Quadratic Control of Stochastic Hybrid Systems with Renewal Transitions

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Abstract

We study the quadratic control of a class of stochastic hybrid systems with linear continuous dynamics for which the lengths of time that the system stays in each mode are independent random variables with given probability distribution functions. We derive a condition for finding the optimal feedback policy that minimizes a discounted infinite horizon cost. We show that the optimal cost is the solution to a set of differential equations with unknown boundary conditions. Furthermore, we provide a recursive algorithm for computing the optimal cost and the optimal feedback policy. The applicability of our result is illustrated through a numerical example, motivated by stochastic gene regulation in biology.

Key words: Markov renewal processes, semi-Markov processes, optimal control, stochastic hybrid systems, renewal transitions

1 Introduction

Hybrid Systems combine continuous dynamics and discrete logic. By introducing randomness in the execution of a hybrid system, one obtains Stochastic Hybrid Systems (SHSs). As surveyed in Pola et al. (2003); Prandini & Lygeros (2010), various models of stochastic hybrid systems have been proposed differing on where randomness comes into play. In most of the models mentioned in these surveys, the solutions are assumed to be unique, however, some researchers have recently proposed modeling tools for a class of uncertain hybrid systems with not necessarily unique solutions Teel (2013). Markov Jump Linear (MJL) systems, can be viewed as a special class of stochastic hybrid systems that has been studied in the control community for the past few years. One can trace the applicability of MJL systems to a variety of processes that involve abrupt changes in their structures (e.g. chemical plants, robotic manipulator systems, solar thermal receiver, biological systems, paper mills, etc. Costa et al. (2005)). In MJL systems, the waiting times between consecutive jumps are assumed to be exponentially distributed Mariton (1990). Thus, over sufficiently small intervals, the probability of transition to another state is roughly proportional to the length of that interval. The memoryless property of the exponential distribution simplifies the analysis of MJL systems, however, in many real world applications, the time intervals between jumps have probability distributions other than the exponential.

We consider a Stochastic Hybrid System with renewal transitions in which the holding times (time between jumps) are independent random variables with given probability distribution functions, and the embedded jump chain is a Markov chain. This can be viewed as a generalization of the Markov Jump Linear systems Mariton (1990) or as a generalization of the Renewal systems Davis (1993) in which there is only one mode. SHSs with renewal transitions cover a wider range of applications, where the transition rates depend on the length of the time residing in the current mode. This work follows the definition of Piecewise Deterministic Markov Processes in Davis (1993), SHS in Hespanha (2005); Bujorianu &
Lygeros (2004), and in particular, the formulation of SHS with renewal transitions in Antunes et al. (2010).

The key challenge in studying SHSs with renewal transitions lies in the fact that the Markov property of MJL systems does not hold. This prevents the direct use of approaches based on Dynkin’s formula Mariton (1990). However, this issue can be overcome by adding a timer to the state of the system that keeps track of the time elapsed since the last transition. This approach was introduced in Davis (1993).

Inspired by the ideas in Swoyer (1980), we consider the quadratic control of SHSs with renewal transitions. We derive an optimal control policy that minimizes a discounted infinite horizon LQR problem by regulating the state of the system around a equilibrium point that may be mode-dependent. We show that the optimal cost is the solution to a set of differential equations (so-called Bellman equations) with unknown boundary conditions. Furthermore, we provide a numerical technique for finding the optimal solution and the corresponding boundary conditions. These are the main contributions of this paper.

While the proofs of our results are inspired by the extended generator approach for Piecewise Deterministic Markov Processes in Davis (1993), we do not require the assumption that the value function belongs to the domain of the extended generator of the closed-loop process. Diverging from Davis (1993), we also do not require the vector field of the process to be bounded in x uniformly over the control signal, which would not hold for linear dynamics. We overcome this issue by deriving a Dynkin’s-like formula for the “stopped process” Kushner (1967), which under appropriate assumptions converges to the original process.

Our main motivation for considering this problem is to study how living organisms respond to environmental fluctuations by orchestrating the expression of sets of genes, Pour Safaie et al. (2012). The model that we use to represent the gene regulation problem in fluctuating environments is a special case of Piecewise Deterministic Markov processes, Davis (1993), and Stochastic Hybrid Systems, Hespanha (2005). We illustrate the applicability of our results through a numerical example motivated by the metabolism of sugar by E. Coli in the presence and absence of lactose in the environment. Considering linear dynamics for enzymes and mRNA, we compute the optimal gene regulator that minimizes the expected square difference between the current states and the one that would be optimal for the current environment, plus the cost of protein production/decay, integrated over the life span of the bacterium.

The paper is organized as follows. Section 2 introduces the mathematical model and the infinite horizon discounted cost to be minimized. In Section 3, a sufficient condition for optimal feedback policy is derived. We derive a set of differential equations (with unknown boundary conditions) to be satisfied by the optimal cost. A numerical algorithm is provided for finding the optimal solution. Section 4 provides a gene regulation example that has motivated us for solving this problem. We consider the metabolism of lactose in E. Coli bacterium in stochastically varying environments. We finally conclude the paper in Section 5 with some final remarks and directions for future research.

Notation. We denote the underlying measurable space by $(\Omega, \mathcal{B})$ and the probability measure by $P : \mathcal{B} \to [0, 1]$. Stochastic processes $x : \Omega \times [0, \infty) \to \mathcal{X} \subset \mathbb{R}^n$ are denoted in boldface. We use “wpo” to denote universal quantification with respect to some subset of $\Omega$ with probability one [as in $x \overset{\text{wpo}}{\in} \mathcal{Z}$ to express that $\exists \omega \in \mathcal{B}$ such that $P(\bar{\omega}) = 1$ and $x(\omega) = \mathcal{Z}(\omega), \forall \omega \in \bar{\omega}$]. Notation $E_{x0}[x(t)]$ indicates expectation of the process $x$ conditioned upon initial condition $x_{0}$. $I_{q, q}$ denotes an indicator function. We also use the notation $t \times s = \min(t, s)$. We denote by $z(t^−)$ and $z(t^+)$ the limits from the left ($\lim_{s \rightarrow t^-} z(s)$) and the right ($\lim_{s \rightarrow t^+} z(s)$), respectively. For a given matrix (vector) A, its transpose is denoted by $A^T$.

2 Problem statement

We consider a class of Stochastic Hybrid Systems (SHSs) with linear dynamics, for which the lengths of the time intervals that the system spends in each mode are independent random variables with given distribution functions. The state space of such system consists of a component $x$ that takes value in the Euclidean space $\mathbb{R}^n$, and a discrete component $q$ that takes value in a finite set $\mathcal{S} = \{q_1, ..., q_N\}$ with $\mathcal{N}$ denoting the number of modes.

A linear stochastic hybrid system with renewal transitions takes the form of

$$\dot{x} = A_q x + B_q u + d_q \quad (x, q) \in \mathbb{R}^n \times \mathcal{S},$$ (1)

where the control input $u(t) \in \mathbb{R}^n$ may depend on $(x(s), q(s))$ for all $s \leq t$ through a causal feedback law and the affine term $d_q \in \mathbb{R}^n$ is a mode-dependent bias term. The causality relation between $u$ and $(x, q)$ can be formalized by requiring $u$ to be adapted to the natural filtration generated by $(x(s), q(s))$, see (Bichteler, 2002, page 39) for the definition of the natural filtration.

Let $\{t_k\}$ denote the sequence of jump times. Given $q(t) = i, i \in \mathcal{S}$, $\forall t \in [t_k, t_{k+1})$, the time intervals between consecutive jumps $h_k := t_{k+1} - t_k$ are assumed to be independent random variables with a given cumulative distribution function $F_t(\tau)$ on a finite support $[0, T_i]$, $0 < T_i < \infty$ where $T_i$ is the smallest value in $\mathbb{R} \geq 0$ for which $F_i$ is one:

$$0 \leq F_i(\tau) < 1 \quad \tau \in [0, T_i), \quad F_i(T_i) = 1. \quad (2)$$
This stochastic hybrid system characterizes a stochastic process in \( \mathbb{R}^n \) called the continuous state, and a jump process \( q \) called the discrete state or mode. Between jump times, the continuous state flows according to (1) while the discrete state remains constant. When a jump happens, the discrete mode is reset according to a time-homogeneous Markov chain

\[
P(q(t_k) = j \mid q(t^-) = i) = P_{ij}, \quad \sum_{j \neq i} P_{ij} = 1 \quad (3)
\]

and \( x \) is reset according to

\[
x(t_k) = H_{ij}x(t^-_k) \quad \text{if} \quad q(t_k) = j, \quad q(t^-_k) = i \quad (4)
\]

with \( H_{ij} \in \mathbb{R}^{n \times n} \) for all \( i, j \in S \). In this paper, we assume that all signals are right continuous, therefore \( x(t^+_k) = x(t_k) \) and \( q(t^+) = q(t) \) at all times \( t \geq 0 \) (including the jump times). Even if we were to set \( u(t) \) to be a deterministic function of \((x(t), q(t))\), the stochastic process \((x(t), q(t))\) might not be a Markov process as the time intervals between jumps are not exponential random variables. The reason is that, at a given time \( t \), the time \( t_{k+1} - t \) until the next jump time \( t_{k+1} \), typically depends on the time \( \tau := t - t_k \) elapsed since the last jump \( t_k \), which can be deduced from past values of the state, but not necessarily from the current state. However, given the elapsed time \( \tau = t - t_k \), no other information about the past has any relevance to the process in future. This is due to the assumption that the future intervals \( t_{k+1}, t_{k+2}, \ldots \) are independent of the past ones \( t_k, t_k-1, \ldots \).

