

Hybrid Control and Switched Systems

Lecture #8 Stability and convergence of hybrid systems (topological view)

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Summary

Lyapunov stability of hybrid systems

Properties of hybrid systems

$\mathcal{X}_{\text{sig}} \equiv$ set of all piecewise continuous signals $x: [0, T) \rightarrow \mathbb{R}^n, T \in (0, \infty]$

$\mathcal{Q}_{\text{sig}} \equiv$ set of all piecewise constant signals $q: [0, T) \rightarrow \mathcal{Q}, T \in (0, \infty]$

Sequence property $\equiv p : \mathcal{Q}_{\text{sig}} \times \mathcal{X}_{\text{sig}} \rightarrow \{\text{false}, \text{true}\}$

E.g.,

$$p(q, x) = \begin{cases} \text{true} & q(t) \in \{1, 3\}, x(t) \geq x(t+3), \forall t \\ \text{false} & \text{otherwise} \end{cases}$$

A pair of signals $(q, x) \in \mathcal{Q}_{\text{sig}} \times \mathcal{X}_{\text{sig}}$ **satisfies** p if $p(q, x) = \text{true}$

A hybrid automaton H **satisfies** p (write $H \models p$) if

$$p(q, x) = \text{true}, \quad \text{for every solution } (q, x) \text{ of } H$$

“**ensemble properties**” \equiv property of the whole family of solutions
(cannot be checked just by looking at isolated solutions)
e.g., continuity with respect to initial conditions...

Lyapunov stability of ODEs (recall)

$$\dot{x} = f(x) \quad x \in \mathbb{R}^n$$

$\mathcal{X}_{\text{sig}} \equiv$ set of all piecewise continuous signals taking values in \mathbb{R}^n

Given a signal $x \in \mathcal{X}_{\text{sig}}, \|x\|_{\text{sig}} := \sup_{t \geq 0} \|x(t)\|$ signal norm

ODE can be seen as an operator

$$T : \mathbb{R}^n \rightarrow \mathcal{X}_{\text{sig}}$$

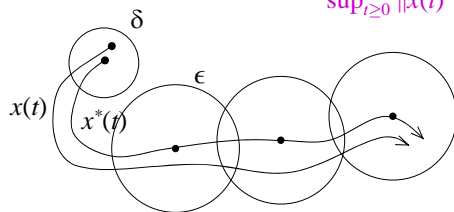
that maps $x_0 \in \mathbb{R}^n$ into the solution that starts at $x(0) = x_0$

Definition (continuity definition):

A solution x^* is **(Lyapunov) stable** if T is continuous at $x_0^* := x^*(0)$, i.e.,

$$\forall \epsilon > 0 \exists \delta > 0 : \|x_0 - x_0^*\| \leq \delta \Rightarrow \|T(x_0) - T(x_0^*)\|_{\text{sig}} \leq \epsilon$$

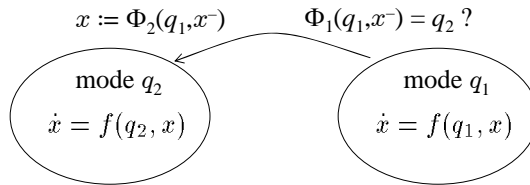
$$\underbrace{\sup_{t \geq 0} \|x(t) - x^*(t)\|}_{\text{signal norm}} \leq \epsilon$$



pend.m

Lyapunov stability of hybrid systems

$$\dot{x} = f(q, x) \quad (q, x) = \Phi(q, x^-) \quad q \in \mathcal{Q}, x \in \mathbb{R}^n$$



$\mathcal{X}_{\text{sig}} \equiv$ set of all piecewise continuous signals $x: [0, T] \rightarrow \mathbb{R}^n, T \in (0, \infty]$
 $\mathcal{Q}_{\text{sig}} \equiv$ set of all piecewise constant signals $q: [0, T] \rightarrow \mathcal{Q}, T \in (0, \infty]$

Hybrid automaton can be seen as an operator

$$T: \mathcal{Q} \times \mathbb{R}^n \rightarrow \mathcal{Q}_{\text{sig}} \times \mathcal{X}_{\text{sig}}$$

that maps $(q_0, x_0) \in \mathcal{Q} \times \mathbb{R}^n$ into the solution that starts at $q(0) = q_0, x(0) = x_0$

Definition (continuity definition):

A solution (q^*, x^*) is **(Lyapunov) stable** if T is continuous at $(q^*(0), x^*(0))$.

To make sense of *continuity* we need ways to measure “distances” in $\mathcal{Q} \times \mathbb{R}^n$ and $\mathcal{Q}_{\text{sig}} \times \mathcal{X}_{\text{sig}}$

Lyapunov stability of hybrid systems

$$\dot{x} = f(q, x) \quad (q, x) = \Phi(q, x^-) \quad q \in \mathcal{Q}, x \in \mathbb{R}^n$$

A few possible “metrics”

$$d((q_1, x_1), (q_2, x_2)) := \begin{cases} \|x_1 - x_2\| & q_1 = q_2 \\ +\infty & \text{otherwise} \end{cases} \quad \left. \begin{array}{l} \text{one cares very much} \\ \text{about the discrete} \\ \text{states matching} \end{array} \right\}$$

$$d((q_1, x_1), (q_2, x_2)) := \|x_1 - x_2\| \quad \left. \begin{array}{l} \text{one does not care at all about the} \\ \text{discrete states matching} \end{array} \right\}$$

