

# LINEAR SYSTEMS THEORY

João P. Hespanha

January 16, 2009

Disclaimer: This is a draft and probably contains a few typos.

Comments and information about typos are welcome.

Please contact the author at [hspanha@ece.ucsb.edu](mailto:hspanha@ece.ucsb.edu).

© Copyright to João Hespanha. Please do not distribute this document without the author's consent.

# Index

- adjoint matrix, 30
- aircraft roll dynamics, 185–188, 207–208, 221
- algebraic equivalence, 33, 36
- algebraic Riccati equation (ARE), 81, 172, 175–180, 199
- applied forces vector, 14
- ARE, *see* algebraic Riccati equation
  
- BIBO, *see* Bounded-input, bounded-output stability
- block diagram, 7–9, 55, 109, 132–133
- bounded-input, bounded-output stability (BIBO), 164, 166
  - continuous time, 73–77
  - discrete time, 78
- Bryson’s rule, 170, 173
  
- canonical basis, 38
- cascade interconnection, 7, 55
- causality, 21, 24, 35, 41
- Cayley–Hamilton theorem, 45, 94, 97, 101, 109, 117
- centrifugal/Coriolis/friction matrix, 14
- certainty equivalence, 197
- characteristic polynomial, 31, 45–46, 116, 143, 152
- cheap control, 188–190
  - closed-loop poles, 188–189
  - LQR cost, 189–190
- closed-loop state estimator, 136
- cofactor, 30, 35
- comparison lemma, 64
- complementary sensitivity function, 183
- complete sequence, 216
- conservative force, 14
- conservative forces vector, 14
- constructibility Gramian, 135, 198
  - continuous time, 124–125
  - discrete time, 125–126
- constructible system
  - continuous time, 123, 124
  - discrete time, 126
- continuous-time system, 5
- controllability Gramian
  - continuous time, 91–95
  - discrete time, 95–98
  - infinite time, 102
- controllability matrix, 93–98, 107, 116, 144
- controllability test
  - controllability matrix, 100
  - eigenvector, 100, 166, 180, 208
  - Lyapunov, 102–104
  - Popov–Belevitch–Hautus (PBH), 101
- controllable canonical form, 32, 33, 35, 105, 117, 143
- controllable decomposition, 107–110
- controllable-from-the-origin subspace, *see* reachable subspace
- controllable state component, 109
- controllable subspace, 104, 108, 133
  - continuous time, 87–92
  - discrete time, 95–98
- controllable system, 111, 117, 139, 166
  - continuous time, 99
- controllable-to-the-origin subspace, *see* controllable subspace
- controlled output, 169, 210, 214, 218
- control specification, 213–215, 219–221
  - convexity, 215, 222
- convolution
  - continuous time, 24
  - discrete time, 26
  - Laplace transform, 27
- coprime
  - fraction, 143
  - polynomials, 143, 149, 152
- CVX, 217–221
  - cvx\_begin, 218
  - cvx\_end, 218
  - minimize, 221
  - variable, 218, 221
  
- degree of a transfer function, 143
- detectability test
  - eigenvector, 134
  - Lyapunov, 135
  - Popov–Belevitch–Hautus (PBH), 135
- detectable system, 134–136, 180
- diagonalizable matrix, 51, 144
- discrete-time system, 6
- disturbance rejection, 192
- domain of the Riccati operator, 175
- duality
  - controllability/observability, 127–128
  - reachability/constructibility, 127–128