Defining a three-component process \((x(t), \tau(t),q(t))\), where \( \tau = 1 \) between jumps, and \( \tau \) is reset to zero after the jumps, the variable \( \tau \) keeps track of the time elapsed since the last jump. It turns out that when the input \( u(t) \) is a deterministic function of \( x(t), \tau(t),q(t) \), the process \((x(t), \tau(t),q(t))\) is now a Markov process (Davis, 1993, Chapter 2).

We assume that cumulative distribution functions \( F_i \) are absolutely continuous and can be written as \( F_i(\tau) = \int_0^{\tau} f_i(s) \, ds \) for some density functions \( f_i(\tau) \geq 0 \). In this case, one can show that the conditional probability of having a jump in the interval \((t, t+dt)\), given that \( \tau(t) = \tau \) is given by

\[
P(q(t+dt) = j \mid q(t) = i, j \neq i) = P_{ij} \lambda_i(\tau) \, dt + o(dt)
\]

for all \( i, j \in S, \tau \in [0, T_1] \), where

\[
\lambda_i(\tau) := \frac{f_i(\tau)}{1 - F_i(\tau)} \quad \tau \in [0, T_1) \quad i \in S,
\]

is called the hazard rate associated with the renewal distribution \( F_i \) (Antunes et al. 2009). The construction of sample paths for this process is similar to that in Hesspanha (2005); Bujorianu & Lygeros (2004). For a given initial condition \( z := (x, \tau, q) \) with \( x \in \mathbb{R}^n, q \in S, \tau \in [0, T_q] \) construct the processes \((x(t), \tau(t),q(t)), t \geq 0 \) as follows:

(i) If \( \tau = T_q \), set \( k = 0 \) and \( t_0 = 0 \).

(ii) If \( \tau < T_q \), obtain the jump interval \( h_0 \) as a realization of the conditional distribution of \( h_0 \) given that \( h_0 > \tau \):

\[
F_q(h_0 \mid h_0 > \tau) = \begin{cases} 0 & \text{if } h_0 < \tau \\ \frac{F_q(h_0) - F_q(\tau)}{1 - F_q(\tau)} & \tau \leq h_0 < T_q \\ 1 & h_0 \geq T_q, \end{cases}
\]

and define \( x(0) = x, q(0) = q, \tau(0) = \tau \). The continuous state of the SHS in the interval \([0, h_0 - \tau] \) flows according to (1), the timer \( \tau \) evolves according to \( \tau = 1 \) and \( q(t) \) remains constant. Set \( k = 1 \) and \( t_1 = h_0 - \tau \). One should note that when \( \tau < T_q \), the event \( t_1 \leq 0 \) (\( h_0 \leq \tau \)) happens with zero probability.

(iii) Reset \( \tau(t_k) = 0 \), update \( q(t_k) \) as a realization of a random variable distributed according to (3) and reset \( x(t_k) \) according to (4).

(iv) Obtain \( h_k \) as a realization of a random variable distributed according to \( F_q(t_{k+1}) \), and set the next jump time \( t_{k+1} = t_k + h_k \).

(v) The continuous state of the SHS in the interval \([t_k, t_{k+1}] \) flows according to (1), the timer \( \tau \) evolves according to \( \tau = 1 \) and \( q(t) \) remains constant.

(vi) Set \( k \to k + 1 \), and jump to (iii).

The above algorithm does not guarantee the existence of sample paths on \([0, \infty) \). This construction can fail if either the stochastic process defined by (1) has a finite escape time (which could only occur with a non-linear control) or if \( \lim_{t \to \infty} t_k \to L < \infty \). Both cases would lead to a “local-in-time solutions”, which we will eventually show that cannot happen for the optimal feedback law.

### 2.1 Quadratic cost function

The goal is to regulate \( x(t) \) around a nominal point \( \hat{x}_q \) that may depend on the current discrete state \( q \), while maintaining the control input \( u \) close to a nominal value \( \bar{u}_q \) that may also depend on \( q \). To this effect, we consider an infinite horizon discounted cost function with a quadratic penalty on state and control excursions of the
where $J_\mu$ given in (7). If there exist $C^1$ and bounded solutions $\Lambda_i : [0, T_i] \to \mathbb{R}^{n \times n}$, $\Gamma_i : [0, T_i] \to \mathbb{R}^n$ and $\Theta_i : [0, T_i] \to \mathbb{R}$, $i \in S$ to the following differential equations with the associated boundary conditions,
\begin{align}
\left\{ \begin{array}{l}
-\frac{d\Lambda_i}{dt} = A_i^\top \Lambda_i + \Lambda_i A_i - \rho \Lambda_i - \Lambda_i B_i R_q^{-1} B_i^\top \Lambda_i + Q_i + \lambda_i(\tau) \sum_{j \neq i} P_{ij} Q_j \Lambda_j(0) H_{ij} - \Lambda_i(\tau) \\
\Lambda_i(T_i) = \sum_{j \neq i} P_{ij} Q_j \Lambda_j(0) H_{ij} + \frac{\rho-\lambda}{\tau} \sum_{i} \Lambda_i(\tau) [H_i^\top \Gamma_i(0) - \Gamma_i(\tau)] + 2 \Lambda_i(B_i \bar{u}_i + d_i) \\
-\frac{d\Gamma_i}{dt} = (A_i^\top - \Lambda_i B_i R_q^{-1} B_i^\top - \rho I) \Gamma_i - 2 Q_i \bar{x}_i + \lambda_i(\tau) \sum_{j \neq i} P_{ij} H_{ij} \Gamma_j(0) - \sum_{j \neq i} \Theta_{ij}(T_i) \sum_{j \neq i} P_{ij} \Theta_j(0) \\
\Theta_{ij}(T_i) = \sum_{j \neq i} P_{ij} \Theta_j(0)
\end{array} \right. \tag{9}
\end{align}

such that for every $q \in S$, $V^*(x, \tau, q) := x^\top \Lambda_q(\tau) x + x^\top \Gamma_q(\tau) + \Theta_q(\tau)$ is a non-negative function, then the feedback policy
\begin{align}
\mu^*(x, \tau, q) := \bar{u}_q - \frac{1}{2} R_q^{-1} B_q^\top (2 \Lambda_q(\tau) x + \Gamma_q(\tau)) \tag{12}
\end{align}
is optimal over all feedback policies of the form (8) for which we have $E_x \{ ||x(t)||^2 \} \leq k_1 e^{-k_2 t} ||x_0||$ for some $k_1 > 0, k_2 > -\rho$. Moreover, the minimal cost is given by $J^*(x_0, \tau_0, q_0) = V^*(x_0, \tau_0, q_0)$. It is worth noting that (9) is a system of ordinary differential Riccati-like equations that should satisfy the given boundary conditions. If we were given initial conditions or terminal conditions for the ODEs in (9)-(11), we could solve the ODE in (9) using the methods developed by Kenney & Leipnik (1985); Rusnak (1988). With the solution to (9), we could then proceed to solve (10) and (11). However, in general we do not know a priori the initial/terminal conditions for the ODEs in (9)-(11). This difficulty shall be overcome in Section 3.1, where we provide a numerical algorithm that computes the $\Lambda_i, \Gamma_i, \Theta_i$ based on a recursive system of differential equations. Moreover, our main focus in this article is to consider feedback policies of the form (8) for which we have $E_x \{ ||x(t)||^2 \} \leq k_1 e^{-k_2 t} ||x_0||$. This implies that we only focus on feedback policies for which the expectations are well-defined.

The following proposition and lemmas are needed to prove Theorem 3.1.

**Proposition 1** \( t_k \to \infty \) as \( k \to \infty \) with probability one.
Proof of Proposition 1. Let \( h_k \) denote the time interval between the jump time \( t_k \) and the subsequent jump \( t_{k+1} \). Suppose \( q(t_k) = i \) which implies that the system is in mode \( i \) during the time interval \([t_k, t_{k+1})\). Since \( h_k \) has the probability distribution \( F_i \) and \( F_i(0) < 1 \) for all \( i \in \mathcal{S} \), we have \( P(h_k > 0) = 1 - F_i(0) > 0. \) Suppose that for all \( h > 0 \), we have \( P(h_k > h) = 0 \). This implies that \( F_i(h) = 1 \) for all \( h \in [0, T_i) \), which contradicts (2). Therefore, there exists \( h_i > 0 \) such that \( P(h_k > h_i) > 0 \). Let \( h = \min_{i \in \mathcal{S}} h_i \). By the second Borel-Cantelli Lemma, see Karlin & Taylor (1975), it follows that with probability one, \( h_k > h \) for infinitely many \( k \). Therefore, \( \lim_{k \to \infty} t_k = \sum_{k=1}^\infty h_k \to \infty \) with probability one. \( \square \)

**Lemma 3.1** Suppose that \( \Lambda_q(\tau) \) and \( \Gamma_i(\tau) \) are bounded \( C^1 \) functions \( \forall \tau \in \mathcal{S} \). Then the stochastic process defined by (1) with the feedback policy \( \mu^* \) in (12) exists on \([0, \infty)\) for almost all realizations of the process.

Proof of Lemma 3.1. Between jumps, the right hand side of (1) with \( u = \mu^*(x, \tau, q) \) can be written as \( \dot{x} = \bar{A}(t)x + \bar{g}(t) \), where \( \bar{A}(t) \) and \( \bar{g}(t) \) are continuous functions on \( t \) and the elements of \( \bar{A}(t) \) are bounded. Thus, by (Khalil, 2002, Theorem 3.2), the solution exists between jumps. When a jump happens, \( x(t_k) \) is reset according to the linear reset map in (4). Therefore, the sample paths of system (1) do not have a finite escape time. Moreover, by Proposition 1, \( t_k \to \infty \) \( \text{wpo} \). Thus, the sample paths of the system exist on \([0, \infty)\) with probability one. \( \square \)

The proof of Theorem 3.1 depends on a Dynkin’s-like formula for a stopped process constructed based on \((x, \tau, q)\). This proof is inspired by the “Poisson driven” process in Kushner (1967) and the Piecewise Deterministic Markov Processes in Davis (1993), but avoids some strong assumptions in these references. We first define the stopped process and then derive a Dynkin-like formula for the stochastic process (1) under a feedback policy (8).