Definition (continuity definition):

A solution (q^*, x^*) is **(Lyapunov) stable** if T is continuous at $(q_0^*, x_0^*) := (q^*(0), x^*(0))$, i.e.,

$$\forall \epsilon > 0 \exists \delta > 0 : d((q^*(0), x^*(0)), (q(0), x(0))) \leq \delta$$

↓

$$\sup_{t \geq 0} d((q^*(t), x^*(t)), (q(t), x(t))) \leq \epsilon$$

1. If the solution starts close to (q^*, x^*) it will remain close to it forever

2. ϵ can be made arbitrarily small by choosing δ sufficiently small

Note: may actually not be metrics on $\mathcal{Q} \times \mathbb{R}^n$ because one may want “zero-distance” between points. However, still define a topology on $\mathcal{Q} \times \mathbb{R}^n$, which is what is really needed to make sense of continuity...

Topological spaces

Given a set X and a collection \mathcal{T}_X of subsets of X

(X, \mathcal{T}_X) is a **topological space** if

1. $\emptyset, X \in \mathcal{T}_X$
2. $\mathcal{A}, \mathcal{B} \in \mathcal{T}_X \Rightarrow \mathcal{A} \cap \mathcal{B} \in \mathcal{T}_X$
3. $\mathcal{A}_1, \mathcal{A}_2, \dots, \mathcal{A}_n \in \mathcal{T}_X \Rightarrow \bigcup_{i=1}^n \mathcal{A}_i \in \mathcal{T}_X (1 \leq n \leq \infty)$

\mathcal{T} is called a topology and the sets in \mathcal{T} are called **open** and their complements are called **closed**

Intuitively: two elements of X are "arbitrarily close" if for every open set one belongs to, the other also belongs to

Examples:

$Q := \{q_1, q_2, \dots, q_n\}$ (finite)

$\mathcal{T}_Q := \{\emptyset, Q\}$ (**trivial topology** – all points are close to each other)

$\mathcal{T}_Q := \{\emptyset\} \cup \{\text{all subsets of } Q\}$ (**discrete topology** – no two distinct points are close to each other)

$\mathcal{T}_Q := \{\emptyset, \{1\}, \{1,2\}\}$ of $\{1,2\}$

$X := \mathbb{R}^n$

$\mathcal{T}_X := \{(\text{possibly infinite}) \text{ union of all open balls}\}$ (**norm-induced topology**)

open ball $\equiv \{x \in \mathbb{R}^n : \|x - x_0\| < \epsilon\}$

A point is only "arbitrarily close" to itself (Hausdorff space)

How to prove that 2. holds? (Hint: \cup and \cap are distributive & intersection of two open balls is in \mathcal{T})

Continuity in Topological spaces

Given a set X and a collection \mathcal{T}_X of subsets of X

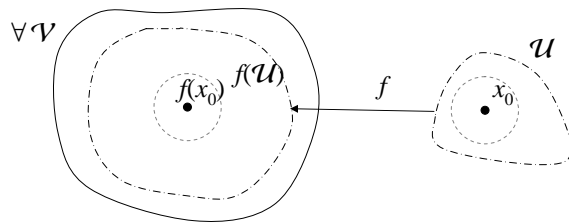
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\mathcal{T} is called a topology and the sets in \mathcal{T} are called **open** and their complements are called **closed**

Given a function $f: X \rightarrow Y$ with X, Y topological space

f is **continuous** at a point x_0 in X if for every neighborhood (i.e., set containing an open set) \mathcal{V} of $f(x_0)$ there is a neighborhood \mathcal{U} of x_0 such that $f(\mathcal{U}) \subset \mathcal{V}$.



Intuitively: "arbitrarily close" points in X are transformed into "arbitrarily close" points in Y

For norm-induced topologies we need only consider balls

$\mathcal{V} := \{y : \|y - f(x_0)\| < \epsilon\}$ and $\mathcal{U} := \{x : \|x - x_0\| < \delta\}$

Continuity in Topological spaces

Given a function $f: X \rightarrow Y$ with X, Y topological space

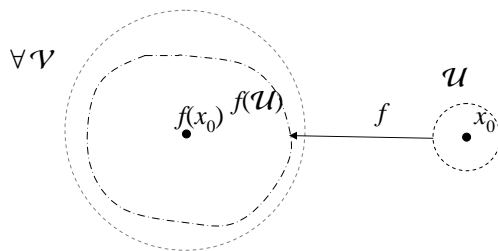
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Examples: $X := \mathbb{R}^n, Y := \mathbb{R}^m$

$\mathcal{T}_X, \mathcal{T}_Y := \{ \text{(possibly infinite) union of all open balls} \}$ (norm-induced top.)
 open ball $\equiv \{ x \in \mathbb{R}^n : \|x - x_0\| < \epsilon \}$

leads to the usual definition of continuity in \mathbb{R}^n : f continuous at x_0 if

$$\forall \epsilon > 0 \exists \delta > 0 : \|x - x_0\| < \delta \Rightarrow \|f(x) - f(x_0)\| < \epsilon$$



Could be restated as: for every ball
 $\mathcal{V} := \{ y : \|y - f(x_0)\| < \epsilon \}$
 there is a ball
 $\mathcal{U} := \{ x : \|x - x_0\| < \delta \}$
 such that
 $x \in \mathcal{U} \Rightarrow f(x) \in \mathcal{V}$
 or equivalently
 $f(\mathcal{U}) \subset \mathcal{V}$