- dual algebraic Riccati equation, 199, 208
- dual ARE, *see* dual algebraic Riccati equation
- eigenvalue assignment, 116
  - output injection, 136
  - state feedback, 115
- eigenvector test
  - controllability, 100
  - detectability, 134
  - observability, 128
  - stabilizability, 112
- equilibrium point
  - continuous time, 11
  - discrete time, 13
- equivalence transformation, *see* similarity transformation
- exogenous input, 210, 214, 218
- feasibility problem, 215, 216
- feedback invariant, 79–81, 170–173
- feedback linearization, 16–19
- forced response, 24, 39
- fundamental theorem of linear equations, 90, 100, 101
- gain margin, 182
- generalized coordinates vector, 14
- generic property, 54
- greatest common divisor (gcd), 150
- Hamiltonian matrix, 83, 175–180, 188–189, 194
- homogeneous response, 23, 29, 39
- homogeneous system, 37, 43, 61, 66
- hovercraft, 15
- Hurwitz matrix, *see* stability matrix
- H-infinity norm, *see* root mean square gain
- image of a matrix, 90
- impulse response, 49
  - continuous time, 23–25, 27, 29–30
  - discrete time, 26, 30
- infinite-dimensional systems, 22
- initial condition, 5–7, 11, 13, 22, 30, 33, 34, 37, 39, 61, 65, 73, 78
- input-output model, 5, 6
- internal stability, *see* Lyapunov stability
- invariant subspace, 100, 104, 133, 180
- invariant zero polynomial, 158
- inverse system, 161–166
  - left inverse, 162
  - right inverse, 162
- inverted pendulum, 15, 18, 19, 69
- irreducible realization, *see* realization
- Jordan normal form, 51–55, 62, 71, 180
- Kalman
  - decomposition theorem, 132–134
  - equality, 181–182, 184, 194
  - filter, 201
  - inequality, 182–183
  - Kalman decomposition, 161
  - kernel of a matrix, 90
  - $L_1$  norm, 214, *see* peak gain
  - $L_2$  norm, 214
  - lag compensator, 193–194
  - Laplace transform, 25, 29, 46
    - convolution, 27
    - derivative, 27
  - lead compensator, 193–194
  - least common denominator (lcd), 152
  - left-inverse matrix, 141
  - Lienard equation, 72
  - linearity, 22, 35, 41
  - linear matrix inequality (LMI), 63, 103, 113
  - linear quadratic Gaussian estimation (LQG), 201
  - linear quadratic regulation (LQR), 79–82, 169–172, 181–188
  - linear time-invariant (LTI) system, 6
  - linear time-varying (LTV) system, 6
  - $L_\infty$  norm, 214
  - LMI, *see* linear matrix inequality
  - local linearization, 11–14
    - around equilibrium point in continuous time, 12, 18, 66–70
    - around equilibrium point in discrete time, 13, 71
    - around trajectory, 14, 18–19
  - loop shaping, 184–185
  - loop transfer recovery (LTR), 202–203, 207–208
  - LQG, *see* linear quadratic Gaussian estimation
  - LQG/LQR controller, 202
    - Q-augmented, *see* Q-augmented LQG/LQR controller
  - LQR, *see* linear quadratic regulation
  - LTI, *see* linear time-invariant system
  - LTR, *see* loop transfer recovery
  - LTV, *see* linear time-varying system
  - Lyapunov equation, 102, 104
    - continuous time, 63, 72, 103
    - discrete time, 66
  - Lyapunov inequality, 114
    - continuous time, 63, 103
    - discrete time, 66
  - Lyapunov stability
    - asymptotic stability, 61, 62, 65, 66, 103, 111, 115, 134, 136
    - continuous time, 60–65
    - discrete time, 65–66
    - exponential stability, 61, 63, 65, 66, 68, 103
    - instability, 61, 65, 66, 69