Suppose \( \mu \) is a feedback policy such that the stochastic process defined by (1) with (8) exists globally (no finite escape time). We denote \( X := (x, \tau, q) \). For every \( q \in \mathcal{S} \), let \( Q^q \) and \( Q\) be two compact subsets of \([0, T_q)\) and \( \mathbb{R}^n \), respectively. Define \( Q^q := Q_x \times Q^q \) which is also compact, see (Willard, 1970, Theorem 17.8) and \( Q := \bigcup_{q \in \mathcal{S}} Q^q \). The \( Q \)-stopped process from the process \( X \) will be defined as following. We define the \( Q \)-stopping time

\[
T_Q := \sup \{ \{ t \in [0, \infty) : \forall \tau \in [0, t) , \ q(t) = q \Rightarrow (x(t), \tau(t)) \in \text{Int}(Q^q) \} \cup \{0\} \}
\]

where \( \text{Int}(Q^q) \) denotes the interior of \( Q^q \). Based on \( T_Q \), we define the \( Q \)- stopped process as

\[
X_Q(t) := \begin{cases} X(t) & t \in [0, T_Q) \\ X(T_Q) & t \in [T_Q, \infty) \end{cases} \tag{13}
\]

and the \( Q \)-stopped jump counter as

\[
N_Q(t) := \begin{cases} N(t) & t \in [0, T_Q) \\ N(T_Q) & t \in [T_Q, \infty) \end{cases} \tag{14}
\]

where \( N(t) := \max\{k : 0 < t_k \leq t\} \) counts the number of jumps up to time \( t \) with the understanding that \( N(0) = 0 \). These stochastic processes are well-defined for all times since the system has no finite escape time and by Proposition 1, \( t_k \to \infty \) with probability one. The following result will be used in deriving the Dynkin’s formula.

**Lemma 3.2** For any compact sets \( Q_1, Q_2 \subseteq \mathbb{R}^n \) and \( \tau_q \subseteq [0, T_q) \) for all \( q \in \mathcal{S} \), the expected value of \( N_Q(t) \) is finite on any finite time interval. In particular,

\[
E\{N_Q(s_2) - N_Q(s_1)\} \leq \lambda_{\max}(s_2 - s_1),
\]

for all \( s_2 \geq s_1 > 0 \) and \( \lambda_{\max} := \sup_{q \in \mathcal{S}} \sup_{\tau \in \tau_q} \lambda_q(\tau) \).

Proof of Lemma 3.2. Since the hazard rate \( \lambda_q(\tau) \) is continuous on \([0, T_q)\), \( \lambda_q(\tau) \) is locally bounded on \([0, T_q)\) for all \( q \in \mathcal{S} \), and \( \lambda_{\max} \) always exists over compact sets \( \tau_q \). Since there are no more jumps after \( T_Q \),

\[
E\{N_Q(s_2) - N_Q(s_1)\} = E\{N_Q(s_2 \wedge T_Q) - N_Q(s_1 \wedge T_Q)\}
\]

\[
= E\{N(s_2 \wedge T_Q) - N(s_1 \wedge T_Q)\}.
\]

Moreover, one can show that \( N(t) = N(\int_0^t \lambda_q(\tau_s) \, ds) \) where \( N(t) \) is the standard Poisson process, see Ap-
penduously differentiable with respect to its first and second

\[\begin{align*}
E\{\mathbf{N}(s_2 \wedge T_{Q}) - \mathbf{N}(s_1 \wedge T_{Q})\} &= E\left\{\int_0^{s_1 \wedge T_{Q}} \lambda_q(\tau(s)) \, ds\right\} \\
&= E\left\{\mathbf{N}\left(\int_0^{s_1 \wedge T_{Q}} \lambda_q(\tau(s)) \, ds\right) \right\} \\
&= E\left\{\mathbf{N}\left(\int_0^{s_1 \wedge T_{Q}} \lambda_q(\tau(s)) \, ds + \int_0^{s_2 \wedge T_{Q}} \lambda_q(\tau(s)) \, ds\right) \right\} \\
&= E\left\{\mathbf{N}\left(\int_0^{s_1 \wedge T_{Q}} \lambda_q(\tau(s)) \, ds\right) \right\} \\
&= \lambda_\max(s_2 - s_1).
\end{align*}\]

Here, we used the fact that for a Poisson process \(\mathbf{N}\), we have \(E\{\mathbf{N}(b) - \mathbf{N}(a)\} = b - a\) for all \(b \geq a > 0\) (Karlin & Taylor, 1975).

The following notations will be used in the proof of Theorem 3.1. For every function \(g(x, \tau, q)\) such that for fixed \(q \in \mathcal{S}\), \(g\) is continuously differentiable with respect to its first and second arguments on \(\mathbb{R}^n \times [0, T_q]\), we define the operators \(L^f_q\) and \(L^g_q\) as

\[\begin{align*}
L^f_q g(x, \tau, q) &= \frac{\partial g(x, \tau, q)}{\partial x} \left(A_q x + B_q \nu\right) + \frac{\partial g(x, \tau, q)}{\partial \tau} \lambda_q(\tau) \\
L^g_q g(x, \tau, q) &= \int_0^\tau \rho_s g(H_q x, 0, j) - g(x, \tau, q) \, ds \\
&+ \lambda_\max(\tau) \left(\sum_{j \neq q} P_{qj} \right) g(H_q x, 0, j) - g(x, \tau, q)
\end{align*}\]

\[\begin{align*}
\forall q \in \mathcal{S} \text{ and all } x \in \mathbb{R}^n, \tau \in [0, T_q], \nu \in \mathbb{R}^m. \text{ When the vector } \nu \text{ is set equal to a function of } (x, \tau, q), \text{ e.g. } \nu = \mu(x, \tau, q), \text{ we use the notation } L^f_{\mu(x, \tau, q)}. \text{ The definition in (15) is inspired by the extended generator in the Markov process literature Davis (1993); Hespanha (2005), but here we further extend this operator to every continuously differentiable function } g \text{ (regardless of what the domain of the extended generator may be).}
\end{align*}\]

Given compact sets \(Q^q \subset \mathbb{R}^n \times [0, T_q]\) \(q \in \mathcal{S}\), we define

\[L^g_q g(x, \tau, q) := \begin{cases} L^g_q g(x, \tau, q) & (x, \tau, q) \in \text{Int}(Q^q) \\
0 & \text{otherwise} \end{cases}\]

\[\forall q \in \mathcal{S} \text{ and all } x \in \mathbb{R}^n, \tau \in [0, T_q], \nu \in \mathbb{R}^m. \text{ Next, we derive a Dynkin’s-formula for the stopped process.}
\]

**Lemma 3.3** Suppose that \(Q_1 \subset \mathbb{R}^n \) and \(Q^q \subset [0, T_q]\), \(\forall q \in \mathcal{S}\) are all compact sets and define \(Q := \bigcup_{q \in \mathcal{S}} Q^q \times Q^q\). Let \(\mu\) in (8) be a feedback policy such that the stochastic process defined by (1) exist globally with probability one. For every function \(V(X) = V(x, \tau, q)\) that is continuously differentiable with respect to its first and second arguments, every initial mode \(q \in \mathcal{S}\), every initial condition \((x, \tau) \in \mathbb{R}^n \times [0, T_q]\) and every \(0 \leq t < \infty\), we have that

\[\begin{align*}
e^{-pt}E_z\{V(X_Q(t))\} &= V(x, \tau, q) \\
&+ I_{(\tau=T_q)}\left(\sum_{j \neq q} P_{qj} V(H_q x, 0, j) - V(x, \tau, q)\right) \\
&+ e^{pt} \left(\int_0^t e^{-pt} L^f_q V(X_Q(s)) - \rho V(X_Q(s)) \, ds\right),
\end{align*}\]

where \(X_Q\) is the stopped process defined by (13) with the process \(X\) constructed according to procedure described in Section 2, initialized with \(z = (x, \tau, q)\).

**Proof of Lemma 3.3.** Let \(N_Q(t)\) denote the \(Q\)-stopped jump timer in (14). One should note that \(N_Q(t)\) does not count a jump that may happen at 0 due to \(\tau = T_q\). Between jump times, \(X = (x, \tau, q)\) is absolutely continuous and evolves according to the vector field of the process, and at a jump time \(t_k\), \(k \leq N(t)\), we have an instantaneous jump from \(X(t_k^-)\) to \(X(t_k) = (x(t_k), 0, q(t_k))\). Therefore, one can conclude that

\[\begin{align*}
e^{-pt}V(X(t \wedge T_Q)) &= V(x, \tau, q) \\
&+ I_{(\tau=T_q)}\left[\rho V(Q_{q0} x, 0, q(0)) - V(x, \tau, q)\right] \\
&+ \int_0^{T_Q} e^{-pt} L^f_q V(X(s)) - \rho V(X(s)) \, ds \\
&+ \sum_{k=1}^{\infty} e^{-pt_k} I_{(k \leq N_Q(t))}\left(V(X(t_k)) - V(X(t_k^-))\right),
\end{align*}\]

with \(I_{(k \leq N_Q(t))}\) denoting the indicator function of the event \(k \leq N_Q(t)\) and \(I_{(\tau=T_q)}\) indicates if a jump happens at time 0. We have \(X_Q(s) = X(s)\) for all \(s \in [0, t \wedge T_Q]\) and these processes are also equal at any jump time in \(t_k\), \(k \leq N_Q(t)\). Moreover, \(X_Q\) remains constant on \([t \wedge T_Q, t)\). Thus, we conclude that

\[\begin{align*}
e^{-pt}V(X_Q(t)) &= V(x, \tau, q) \\
&+ I_{(\tau=T_q)}\left[\rho V(Q_{q0} x, 0, q(0)) - V(x, \tau, q)\right] \\
&+ \int_0^{t \wedge T_Q} e^{-pt} L^f_q V(X_Q(s)) - \rho V(X_Q(s)) \, ds \\
&+ \sum_{k=1}^{\infty} e^{-pt_k} I_{(k \leq N_Q(t))}\left(V(X_Q(t_k)) - V(X_Q(t_k^-))\right),
\end{align*}\]

where for every \(q \in \mathcal{S}\),

\[\begin{align*}
L^f_{\mu} g(x, \tau, q) &= \begin{cases} L^f_q g(x, \tau, q) & (x, \tau, q) \in \text{Int}(Q^q) \\
0 & \text{otherwise} \end{cases}
\end{align*}\]

and \(Q^q = Q_1 \times Q^q\). We take the expected value of the both sides of (17), conditioned upon the initial condition \(z = (x, \tau, q)\). Starting with the expectation of the second
Here, we used the fact that

