Designing transparent stabilizing haptic controllers

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***Abstract*— This paper addresses the design of controllers for haptic systems that are both stable and transparent. Stability refers to the property that the control system should not introduce oscillations felt in the haptic device by the human operator. Transparency refers to the property that the haptic system should faithfully reproduce the dynamics of the virtual environment. We propose a design methodology that converts the problem of maximizing transparency, while guaranteeing stability, into a set of matrix inequalities. Then we propose a numerical procedure to solve the matrix inequalities based on the cone complementarity algorithm. With this formulation one can also incorporate other design tasks, such as robustness to parameter variations, by introducing additional norm conditions and solving H_∞ or mixed norm problems. A numerical example is used to illustrate the design methodology.**

I. Introduction

Haptic systems attempt to emulate the interaction between a human and an environment populated by virtual objects. In practice, the human operator holds a controlled robotic device (called the haptic interface) that exerts on the human the reaction forces that the virtual objects would, thus leading the human to believe that she is actually touching real objects. We address a problem that is related to the design of the low-level controllers that compute the actuation signal sent to the haptic interface and that will translate into the force felt by the operator. For simplicity we concentrate on the *virtual wall* problem, in which the haptic system attempts to simulate the interaction of the human with a virtual planar object. When the virtual wall is penetrated, the human should feel a reaction force equal to

$$-Kx - B\dot{x}, \tag{1}$$

where x denotes the penetration depth, $\dot{x}$ the speed, K a stiffness coefficient, and B a damping coefficient.

If one were to naively control the motors of the haptic interface to apply a force equal to (1) when the virtual wall is penetrated, the human would often feel the virtual wall oscillating or "buzzing" upon touch, especially for high values of the stiffness coefficient K. This phenomenon results from *loop-instability* mainly due to the existence of a feedback loop between the human and the haptic system, compounded by the fact that the haptic controller is implemented digitally with finite sampling rate. To avoid instability one generally does not simply apply the force in (1) to the haptic interface. In practice, this means that there will be a difference between the force that the human *will* feel and the one that she *should* feel. This is known as haptic distortion. A haptic system is said to be *transparent* if the haptic distortion is small. Thus two main concerns dictate the design of the low-level controllers for haptic systems: stability and transparency.

We address the design of controllers for haptic systems that are maximally transparent under the constraint that they should stabilize the system. By stability we mean velocity of haptic device converges to zero, hence the system is oscillation free and all other states such as haptic device position remain bounded. When one attempts to simulate virtual objects with high stiffness or when the controller sampling rate is low, previous controller design methods that guarantee stability but ignore distortion, typically lead to haptic systems with poor transparency.

Fig. 1 depicts the haptic system under consideration. It consists of four main blocks: the human operator $H(s)$, the haptic interface $M(s)$, the virtual environment (VE), and the virtual coupling $C(z)$. Here we follow the assumption made by [1] and others, under which the human behaves passively when regarded as an operator from velocity to applied force. The block $M(s)$ models the dynamics of the haptic interface as given by Newton's law. The simplest form is a first order transfer function of the form

$$M(s) = \frac{1}{ms + b}, \tag{2}$$

where m and b are the mass and physical damping coefficient, respectively. However we do not restrict $M(s)$ to the first order model throughout this paper. The applied force has two components: the one applied by the human on the haptic interface and the one applied by its own motors. The position of the interface is the integral of its velocity and we assume here that only position is available for feedback.

While the human operator and the haptic interface have continuous models, the rest of the system is implemented digitally and is therefore modeled by discrete-time blocks. The discrete and continuous components are connected via sample and hold blocks with a sampling time T. In haptic systems with complex virtual environments, collision detection and graphic rendering are typically computationally very intense and pose severe constrains on the sampling rate. In particular, to achieve accurate haptic rendering the sampling time should not be smaller than the time it takes for the haptic system to detect the presence or absence of collisions in complex virtual environments. The virtual environment (VE) in Fig. 1 is represented by the nonlinear function:

$$\psi(x_k, v_k) := \begin{cases} Kx_k + Bv_k & x_k > 0 \\ 0 & x_k \le 0 \end{cases}$$