- marginal stability, 61, 65
  - stability margin, 72
- Lyapunov stability theorem
  - continuous time, 62–65, 103
  - discrete time, 66
- Lyapunov test
  - controllability, 102
  - detectability, 135
  - observability, 128
  - stabilizability, 113
- Markov parameters, 139–142
- mass matrix, 14
- MATLAB
  - append, 9
  - bode, 186, 208
  - ctrbf, 110
  - ctrb, 98
  - dlyap, 71
  - eig, 31, 62, 71
  - expm, 48
  - feedback, 9
  - freqresp, 220
  - ilaplace, 48
  - impulse, 220
  - iztrans, 49
  - jacobian, 48
  - jordan, 52
  - kalman, 207, 208
  - laplace, 48
  - linmod, 218
  - lqr, 82, 172, 186
  - lyap, 71
  - minreal, 144, 218
  - norm, 59
  - nyquist, 190
  - obsvf, 137
  - obsv, 130
  - parallel, 9
  - pinv, 144
  - place, 115
  - poly, 31
  - reg, 207, 208
  - series, 9
  - sigma, 190
  - ssdata, 218
  - ss, 6, 34, 35, 186, 208, 217
  - step, 219
  - svd, 59, 98
  - syms, 47
  - tf, 34, 35, 217
  - zpk, 34, 35
  - ztrans, 49
  - Simulink, 8
  - symbolic computation, 47–49
- matrix exponential, 43–46, 49, 53–54, 71
- matrix norm, 59–60, 71
  - submultiplicative, 60, 71
- matrix power, 47, 49, 52–53
- McMillan degree, 152, 160, 164
- measured output, 169
- mechanical systems, 14–16
  - fully actuated, 16
- MEE, *see* minimum-energy estimation
- memoryless system, 5
- MI, *see* multiple-input system
- MIMO, *see* multiple-input, multiple-output system
- minimal realization, *see* realization
- minimum-energy
  - control, 92–93
  - estimation (MEE), 198–201
- minimum-phase system, 164
- MISO, *see* multiple-input, single-output system
- MO, *see* multiple-output system
- monic polynomial, 143
- multiple-input, multiple-output system (MIMO), 5
- multiple-input, single-output system (MISO), 5
- multiple-input system (MI), 5
- multiple-output system (MO), 5
- negative-definite matrix, 62
- negative-feedback interconnection, 8
- negative-semidefinite matrix, 62
- noise rejection, 193
- norm
  - $\infty$ -norm, 59
  - Frobenius norm, 59
  - H-infinity, *see* root mean square gain
  - $L_1$ , 214, *see* peak gain
  - $L_2$ , 214
  - $L_\infty$ , 214
  - one norm, 59
  - two norm, 59
- nullity of a matrix, 90
- null space of a matrix, *see* kernel of a matrix
- Nyquist
  - plot, 182, 191
  - stability criterion, 191
- observability Gramian, 135
  - continuous time, 124–125
  - discrete time, 125–126
- observability matrix, 128, 131, 144
- observability test
  - eigenvector, 128
  - Lyapunov, 128
  - Popov-Belevitch-Hautus (PBH), 128
- observable canonical form, 33, 35, 36

- observable decomposition, 131–132
- observable state component, 132
- observable system, 134, 136, 139, 166
  - continuous time, 122, 124
  - discrete time, 126
- open-loop state estimator, 135
- orthogonal complement, 90
- orthogonal matrix, 98, 190, 194
- output feedback, 121, 135
- output injection, 136
- overactuated system, 204
- overshoot, 192, 214, 219, 221
  
- parallel interconnection, 7, 55, 123
- peak gain, 215, 221
- Peano-Baker series, 37, 43
- phase margin, 183, 192
- poles, 152, 157, 160
  - SISO, 143, 149
- pole placement, *see* eigenvalue assignment
- pole polynomial, *see* characteristic polynomial
- Popov-Belevitch-Hautus (PBH) test
  - controllability test, 101
  - detectability test, 135
  - observability test, 128
  - stabilizability, 113
- positive-definite matrix, 62
- positive-semidefinite matrix, 62
- prefilter, 213
- proofs
  - by contradiction, 64, 75
  - by contraposition, 75
  - contraposition, 101
  - direct, 31
  - equivalence, 31, 63, 101
  - set equality, 91
- proper rational function, 31
- proportional-derivative (PD) control, 16
- proportional (P) control, 194
- pseudoinverse matrix, 141, 204
  
- Q-augmented LQG/LQR controller, 209–213, 218
- Q design, 215–222
- Q parameterization, 209–212
- Q system, 209, 215
  