Moreover, using the density functions of the inter-jump intervals. We also define

This is due to steps (i) and (iii) in the construction of the sample paths. We also define

Next, the conditional expectation of the summation in (17) can be computed as

\[ E_x \left\{ \sum_{k=1}^{\infty} e^{-\rho s} I_{(k \leq N_Q(t))} (V (X_Q(t_k)) - V (X_Q(t_k^-))) \right\} = E_x \left\{ \sum_{k=1}^{\infty} e^{-\rho s} I_{(k \leq N_Q(t))} h (X_Q(t_k^-)) \right\}, \]

due to step (iii) in the construction of the sample paths (regardless of whether \( \tau = T_q \) or \( \tau < T_q \)). We use the Dominated Convergence Theorem (Klebaner, 2005) to move the series inside the expectation: To verify that the theorem is applicable, note that for \( k \leq N_Q(t) \) and \( q_j \in \mathbb{S} \), \( (X_Q(t_k^-), \tau, q_j) \in \mathcal{Q}^0 \) with probability one, and we have

\[ |e^{-\rho s} I_{(k \leq N_Q(t))} h (X_Q(t_k^-))| \xrightarrow{w.p.o.} \leq h_Q \]

\[ h_Q := \sup_{q \in \mathbb{S}} \sup_{(x, \tau, q) \in \mathcal{Q}^0} |h(X)| < \infty. \]

Here, we used the fact that \( V(x, \tau, q) \) is \( C^1 \) and therefore \( h(x, \tau, q) \) is locally bounded for any \( q \in \mathbb{S} \). Moreover, since the number of jump times \( t_k \) with \( k \leq N_Q(t) \) is upper bounded by \( N_Q(t) \), with probability one, we conclude that for every finite integer \( M \)

\[ \left| \sum_{k=1}^{M} e^{-\rho s} I_{(k \leq N_Q(t))} (X_Q(t_k^-)) \right| \xrightarrow{w.p.o.} \leq \sum_{k=1}^{\infty} I_{(k \leq N_Q(t))} |h(X_Q(t_k^-))| \leq h_Q N_Q(t) \]

and by Lemma 3.2, we know that \( E_x \{ h_Q N_Q(t) \} \leq h_Q \lambda_{\max} t < \infty \) for every \( t < \infty \). We therefore conclude from the Dominated Convergence Theorem (Klebaner, 2005) that

\[ E_x \left\{ \sum_{k=1}^{\infty} e^{-\rho s} I_{(k \leq N_Q(t))} h (X_Q(t_k^-)) \right\} = E_x \left\{ \sum_{k=1}^{\infty} e^{-\rho s} I_{(k \leq N_Q(t))} h (X_Q(t_k^-)) \right\}. \]

Moreover, using the density functions of the inter-jump intervals and defining \( t_0 := 0 \), one can show that (Appendix B)

\[ \sum_{k=1}^{\infty} E_x \left\{ e^{-\rho s} I_{(k \leq N_Q(t))} h (X_Q(t_k^-)) \right\} = \sum_{k=1}^{\infty} E_x \left\{ \int_{t_k}^{t_{k+1}} e^{-\rho s} (h \lambda_Q (X_Q(s))) ds \right\} \]

with \( (h \lambda_Q (x, \tau, q) := h(x, \tau, q) \lambda_q (\tau) \) inside \( Q^0 \) and zero otherwise. We are able to use the result of Appendix B, because \( V \) is continuous, the solutions exist globally, and therefore \( t \rightarrow h(X(t)) \) is integrable on any finite time interval \([0, t]\). We now use the Dominated Convergence Theorem to move the series inside the expectation in (19): To verify that this is allowed, note that by defining \( \lambda_{\max} := \sup_{q \in \mathbb{S}} \sup_{\tau \in \mathcal{Q}^0} \lambda_q (\tau) \) as before, any finite summation is dominated by

\[ \left| \sum_{k=1}^{M} \int_{t_k}^{t_{k+1}} e^{-\rho s} (h \lambda_Q (X_Q(s))) ds \right| \leq h_Q \lambda_{\max} t \]

for all \( M \geq 1 \) and \( t < \infty \). By Proposition 1, \( t_M \) becomes larger than any \( t < \infty \) with probability one, as \( M \) tends to infinity, therefore

\[ \sum_{k=1}^{\infty} \int_{t_k}^{t_{k+1}} e^{-\rho s} (h \lambda_Q (X_Q(s))) ds \xrightarrow{w.p.o.} \lim_{M \rightarrow \infty} \int_0^t e^{-\rho s} (h \lambda_Q (X_Q(s))) ds \]

We therefore conclude from the Dominated Convergence Theorem that, one can move the series inside the expectation in the left-hand side of (19) and obtain

\[ E_x \left\{ \sum_{k=1}^{\infty} e^{-\rho s} I_{(k \leq N_Q(t))} (V (X_Q(t_k)) - V (X_Q(t_k^-))) \right\} = E_x \left\{ \int_0^t e^{-\rho s} (h \lambda_Q (X_Q(s))) ds \right\}. \]

Using this equality and (18), when we take the expectation of both sides of (17) conditioned upon initial condition, we obtain

\[ e^{-\rho t} E_x \{ V (X_Q(t)) \} = V (x, \tau, q) + \int_{t=0}^{t} \left( \sum_{j \neq q} P_{j} V (H_{j}(x, 0, j) - V (x, \tau, q)) \right) + E_x \left\{ \int_0^t e^{-\rho s} \left( \int_{t_k}^{t_{k+1}} e^{-\rho s} (h \lambda_Q (X_Q(s))) ds \right) \right\}. \]
Therefore,

$$
e^{-\rho t}E_z \{ V(X_{Q}(t)) \} = V(x, \tau, q)
$$

$$
+ I_{(\tau \in T_q)} \left( \sum_{j \neq q} P_{qj} V(H_{qj}x, 0, j) - V(x, \tau, q) \right)
$$

$$
+ E_z \left\{ \int_0^t e^{-\rho s} \left( \mathcal{L}^\mu V(X_{Q}(s)) - \rho V(X_{Q}(s)) \right) \, ds \right\}. \Box
$$

We introduce the following notation which is used in the proof of Theorem 3.1. For a given feedback policy \( \mu \), for which the stochastic process defined by (1) exists globally, let \( J_\mu \) be the corresponding cost:

$$
J_\mu(x, \tau, q) = E_z \left\{ \int_0^\infty e^{-\rho s} I_\mu(x, \tau, q) \, ds \right\} \tag{20}
$$

conditioned upon the initial conditions \( z = (x, \tau, q) \in \mathbb{R}^n \times [0, T_q] \times \mathcal{S} \), and define

$$
\ell_\mu(x, \tau, q) := (x - \hat{x}_q)Q_q(x - \hat{x}_q) + (\mu(x, \tau, q) - \bar{u}_q)R_q(\mu(x, \tau, q) - \bar{u}_q).
$$

We are now ready to prove Theorem 3.1.

**Proof of Theorem 3.1.** For given \( q \in \mathcal{S} \), \( x \in \mathbb{R}^n \) and \( \tau \in [0, T_q] \), consider the following equation:

$$
\min_{\nu \in \mathbb{R}^n} \{ \mathcal{L}_{\nu} V(x, \tau, q) - \rho V(x, \tau, q) \}
$$

$$
+ (x - \hat{x}_q)Q_q(x - \hat{x}_q) + (\nu - \bar{u}_q)R_q(\nu - \bar{u}_q) = 0. \tag{21}
$$

It is straightforward to show that \( V^* := \mu^*(x, \tau, q) \)

defined in (12) achieves the minimum in (21). Since \( A_q(\tau), \Gamma_q(\tau), \Theta_q(\tau) \) satisfy (9)-(11), one can show that defining \( V^*(x, \tau, q) = x^T \hat{A}_q(\tau)x + x^T \hat{T}_q(\tau) + \Theta_q(\tau) \), this function is a solution to (21) and satisfies the boundary condition

$$
V^*(x, T_q, q) = \sum_{j \neq q} P_{qj} V^*(H_{qj}x, 0, j). \tag{22}
$$

We first show that \( V^*(x, \tau, q) \) is the cost due to the feedback policy \( \mu^* \) and then that \( \mu^* \) is indeed the optimal policy. One should note that by Lemma 3.1, the stochastic process defined by (1) with feedback policy \( \mu^* \) exists globally with probability one.

Denoting \( X := (x, \tau, q) \), the function \( V^*(X) = V^*(x, \tau, q) \) is non-negative, \( C^1 \) with respect to \( (x, \tau) \), and uniformly bounded in \( \tau \) over \( [0, T_q] \) for a fixed value of \( x \). For \( m \geq 0 \), define \( Q_q(m) := \{ x \in \mathbb{R}^n : \sup_{q \in \mathcal{S}, \tau \in [0, T_q]} V(x, \tau, q) \leq m \} \), \( Q^*_q(m) := [0, (1 - e^{-m})T_q] \) and \( Q^m := \cup_{q \in \mathcal{S}} Q_q(m) \times Q^*_q(m) \).

