Hybrid Control and Switched Systems

Lecture #9
Analysis tools for hybrid systems: Impact maps

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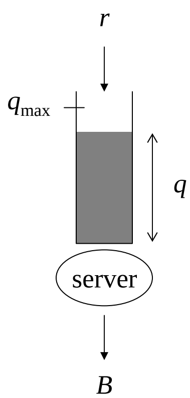
Summary

Analysis tools for hybrid systems–Impact maps

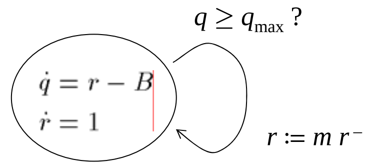
- Fixed-point theorem
- Stability of periodic solutions

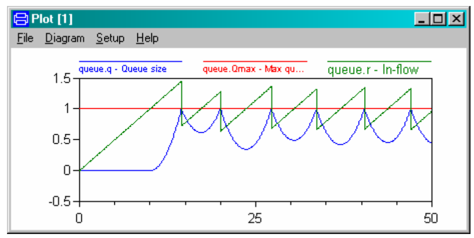
Example #7: Server system with congestion control

incoming rate



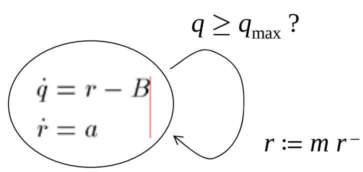
rate of service (bandwidth)





there is an asymptotically stable periodic solution

Example #7: Server system with congestion control



For given time t_0, and initial conditions q_0, r_0

$$r(t) = r_0 + a(t - t_0)$$

$$q(t) = q_0 + (r_0 - B)(t - t_0) + \frac{a(t - t_0)^2}{2}$$

If a transition occurred at time t_0, when will the next one occur?

$$q(t_0) = q^-(t_1) = q_{\max} \quad \Rightarrow \quad \begin{cases} t_1 = \frac{2(B - r_0)}{a} + t_0 \\ r^-(t_1) = 2B - r_0 \\ r(t_1) = m(2B - r_0) \end{cases}$$

right after one jump

$$q = q_{\max}$$
$$r = r_0$$

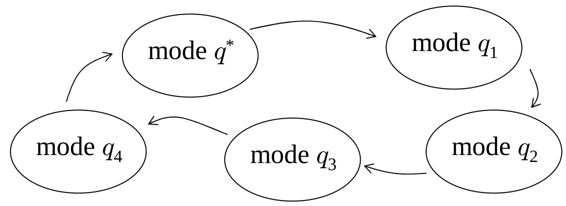
impact, return, or Poincaré map $\longrightarrow$

right after next jump

$$q = q_{\max}$$
$$r = m(2B - r_0)$$

Impact maps

$$\dot{x} = f(q, x) \qquad (q, x) = \Phi(q^-, x^-) \qquad q \in \mathcal{Q},\ x \in \mathbb{R}^n$$



Recurring mode $\equiv q^* \in \mathcal{Q}$ such that for every initialization there are infinitely many transitions into q^*, i.e., $\Box \Diamond \{ q^* = q,\ (q,x) \neq \Phi(q^-, x^-) \}$

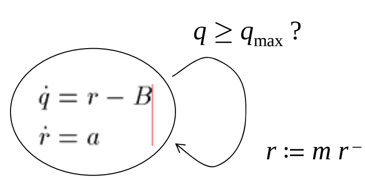
always eventually — jump into mode q^* (not necessarily from a different mode)

Definition: ***Impact, return, or Poincaré map*** (***Poincaré*** from ODEs)

Function $F : \mathbb{R}^n \to \mathbb{R}^n$ such that if t_k and t_{k+1} are consecutive times for which there is a transition into q^* then $x(t_{k+1}) = F(x(t_k))$

time-to-impact map $\equiv$ function $\Pi : \mathbb{R}^n \to (0,\infty)$ such that $t_{k+1} - t_k = \Pi(x(t_k))$

Example #7: Server system with congestion control



If a transition occurred at time t_0, when will the next one occur?