Continuity in Topological spaces

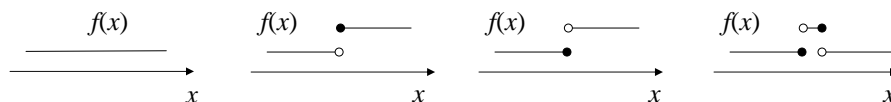
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Examples: $Q := \{ q_1, q_2, \dots, q_n \}$ (finite)

1. $\mathcal{T}_Q := \{ \emptyset, Q \}$ (**trivial topology** – all points are close to each other)
2. $\mathcal{T}_Q := \{ \emptyset \} \cup \{ \text{all subsets of } Q \}$ (**discrete topology** – no two distinct points are close to each other)
3. $\mathcal{T}_Q := \{ \emptyset, \{1\}, \{1,2\} \}$ of $\{1,2\}$

Is any of these functions $f: \mathbb{R} \rightarrow Q$ continuous? (usual norm-topology in \mathbb{R})



Continuity in Topological spaces

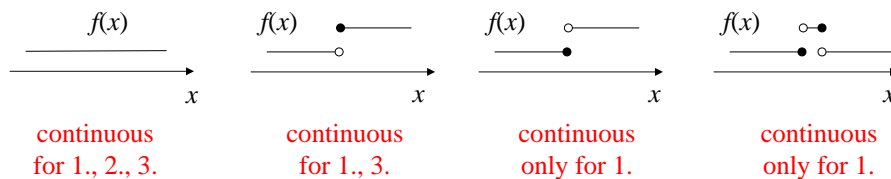
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Examples: $Q := \{q_1, q_2, \dots, q_n\}$ (finite)

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3. $\mathcal{T}_Q := \{\emptyset, \{1\}, \{1,2\}\}$ of $\{1,2\}$ (2 is ?close? to 1 but 1 is not ?close? to 2)

Is any of these functions $f: \mathbb{R} \rightarrow Q$ continuous? (usual norm-topology in \mathbb{R})



(for those that don't want to leave anything to the imagination...)

Given a sets Q, X with topologies \mathcal{T}_Q and \mathcal{T}_X

One can construct a topology $\mathcal{T}_{Q \times X}$ on $Q \times X$:

$$\mathcal{T}_{Q \times X} := \{ \mathcal{A} \times \mathcal{B} : \mathcal{A} \in \mathcal{T}_Q, \mathcal{B} \in \mathcal{T}_X \}$$

Example: $Q := \{1, 2\}, \mathcal{T}_Q := \{\emptyset, \{1\}, \{1,2\}\}$

$X := \mathbb{R}, \mathcal{T}_{\mathbb{R}} \equiv$ norm-induced topology

some open sets: $\emptyset, \{(1,x) : x \in (1,2)\}, \{(q,x) : q=1,2, x \in (1,\infty)\}$

not open sets: $\{(1,x) : x \in (1,2]\}, \{(2,x) : x \in (1,2)\}$

One can construct a topology $\mathcal{T}_{Q_{\text{sig}}}$ on the set Q_{sig} of signals $q: [0,T] \rightarrow Q, T \in (0,\infty]$:

$$\mathcal{T}_{Q_{\text{sig}}} := \text{sets of the form } \mathcal{A} := \{ q \in Q_{\text{sig}} : q(t) \in \mathcal{A}(t) \forall t \}$$

where the $\mathcal{A}(t)$ are a collection of open sets

Example: $Q := \{1, 2\}, \mathcal{T}_Q := \{\emptyset, \{1\}, \{1,2\}\}$

some open sets: $\{ q : q(t) = 1 \forall t \leq 1 \}, \{ q : q(t) = 1 \forall t \in \mathbb{Q} \}$

not open sets: $\{ q : q(t) = 2 \forall t \leq 1 \}$

$X := \mathbb{R}, \mathcal{T}_{\mathbb{R}} \equiv$ norm-induced topology

some open sets: $\{ x : x(t) < 0 \forall t \}, \{ x : |x(t)| < 1 \forall t \}$

non open sets: $\{ x : \int_0^\infty x(t) dt < 1 \}$

Back to hybrid systems...

$$\dot{x} = f(q, x) \quad (q, x) = \Phi(q, x^-) \quad q \in \mathcal{Q}, x \in \mathbb{R}^n$$

Hybrid automaton can be seen as an operator

$$T : \mathcal{Q} \times \mathbb{R}^n \rightarrow \mathcal{Q}_{\text{sig}} \times \mathcal{X}_{\text{sig}}$$

that maps $(q_0, x_0) \in \mathcal{Q} \times \mathbb{R}^n$ into the solution that starts at $q(0) = q_0, x(0) = x_0$

Definition (continuity definition):

A solution (q^*, x^*) is *(Lyapunov) stable* if T is continuous at $(q_0^*, x_0^*) := (q^*(0), x^*(0))$, i.e., for every neighborhood \mathcal{V} of $T(q_0^*, x_0^*)$ there is a neighborhood \mathcal{U} of (q_0^*, x_0^*) such that $T(\mathcal{U}) \subset \mathcal{V}$

Case 1: domain of T: $\mathcal{T}_{\mathcal{Q}} \equiv$ trivial topology (all points close to each other)

$\mathcal{T}_{\mathbb{R}^n} \equiv$ usual topology induced from Euclidean norm

co-domain of T: $\mathcal{T}_{\mathcal{Q}_{\text{sig}}} \equiv$ trivial topology (all signals close to each other)