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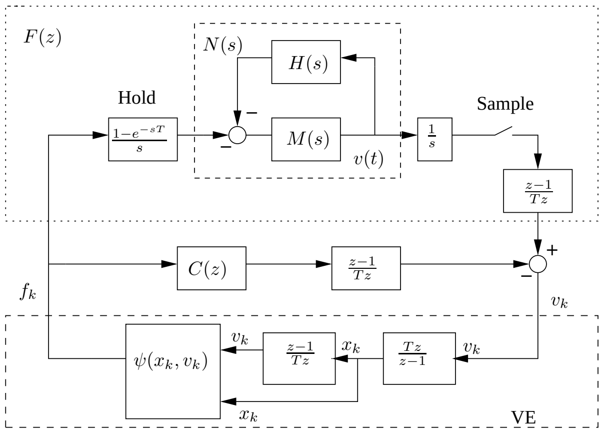

Fig. 1. Block diagram of haptic system

and is consistent with the model for the virtual wall presented above: When there is contact the transfer function of VE is equal to $K + B\frac{z-1}{Tz}$, which corresponds to the springer/damper force in (1); and in the absence of contact there is no force. Note that the block $\frac{Tz}{z-1}$ on the right of the VE block cancels the two block $\frac{z-1}{Tz}$ that enter the summation point so for now we can simply ignore these three blocks. They will be convenient for analysis later.

As mentioned above, stability of the overall system is an important requirement because oscillations can cause injury to the user. Specifically, we want the states of the overall system to remain bounded and also that the velocity $v(t)$ of the interface converge to zero as $t \to \infty$. When this happens we say that the haptic system is *stable and oscillation free*. Early work on the stability of haptic systems focused on finding the range of parameters that guarantee stability of the system in the absence of the virtual coupling block $C(z)$. Colgate et al. [1] showed that without this block and $M(s)$ given by (2) stability is achieved as long as

$$b > \frac{KT}{2} + |B|. \tag{3}$$

This relationship reveals that, at least in the absence of virtual coupling, for lightly damped devices (small b) stability is guaranteed with a small damping coefficient B as well as either a small stiffness K or a small sampling period T.

The condition (3) turns out to be very restrictive, because there are practical limitations that prevent one from reducing the sampling rate T sufficiently small to make (3) hold. This led to the idea of introducing the virtual coupling block $C(z)$. With the aid of the extra degrees of freedom introduced by virtual coupling it is possible to design stable haptic systems for a much wider range of sampling rates and classes of virtual environments. Adams et al. [2] used arguments based on absolute stability to design stable controllers for haptic systems represented as two-port networks with a passive virtual environment and a passive human as end-terminals. Miller et al. [3] used passivity-based constructions to design virtual coupling blocks that guarantee stability for virtual environments modeled by static sector nonlinearity. In Section II, we adapt the results in [3] to our setup and write a condition on the virtual coupling transfer function that, when satisfied, guarantees that the overall system is stable and oscillation free. More recently Miller et al. [4] extended their original results to virtual mass/spring/damper virtual environments with computational delay. We will ignore delay but our results could be extended along the lines of [4].

Virtual coupling provides an effective method to achieve stability. Moreover, the "derivate" block $\frac{z-1}{Tz}$ in series with the virtual coupling controller $C(z)$ guarantees that in steady-state (i.e., when the force converges to a constant and the velocity drops to zero) the effect of virtual coupling disappears. However, until steady-state is reached it alters the dynamics of the virtual environment "felt" by the human. Eom et al. [5] introduced the idea of designing a controller that not only stabilizes the haptic feedback system, but also minimizes the error between the ideal virtual wall admittance and the real virtual wall admittance with a virtual coupling controller in feedback. When this error is small the controller has minimum effect on the dynamics and the human "feels" the true virtual environment through a "transparent" virtual coupling. This idea inspired the basic problem that:

Problem 1: Among all the virtual coupling controllers that stabilize the haptic system, compute the one that maximizes transparency.

We show in Section III that this problem can be reformulated as an H_∞ minimization subject to a positive realness constrain. Such problems are typically hard to solve [6], [7] because they result in a non-convex matrix inequalities (specially the condition on the controller). However, we show that it can be solved with the recently introduced cone complementarity algorithm. Then through an example we illustrate the use of our method.

II. STABILITY

The main objective of the virtual coupling controller is to guarantee stability. In this section we establish a condition on this controller that guarantees stable and oscillation free contact. We will later search among all controllers that satisfy this condition for the one that maximizes transparency.