$$q(t_0) = q^-(t_1) = q_{\max} \quad \Rightarrow \quad \begin{cases} t_1 = \frac{2(B - r_0)}{a} + t_0 \\ r^-(t_1) = 2B - r_0 \\ r(t_1) = m(2B - r_0) \end{cases}$$

right after one jump $\qquad\longrightarrow\qquad$ right after next jump

$$\begin{matrix} q = q_{\max} \\ r = r_0 \end{matrix} \quad \longmapsto \quad \begin{matrix} q = q_{\max} \\ r = m(2B - r_0) \end{matrix}$$

impact map

$$F\left(\begin{bmatrix} q \\ r \end{bmatrix}\right) = \begin{bmatrix} q_{\max} \\ m(2B - r) \end{bmatrix}$$

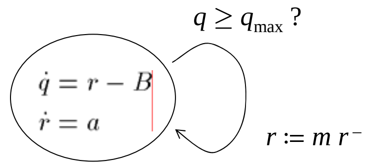
time-to-impact map

$$\Pi\left(\begin{bmatrix} q \\ r \end{bmatrix}\right) = \frac{2(B - r)}{a}$$

average rate?

fixpoints.nb

Example #7: Server system with congestion control

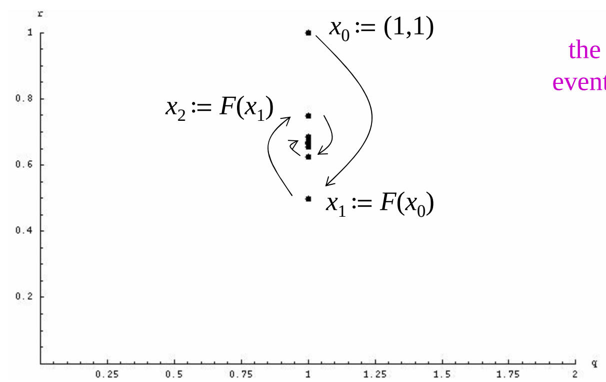


impact map

$$F\left(\begin{bmatrix} q \\ r \end{bmatrix}\right) = \begin{bmatrix} q_{max} \\ m(2B - r) \end{bmatrix}$$

time-to-impact map

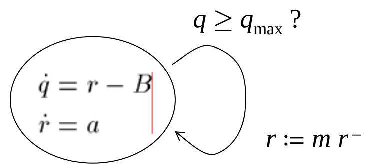
$$\Pi\left(\begin{bmatrix} q \\ r \end{bmatrix}\right) = \frac{2(B-r)}{a}$$



the impact points eventually converge

fixpoints.nb

Example #7: Server system with congestion control



impact map

$$F\left(\begin{bmatrix} q \\ r \end{bmatrix}\right) = \begin{bmatrix} q_{max} \\ m(2B - r) \end{bmatrix}$$

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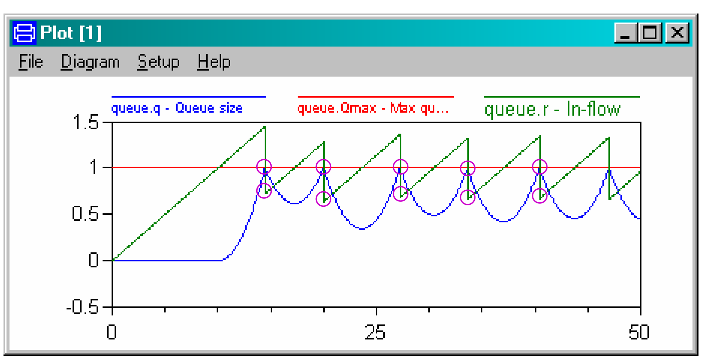


the impact points eventually converge

congestion1.nb

Contraction Mapping Theorem

Contraction mapping ≡ function $F : \mathbb{R}^n \to \mathbb{R}^n$ for which $\exists\, \gamma \in [0,1)$ such that

$$\|F(x) - F(x')\| \leq \gamma \|x - x'\| \qquad \forall x, x' \in \mathbb{R}^n$$

Lipschitz coefficient

Contraction mapping Theorem:

If $F : \mathbb{R}^n \to \mathbb{R}^n$ is a contraction mapping then — fixed-point of F

1. there is one and only point $x^* \in \mathbb{R}^n$ such that $F(x^*) = x^*$
2. for every $x_0 \in \mathbb{R}^n$, the sequence $x_{k+1} = F(x_k)$, $k \geq 0$ converges to x^* as $k \to \infty$

Why?