$\mathcal{T}_{\mathcal{X}_{\text{sig}}} \equiv$ usual topology induced from sup-norm

$$\forall \epsilon > 0 \exists \delta > 0 : \forall (q_0, x_0) \in \mathcal{U}, \|x^*(0) - x(0)\| < \delta \Rightarrow \forall t \ \|x^*(t) - x(t)\| < \epsilon$$

one does not care at all about the discrete states matching

$$d((q_1, x_1), (q_2, x_2)) := \|x_1 - x_2\|$$

Back to hybrid systems...

$$\dot{x} = f(q, x) \quad (q, x) = \Phi(q, x^-) \quad q \in \mathcal{Q}, x \in \mathbb{R}^n$$

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Case 2: domain of T: $\mathcal{T}_{\mathcal{Q}} \equiv$ discrete topology (all points far from each other)

$\mathcal{T}_{\mathbb{R}^n} \equiv$ usual topology induced from Euclidean norm

co-domain of T: $\mathcal{T}_{\mathcal{Q}_{\text{sig}}} \equiv$ discrete topology (all signals far from each other)

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$$\forall \epsilon > 0 \exists \delta > 0 : q^*(0) = q(0), \|x^*(0) - x(0)\| < \delta \Rightarrow \forall t \ q^*(t) = q(t), \|x^*(t) - x(t)\| < \epsilon$$

one cares very much about the discrete states matching

$$d((q_1, x_1), (q_2, x_2)) := \begin{cases} \|x_1 - x_2\| & q_1 = q_2 \\ +\infty & \text{otherwise} \end{cases}$$

Back to hybrid systems...

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Hybrid automaton can be seen as an operator

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Case 3: domain of T: $\mathcal{T}_{\mathcal{Q}} \equiv$ discrete topology (all points far from each other)

$\mathcal{T}_{\mathbb{R}^n} \equiv$ usual topology induced from Euclidean norm

co-domain of T: $\mathcal{T}_{\mathcal{Q}_{\text{sig}}} \equiv$ trivial topology (all signals close to each other)

$\mathcal{T}_{\mathcal{X}_{\text{sig}}} \equiv$ usual topology induced from sup-norm

$$\forall \epsilon > 0 \exists \delta > 0 : \underbrace{q^*(0) = q(0), \|x^*(0) - x(0)\|}_{\mathcal{U}} < \delta \Rightarrow \forall t \underbrace{\|x^*(t) - x(t)\|}_{\mathcal{V}} < \epsilon$$

one cares very much about the discrete states matching

Back to hybrid systems...

$$\dot{x} = f(q, x) \quad (q, x) = \Phi(q, x^-) \quad q \in \mathcal{Q}, x \in \mathbb{R}^n$$

Hybrid automaton can be seen as an operator

$$T : \mathcal{Q} \times \mathbb{R}^n \rightarrow \mathcal{Q}_{\text{sig}} \times \mathcal{X}_{\text{sig}}$$

that maps $(q_0, x_0) \in \mathcal{Q} \times \mathbb{R}^n$ into the solution that starts at $q(0) = q_0, x(0) = x_0$

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Case 4: domain of T: $\mathcal{T}_{\mathcal{Q}} \equiv \{\emptyset, \{1\}, \{1,2\}\}, \mathcal{Q} := \{1,2\}$

$\mathcal{T}_{\mathbb{R}^n} \equiv$ usual topology induced from Euclidean norm

co-domain of T: $\mathcal{T}_{\mathcal{Q}_{\text{sig}}} \equiv$ trivial topology (all signals close to each other)

$\mathcal{T}_{\mathcal{X}_{\text{sig}}} \equiv$ usual topology induced from sup-norm

small perturbation in x (but no perturbation in q) leads to small change in x

$$\text{for } q^*(0) = 1: \forall \epsilon > 0 \exists \delta > 0 : q(0) = 1, \|x^*(0) - x(0)\| < \delta \Rightarrow \forall t \|x^*(t) - x(t)\| < \epsilon$$

$$\text{for } q^*(0) = 2: \forall \epsilon > 0 \exists \delta > 0 : \forall q(0), \|x^*(0) - x(0)\| < \delta \Rightarrow \forall t \|x^*(t) - x(t)\| < \epsilon$$

small perturbation in x leads to small change in x , regardless of $q(0)$

Back to hybrid systems...

$$\dot{x} = f(q, x) \quad (q, x) = \Phi(q, x^-) \quad q \in \mathcal{Q}, x \in \mathbb{R}^n$$

Hybrid automaton can be seen as an operator

$$T : \mathcal{Q} \times \mathbb{R}^n \rightarrow \mathcal{Q}_{\text{sig}} \times \mathcal{X}_{\text{sig}}$$

that maps $(q_0, x_0) \in \mathcal{Q} \times \mathbb{R}^n$ into the solution that starts at $q(0) = q_0, x(0) = x_0$

Definition (continuity definition):

