Lecture #7
Stability and convergence of ODEs

Lyapunov stability of ODEs
• epsilon-delta and beta-function definitions
• Lyapunov’s stability theorem
• LaSalle’s invariance principle
• Stability of linear systems
Properties of hybrid systems

\[ X_{\text{sig}} \equiv \text{set of all piecewise continuous signals } x: [0, T) \to \mathbb{R}^n, \ T \in (0, \infty) \]
\[ Q_{\text{sig}} \equiv \text{set of all piecewise constant signals } q: [0, T) \to \mathbb{Q}, \ T \in (0, \infty) \]

**Sequence property** \( \equiv p: Q_{\text{sig}} \times X_{\text{sig}} \to \{\text{false}, \text{true}\} \)

E.g.,

\[
p(q, x) = \begin{cases} 
\text{true} & q(t) \in \{1, 3\}, \ x(t) \geq x(t + 3), \ \forall t \\
\text{false} & \text{otherwise}
\end{cases}
\]

A pair of signals \((q, x) \in Q_{\text{sig}} \times X_{\text{sig}}\) *satisfies* \(p\) if \(p(q, x) = \text{true}\)

A hybrid automaton \(H\) *satisfies* \(p\) (write \(H \models p\)) if \(p(q, x) = \text{true}, \ \forall (q, x) \) of \(H\)

“**ensemble properties**” \(\equiv\) property of the whole family of solutions

(cannot be checked just by looking at isolated solutions)

E.g., continuity with respect to initial conditions…

Lyapunov stability (ODEs)

\[
\dot{x} = f(x) \quad x \in \mathbb{R}^n
\]

**equilibrium point** \(\equiv x_{eq} \in \mathbb{R}^n\) for which \(f(x_{eq}) = 0\)

thus \(x(t) = x_{eq} \ \forall \ t \geq 0\) is a solution to the ODE

E.g., pendulum equation

\[
\dot{x}_1 = x_2 \\
\dot{x}_2 = -\frac{g}{l} \sin x_1 - \frac{k}{m} x_2
\]

two equilibrium points:

- \(x_1 = 0, x_2 = 0\) (down)
- \(x_1 = \pi, x_2 = 0\) (up)

\[ k \equiv \text{friction coefficient} \]

\[ x_1 \equiv \theta \quad x_2 \equiv \dot{\theta} \]

[Diagram of pendulum with labels for \(k\), \(l\), \(\theta\), and \(m\).]
Lyapunov stability (ODEs)

\[ \dot{x} = f(x) \quad x \in \mathbb{R}^n \]

**equilibrium point** $\equiv x_{eq} \in \mathbb{R}^n$ for which $f(x_{eq}) = 0$

thus $x(t) = x_{eq} \quad \forall t \geq 0$ is a solution to the ODE

**Definition** ($\epsilon$–$\delta$ definition):
The equilibrium point $x_{eq} \in \mathbb{R}^n$ is (Lyapunov) stable if

\[ \forall \epsilon > 0 \exists \delta > 0 : \|x(t_0) - x_{eq}\| \leq \delta \implies \|x(t) - x_{eq}\| \leq \epsilon \quad \forall t \geq t_0 \geq 0 \]

1. if the solution starts close to $x_{eq}$ it will remain close to it forever
2. $\epsilon$ can be made arbitrarily small by choosing $\delta$ sufficiently small

**Example #1: Pendulum**

\[ \begin{align*}
\dot{x}_1 &= x_2 \\
\dot{x}_2 &= -\frac{g}{l} \sin x_1 - \frac{k}{m} x_2
\end{align*} \]

$k \equiv$ friction coefficient

\[ \begin{align*}
x_1 &\equiv \theta \\
x_2 &\equiv \dot{\theta}
\end{align*} \]
**Lyapunov stability – continuity definition**

\[ \dot{x} = f(x) \quad x \in \mathbb{R}^n \]

\[ \mathcal{X}_{\text{sig}} \equiv \text{set of all piecewise continuous signals taking values in } \mathbb{R}^n \]