As noted above, \( V^* \) is a solution to (21), so we have that

$$
\mathcal{L}_{\mu^*} V^* - \rho V^* = -\ell_{\mu^*} \leq 0 \tag{23}
$$

with \( \mu^* \) given in (12). Since \( \mathcal{L}_{\mu^*} V^*, \mathcal{L}_{\mu^*} V \) and \( \ell_{\mu^*} \) are quadratic functions in \( x \) and \( \Lambda_i(\tau), \Gamma_i(\tau), \Theta_i(\tau) \) are bounded in \( \tau \) for every \( i \in \mathcal{S} \), there exists a positive constant \( c \) such that \( \mathcal{L}_{\mu^*} V^* - \rho V^* \leq -cV^* \). Our goal is to use (16) but with \( X_{Q^m} \) replaced by \( X \), which we will achieve by making \( m \to \infty \). From the definition of the stopped process, we have \( X(t \wedge T_{Q^m}) = X_{Q^m}(t), \forall t \geq 0 \), and using (16), we get

$$
e^{-\rho t}E_z \{ V^*(X_{Q^m}(t)) \} = V^*(x, \tau, q)
$$

$$
+ I_{(\tau = T_q)} \left( \sum_{j \neq q} P_{qj} V^*(H_{qj}x, 0, j) - V^*(x, \tau, q) \right)
$$

$$
+ E_z \left\{ \int_0^t e^{-\rho s} \mathcal{L}_{\mu^*} V^*(X_{Q^m}(s)) - \rho V^*(X_{Q^m}(s)) \, ds \right\} \tag{24}
$$

for all \( t < \infty \). We show next that, for the function \( V^* \) considered here, the second term on the right in (24) is always zero. For \( \tau \leq T_q \), we have \( I_{(\tau = T_q)} = 0 \), and for \( \tau = T_q \), one can use (22) to conclude that

$$
I_{(\tau = T_q)} \left( \sum_{j \neq q} P_{qj} V^*(H_{qj}x, 0, j) - V^*(x, \tau, q) \right) = 0, \tag{25}
$$

for all \( q \in \mathcal{S} \) and \( (x, \tau) \in \mathbb{R}^n \times [0, T_q] \). Using Fubini’s Theorem (Klebaner, 2005), one can interchange the expectation and the integration in (24), and obtain

$$
e^{-\rho t}E_z \{ V^*(X_{Q^m}(t)) \} = V^*(x, \tau, q)
$$

$$
+ \int_0^t e^{-\rho s} E_z \left\{ \mathcal{L}_{\mu^*} V^*(X_{Q^m}(s)) - \rho V^*(X_{Q^m}(s)) \right\} \, ds \tag{26}
$$

for all \( t < \infty \). Hence \( t \to E_z \{ e^{-\rho t} V^*(X_{Q^m}(t)) \} \) is absolutely continuous and therefore differentiable almost everywhere with

$$
\frac{d}{dt} E_z \{ e^{-\rho t} V^*(X_{Q^m}(t)) \}
$$

$$
= e^{-\rho t} E_z \{ \mathcal{L}_{\mu^*} V^*(X_{Q^m}(t)) - \rho V^*(X_{Q^m}) \}. \tag{27}
$$

We therefore have

$$
\frac{d}{dt} E_z \left\{ e^{(c-\rho)t} V^*(X_{Q^m}(t)) \right\}
$$

$$
= e^{(c-\rho)t} E_z \{ \mathcal{L}_{\mu^*} V^*(X_{Q^m}(t)) \}
$$

$$
+ (c - \rho) e^{(c-\rho)t} E_z \{ V^*(X_{Q^m}(t)) \}
$$

$$\leq 0.
$$

From this, we conclude that

$$
0 \leq E_z \{ e^{-\rho t} V^*(X_{Q^m}(t)) \} \leq e^{-\rho t} V^*(x, \tau, q). \tag{27}
$$
Since \( \lim_{t \to \infty} \lim_{m \to \infty} e^{-ct}V^*(x, \tau, q) = 0 \), we conclude from (27) that
\[
\lim_{t \to \infty} \lim_{m \to \infty} E_z \{e^{-\rho t}V^*(X_{Q_m}(t))\} = 0. \tag{28}
\]

Moreover, for the feedback policy \( \mu^* \), the integral term on the right-hand side of (24) can be bounded by
\[
\left| \int_0^t e^{-\rho s} \left( \mathcal{L}^{Q_m^*} \nu^*(X_{Q_m}(s)) - \rho V^*(X_{Q_m}(s)) \right) \, ds \right| \\
\leq \int_0^t e^{-\rho s} \left| \mathcal{L}^{\mu^*} \nu^*(X(s)) - \rho V^*(X(s)) \right| \, ds,
\]
and by the Dominated Convergence Theorem, we get
\[
\lim_{m \to \infty} E_z \left\{ \int_0^t e^{-\rho s} (\mathcal{L}^{\mu^*} \nu^*(X(s)) - \rho V^*(X(s))) \, ds \right\} = \lim_{m \to \infty} E_z \left\{ \int_0^t e^{-\rho s} (\mathcal{L}^{\mu^*} \nu^*(X(s)) - \rho V^*(X(s))) \, ds \right\} = 0. \tag{29}
\]

Now, we take the limit of (24) as \( m \to \infty \) and \( t \to \infty \). By (28), the limit on the left of (24) is zero and from (29) and (23), we conclude
\[
V^*(x, \tau, q) = E_z \left\{ \int_0^\infty e^{-\rho s} \ell_{\mu^*}(x(s), \tau(s), q(s)) \, ds \right\}
\]
for all \( q \in S \) and \( x \in \mathbb{R}^n, \tau \in [0, T_q] \). This implies that \( V^*(x, \tau, q) \) is the cost due to the feedback policy \( \mu^* \), conditioned upon the initial condition \( z = (x, \tau, q) \).

Now let \( \mu \) be an arbitrary feedback control for which the process \( X(t) \) exists globally and we have \( E_z \{||x(t)||^2\} \leq k_1 e^{-k_2 t} ||x_0|| \) for some \( k_1 > 0, k_2 > -\rho \). For such a process, we still have
\[
\lim_{t \to \infty} \lim_{m \to \infty} e^{-\rho t} E_z \{V^*(X_{Q_m}(t))\} = 0,
\]
but since \( \mu \) typically does not minimize (21), we have \( \mathcal{L}^{\mu} V^*(x, \tau, q) - \rho V^* \geq -\ell_{\mu}(x, \tau, q) \) instead of (23). In this case, the argument above applies but with the appropriate equalities replaced by inequalities, leading to
\[
V^*(x, \tau, q) \leq \lim_{t \to \infty} \lim_{m \to \infty} e^{-\rho t} E_z \{V^*(X_{Q_m}(t))\} \\
+ E_z \left\{ \int_0^\infty e^{-\rho s} \ell_{\mu}(x(s), \tau(s), q(s)) \, ds \right\} = E_z \left\{ \int_0^\infty e^{-\rho s} \ell_{\mu}(x(s), \tau(s), q(s)) \, ds \right\}. \tag{30}
\]

Therefore, the cost \( V^* \) associated with \( \mu^* \) is always no larger than the cost \( J_\mu \) associated with another policy \( \mu \), which proves the optimality of \( \mu^* \).

### 3.1 Recursive Computations

The main challenge in solving (9)-(11) in Theorem 3.1 is due to the unknown boundary conditions, but the following algorithm can be used to compute the optimal cost and the optimal policy.

**Algorithm 1**

(a) Set \( k = 0 \), \( \Lambda^0_q(\tau) = 0 \times 0 \times \mathbb{R}^n \), \( \Gamma^0_q(\tau) = 0 \times 1 \), and \( \Theta^0_q(\tau) = c_q \in [0, \infty) \), \( \forall \tau \in [0, T_q] \) and \( \forall q \in S \).

(b) Compute solutions that are \( C^1 \) in the interval \([0, T_q]\) for the following ODEs

\[
\begin{aligned}
-\frac{d\Lambda^k_q(\tau)}{d\tau} &= \Lambda^k_q(\tau) + \Lambda^k_q(\tau) A_q - \rho \Lambda^k_q(\tau) \\
&- \Lambda^k_q(\tau) B_q R_q^{-1} B_q^\top \Lambda^k_q(\tau) + Q_q \\
&+ \lambda_q(\tau) \sum_{j \neq q} P_{jq}(H_q^k \Lambda^k_j(0) H_{qj} - \Lambda^k_j(\tau)) \\
\Lambda^k_q(T_q) &= \sum_{j \neq q} P_{jq} \Lambda^k_j(0) H_{qj}
\end{aligned}
\tag{31}
\]

\[
\begin{aligned}
-\frac{d\Gamma^k_q(\tau)}{d\tau} &= (A_q' - \Lambda^k_q(\tau) B_q R_q^{-1} B_q' - \rho I) \Gamma^k_q(\tau) \\
&- 2Q_q \bar{x}_q + 2A_q'(\tau) (B_q \bar{u}_q + d_q) \\
&+ \lambda_q(\tau) \sum_{j \neq q} P_{jq}(H_q^k \Gamma^k_j(0) - \Gamma^k_q(\tau)) \\
\Gamma^k_q(T_q) &= \sum_{j \neq q} P_{jq} \Gamma^k_j(0)
\end{aligned}
\tag{32}
\]

\[
\begin{aligned}
-\frac{d\Theta^k_q(\tau)}{d\tau} &= \frac{1}{2} \Gamma^k_q(\tau) B_q R_q^{-1} B_q' \Gamma^k_q(\tau) + \bar{x}_q \Gamma^k_q(\tau) \\
&- \Gamma^k_q(\tau) (B_q \bar{u}_q + d_q) - \rho \Theta^k_q(\tau) \\
&+ \lambda_q(\tau) \sum_{j \neq q} P_{jq} (\Theta^k_j(0) - \Theta^k_q(\tau)) \\
\Theta^k_q(T_q) &= \sum_{j \neq q} P_{jq} \Theta^k_j(0)
\end{aligned}
\tag{33}
\]

(c) Set \( k \to k + 1 \) and go back to step (b).