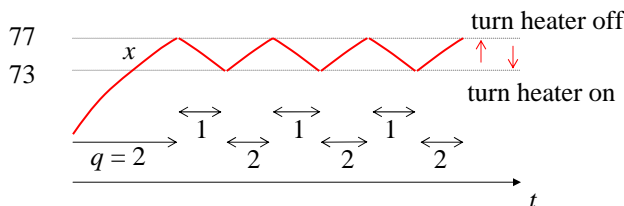
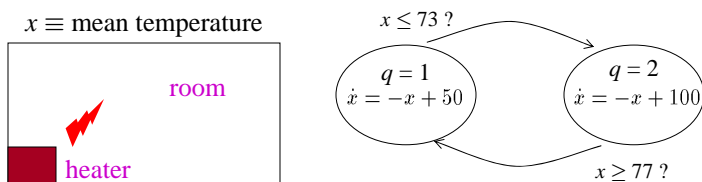
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Case 4: domain of T: $\mathcal{T}_{\mathcal{Q}} \equiv$ discrete topology (all points far from each other)
 $\mathcal{T}_{\mathbb{R}^n} \equiv$ usual topology induced from Euclidean norm
 co-domain of T: $\mathcal{T}_{\mathcal{Q}} \equiv \{\emptyset, \{1\}, \{1,2\}\}, \mathcal{Q} := \{1,2\}$ (signal version...)
 $\mathcal{T}_{\mathcal{X}_{\text{sig}}} \equiv$ usual topology induced from sup-norm

$$\forall \epsilon > 0 \exists \delta > 0 : q^*(0) = q(0), \|x^*(0) - x(0)\| < \delta \Rightarrow \forall t \ q^*(t) = 2 \text{ or } q^*(t) = q(t), \\ \|x^*(t) - x(t)\| < \epsilon$$

small perturbation in x (but no perturbation in q) leads to small change in x ,
 q^* and q may differ only when $q^* = 2$

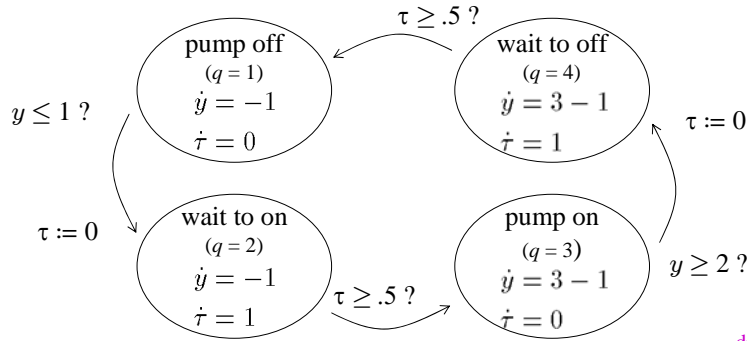
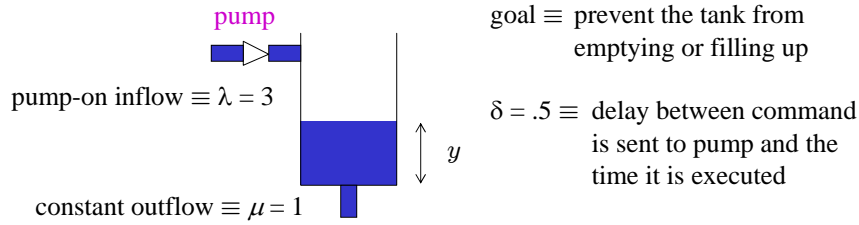
Example #2: Thermostat



Why?

no trajectory is stable some trajectories are stable others unstable all trajectories are stable

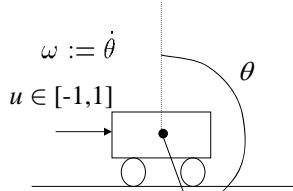
Example #5: Tank system



this topology only distinguishes between modes based on the state of the pump

A possible topology for $Q: \mathcal{T}_Q := \{\emptyset, \{1,2\}, \{3,4\}, \{1,2,3,4\}\}$

Example #4: Inverted pendulum swing-up



$$\ddot{\theta} = \sin \theta - u \cos \theta \quad E := \frac{1}{2} \omega^2 + (\cos \theta - 1)$$

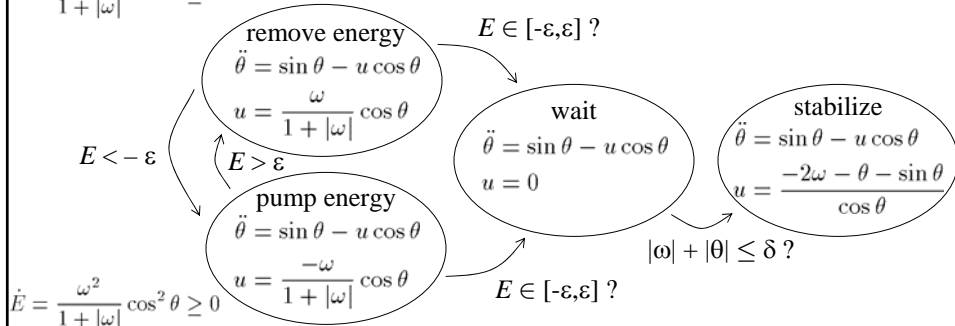
Hybrid controller:

1st pump/remove energy into/from the system by applying maximum force, until $E \approx 0$ (energy control) $\dot{E} = -u\omega \cos \theta$