- range of a matrix, *see* image of a matrix
- rank of a matrix, 90, 141
- reachability Gramian
  - continuous time, 91–95
  - discrete time, 95–98
  - infinite time, 102
- reachable subspace
  - continuous time, 87–92
  - discrete time, 95–98
- reachable system
  - continuous time, 99
- realization, 30
  - minimal, 83, 139–144, 159–161, 166
  - order of, 139, 159–161
  - SISO, 32, 35
- real polynomial matrix, 150
  - determinantal divisors, 150
  - invariant factors, 151
  - minors, 150
  - rank, 150
  - unimodular, 151
- real rational matrix, 151
- reference tracking, 192
- Riemann integral, 27
- right-inverse matrix, 141
- Ritz approximation, 216–217
- RMS gain, *see* root mean square gain
- robot arm, 15
- roll-off rate, 185
- root locus, 189
- root mean square gain, 215, 221
- Rosenbrock's system matrix, 158, 204
  
- satellite, 105
- Schur stable matrix, 47, 53, 66, 128
- semigroup property, *see* state transition matrix
- semisimple matrix, *see* diagonalizable matrix
- sensitivity function, 183
- separation principle, 137, 202, 208
- separation theorem, *see* separation principle
- set-point control, 187, 203–206, 208
- settling time, 214, 219
- SI, *see* single-input system
- similarity transformation, 33, 107–110, 131–134, 141–142
- SIMO, *see* single-input, multiple-output system
- Simulink, *see* MATLAB
- single-input, multiple-output system (SIMO), 5
- single-input, single-output system (SISO), 5
- single-input system (SI), 5
- single-output system (SO), 5
- SISO, *see* single-input, single-output system
- Smith
  - factorization, 151
  - form, 150–151, 158, 160
- Smith-McMillan
  - factorization, 152
  - form, 151–155, 157, 160, 163
- SO, *see* single-output system
- square completion, 81, 114, 115, 172
- stability matrix, 46, 54, 61, 63, 71, 102, 103, 128, 172
- stabilizability test
  - eigenvector, 112

- Lyapunov, 113–115
  - Popov-Belevitch-Hautus (PBH), 113
- stabilizable system, 111–115
- stabilization
  - output feedback, 136–137
  - state feedback, 103–104, 114–115
- stabilizing solution to the ARE, 176
- stable subspace, 176
- standard form for
  - uncontrollable systems, 108, 111
  - unobservable systems, 132, 134
- state estimation, 135–136
- state estimation error, 136
- state feedback, 121, 135
- state reconstruction
  - Gramian-based in continuous time, 124–125
  - Gramian-based in discrete time, 126
- state transition matrix, 41
  - continuous time, 37–39
  - discrete time, 40
  - inverse, 39, 40
  - semigroup property, 38, 40
- Stein equation, *see* Lyapunov equation, discrete time
- step response, 214, 219
- strict feedback form, 19
- strictly proper rational function, 31
- strict feedback form, 17–18
  
- time invariance, 22, 24, 35
- time reversibility
  - controllability/reachability, 93
  - observability/constructibility, 127
- time scaling
  - controllability and reachability, 93
  - observability/constructibility, 127
- time shifting, 22
- tracking error, 192
- transfer function
  - continuous time, 25–26, 29–30
  - discrete time, 26, 30
- transmission-blocking property
  - invariant zeros, 159
  - transmission zeros, 153–155
- transmission blocking property, 204
- transpose of a vector/matrix, 7
  
- unconstructible subspace
  - continuous time, 123–125
  - discrete time, 125–126
- uncontrollable state component, 109
- uncontrollable system
  - continuous time, 99
- under-actuated system, 203
- undershoot, 214, 219
  
- unicycle, 18
- uniformly-bounded signal, 74
- unobservable state component, 132
- unobservable subspace, 131
  - continuous time, 122–125
  - discrete time, 125–126
  
- variation of constants formula
  - continuous time, 39, 44, 87, 122, 134
  - discrete time, 40, 47
  
- $\mathcal{Z}$ -transform, 26, 35
- zero-state equivalence, 30, 33, 36, 140
- zero-state response, 24
- zeros
  - invariant, 158–160, 165, 204
  - SISO, 143, 149
  - transmission, 149, 152–155, 158–160, 164–166, 189, 190, 203, 204
- zero polynomial, 152