Given a signal \( x \in \mathcal{X}_{\text{sig}} \), \( ||x||_{\text{sig}} := \sup_{t \geq 0} ||x(t)|| \)

ODE can be seen as an operator

\[ T : \mathbb{R}^n \rightarrow \mathcal{X}_{\text{sig}} \]

that maps \( x_0 \in \mathbb{R}^n \) into the solution that starts at \( x(0) = x_0 \)

**Definition** (continuity definition):

The equilibrium point \( x_{eq} \in \mathbb{R}^n \) is (Lyapunov) stable if \( T \) is continuous at \( x_{eq} \):

\[ \forall \epsilon > 0 \exists \delta > 0 : ||x_0 - x_{eq}|| \leq \delta \Rightarrow \|T(x_0) - T(x_{eq})\|_{\text{sig}} \leq \epsilon \]

\[ \sup_{t \geq 0} ||x(t) - x_{eq}|| \leq \epsilon \]

**Stability of arbitrary solutions**

\[ \dot{x} = f(x) \quad x \in \mathbb{R}^n \]

\[ \mathcal{X}_{\text{sig}} \equiv \text{set of all piecewise continuous signals taking values in } \mathbb{R}^n \]

Given a signal \( x \in \mathcal{X}_{\text{sig}} \), \( ||x||_{\text{sig}} := \sup_{t \geq 0} ||x(t)|| \)

ODE can be seen as an operator

\[ T : \mathbb{R}^n \rightarrow \mathcal{X}_{\text{sig}} \]

that maps \( x_0 \in \mathbb{R}^n \) into the solution that starts at \( x(0) = x_0 \)

**Definition** (continuity definition):

A solution \( x^* : [0,T) \rightarrow \mathbb{R}^n \) is (Lyapunov) stable if \( T \) is continuous at \( x^*_{0} = x^*(0) \), i.e.,

\[ \forall \epsilon > 0 \exists \delta > 0 : ||x_0 - x^*_{0}|| \leq \delta \Rightarrow \|T(x_0) - T(x^*_{0})\|_{\text{sig}} \leq \epsilon \]

\[ \sup_{t \geq 0} ||x(t) - x^*(t)|| \leq \epsilon \]
Example #2: Van der Pol oscillator

\[ \begin{align*}
\dot{x}_1 &= x_2 \\
\dot{x}_2 &= -x_1 + 5(1-x_1^2)x_2
\end{align*} \]

Stability of arbitrary solutions

E.g., Van der Pol oscillator

\[ \begin{align*}
\dot{x}_1 &= x_2 \\
\dot{x}_2 &= -x_1 + 5(1-x_1^2)x_2
\end{align*} \]
Lyapunov stability

\[ \dot{x} = f(x) \quad x \in \mathbb{R}^n \]

equilibrium point \( \equiv x_{eq} \in \mathbb{R}^n \) for which \( f(x_{eq}) = 0 \)

class \( \mathcal{K} \equiv \) set of functions \( \alpha : [0, \infty) \rightarrow [0, \infty) \) that are
1. continuous
2. strictly increasing
3. \( \alpha(0) = 0 \)

Definition (class \( \mathcal{K} \) function definition):
The equilibrium point \( x_{eq} \in \mathbb{R}^n \) is (Lyapunov) stable if \( \exists \alpha \in \mathcal{K} : \)
\[ ||x(t) - x_{eq}|| \leq \alpha(||x(t_0) - x_{eq}||) \quad \forall t \geq t_0 \geq 0 \]
\[ ||x(t) - x_{eq}|| \leq c \]

the function \( \alpha \) can be constructed directly from the \( \delta(\epsilon) \) in the \( \epsilon-\delta \) (or continuity) definitions

Asymptotic stability

\[ \dot{x} = f(x) \quad x \in \mathbb{R}^n \]

equilibrium point \( \equiv x_{eq} \in \mathbb{R}^n \) for which \( f(x_{eq}) = 0 \)

class \( \mathcal{K} \equiv \) set of functions \( \alpha : [0, \infty) \rightarrow [0, \infty) \) that are
1. continuous
2. strictly increasing
3. \( \alpha(0) = 0 \)