2nd wait until pendulum is close to the upright position

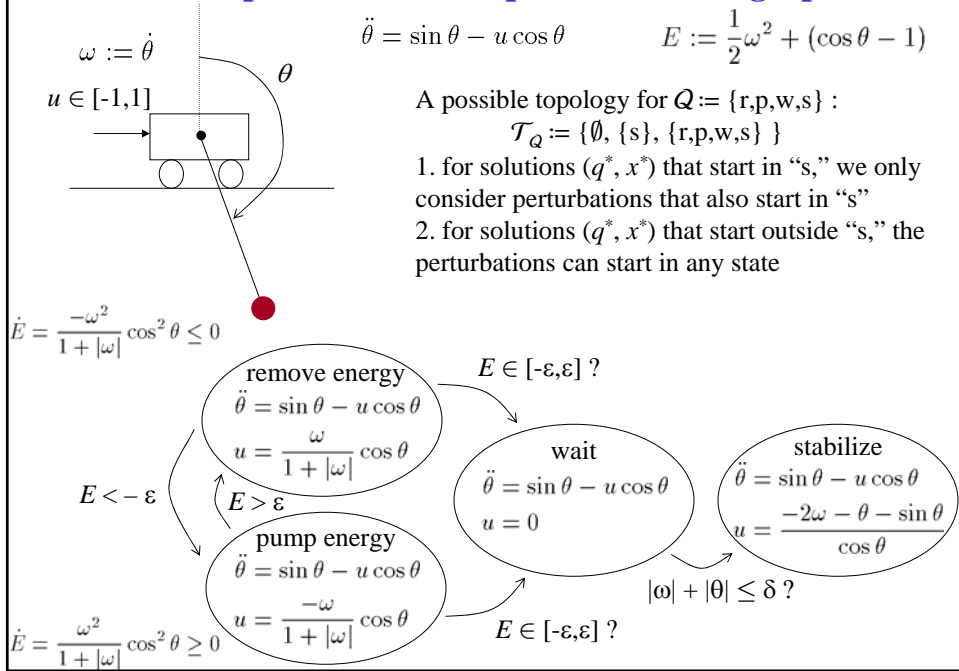
3rd next to upright position use feedback linearization controller

$$\dot{E} = \frac{-\omega^2}{1 + |\omega|} \cos^2 \theta \leq 0$$



$$\dot{E} = \frac{\omega^2}{1 + |\omega|} \cos^2 \theta \geq 0$$

Example #4: Inverted pendulum swing-up



Asymptotic stability for hybrid systems

$$\dot{x} = f(q, x) \quad (q, x) = \Phi(q, x^-) \quad q \in Q, x \in \mathbb{R}^n$$

Hybrid automaton can be seen as an operator

$$T : Q \times \mathbb{R}^n \rightarrow Q_{\text{sig}} \times X_{\text{sig}}$$

that maps $(q_0, x_0) \in Q \times \mathbb{R}^n$ into the solution that starts at $q(0) = q_0, x(0) = x_0$

Definition (continuity definition):

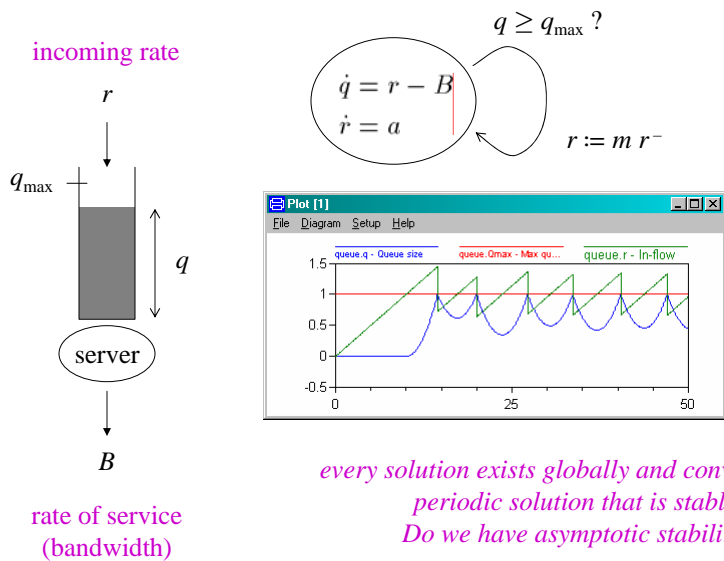
A solution (q^*, x^*) is **(Lyapunov) stable** if T is continuous at $(q_0^*, x_0^*) := (q^*(0), x^*(0))$, i.e., for every neighborhood \mathcal{V} of $T(q_0^*, x_0^*)$ there is a neighborhood \mathcal{U} of (q_0^*, x_0^*) such that $T(\mathcal{U}) \subset \mathcal{V}$

Definition:

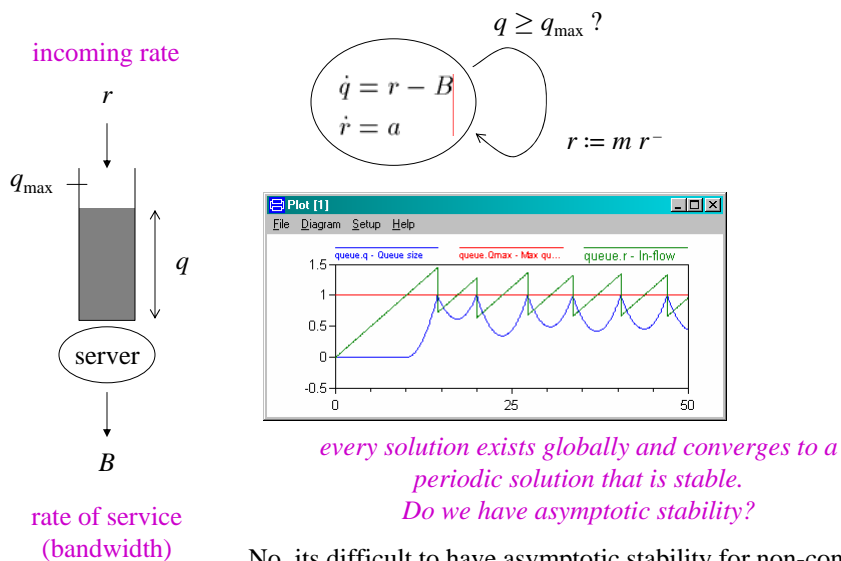
A solution (q^*, x^*) is **asymptotically stable** if it is stable, every solution (q, x) exists globally, and $q \rightarrow q^*, x \rightarrow x^*$ as $t \rightarrow \infty$

