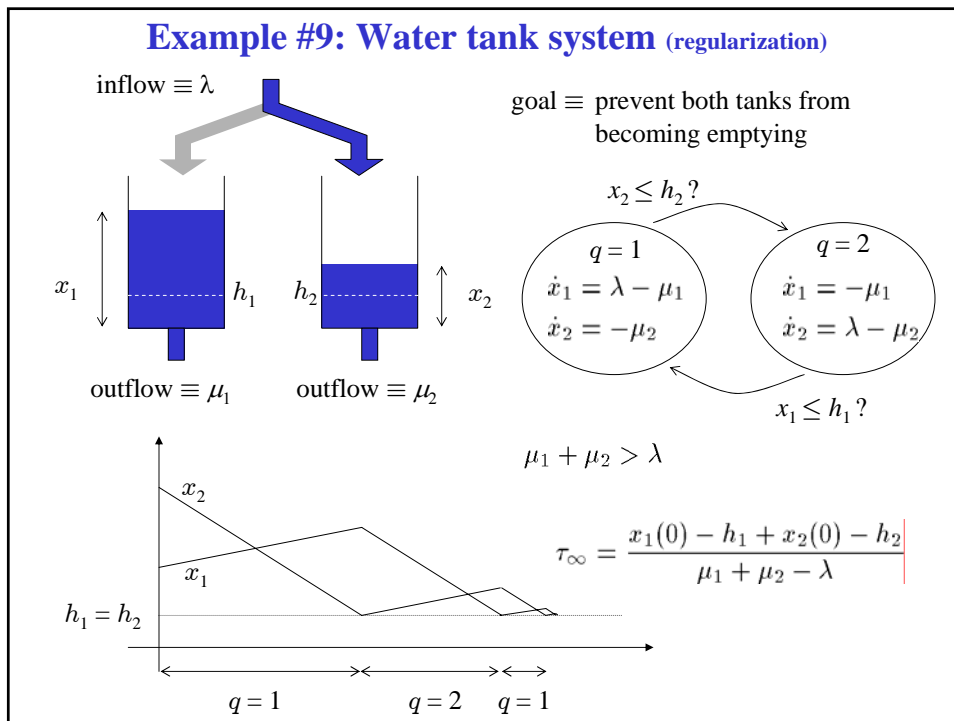
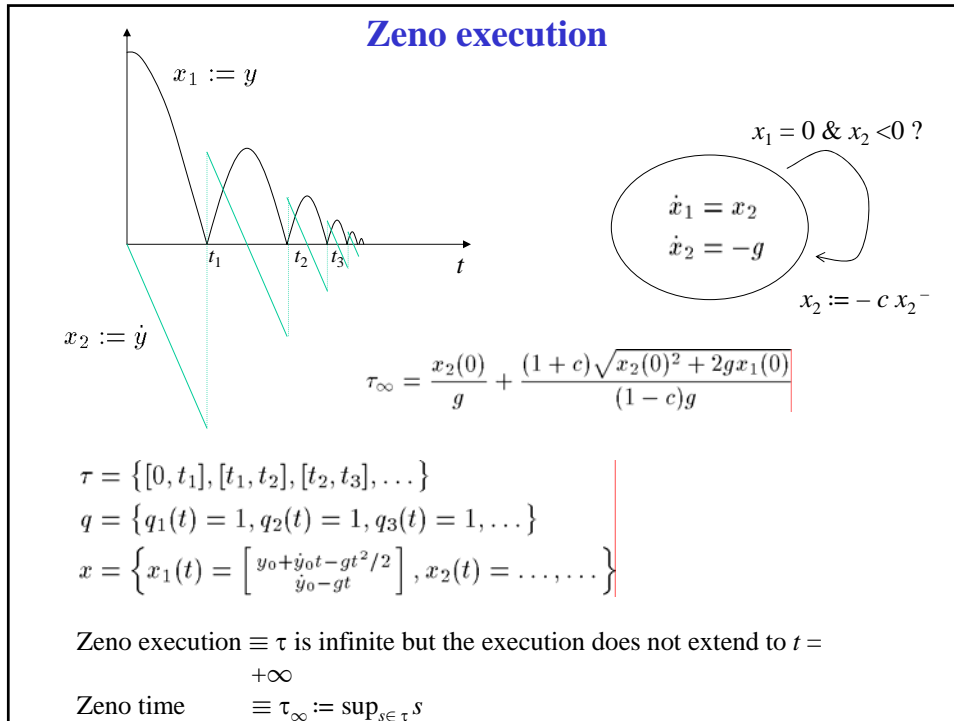
Free fall $\equiv \ddot{y} = -g$