Definition:
The equilibrium point \( x_{eq} \in \mathbb{R}^n \) is (globally) asymptotically stable if
it is Lyapunov stable and for every initial state the solution exists on \([0, \infty)\) and
\( x(t) \to x_{eq} \) as \( t \to \infty \).
Asymptotic stability

\[ \dot{x} = f(x) \quad x \in \mathbb{R}^n \]

equilibrium point \( x_{eq} \in \mathbb{R}^n \) for which \( f(x_{eq}) = 0 \)

class \( \mathcal{KL} \equiv \) set of functions \( \beta: [0, \infty) \times [0, \infty) \rightarrow [0, \infty) \) s.t.
1. for each fixed \( t \), \( \beta(s,t) \in \mathcal{K} \)
2. for each fixed \( s \), \( \beta(s, \cdot) \) is monotone decreasing and \( \beta(s,t) \rightarrow 0 \) as \( t \rightarrow \infty \)

**Definition (class \( \mathcal{KL} \) function definition):**
The equilibrium point \( x_{eq} \in \mathbb{R}^n \) is (globally) asymptotically stable if \( \exists \beta \in \mathcal{KL}: \)
\[
\|x(t) - x_{eq}\| \leq \beta(\|x(t_0) - x_{eq}\|, |t - t_0|) \quad \forall t \geq t_0 \geq 0
\]

We have exponential stability when
\[
\beta(s,t) = c e^{-\lambda t} \quad s
\]

with \( c, \lambda > 0 \)
linear in \( s \) and negative exponential in \( t \)

Example #1: Pendulum

\[
\begin{align*}
\dot{x}_1 &= x_2 \\
\dot{x}_2 &= \frac{g}{l} \sin(x_1) - \frac{k}{m} x_2
\end{align*}
\]

\( k > 0 \) (with friction) \hspace{1cm} \( k = 0 \) (no friction)

\( x_{eq} = (0,0) \)
asymptotically stable

\( x_{eq} = (\pi,0) \)
unstable

\( x_{eq} = (0,0) \)
stable but not asymptotically

\( x_{eq} = (\pi,0) \)
unstable

pend.m
Example #3: Butterfly

Convergence by itself does not imply stability, e.g.,

\[
\begin{align*}
\dot{x}_1 &= x_1^2 - x_2^2 \\
\dot{x}_2 &= 2x_1x_2
\end{align*}
\]

equilibrium point \( \equiv (0,0) \)

all solutions converge to zero but \( x_{eq} = (0,0) \) system is not stable

Lyapunov’s stability theorem

\[
x = f(x) \quad x \in \mathbb{R}^n
\]

Definition (class \( \mathcal{K} \) function definition):
The equilibrium point \( x_{eq} \in \mathbb{R}^n \) is (Lyapunov) stable if \( \exists \alpha \in \mathcal{K} \):

\[
||x(t) - x_{eq}|| \leq \alpha(||x(t_0) - x_{eq}||) \quad \forall \ t \geq t_0 \geq 0, \ ||x(t_0) - x_{eq}|| \leq c
\]

Suppose we could show that \( ||x(t) - x_{eq}|| \) always decreases along solutions to the ODE. Then

\[
||x(t) - x_{eq}|| \leq ||x(t_0) - x_{eq}|| \quad \forall \ t \geq t_0 \geq 0
\]

we could pick \( \alpha(s) = s \Rightarrow \text{Lyapunov stability} \)