in the sense of the topology on Q_{sig}

Example #7: Server system with congestion control

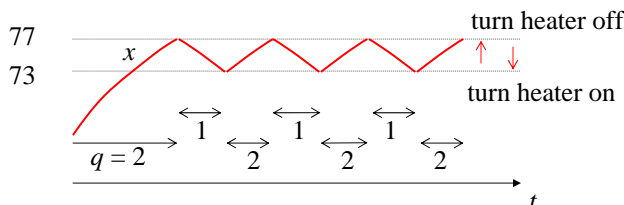
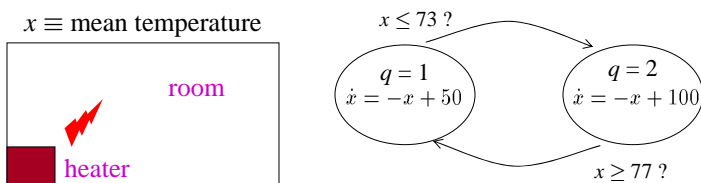


Example #7: Server system with congestion control



No, its difficult to have asymptotic stability for non-constant solutions due to the “synchronization” requirement.
(not even stability... Always?)

Example #2: Thermostat



no trajectory is stable
 for discrete topology on \mathcal{Q}
 (all points far from each other)

all trajectories are stable
 but not asymptotically
 for trivial topology on \mathcal{Q}
 (all points close to each other)

Why?

Stability of sets

$$\dot{x} = f(q, x) \quad (q, x) = \Phi(q^-, x^-) \quad q \in \mathcal{Q}, x \in \mathbb{R}^n$$

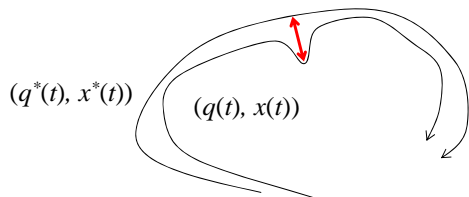
Hybrid automaton can be seen as an operator

$$T : \mathcal{Q} \times \mathbb{R}^n \rightarrow \mathcal{Q}_{\text{sig}} \times \mathcal{X}_{\text{sig}}$$

that maps $(q_0, x_0) \in \mathcal{Q} \times \mathbb{R}^n$ into the solution that starts at $q(0) = q_0, x(0) = x_0$

Poincaré distance between $(q, x), (q^*, x^*) \in \mathcal{Q}_{\text{sig}} \times \mathcal{X}_{\text{sig}}$ after t_0

$$d_P((q, x), (q^*, x^*); t_0) := \sup_{t \geq t_0} \inf_{\tau \geq t_0} d_T((q(t), x(t)), (q^*(\tau), x^*(\tau)))$$



distance at the point t
 where the $(q(t), x(t))$ is the
 furthest apart from (q^*, x^*)

can also be viewed as the
 distance from the trajectory
 (q, x) to the set
 $\{(q^*(t), x^*(t)) : t \geq t_0\}$

For constant trajectories (q^*, x^*) its just the sup-norm:

$$d_P((q, x), (q^*, x^*); t_0) := \sup_{t \geq t_0} d_T((q(t), x(t)), (q^*, x^*))$$

Stability of sets

$$\dot{x} = f(q, x) \quad (q, x) = \Phi(q^-, x^-) \quad q \in \mathcal{Q}, x \in \mathbb{R}^n$$

Hybrid automaton can be seen as an operator

$$T : \mathcal{Q} \times \mathbb{R}^n \rightarrow \mathcal{Q}_{\text{sig}} \times \mathcal{X}_{\text{sig}}$$

that maps $(q_0, x_0) \in \mathcal{Q} \times \mathbb{R}^n$ into the solution that starts at $q(0) = q_0, x(0) = x_0$

Poincaré distance between $(q, x), (q^*, x^*) \in \mathcal{Q}_{\text{sig}} \times \mathcal{X}_{\text{sig}}$ after t_0

$$d_P((q, x), (q^*, x^*); t_0) := \sup_{t \geq t_0} \inf_{\tau \geq t_0} d_T((q(t), x(t)), (q^*(\tau), x^*(\tau)))$$

Definition: A solution (q^*, x^*) is **Poincaré stable** if T is continuous at $(q_0^*, x_0^*) := (q^*(0), x^*(0))$ for the topology on \mathcal{X}_{sig} induced by the Poincaré distance,

$$\forall \epsilon > 0 \exists \delta > 0 : d_0((q(0), x(0)), (q^*(0), x^*(0))) \leq \delta$$

↓

$$d_P((q^*, x^*), (q, x); 0) = \sup_{t \geq 0} \inf_{\tau \geq 0} d_T((q(t), x(t)), (q^*(\tau), x^*(\tau))) \leq \epsilon$$

in more modern terminology one would say that the following **set is stable**

$$\{ (q^*(t), x^*(t)) : t \geq 0 \} \subset \mathcal{Q} \times \mathcal{X}$$

(open sets are unions of open Poincaré balls $\{ x \in \mathcal{X}_{\text{sig}} : d_P(x - x_0) < \epsilon \}$. Show this is a topology...)

