

LINEAR SYSTEMS THEORY

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Disclaimer: This is a draft and probably contains a few typos.

Comments and information about typos are welcome.

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