1. Consider sequence: $x_{k+1} = F(x_k)$, $k \geq 0$. After some work (induction …)
$$\|x_m - x_k\| \leq \frac{\gamma^m}{1-\gamma} \|x_1 - x_2\| \qquad \forall m \leq k$$
2. This means that the sequence $x_{k+1} = F(x_k)$, $\gamma \geq 0$ is Cauchy, i.e.,
$$\forall \epsilon > 0 \ \ \exists N \ \ \forall m, k > N : \|x_m - x_k\| \leq \epsilon$$
and therefore x_k converges as $k \to \infty$.
3. Let x^* be the limit. Then
$$F(x^*) = F\big(\lim x_k\big) = \lim F(x_k) = \lim x_{k+1} = \lim x_k = x^*$$
F is continuous

Contraction Mapping Theorem

Contraction mapping ≡ function $F : \mathbb{R}^n \to \mathbb{R}^n$ for which $\exists\, \gamma \in [0,1)$ such that

$$\|F(x) - F(x')\| \leq \gamma \|x - x'\| \qquad \forall x, x' \in \mathbb{R}^n$$

Lipschitz coefficient

Contraction mapping Theorem:

If $F : \mathbb{R}^n \to \mathbb{R}^n$ is a contraction mapping then — fixed-point of F

1. there is one and only point $x^* \in \mathbb{R}^n$ such that $F(x^*) = x^*$
2. for every $x_0 \in \mathbb{R}^n$, the sequence $x_{k+1} = F(x_k)$, $k \geq 0$ converges to x^* as $k \to \infty$

Why?

So far $x^* := \lim F(x_k)$ exists and $F(x^*) = x^*$ (unique??)

4. Suppose y^* is another fixed point:
$$\|x^* - y^*\| = \|F(x^*) - F(y^*)\| \leq \gamma \|x^* - y^*\| \quad \Rightarrow \quad (1-\gamma)\|x^* - y^*\| \leq 0$$

x^* must be equal to y^*

Contraction Mapping Theorem

Contraction mapping ≡ function $F : \mathbb{R}^n \to \mathbb{R}^n$ for which $\exists\, \gamma \in [0,1)$ such that

$$\|F(x) - F(x')\| \le \gamma \|x - x'\| \qquad \forall x, x' \in \mathbb{R}^n$$

Lipschitz coefficient

Contraction mapping Theorem:

If $F : \mathbb{R}^n \to \mathbb{R}^n$ is a contraction mapping then (fixed-point of F)

1. there is one and only point $x^* \in \mathbb{R}^n$ such that $F(x^*) = x^*$
2. for every $x_0 \in \mathbb{R}^n$, the sequence $x_{k+1} = F(x_k)$, $k \ge 0$ converges to x^* as $k \to \infty$

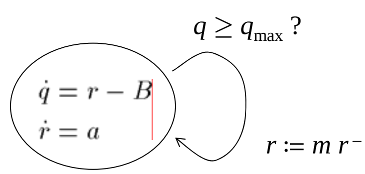
Example:
$$F\left(\begin{bmatrix} q \\ r \end{bmatrix}\right) = \begin{bmatrix} q_{\max} \\ m(2B - r) \end{bmatrix}$$

contraction as long as $m < 1$

$$\left\| F\left(\begin{bmatrix} q \\ r \end{bmatrix}\right) - F\left(\begin{bmatrix} q' \\ r' \end{bmatrix}\right) \right\| = \left\| \begin{bmatrix} 0 \\ m(r' - r) \end{bmatrix} \right\| = m|r' - r| \le m \left\| \begin{bmatrix} q' - q \\ r' - r \end{bmatrix} \right\|$$

$$F\left(\begin{bmatrix} q \\ r \end{bmatrix}\right) = \begin{bmatrix} q \\ r \end{bmatrix} \quad \Rightarrow \quad q = q_{\max},\, r = m(2B - r) \quad \Rightarrow \quad q = q_{\max},\, r = \frac{2mB}{1+m}$$