Before proceeding it is important to emphasize that oscillation free, which we recall means that the *velocity* $v(t)$ of the haptic interface converges to zero as $t \to \infty$, does not imply asymptotic stability of the origin. In particular, the *position* of the haptic interface will in general not converge to the origin (which in our setup coincides with the location of the virtual wall). Clearly, if the haptic interface starts at rest far away from the wall there is no reason why it should be attracted to wall and eventually come to rest there. However, regardless of the initial condition it should eventually come to rest ($v(t) \to 0$) as long as the human does not exert a non-passive external force.

Inspired by the results in [3] we use a passivity-based argument to determine a stability condition on the virtual coupling controller $C(z)$. A dynamical system with input $u \in \mathbb{R}$, output $y \in \mathbb{R}$, and state $x \in \mathbb{R}^n$ is *passive* if there exists a continuously differentiable positive semi-definite function $V : \mathbb{R}^n \to \mathbb{R}$, called the *storage function*, such that

$$V\big(x(t)\big) \le V\big(x(0)\big) + \int_0^t y(\tau)u(\tau)d\tau - \rho\int_0^t y^2(\tau)d\tau \\ -\varphi\int_0^t u^2(\tau)d\tau, \quad \forall x \in \mathbb{R}^n,\ u,y \in \mathbb{R},\ t \ge 0, \quad (4)$$

where ρ and $\varphi \ge 0$ denote nonnegative constants. For linear time-invariant systems with minimal realization, passivity is equivalent to positive-realness of the transfer function. The discrete-time version of passivity is defined by replacing the integrals in (4) by summations. Defining $\Delta V(k) := V(x_{k+1}) - V(x_k)$, we obtain the following incremental version of discrete-time passivity:

$$\triangle V(k) \le y_k u_k - \rho y_k^2 - \varphi u_k^2.$$

For discrete-time linear time-invariant systems the frequency-domain condition for passivity is the same as in continuous-time, provided that positive realness is understood in the discrete-time sense: A real rational transfer function $H_d(z)$ is *discrete-time positive real* (abbreviated as $H_d(z) \in DPR$) if $H_d(z)$ is analytic in the open exterior of the unit disk $|z| > 1$, it has no poles on the unit circle $|z| = 1$ with multiplicity larger than one, and $H_d(e^{j\Omega}) + H_d(e^{j\Omega}) \ge 0$, $\forall\Omega \in [0, 2\pi)$. The transfer function $H_d(z)$ is *Discrete-time Strictly Positive Real* (abbreviated as $H_d(z) \in DSPR$) if it is analytic in the close exterior of the unit disk $|z| \ge 1$ and $H_d(e^{j\Omega}) + H_d(e^{j\Omega}) > 0$, $\forall\Omega \in [0, 2\pi)$. The reader is referred to [8], [9], [10] for a detailed treatment of passivity for continuous, discrete, and sampled-data systems.

To establish the stability of the overall haptic system in Fig. 1, it is convenient to decompose it into two subsystems: the virtual environment block VE at the bottom and the rest of the system composed by the human operator, haptic device, and virtual controller. We will show that the VE subsystem is passive and that, under appropriate conditions on the controller $C(z)$, the rest of the system is DSPR. From this we will be able to conclude that the overall system is stable and oscillation free. The following result proved in the Appendix establishes the passivity of the virtual environment block with input v_k and output f_k:

Proposition 1: The virtual environment is passive and has a storage function

$$V_E(x) = \begin{cases} \frac{K}{2T}x^2 & x \ge 0 \\ 0 & x < 0 \end{cases}$$

for which $V_E(x_k) - V_E(x_{k-1}) \le v_k f_k$.

Proof is straight forward and can be found in [11]. We can now derive a condition on the virtual controller $C(z)$ that guarantees stability:

Theorem 1: Assuming that the human has a passive real rational transfer function and that

$$\frac{z-1}{Tz}\big(C(z) - \frac{T}{2b}\big) \in DSPR, \quad (5)$$

the haptic system in Fig. 1 is stable and oscillation free.

The proof of this result is inspired by the one in [3], except that here we cannot use an argument based on LaSalle's invariant principle because the storage function is not a positive definite radially unbounded Lyapunov function.