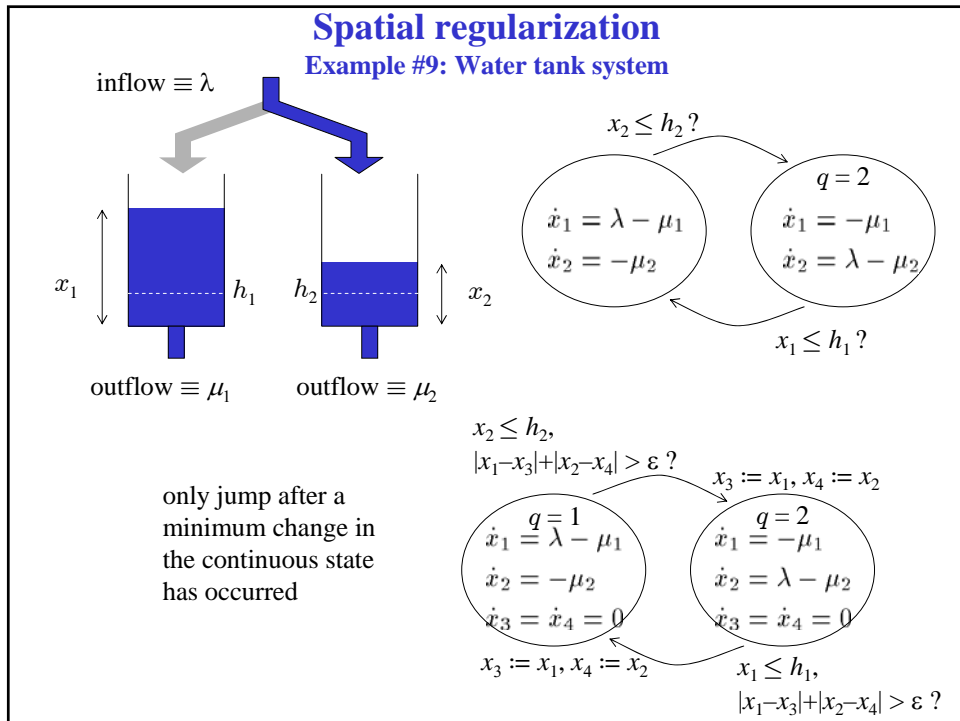
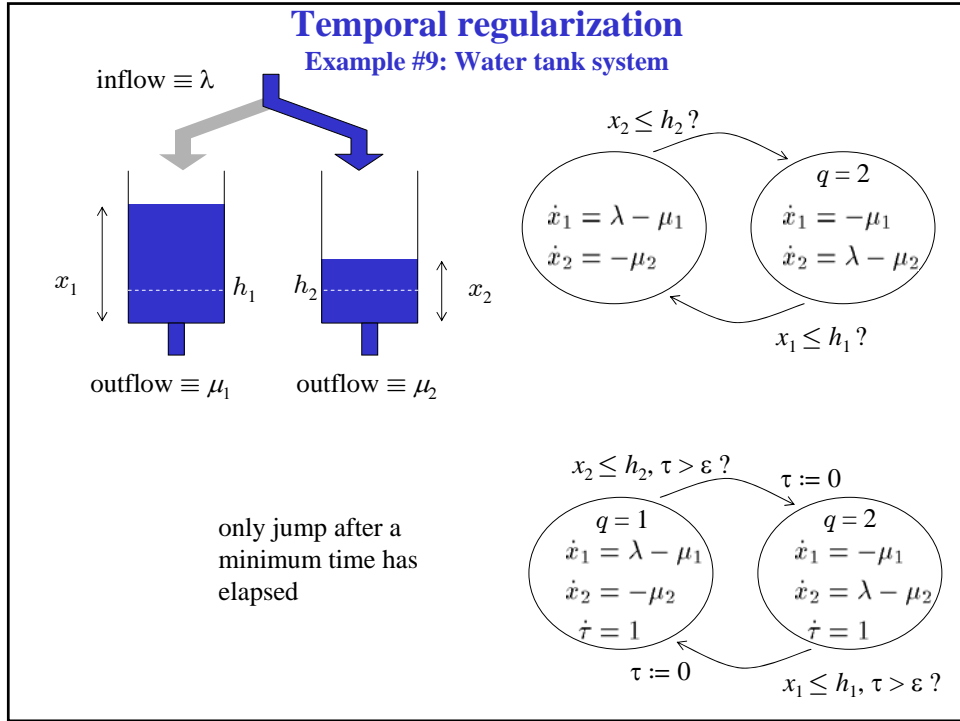
Collision $\equiv \begin{cases} y(t) = y^-(t) = 0 \\ \dot{y}(t) = -c\dot{y}^-(t) \end{cases}$