Example #7: Server system with congestion control

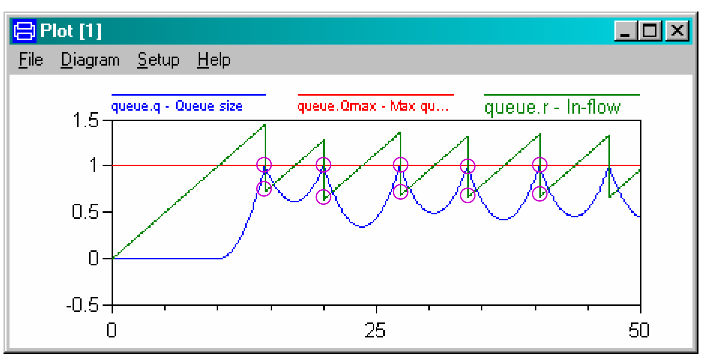


impact map

$$F\left(\begin{bmatrix} q \\ r \end{bmatrix}\right) = \begin{bmatrix} q_{\max} \\ m(2B - r) \end{bmatrix}$$

time-to-impact map

$$\Pi\left(\begin{bmatrix} q \\ r \end{bmatrix}\right) = \frac{2(B - r)}{a}$$



the impact points eventually converge to

$$q = q_{\max},\, r = \frac{2mB}{1+m}$$

congestion1.nb

Impact maps

$$\dot{x} = f(q,x) \qquad (q,x) = \Phi(q^-, x^-) \qquad q \in \mathcal{Q},\ x \in \mathbb{R}^n$$

Recurring mode $\equiv q^* \in \mathcal{Q}$ such that for every initialization there are infinitely many transitions into q^*, i.e., $\Box \Diamond \{ q^* = q, (q,x) \neq \Phi(q^-,x^-) \}$

Definition: ***Impact map***

Function $F : \mathbb{R}^n \to \mathbb{R}^n$ such that if t_k and t_{k+1} are consecutive times for which there is a transition into q^* then $x(t_{k+1}) = F(x(t_k))$

Theorem:

Suppose the hybrid system has a recurring mode $q^* \in \mathcal{Q}$ and

1. the corresponding impact map is a contraction
2. the interval map is nonzero on a neighborhood of the fixed point x^* of the impact map

then

a) it has a periodic solution (may be constant)

b) the impact points converge to the unique fixed point of F

Impact maps

$$\dot{x} = f(q,x) \qquad (q,x) = \Phi(q^-, x^-) \qquad q \in \mathcal{Q},\ x \in \mathbb{R}^n$$

Theorem:

Suppose the hybrid system has a recurring mode $q^* \in \mathcal{Q}$ and

1. the corresponding impact map is a contraction
2. the interval map is nonzero on a neighborhood of the fixed point x^* of the impact map

then

a) it has a (global) periodic solution with period $T := \Pi(x^*)$ (may be constant)

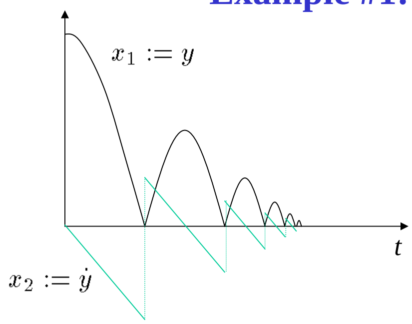
b) the impact points converge to the unique fixed point of F

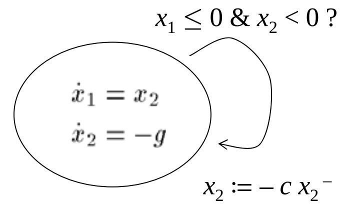
Why?

a) Since F is a contraction it has a fixed point $x^* \in \mathbb{R}^n$
Take any t_0. Since $F(x^*) = x^*$,
$$q(t_0) = q^*, x(t_0) = x^* \quad \Rightarrow \quad q(t_0+T) = q^*, x(t_0+T) = F(x^*) = x^*$$

b) Impact points are defined by the sequence $x(t_{k+1}) = F(x(t_k))$, which converges to the fixed point x^*