We can draw the same conclusion by using other measures of how far the solution is from \( x_{eq} \):

\[
V: \mathbb{R}^n \rightarrow \mathbb{R} \text{ positive definite } \equiv V(x) \geq 0 \quad \forall \ x \in \mathbb{R}^n \text{ with } = 0 \text{ only for } x = 0
\]

\[
V: \mathbb{R}^n \rightarrow \mathbb{R} \text{ radially unbounded } \equiv x \rightarrow \infty \Rightarrow V(x) \rightarrow \infty
\]

\[
V(x - x_{eq}) = \begin{cases} 
0 & x = x_{eq} \\
> 0 & x \neq x_{eq} \\
\rightarrow \infty & ||x - x_{eq}|| \rightarrow \infty
\end{cases}
\]

provides a measure of
how far \( x \) is from \( x_{eq} \)
(not necessarily a metric–may not satisfy triangular inequality)
Lyapunov’s stability theorem

\[ \dot{x} = f(x) \quad x \in \mathbb{R}^n \]

\( V: \mathbb{R}^n \rightarrow \mathbb{R} \) positive definite \( \equiv V(x) \geq 0 \ \forall \ x \in \mathbb{R}^n \) with \( = 0 \) only for \( x = 0 \)

\[ V(x - x_{eq}) = \begin{cases} 0 & x = x_{eq} \\ > 0 & x \neq x_{eq} \end{cases} \]

provides a measure of how far \( x \) is from \( x_{eq} \) (not necessarily a metric–may not satisfy triangular inequality)

**Q: How to check if \( V(x(t) - x_{eq}) \) decreases along solutions?**

\[
\frac{d}{dt} V(x(t) - x_{eq}) = \frac{\partial V}{\partial x}(x(t) - x_{eq}) \dot{x}(t) \\
= \frac{\partial V}{\partial x}(x(t) - x_{eq}) f(x(t))
\]

**A: \( V(x(t) - x_{eq}) \) will decrease if**

\[ \frac{\partial V}{\partial x}(z - x_{eq}) f(z) \leq 0 \quad \forall z \in \mathbb{R}^n \]

can be computed without actually computing \( x(t) \) (i.e., solving the ODE)

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Lyapunov’s stability theorem

\[ \dot{x} = f(x) \quad x \in \mathbb{R}^n \]

**Definition** (class \( \mathcal{K} \) function definition):

The equilibrium point \( x_{eq} \in \mathbb{R}^n \) is \textbf{(Lyapunov) stable} if \( \exists \ \alpha \in \mathcal{K} \):

\[ \|x(t) - x_{eq}\| \leq \alpha(\|x(t_0) - x_{eq}\|) \ \forall \ t \geq t_0 \geq 0, \|x(t_0) - x_{eq}\| \leq c \]

**Theorem** (Lyapunov):

Suppose there exists a continuously differentiable, positive definite function \( V: \mathbb{R}^n \rightarrow \mathbb{R} \) such that

\[ \frac{\partial V}{\partial x}(z - x_{eq}) f(z) \leq 0 \quad \forall z \in \mathbb{R}^n \]

Then \( x_{eq} \) is a \textbf{Lyapunov stable equilibrium}.

Why?

\( V \) non increasing \( \Rightarrow V(x(t) - x_{eq}) \leq V(x(t_0) - x_{eq}) \ \forall \ t \geq t_0 \)

Thus, by making \( x(t_0) - x_{eq} \) small we can make \( V(x(t) - x_{eq}) \) arbitrarily small \( \forall \ t \geq t_0 \)

So, by making \( x(t_0) - x_{eq} \) small we can make \( x(t) - x_{eq} \) arbitrarily small \( \forall \ t \geq t_0 \)

(we can actually compute \( \alpha \) from \( V \) explicitly and take \( c = +\infty \)).
Example #1: Pendulum

\[
\begin{align*}
\dot{x}_1 &= x_2 \\
\dot{x}_2 &= -\frac{g}{l} \sin x_1 - \frac{k}{m} x_2 \\
k &\equiv \text{friction coefficient} \\
x_1 &\equiv \theta \\
x_2 &\equiv \dot{\theta}
\end{align*}
\]

positive definite because \( V(x) = 0 \) only for \( x_1 = 2k\pi, k \in \mathbb{Z} \) & \( x_2 = 0 \) (all these points are really the same because \( x_1 \) is an angle)