The following Theorem implies that as \( k \) increases, one can achieve a cost that can be made arbitrarily close to the optimal cost by selecting large \( k \).

**Theorem 3.2** Suppose that for every \( k > 0 \) in Algorithm 1, the functions \( \Lambda^k_q : [0, T_q] \to \mathbb{R}^n \times \mathbb{R}^n \), \( \Gamma^k_q : [0, T_q] \to \mathbb{R}^n \), \( \Theta^k_q : [0, T_q] \to \mathbb{R}^n \), \( q \in S \) are solutions to (31)-(33) and have the property that the function \( G^k(x, \tau, q) := x' \Lambda^k_q(\tau) x + x' \Gamma^k_q(\tau) + \Theta^k_q(\tau) \) is non-negative. Then, for each initial mode \( q \in S \) and every initial condition \( (x, \tau) \in [0, T_q] \),

\[
G^k(x, \tau, q) = \min_{\mu} E_z \left\{ \int_0^{T_k} e^{-\rho s} \ell_{\mu}(x(s), \tau(s), q(s)) \, ds \right\}
\tag{34}
\]
Moreover, for every $\epsilon > 0$, there exists a sufficiently large $k$ so that the feedback law
\[
\mu^*_{\epsilon}(x, \tau, q) := \bar{u}_q - \frac{1}{2} R_q^{-1} E_q'(2\Lambda_q^{(k)}(\tau) x + \Gamma_q^{(k)}(\tau))
\]  
leads to a cost that is above the optimal cost $V^*$ associated with the optimal policy $\mu^*$ (12) by less than $\epsilon$.

Theorem 3.2 implies that the minimum cost in (7) is given by $\lim_{k \to \infty} G^k(x, \tau, q)$ and such a cost is achieved by $\mu^*_{\epsilon}(x, \tau, q)$.

The following result will be used in the Proof of Theorem 3.2.

**Lemma 3.4** For every function $\psi(\cdot, \cdot, \cdot) : \mathbb{R}^n \times [0, T_q] \to \mathbb{R}$ that is continuous with respect to its first and second arguments, we have
\[
E_z \left\{ e^{-r\tau_{1}} \psi(x(t_1), \tau(t_1), q(t_1)) \right\} =
I_{(\tau = T_q)} E_z \left\{ \psi(x(t_0), \tau(t_0), q(t_0)) \right\}
+ E_z \left\{ \int_0^{T_1} e^{-r\lambda_q}\psi(x(s), \tau(s), q(s)) \, ds \right\}
\]
for every initial mode $q \in \mathcal{S}$ and every initial condition $(x, \tau) \in \mathbb{R}^n \times [0, T_q]$ with $z = (x, \tau, q)$.

Proof of Lemma 3.4. We start by computing the expectation of the last term on the right-hand side of (36) when $\tau < T_q$. Until the first jump, we have $\tau(t) = t + \tau$, so by using (5), we get
\[
E_z \left\{ \int_0^{T_1} e^{-r\lambda_q}\psi(x(s), \tau(s), q(s)) \, ds \right\}
= \int_0^{T_0} \int_0^{h_0 - \tau_0} e^{-r\lambda_q}\psi(x(s), \tau(s), q(s)) \, ds \, dh_0
= \int_0^{T_0} \int_0^w e^{-r\lambda_q}\psi(x(s), \tau(s), q(s)) \, ds \, dw
\]
where we made the change of integration variable $w = h_0 - \tau$ with $f_{h_0}(w)$ denoting the probability density function of the jump interval in mode $q$. By changing the order of integration, we have that
\[
E_z \left\{ \int_0^{T_1} e^{-r\lambda_q}\psi(x(s), \tau(s), q(s)) \, ds \right\}
= \int_0^{T_0} \int_0^{T_h - \tau} e^{-r\lambda_q}\psi(x(s), \tau(s), q(s)) \, dw \, ds
= \int_0^{T_0} \int_0^{T_h - \tau} e^{-r\lambda_q}\psi(x(s), \tau(s), q(s)) \, dw \, ds
= \int_0^{T_0} \int_0^{T_h - \tau} e^{-r\lambda_q}\psi(x(s), \tau(s), q(s)) \, dw \, ds
= E_z \left\{ \int_0^{T_1} e^{-r\lambda_q}\psi(x(s), \tau(s), q(s)) \, ds \right\}.
\]

The proof of the equation (37) follows along the lines of the proof of equation (45). Moreover, for $\tau = T_q$, we have
\[
E_z \left\{ e^{-r\tau_{1}} \psi(x(t_1), \tau(t_1), q(t_1)) \right\} = E_z \left\{ \psi(x(t_0), \tau(t_0), q(t_0)) \right\}
+ E_z \left\{ \int_0^{T_1} e^{-r\lambda_q}\psi(x(s), \tau(s), q(s)) \, ds \right\}
\]
with $z_{t_0} = (x(t_0), \tau(t_0), q(t_0))$. Similar to (37), one can show that
\[
E_z \left\{ \int_0^{T_1} e^{-r\lambda_q}\psi(x(s), \tau(s), q(s)) \, ds \right\} =
I_{(\tau = T_q)} E_z \left\{ \psi(x(t_0), \tau(t_0), q(t_0)) \right\}
+ E_z \left\{ \int_0^{T_1} e^{-r\lambda_q}\psi(x(s), \tau(s), q(s)) \, ds \right\}.
\]

Hence, from (37)-(38), we get
\[
E_z \left\{ e^{-r\tau_{1}} \psi(x(t_1), \tau(t_1), q(t_1)) \right\} =
I_{(\tau = T_q)} E_z \left\{ \psi(x(t_0), \tau(t_0), q(t_0)) \right\}
+ E_z \left\{ \int_0^{T_1} e^{-r\lambda_q}\psi(x(s), \tau(s), q(s)) \, ds \right\}.
\]

We are now ready to prove Theorem 3.2.

**Proof of Theorem 3.2.** Similar to the proof of Theorem 3.1, for given $k > 0$, $q \in \mathcal{S}$, $x \in \mathbb{R}^n$ and $\tau \in (0, T_q)$, consider the following equation
\[
\min_{\nu \in \mathbb{R}^n} \left\{ \frac{\partial G_k}{\partial x}(A_q x + B_q \nu) + \frac{\partial G_k}{\partial \nu} - \rho G_k(x, \tau, q) \right\}
+ \lambda_q(\tau) \sum_{j \neq q} P_{jq} (G^{k-1}(H_{jq} x, 0, j) - G^k(x, \tau, q))
+ (x - \bar{x}_q)'Q_{jq} (x - \bar{x}_q) + (\nu - \bar{u}_q)'R_q (\nu - \bar{u}_q) = 0.
\]

It is straightforward to show that $\nu^* := \mu^*_\epsilon(x, q, \tau)$ defined in (35) achieves the minimum in (39). Moreover, since $A_q^{(k)}, \Gamma_q^{(k)}, \Theta_q^{(k)}$ satisfy (31)-(33), one can show that $G^k(x, \tau, q) = x' A_q^{(k)}(\tau) x + x' \Gamma_q^{(k)}(\tau) + \Theta_q^{(k)}(\tau)$ is a solution to (39). Moreover, for every $q \in \mathcal{S}$, $G^k$ satisfies the boundary condition
\[
G^k(x, T_q, q) = \sum_{j \neq q} P_{jq} G^{k-1}(H_{jq} x, 0, j).
\]
Since $A_q^{(k)}(\tau)$ and $\Gamma_q^{(k)}(\tau)$ are bounded $C^1$ functions of $\tau$, by Lemma 3.1, one can conclude that the stochastic process (1) with the feedback policy $\mu^*_\epsilon(x, \tau, q)$ exists globally with probability one.

\footnote{We used the conditional expectation property $E\{E(X|\mathcal{F})|\mathcal{G}\} = E\{X|\mathcal{G}\}$ for $\sigma$-algebras $\mathcal{G} \subset \mathcal{F}$.}
Let \((x(t), \tau(t), q(t))\) denote the stochastic process defined by (1) at time \(t \geq 0\), starting from the initial mode \(q \in S\) and the initial condition \((x, \tau) \in \mathbb{R}^n \times [0, T]\) with the feedback control \(\mu^*(x, \tau, q)\). For every \(t \in (0, T)\), the value of the derivative \(\frac{d}{dt} G^k(x(t), \tau(t), q(t))\) along the solution of the system is given by

\[
\frac{d}{dt} G^k(x(t), \tau(t), q(t)) \equiv \frac{\partial G^k}{\partial x} (Aq x + Bq \mu^*_x(x, \tau, q)) + \frac{\partial G^k}{\partial \tau} - \lambda_q(\tau) \sum_{j \neq q} P_{qj} (G^{k-1}(H_{qj}x, 0, j) - G^k(x, \tau, q))
\]

where the second equality results from the fact that \(G^k\) is a solution to (39). Therefore

\[
e^{-\rho t} G^k(x(t^-_1), \tau(t^-_1), q(t^-_1)) - G^k(x, \tau, q) \quad \text{w.p.1}
\]

\[
I_{(\tau=T)} G^k(x(t_0), \tau(t_0), q(t_0)) - G^k(x, \tau, q)
\]

\[
- \int_0^{t_1} e^{-\rho s} \lambda_q(x(s), \tau(s), q(s)) \, ds - \int_0^{t_1} \sum_{j \neq q} P_{qj} e^{-\rho s} \lambda_q(\tau(s)) \left( G^{k-1}(H_{qj}x(s), 0, j) - G^k(x(s), \tau(s), q(s))\right) \, ds
\]