Example #1: Bouncing ball





impact map

$$F\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} 0 \\ c x_2 \end{bmatrix}$$

contraction mapping for $c < 1$

time-to-impact map

$$\Pi\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \frac{2x_2}{g} \qquad x_2 > 0$$

time-to-impact map is not bounded below

Impact maps

$$\dot{x} = f(q, x) \qquad (q, x) = \Phi(q^-, x^-) \qquad q \in \mathcal{Q},\ x \in \mathbb{R}^n$$

Theorem:

Suppose the hybrid system has a recurring mode and

1. the corresponding impact map is a contraction
2. the interval map is nonzero on a neighborhood of the fixed point x^* of the impact map

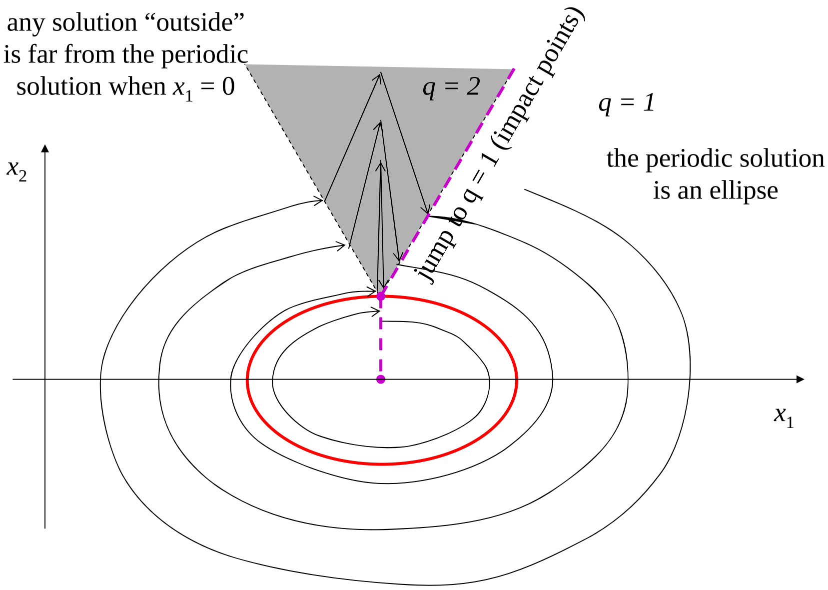
then

a) it has a periodic solution (may be constant)
b) the impact points converge to the unique fixed point of F

Is the periodic solution unique?
yes/no? in what sense

Does not necessarily mean that the periodic solution is stable (Lyapunov or Poincaré sense)

Impact maps



Impact maps

$$\dot{x} = f(q, x) \qquad (q, x) = \Phi(q^-, x^-) \qquad q \in \mathcal{Q},\ x \in \mathbb{R}^n$$

Theorem:

Suppose the hybrid system has a recurring mode and

1. the corresponding impact map is a contraction
2. the interval map is nonzero on a neighborhood of the fixed point x^* of the impact map

then

a) it has a periodic solution (may be constant)

b) the impact points converge to the unique fixed point of F

Moreover, if

1. the sequence of jumps between consecutive transitions to q^* is the same for every initial condition
2. f is locally Lipschitz with respect to x
3. Φ_2 (continuous-state reset) is continuous with respect to x
4. interval map is bounded in a neighborhood of x^*

then

a) any solution converges to a periodic solution

b) every periodic solution is Poincaré asymptotically stable

Impact maps

$$\dot{x} = f(q, x) \qquad (q, x) = \Phi(q^-, x^-) \qquad q \in \mathcal{Q},\ x \in \mathbb{R}^n$$

Moreover, if

1. the sequence of jumps between consecutive transitions to q^* is the same for every initial condition
2. f is locally Lipschitz with respect to x
3. Φ_2 (continuous-state reset) is continuous with respect to x
4. interval map is bounded in a neighborhood of x^*

then

a) any solution converges to a periodic solution
b) every periodic solution is Poincaré asymptotically stable