For \( x_{eq} = (0,0) \)

\[
\frac{\partial V}{\partial x}(x - x_{eq}) f(x) = \left[ \begin{array}{cc} \frac{g}{l} \sin x_1 & x_2 \\ -\frac{g}{l} \sin x_1 & -\frac{k}{m} \end{array} \right] \left[ \begin{array}{c} x_2 \\ -\frac{g}{l} \sin x_1 - \frac{k}{m} x_2 \end{array} \right]
\]

\[= -\frac{k}{m} x_2^2 \leq 0 \quad \forall x \in \mathbb{R}^n\]

Therefore \( x_{eq} = (0,0) \) is Lyapunov stable

Example #1: Pendulum

\[
\begin{align*}
\dot{x}_1 &= x_2 \\
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positive definite because \( V(x) = 0 \) only for \( x_1 = 2k\pi, k \in \mathbb{Z} \) & \( x_2 = 0 \) (all these points are really the same because \( x_1 \) is an angle)

For \( x_{eq} = (\pi,0) \)

\[
\frac{\partial V}{\partial x}(x - x_{eq}) f(x) = \left[ \begin{array}{cc} \frac{g}{l} \sin(x_1 - \pi) & x_2 \\ -\frac{g}{l} \sin x_1 & -\frac{k}{m} \end{array} \right] \left[ \begin{array}{c} x_2 \\ -\frac{g}{l} \sin x_1 - \frac{k}{m} x_2 \end{array} \right]
\]

\[= -\frac{2g}{l} x_2 \sin x_1 - \frac{k}{m} x_2^2 \leq 0 \]

Cannot conclude that \( x_{eq} = (\pi,0) \) is Lyapunov stable (in fact it is not!)
Lyapunov’s stability theorem

\[ \dot{x} = f(x) \quad x \in \mathbb{R}^n \]

**Definition** (class \(\mathcal{K}\) function definition):
The equilibrium point \(x_{eq} \in \mathbb{R}^n\) is **(Lyapunov) stable** if \(\exists \alpha \in \mathcal{K}:
\|x(t) - x_{eq}\| \leq \alpha(\|x(t_0) - x_{eq}\|) \quad \forall t \geq t_0 \geq 0, \|x(t_0) - x_{eq}\| \leq c)\)

**Theorem** (Lyapunov):
Suppose there exists a continuously differentiable, positive definite, radially unbounded function \(V: \mathbb{R}^n \rightarrow \mathbb{R}\) such that
\[ \frac{\partial V}{\partial x}(z - x_{eq})f(z) \leq 0 \quad \forall z \in \mathbb{R}^n \]
Then \(x_{eq}\) is a Lyapunov stable equilibrium and the solution always exists globally. Moreover, if \(f = 0\) only for \(z = x_{eq}\) then \(x_{eq}\) is a (globally) asymptotically stable equilibrium.

Why?
\(V\) can only stop decreasing when \(x(t)\) reaches \(x_{eq}\)
but \(V\) must stop decreasing because it cannot become negative
Thus, \(x(t)\) must converge to \(x_{eq}\)

Lyapunov’s stability theorem

\[ \dot{x} = f(x) \quad x \in \mathbb{R}^n \]

**Definition** (class \(\mathcal{K}\) function definition):
The equilibrium point \(x_{eq} \in \mathbb{R}^n\) is **(Lyapunov) stable** if \(\exists \alpha \in \mathcal{K}:
\|x(t) - x_{eq}\| \leq \alpha(\|x(t_0) - x_{eq}\|) \quad \forall t \geq t_0 \geq 0, \|x(t_0) - x_{eq}\| \leq c)\)

**Theorem** (Lyapunov):
Suppose there exists a continuously differentiable, positive definite, radially unbounded function \(V: \mathbb{R}^n \rightarrow \mathbb{R}\) such that
\[ \frac{\partial V}{\partial x}(z - x_{eq})f(z) \leq 0 \quad \forall z \in \mathbb{R}^n \]
Then \(x_{eq}\) is a Lyapunov stable equilibrium and the solution always exists globally. Moreover, if \(f = 0\) only for \(z = x_{eq}\) then \(x_{eq}\) is a (globally) asymptotically stable equilibrium.