$c \in [0,1) \equiv$ energy absorbed at impact









Continuity with respect to initial conditions

$$\dot{x} = f(x)$$

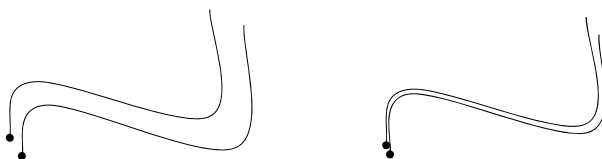
Theorem [Uniqueness & continuity of solution]

If $f: \mathbb{R}^n \rightarrow \mathbb{R}^n$ is **Lipschitz continuous**, then $\forall x_0 \in \mathbb{R}^n$ there a single solution with $x(0) = x_0$, defined on some interval $[0, \varepsilon)$

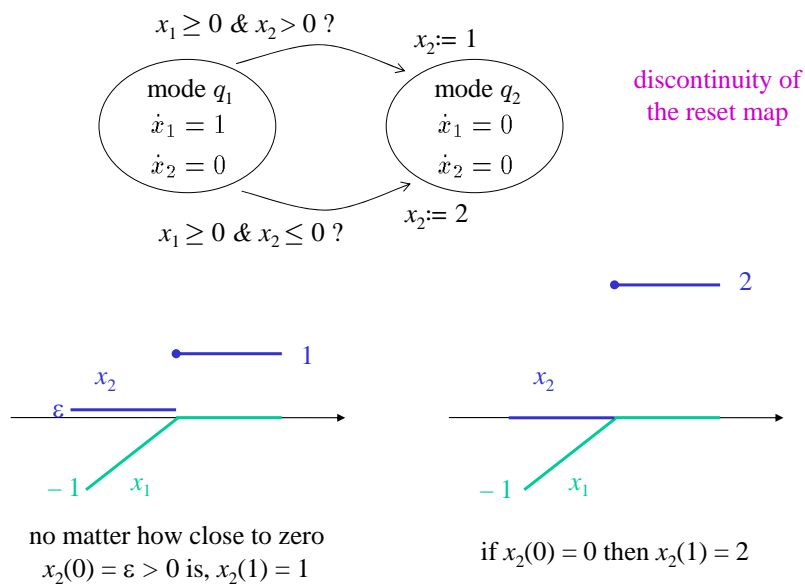
Moreover, given any $T < \infty$, and two solutions x_1, x_2 that exist on $[0, T]$:

$$\forall \varepsilon > 0 \quad \exists \delta > 0 : \|x_1(0) - x_2(0)\| \leq \delta \Rightarrow \|x_1(t) - x_2(t)\| \leq \varepsilon \quad \forall t \in [0, T]$$

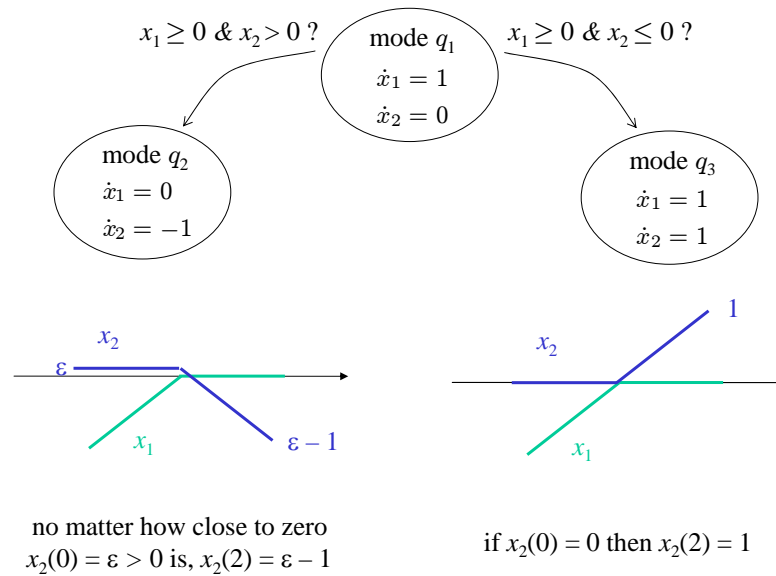
value of the solution on the interval $[0, T]$ is continuous with respect to the initial conditions



Discontinuity with respect to initial conditions



Discontinuity with respect to initial conditions



problem arises from discontinuity of the transition function

Next class...

1. Numerical simulation of hybrid automata

- simulations of ODEs
- zero-crossing detection

2. Simulators

- Simulink
- Stateflow
- SHIFT
- Modelica

Follow-up homework

- Find conditions for the existence of solution to a hybrid system