By the result of Lemma 3.4 with \(\psi = G^k\), we have that

\[
E_z \left\{ e^{-\rho t_1} G^k(x(t_1^-), \tau(t_1^-), q(t_1^-)) \right\} = E_z \left\{ I_{(\tau=T)} G^k(x(t_0), \tau(t_0), q(t_0)) \right\} + E_z \left\{ \int_0^{t_1} e^{-\rho s} \lambda_q(\tau(s)) G^k(x(s), \tau(s), q(s)) \, ds \right\}.
\]

By taking the expectation of both sides of (41), conditioned to the initial condition and using (42), we obtain

\[
-G^k(x, \tau, q) = -I_{(\tau=T)} G^k(x, T, q)
\]

\[-E_z \left\{ \int_0^{t_1} e^{-\rho s} \lambda_q(x(s), \tau(s), q(s)) \, ds \right\}
\]

\[-E_z \left\{ \sum_{j \neq q} P_{qj} \int_0^{t_1} e^{-\rho s} \lambda_q(\tau(s)) G^{k-1}(H_{qj}x(s), 0, j) \, ds \right\}
\]

Due to step (iii) in the construction of the sample paths, for any \(t \in [0, T]\), we get

\[
E_z \left\{ e^{-\rho t_1} G^{k-1}(x(t_1^-), \tau(t_1^-), q(t_1^-)) \right\} = E_z \left\{ \sum_{j \neq q} P_{qj} e^{-\rho t_1} G^{k-1}(H_{qj} x(t_1^-), 0, j) \right\}.
\]

To compute the right-hand side of (44), we consider two different cases when \(\tau < T_q\) and \(\tau = T_q\). Similar to (37), for \(\tau < T_q\), one can obtain

\[
E_z \left\{ \int_0^{t_1} e^{-x} \lambda_q(\tau(s)) G^{k-1}(H_{qj}x(s), 0, j) \right\} = \int_0^{T_q} \int_0^w e^{-\rho s} \lambda_q(\tau(s)) G^{k-1}(H_{qj}x(s), 0, j)
\]

\[
= \int_0^{T_q} \int_0^w \frac{e^{-\rho s} \lambda_q(\tau(s)) G^{k-1}(H_{qj}x(s), 0, j)}{1 - F_q(\tau)} \, ds \, dw
\]

\[
= \int_0^{T_q} \int_0^w e^{-\rho s} \lambda_q(\tau(s)) G^{k-1}(H_{qj}x(s), 0, j) \left( \frac{1 - F_q(\tau)}{1 - F_q(\tau)} \right) \, ds \, dw
\]

\[
E_z \left\{ e^{-\rho t_1} G^{k-1}(H_{qj} x(t_1^-), 0, j) \right\}.
\]

Moreover, for \(\tau = T_q\), following the lines of the proof of (38), we get

\[
E_z \left\{ \int_0^{t_1} e^{-\rho s} \lambda_q G^{k-1}(H_{qj}x(s), 0, j) \right\}
\]

\[
= E_z \left\{ \sum_{j \neq q} P_{qj} \int_0^{t_1} e^{-\rho s} \lambda_q G^{k-1}(H_{qj}x(s), 0, j) \right\} + \sum_{j \neq q} P_{qj} G^{k-1}(H_{qj}x, 0, j).
\]

Combining (45) and (46), we get

\[
E_z \left\{ \int_0^{t_1} e^{-\rho s} \lambda_q G^{k-1}(H_{qj}x(s), 0, j) \right\}
\]

\[
= E_z \left\{ \sum_{j \neq q} P_{qj} \int_0^{t_1} e^{-\rho s} \lambda_q G^{k-1}(H_{qj}x(s), 0, j) \right\}
\]

\[
+ I_{(\tau=T_q)} \sum_{j \neq q} P_{qj} G^{k-1}(H_{qj}x, 0, j).
\]

Thus, we conclude from (39), (43) and (47) that

\[
G^k(x, \tau, q) = E_z \left\{ e^{-\rho t_1} G^{k-1}(x(t_1), \tau(t_1), q(t_1)) \right\}
\]

\[
+ E_z \left\{ \int_0^{t_1} e^{-\rho s} \lambda_q(x(s), \tau(s), q(s)) \, ds \right\}
\]

Now let \(\mu\) be an arbitrary feedback policy for which the process \((x(t), \tau(t), q(t))\) exists globally. Since \(\mu\) typically does not minimize (39), we have

\[
\frac{\partial G^k}{\partial x}(Aq x + Bq \mu(x, \tau, q)) + \frac{\partial G^k}{\partial \tau} \geq -\lambda_q(x, \tau, q) + \rho G^k(x, \tau, q)
\]

\[-\lambda_q \sum_{j \neq q} P_{qj} (G^{k-1}(H_{qj}x, 0, j) - G^k(x, \tau, q))
\]

instead of (39). In this case, the argument above applies but with the appropriate equalities replaced by inequalities, leading to

\[
G^k(x, \tau, q) \leq E_z \left\{ \int_0^{t_1} e^{-\rho s} \lambda_q(x(s), \tau(s), q(s)) \, ds \right\}
\]

\[+ E_z \left\{ e^{-\rho t_1} G^{k-1}(x(t_1), \tau(t_1), q(t_1)) \right\}.
\]

Therefore, the cost \(G^k\) associated with \(\mu^*_q\) is always no larger than the cost associated with another policy \(\mu\),
which proves the optimality of \( \mu^*_k \):

\[
G^k(x, \tau, q) = \min_{\mu} E_z \left\{ \int_0^{t_1} e^{-\rho t} \ell_{\mu}(x(s), \tau(s), q(s)) \, ds + e^{-\rho t_1} G^{k-1}(x(t_1), \tau(t_1), q(t_1)) \right\}.
\]

(48)

We now prove (34) for all \( k \geq 1 \) by induction on \( k \). The base of induction \( k = 1 \) follows directly from (48). Assuming now that (34) holds for some \( k > 1 \), we shall show that it holds for \( k + 1 \):

\[
G^{k+1}(x, \tau, q) = \min_{\mu} E_z \left\{ \int_0^{t_1} e^{-\rho t} \ell_{\mu}(x(s), \tau(s), q(s)) \, ds + e^{-\rho t_1} G^{k}(x(t_1), \tau(t_1), q(t_1)) \right\}.
\]

(49)

From Proposition 1, we know that \( \lim_{k \to \infty} E_z \left\{ \int_0^{t_1} e^{-\rho t} \ell_{\mu}(x(s), \tau(s), q(s)) \, ds + e^{-\rho t_1} G^{k}(x(t_1), \tau(t_1), q(t_1)) \right\} = 0 \) with probability one, and by Bounded Convergence Theorem \( E_z \left\{ \int_0^{t_1} e^{-\rho t} \ell_{\mu}(x(s), \tau(s), q(s)) \, ds + e^{-\rho t_1} G^{k}(x(t_1), \tau(t_1), q(t_1)) \right\} \to 0 \). Since the integral on the right of (34) is an increasing function of \( k \), by the Monotone Convergence Theorem (Rudin, 1987), we get:

\[
\lim_{k \to \infty} G^k(x, \tau, q) = \min_{\mu} E_z \left\{ \int_0^{t_1} e^{-\rho t} \ell_{\mu}(x(s), \tau(s), q(s)) \, ds \right\}.
\]

Thus, for every initial condition \( z = (x, \tau, q) \), as \( k \) increases the policy \( \mu^*_k \) achieves a cost \( G^k \) that can be made arbitrarily close to the optimal cost by selecting large \( k \).

\[ \square \]

It is worth noting that Theorems 3.1 and 3.2 require that \( V^* \) and \( G^k \) are non-negative functions. Checking such conditions in practice might be challenging as the dimension of the system becomes larger.

4 Example

Living organisms respond to changes in their surroundings by sensing the environmental context and by orchestrating the expression of sets of genes to utilize available resources and to survive stressful conditions (Dekel et al., 2005; Shahrezaei & Swain, 2008; Pour Safaei et al., 2012). We consider a model for the lac operon regulatory network in E. Coli bacterium. E. Coli regulates the expression of many of its genes according to the food sources that are available to it. In the absence of lactose, the Lac repressor in E. Coli binds to the promoter region of the lac gene and keeps it from transcribing the protein. If the bacteria expressed lac genes when lactose was not present, there would likely be an energetic cost of producing an enzyme that was not in use. However, when lactose is available, the lac genes are expressed because allolactose binds to the Lac repressor protein and keeps it from binding to the promoter region. As a result of this change, the repressor can no longer bind to the promoter region and falls off. RNA polymerase can then bind to the promoter and transcribe the lac genes. Therefore, depending on the presence or absence of lactose, E. Coli regulates the production/decay of some specific proteins.

Inspired by the model of lac operon in Farina & Prandini (2007), we consider the case of two specific enzymes in E. Coli: the lactose permease and β-galactosidase. The first enzyme allows the bacterium to allow external lactose to enter the cytoplasm of the cell, while the latter one is used for degrading lactose into glucose which is its main source of energy. We denote the concentration of β-galactosidase and lactose permease by \( x(t) \) and \( y(t) \), respectively; and we denote the concentration of the RNA molecules resulted from transcription of lac genes by \( z(t) \).