Why? (since the sequence of jumps is the same we don't need to worry about Q)

a) i) 1–3 guarantee continuity with respect to initial conditions (on finite interval)
ii) Therefore, if at an impact time t_k, x is close to the fixed point x^* it will remain close to the periodic solution until the next impact
(time between impacts is bounded because of 4)
iii) Moreover, if $x(t_k)$ converges to x^* the whole solution converges to the periodic solution

b) Stability from ii)
convergence from iii) plus the fact that, from a Poincaré perspective, all periodic solutions are really the same (by uniqueness of fixed point)

Proving that a function is a contraction

Contraction mapping ≡ function $F : \mathbb{R}^n \to \mathbb{R}^n$ for which $\exists\, \gamma \in [0,1)$ such that

$$\|F(x) - F(x')\| \le \gamma \|x - x'\| \qquad \forall x, x' \in \mathbb{R}^n$$

A good tool to prove that a mapping is contraction…

Mean Value Theorem:

$$\| F(x) - F(x') \| \le \gamma \| x - x' \| \qquad \forall\, x, x' \in \mathbb{R}^n$$

where

$$\gamma := \sup_{z \in \mathbb{R}^n} \left\| \frac{\partial F}{\partial x}(z) \right\| \qquad \frac{\partial F}{\partial x} := \left[\frac{\partial F_i}{\partial x_j} \right]_{i,j}$$

(constant matrix for a linear or affine F)

A mapping may be contracting for one norm but not for another
– often most of the effort is spent in finding the "right" norm

$$\|x\|_2 := \sqrt{x_1^2 + x_2^2} \qquad \|x\|_1 := |x_1| + |x_2| \qquad \|x\|_\infty := \max\{|x_1|, |x_2|\}$$

$$F\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) := \begin{bmatrix} .99 x_1 \\ .49(x_1 + x_2) \end{bmatrix}$$

contraction for $\| \cdot \|_\infty$

not a contraction for $\| \cdot \|_2$

Stability of difference equations

Given a discrete-time system

$$x_{k+1} = F(x_k) \qquad k \geq 0, x_k \in \mathbb{R}^n$$

equilibrium point $\equiv x_{eq} \in \mathbb{R}^n$ for which $F(x_{eq}) = x_{eq}$

thus $x_k = x_{eq}$ $\forall\, k \geq 0$ is a solution to the difference equation

Definition (ϵ–δ definition):
The equilibrium point $x_{eq} \in \mathbb{R}^n$ is ***(Lyapunov) stable*** if

$$\forall \epsilon > 0\ \exists \delta > 0 : ||x_{k_0} - x_{eq}|| \leq \delta \quad \Rightarrow \quad ||x_k - x_{eq}|| \leq \epsilon \quad \forall k \geq k_0 \geq 0$$

The equilibrium point $x_{eq} \in \mathbb{R}^n$ is ***(globally) asymptotically stable*** if
it is Lyapunov stable and for every initial state $x_k \to x_{eq}$ as $k \to \infty$.

(when F is a contraction we automatically have that
an equilibrium point exists & global asymptotic
stability with exponential convergence)

Impact maps

$$\dot{x} = f(q,x) \qquad (q,x) = \Phi(q^-, x^-) \qquad q \in \mathcal{Q},\ x \in \mathbb{R}^n$$

Theorem:

Suppose the hybrid system has a recurring mode and

1. the discrete-time impact system $x_{k+1} = F(x_k)$ has an asymptotically stable equilibrium point x_{eq}
2. the interval map is nonzero on a neighborhood of the equilibrium point x_{eq} of the impact system

then

a) it has a periodic solution (may be constant)
b) the impact points converge to x_{eq}

Moreover, if

1. the sequence of jumps between consecutive transitions to q^* is the same for every initial condition
2. f is locally Lipschitz with respect to x
3. Φ_2 (continuous-state reset) is continuous with respect to x
4. interval map is bounded in a neighborhood of x_{eq}

then

a) any solution converges to a periodic solution
b) every periodic solution is Poincaré asymptotically stable

Next lecture…

Decoupling between continuous and discrete dynamics

- Switched systems
- Supervisors

Stability of switched systems

- Stability under arbitrary switching