What if \(f = 0\) for other \(z\) then \(x_{eq}\)? Can we still claim some form of convergence?
Example #1: Pendulum

\[ \dot{x}_1 = x_2 \]
\[ \dot{x}_2 = -\frac{q}{l} \sin x_1 - \frac{k}{m} x_2 \]

\( k \equiv \text{friction coefficient} \)

\( x_1 \equiv \theta \]
\( x_2 \equiv \dot{\theta} \]

\( V(x) := \frac{q}{l} (1 - \cos x_1) + \frac{x_2^2}{2} \geq 0 \)

For \( x_{eq} = (0,0) \)

\[ \frac{\partial V}{\partial x}(x - x_{eq}) f(x) = \begin{bmatrix} \frac{q}{l} \sin x_1 & x_2 \\ -\frac{q}{l} \sin x_1 - \frac{k}{m} x_2 \end{bmatrix} \]
\[ = -\frac{k}{m} x_2 \leq 0 \quad \forall x \in \mathbb{R}^n \]

not strict for \((x_1 \neq 0, x_2=0 !)\)

LaSalle’s Invariance Principle

\[ \dot{x} = f(x) \quad x \in \mathbb{R}^n \]

\( M \in \mathbb{R}^n \) is an invariant set \( \equiv x(t_0) \in M \Rightarrow x(t) \in M \forall t \geq t_0 \)

(in the context of hybrid systems: Reach(M) \( \subset M \)

**Theorem** (LaSalle Invariance Principle):

Suppose there exists a continuously differentiable, positive definite, radially unbounded function \( V: \mathbb{R}^n \rightarrow \mathbb{R} \) such that

\[ \frac{\partial V}{\partial x}(z - x_{eq}) f(z) \leq W(z) \leq 0 \quad \forall z \in \mathbb{R}^n \]

Then \( x_{eq} \) is a Lyapunov stable equilibrium and the solution always exists globally. Moreover, \( x(t) \) converges to the largest invariant set \( M \) contained in

\( E := \{ z \in \mathbb{R}^n : W(z) = 0 \} \)

Note that:

1. When \( W(z) = 0 \) only for \( z = x_{eq} \) then \( E = \{ x_{eq} \} \).
   
   Since \( M \subset E, M = \{ x_{eq} \} \) and therefore \( x(t) \rightarrow x_{eq} \Rightarrow \text{asympt. stability} \)

2. Even when \( E \) is larger then \( \{ x_{eq} \} \) we often have \( M = \{ x_{eq} \} \)
   
   and can conclude asymptotic stability.
Example #1: Pendulum

\[ \begin{align*}
\dot{x}_1 &= x_2 \\
\dot{x}_2 &= -\frac{g}{l} \sin x_1 - \frac{k}{m} x_2 \\
k &\equiv \text{friction coefficient} \\
x_1 &\equiv \theta \\
x_2 &\equiv \dot{\theta}
\end{align*} \]

For \( x_{eq} = (0,0) \)

\[ V(x) := \frac{g}{l} (1 - \cos x_1) + \frac{x^2_2}{2} \geq 0 \]

For \( x_{eq} = (0,0) \)

\[ \frac{\partial V}{\partial x} (x - x_{eq}) f(x) = -\frac{k}{m} x^2_2 \leq 0 \quad \forall x \in \mathbb{R}^n \]

\[ E := \{ (x_1, x_2): x_1 \in \mathbb{R}, x_2 = 0 \} \]

Inside \( E \), the ODE becomes

\[ \begin{align*}
\dot{x}_1 &= x_2 = 0 \\
0 &= \dot{x}_2 = -\frac{g}{l} \sin x_1 - \frac{k}{m} x_2 = -\frac{g}{l} \sin x_1
\end{align*} \]