Assume that the cell experiences two different environmental conditions \( (0, 1) \) where 0 denotes the absences of lactose and 1 corresponds to the presence of lactose and absence of glucose. At each point in time, we assume that the cost of deviating the state from its optimal level for the current environment is a quadratic function of the difference between these values. We also consider a term in the cost function that reflects the energetic costs of producing/decaying mRNA and proteins (Wanger, 2005). If the life span of the cell is modeled by an exponential random variable with mean \( 1/\rho \), one can model the total expected life-time cost of the cell similar to (7), as discussed in (Pour Safaei et al., 2012) and references in.
For the concentration of the enzymes and mRNA, we use the linear model given in Tournier & Farcot (2002)

\[
\begin{align*}
\dot{z} &= u - 0.716z \\
\dot{x} &= 9.4z - 0.033x \\
\dot{y} &= 18.8z - 0.033y
\end{align*}
\]

(49)

where \( u \) denotes the transcription rate of \( z \) and is allowed to depend on the environmental condition, and \( \mathbf{x} \) and \( \mathbf{y} \) are independent random variables with given probability distribution functions on finite supports. We derived a Bellman-like equation for this problem and showed that the solution is likely to exhibit specialization to the statistics that determine the changes in the environment.

Using the result of Theorem 3.2, we compute the optimal cost and the optimal feedback policy. Figure 1 illustrates a sample path of this stochastic process using the optimal feedback policy when \( a = b = 40 \), i.e., the environmental change is almost periodic with period 0.5 unit of time. In our simulations, we have chosen \( \rho = 0.1 \), \( H_{ij} = I \),

\[
Q_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 10^{-3} & 0 \\ 0 & 0 & 10^{-3} \end{bmatrix}, \quad Q_2 = \begin{bmatrix} 5 & 0 & 0 \\ 0 & 10^{-3} & 0 \\ 0 & 0 & 10^{-3} \end{bmatrix},
\]

\[
R_1 = 0.1 \quad \text{and} \quad R_2 = 0.1/5\]

with the initial condition \( [z(0), x(0), y(0)] = [4e-10, 6e-8, 12e-8] \) and \( q(0) = 0 \). Fig. 2 illustrates the values of the optimal control for the sample path of Fig. 1. As one can see in Fig. 1, the controller that is optimal for the changing environment tries to anticipate the change of the environment instead of simply reacting to changes in \( q(t) \) and that the the optimal control law depends on the value of the timer \( \tau(t) \) that keeps track of the time elapsed since the last environmental change. The required cell machinery to “implement” such a control law is a topic for future research.

5 Conclusions

We studied quadratic control of stochastic hybrid systems with renewal transitions for which the lengths of time intervals that the system spends in each mode are independent random variables with given probability distribution functions on finite supports. We derived a Bellman-like equation for this problem and showed that if the solution \( V^* \) to this equation satisfies a boundary condition on \( \tau \), then \( V^* \) is the minimum value of (7). A recursive algorithm was provided for computing the optimal cost since the boundary conditions are unknown. The applicability of our result was illustrated through a numerical example, motivated by stochastic gene regulation in biology.
A topic for future research includes the derivation of conditions that guarantee existence of the solution to (9)-(11). Furthermore, we plan to consider the $H_{\infty}$ control of SHS with renewal transitions. In this problem, one would like to characterize a feedback controller for disturbance attenuation in terms of a set of differential matrix inequalities.

Another topic for future research is to consider the case in which the cumulative distribution functions $F_i$ do not necessarily have finite supports.

A Appendix

We show how the jumps counter $N(t) = \max\{k : 0 < t_k \leq t\}$ can be related to the standard Poisson process, $N(t)$, through the following intensity-dependent time scaling

$$N(t) = \bar{N} \left( \int_0^t \lambda_q(\tau(s)) \, ds \right), \quad \forall t \in [0, \infty).$$

Denoting by $t_k := \sum_{i=1}^k h_i$, the event times of the standard Poisson process $N(t), t \geq 0$, we have that

$$\bar{N} \left( \int_0^t \lambda_q(\tau(s)) \, ds \right) = \max\{k : 0 < t_k \leq t\} \leq \sum_{i=1}^k h_i \leq \int_0^t \lambda_q(\tau(s)) \, ds.$$  \hspace{1cm} (A.1)

Our goal is to show that this expression is equal to $N(t)$. To this effect, take an arbitrary jump time $t_k$. Since the hazard rate is non-negative, if $t_k \leq t$, then

$$\int_0^{t_k} \lambda_q(\tau(s)) \, ds \leq \int_0^t \lambda_q(\tau(s)) \, ds \Rightarrow \sum_{i=1}^k h_i \leq \int_0^t \lambda_q(\tau(s)) \, ds,$$

where we used the fact that

$$\int_0^{t_k} \lambda_q(\tau(s)) \, ds = \sum_{i=1}^k \int_{t_{i-1}}^{t_i} \lambda_q(\tau(s)) \, ds = \sum_{i=1}^k h_i.$$  \hspace{1cm} (A.2)

Since $\{k : 0 < t_k \leq t\} \subset \{k : \sum_{i=1}^k h_i \leq \int_0^t \lambda_q(\tau(s)) \, ds\}$, we conclude that

$$N(t) = \max\{k : 0 < t_k \leq t\} \leq \max\{k : \sum_{i=1}^k h_i \leq \int_0^t \lambda_q(\tau(s)) \, ds\} = \bar{N} \left( \int_0^t \lambda_q(\tau(s)) \, ds \right).$$

To prove that we actually have equality, assume by contradiction that

$$\max\{k : 0 < t_k \leq t\} < \max\{k : \sum_{i=1}^k h_i \leq \int_0^t \lambda_q(\tau(s)) \, ds\}$$

which means that there exists a $k^*$ such that $t_{k^* - 1} \leq t < t_{k^*}$, but using (A.1), we can show

$$\sum_{i=1}^{k^*} h_i = \int_0^{t_{k^*}} \lambda_q(\tau(s)) \, ds \leq \int_0^t \lambda_q(\tau(s)) \, ds \Rightarrow \int_0^{t_{k^*}} \lambda_q(\tau(s)) \, ds \leq 0.$$  \hspace{1cm} (A.2)

However, for $t_{k^*} > t$ to be a jump time, we must have

$$\int_0^{t_{k^*}} \lambda_q(\tau(s)) \, ds = \int_0^t \lambda_q(\tau(s)) \, ds + \int_{t_{k^*}}^{t_{k^*}} \lambda_q(\tau(s)) \, ds = h_{k^*}$$

$$\Rightarrow \int_0^{t_{k^*}} \lambda_q(\tau(s)) \, ds \geq h_{k^*},$$

which means that $t_{k^*} \leq t$ and thus contradicts the assumption. Therefore, we actually have equality in (A.2).

B Appendix

We use the (conditional) probability density functions of the inter-jump interval to show that

$$\sum_{k=1}^{\infty} E_z \left\{ e^{-\rho t_k I(t_k \leq N(t))} h(\text{X}_Q(t_{k-})) \right\} = E_z \left\{ \int_0^{t_{k-1}} e^{-\rho t} (\lambda h \lambda Q) (\text{X}_Q(t)) \, dt \right \} + \sum_{k=1}^{\infty} E_z \left\{ \int_{t_{k-1}}^{t_k} e^{-\rho t} (\lambda h \lambda Q) (\text{X}_Q(t)) \, dt \right \}.  \hspace{1cm} (B.1)$$

From the construction of the SHS, $\forall t \in (t_{k-1}, \infty)$ and $k > 1$, the probability density functions of the event ($t_k < t$) conditioned to the natural filtration $\mathfrak{F}_{t_{k-1}}$ is given by

$$f_k(t|\mathfrak{F}_{t_{k-1}}) = \frac{d}{dt} P(t_k \leq t|\mathfrak{F}_{t_{k-1}})$$

$$= \frac{d}{dt} P(h_{k-1} \leq t - t_{k-1}|\mathfrak{F}_{t_{k-1}})$$

$$= \frac{d}{dt} \lambda_q(t_{k-1})(1 - F_q(t_{k-1})(\tau(t)))$$

$$= \lambda_q(t_{k-1})(\tau(t)) \mathcal{E}\{ t_{k-1} > t | \mathfrak{F}_{t_{k-1}} \}$$

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can be computed as

\[
E \{ I_{\{k \leq N_Q(t)\}} \} = E \{ I_{\{k \leq N_Q(t)\}} e^{-\rho t_k} h \{ X_Q(t_k) \} \} | \mathcal{F}_{k-1}
\]

with \( h(x, \tau, q) := h(x, \tau, q) \lambda_q(\tau) \). Here \( T_Q(t_{k-1}) \) denotes the first time that the solution of the system leaves the interior of \( Q \) after the jump \( t_{k-1} \). Moreover, since \( \forall t \in [t_{k-1}, t_k \wedge T_Q(t_{k-1})] \),

\[
X_Q(t) = \begin{cases} 
X(t) & X_Q(t_{k-1}) \in \text{Int}(Q), \\
X_Q(t_{k-1}) & X_Q(t_{k-1}) \not\in \text{Int}(Q),
\end{cases}
\]

we can re-write the right-hand side of (B.2) compactly as

\[
E \{ I_{\{k \leq N_Q(t)\}} e^{-\rho t_k} h \{ X_Q(t_k) \} \} | \mathcal{F}_{k-1} = E \left\{ \int_{TA_k} e^{-\rho t} (h(\lambda)) (X_Q(t)) d\mathcal{F}_{k-1} \right\}
\]

Combining (B.3) and (B.4), the left-hand-side of (B.1) can be written as

\[
\sum_{k=1}^{\infty} E \{ e^{-\rho t_k} I_{\{k \leq N_Q(t)\}} h \{ X_Q(t_k) \} \} = E \{ e^{-\rho t_k} I_{\{k \leq N_Q(t)\}} h \{ X_Q(t_k) \} \} + \sum_{k=2}^{\infty} E \{ e^{-\rho t_k} I_{\{k \leq N_Q(t)\}} h \{ X_Q(t_k) \} \} d|\mathcal{F}_{k-1} \}
\]

References