Stability of sets

$$\dot{x} = f(q, x) \quad (q, x) = \Phi(q^-, x^-) \quad q \in \mathcal{Q}, x \in \mathbb{R}^n$$

Hybrid automaton can be seen as an operator

$$T : \mathcal{Q} \times \mathbb{R}^n \rightarrow \mathcal{Q}_{\text{sig}} \times \mathcal{X}_{\text{sig}}$$

that maps $(q_0, x_0) \in \mathcal{Q} \times \mathbb{R}^n$ into the solution that starts at $q(0) = q_0, x(0) = x_0$

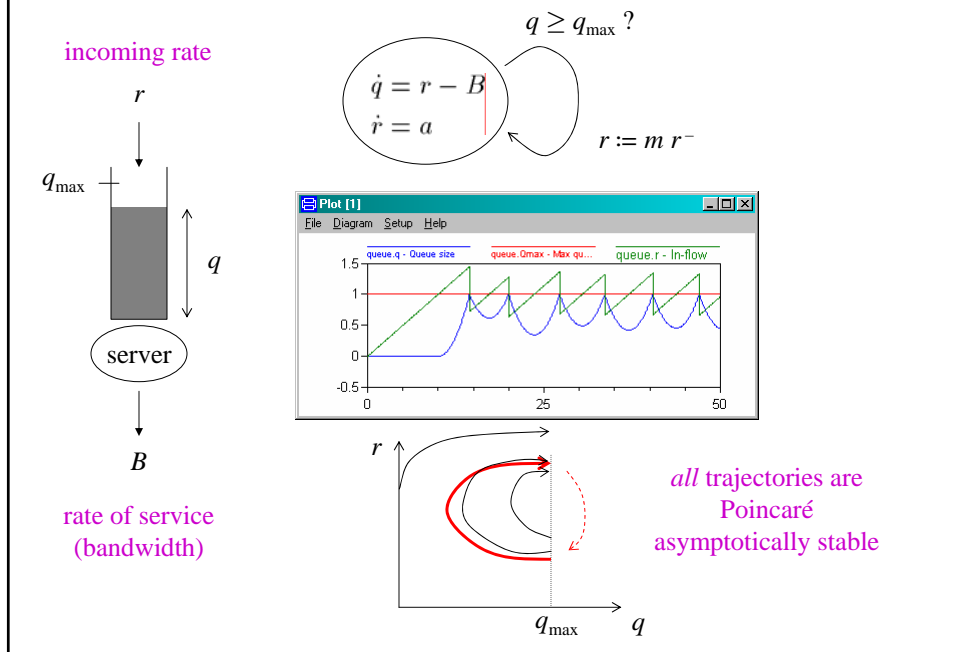
Poincaré distance between $(q, x), (q^*, x^*) \in \mathcal{Q}_{\text{sig}} \times \mathcal{X}_{\text{sig}}$ after t_0

$$d_P((q, x), (q^*, x^*); t_0) := \sup_{t \geq t_0} \inf_{\tau \geq t_0} d_T((q(t), x(t)), (q^*(\tau), x^*(\tau)))$$

Definition: A solution (q^*, x^*) is **Poincaré asymptotically stable** if it is Poincaré stable, every solution (q, x) exists globally, and $d_P((q, x), (q^*, x^*); t) \rightarrow 0$ as $t \rightarrow \infty$

in more modern terminology one would say that the following **set is asymptotically stable**: $\{ (q^*(t), x^*(t)) : t \geq 0 \} \subset \mathcal{Q} \times \mathcal{X}$

Example #7: Server system with congestion control



To think about ...

1. With hybrid systems there are many possible notions of stability. (especially due to the topology imposed on the discrete state)
WHICH ONE IS THE BEST?
(engineering question, not a mathematical one)

What type of perturbations do you want to consider on the initial conditions?

(this will define the topology on the initial conditions)

What type of changes are you willing to accept in the solution?

(this will define the topology on the signals)

2. Even with ODEs there are several alternatives: e.g.,
 - $\forall \epsilon > 0 \exists \delta > 0 : \|x_0 - x_{eq}\| \leq \delta \Rightarrow \sup_{t \geq 0} \|x(t) - x_{eq}\| \leq \epsilon$ Lyapunov
 - or
 - $\forall \epsilon > 0 \exists \delta > 0 : \|x_0 - x_{eq}\| \leq \delta \Rightarrow \int_0^\infty \|x(t) - x_{eq}\| dt \leq \epsilon$ integral
 - or
 - $\forall \epsilon > 0 \exists \delta > 0 : \|x_0 - x_0^*\| \leq \delta \Rightarrow d_p(x, x^*; 0) \leq \epsilon$ Poincaré

(even for linear systems these definitions may differ: Why?)

Next lecture...

Analysis tools for hybrid systems

1. Impact maps
 - Fixed-point theorem
 - Stability of periodic solutions
2. Decoupling
 - Switched systems
 - Supervisors