Therefore \( x \) converges to \( M := \{ (x_1, x_2): x_1 = k \pi, x_2 = 0 \} \)

However, the equilibrium point \( x_{eq} = (0,0) \) is not (globally) asymptotically stable because if the system starts, e.g., at \((\pi, 0)\) it remains there forever. 

\[ \text{pend.m} \]

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Linear systems

\[ \dot{x} = Ax \quad x \in \mathbb{R}^n, A \in \mathbb{R}^{n \times n} \]

Solution to a linear ODE:

\[ x(t) = e^{At - t_0} x(t_0) \quad t \geq t_0 \quad e^{At} := \sum_{k=0}^{\infty} \frac{t^k}{k!} A^k \]

**Theorem:** The origin \( x_{eq} = 0 \) is an equilibrium point. It is

1. **Lyapunov stable** if and only if all eigenvalues of \( A \) have negative or zero real parts and for each eigenvalue with zero real part there is an independent eigenvector.
2. **Asymptotically stable** if and only if all eigenvalues of \( A \) have negative real parts. In this case the origin is actually exponentially stable.
**Linear systems**

Solution to a linear ODE:

\[
x(t) = e^{A(t-t_0)}x(t_0) \quad t \geq t_0
\]

\[
e^{At} := \sum_{k=0}^{\infty} \frac{t^k}{k!} A^k
\]

**Theorem:** The origin \( x_{eq} = 0 \) is an equilibrium point. It is asymptotically stable if and only if for every positive symmetric definite matrix \( Q \) the equation

\[
A^* P + PA = -Q
\]

has a unique solutions \( P \) that is symmetric and positive definite

**Lyapunov equation**

Recall: given a symmetric matrix \( P \)

- \( P \) is positive definite \( \equiv \) all eigenvalues are positive
  - \( P \) positive definite \( \Rightarrow \) \( x^* P x > 0 \) \( \forall x \neq 0 \)
- \( P \) is positive semi-definite \( \equiv \) all eigenvalues are positive or zero
  - \( P \) positive semi-definite \( \Rightarrow \) \( x^* P x \geq 0 \) \( \forall x \)
**Lyapunov equation**

\[ \dot{x} = Ax \quad x \in \mathbb{R}^n, A \in \mathbb{R}^{n \times n} \]

Solution to a linear ODE:

\[ x(t) = e^{A(t-t_0)}x(t_0) \quad t \geq t_0 \quad e^{At} := \sum_{k=0}^{\infty} \frac{t^k}{k!} A^k \]

**Theorem:** The origin \( x_{eq} = 0 \) is an equilibrium point. It is asymptotically stable if and only if for every positive symmetric definite matrix \( Q \) the equation

\[ A^T P + PA = -Q \]

has a unique solutions \( P \) that is symmetric and positive definite

\[ Lyapunov \ equation \]

Why?

1. asympt. stable \( \Rightarrow P \) exists and is unique (constructive proof)

\[
P := \lim_{T \to \infty} \int_0^T e^{A^T \tau} Q e^{A \tau} d\tau = \lim_{T \to \infty} \int_0^T e^{A(T-s)} Q e^{A(T-s)} ds
\]

\[ A \ is \ asympt. \ stable \Rightarrow e^{At} \ decreases \ to \ zero \ exponentially \ fast \Rightarrow P \ is \ well \ defined \ (limit \ exists \ and \ is \ finite) \]

\[ \text{change of integration variable } \tau = T - s \]

2. \( P \) exists \( \Rightarrow \) asymp. stable

Consider the quadratic Lyapunov equation: \( V(x) = x' P x \)

\( V \) is positive definite & radially unbounded because \( P \) is positive definite

\( V \) is continuously differentiable:

\[
\frac{\partial V}{\partial x}(x) = 2x' P
\]

\[
\frac{\partial V}{\partial x}(x) Ax = x'(A'P + PA)x = -x'Qx < 0 \quad \forall x \neq 0
\]

thus system is asymptotically stable by Lyapunov Theorem
Next lecture…

Lyapunov stability of hybrid systems