Practice Mid-term Exam
Linear Systems

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Please explain all you answers.

1. The following equation models the motion of a pendulum:

$$\ddot{\theta} + k \sin\theta = \tau$$

where $\theta \in \mathbb{R}$ is the angle of the pendulum with the vertical, $\tau \in \mathbb{R}$ an applied torque, and k a positive constant.

   (a) Compute a state-space model for the system when $u := \tau$ is viewed as the input and $y := \theta$ as the output.

   (b) Compute the linearization of the system around the solution $\tau(t) = \theta(t) = \dot{\theta}(t) = 0$, $t \geq 0$.

2. Consider the homogeneous linear time-varying system

$$\dot{x} = A(t)x, \qquad x(0) = x_0$$

with state transition matrix $\Phi(t,\tau)$. Consider also the non-homogeneous system

$$\dot{z} = A(t)z + x(t), \qquad z(0) = z_0.$$

whose input $x(t)$ is the state of the homogeneous system.

   (a) Compute $x(t)$ and $z(t)$ as a function of x_0, z_0, and Φ. Hint: no integrals should appear in your answer.

   (b) What must be true of z_0 to have $z(T) = 0$ for some particular time $T > 0$.

3. Compute e^{At} for

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -2 \end{bmatrix}.$$

Is the system $\dot{x} = Ax$ asymptotically stable? What about marginally stable?

4. Compute the transfer function of the system

$$\dot{x} = Ax + bu$$
$$y = cx + u,$$

where

$$A := \begin{bmatrix} -2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix}, \qquad b := \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, \qquad c := \begin{bmatrix} 1 & 1 & 1 \end{bmatrix}.$$

Is this system BIBO stable?

5. Can the inverse of a $n \times n$ nonsingular matrix A be written as a linear combination of the matrices $I, A, A^2, \ldots, A^{n-1}$? Carefully justify your answer. Hint: Mr. Cayley and his friend Mr. Hamilton would know how to do this with their hands tied behind their backs.