Proof of Theorem 1. We start by establishing that the system composed by the human operator/haptic device/controller, with input f_k and output v_k is discrete-time strictly positive real transfer function. The transfer function $L(z)$ of this system is given by

$$L(z) = F(z) + \frac{z-1}{Tz}C(z),$$

with $F(z) := \frac{z-1}{Tz}\mathcal{Z}\Big\{\frac{1-e^{-sT}}{s^2}N(s)\Big\}$, $N(s) := \frac{M(s)}{1+H(s)M(s)}$, $M(s) := \frac{1}{ms+b}$. where, given a continuous-time transfer function $G(s)$, we denote by $G_d(z) := \mathcal{Z}\{G(s)\}$ the discrete time sampled transfer function defined by

$$G_d(e^{sT}) = \frac{1}{T}\sum_{n=-\infty}^{n=\infty} G(s + j\frac{2\pi n}{T}).$$

It is straightforward to show that, when $H(s)$ is a positive real rational transfer function,

$$F(z) + \frac{1}{2b}\frac{z-1}{z} \in DPR, \quad (6)$$

because this function is analytic in the open exterior of the unit disk, it has no poles on the unit circle with multiplicity larger than one, and its real part on the unit circle is nonnegative. If we select $C(z)$ such that (5) holds, $L(z) \in DSPR$ because it can be viewed as the sum of the DPR block (6) and the DSPR block $\frac{z-1}{Tz}\big(C(z) - \frac{T}{2b}\big)$. Let us denote by

$$z_{k+1} = Az_k + Bf_k, \quad v_k = Cz_k + Df_k, \quad k \ge 0, \quad (7)$$

a minimal realization of $L(z)$. We conclude from Lemma 2 in the Appendix that there exists a positive-definite function V_L and positive constants μ, φ, ρ such that

$$\Delta V_L(k) \le -\mu V_L(z_k) - \varphi\|f_k\|^2 - \rho\|v_k\|^2 - f_k v_k,$$

along solutions to (7). On the other hand, from Proposition 1, we have that $\Delta V_E(k-1) \le f_k v_k$. Defining $W(k) := V_L(z_k) + V_E(x_{k-1})$, we conclude that

$$\Delta W(k) \le -\mu V_L(z_k) - \varphi\|f_k\|^2 - \rho\|v_k\|^2,$$

from which one concludes that $W(k)$ is bounded and z_k, f_k, and v_k all converge to zero as $k \to \infty$, proving that the system is indeed oscillation free. The above argument does not suffice to show that x_k is bounded because V_E is not radially unbounded. Since $V_E(x) \to +\infty$ as $x \to +\infty$, from the boundedness of $V_E(x_k)$ we conclude that x_k must be bounded above. However $V_E(x) \to 0$ as $x \to -\infty$ so conceivably x_k could be not bounded below. To show that this is not the case suppose than $x_{k_0} \ge 0$ and $x_{k_0+1} < 0$ for some integer k_0. Assuming that x_k remains negative until some time $k_1 > k_0$, we have

$$x_{k_1} = x_{k_0} + T\sum_{k=k_0+1}^{k_1} v_k = x_{k_0} + T\sum_{k=k_0+1}^{k_1} Cz_k,$$

because $f_k = 0$ while $x_k < 0$. Therefore $x_{k_1} \geq x_{k_0} - T\|C\| \sum_{k=k_0+1}^{+\infty} \|z_k\|$. But during this interval z_k is the state of an exponentially stable linear system so it converges to zero exponentially fast. So, the right-hand-side of the above inequality is finite and therefore x_k must be bounded below.

III. Transparency of virtual coupling

Ideally, we would like to eliminate virtual coupling because it changes the characteristics of the virtual environments. However, if it is removed the overall haptic systems may not be stable. We thus propose to construct the virtual coupling controller $C(z)$ so as to minimize its effect on the dynamics of the virtual environment as felt by the human operator. We focus on the case in which there is contact (i.e., when $E(x_k) = x_k$), because in the absence of contact the sampled force f_k is null and virtual coupling has no effect regardless of how $C(z)$ is selected.

The *admittance* of a haptic control system is the transfer function from the velocity of the interface to the force applied by its motors. We assume the virtual environment is in contact and therefore behaves as a linear system. For the haptic system in Fig. 1, the ideal admittance is

$$G(z) := B\frac{z-1}{Tz} + K \tag{8}$$

as this would be the discrete-time equivalent of the spring/damper virtual wall model in (1). However, virtual coupling leads to an admittance of

$$\frac{G(z)}{1 + G(z)C(z)}, \tag{9}$$

resulting in haptic distortion. A reasonable measure of distortion is the norm of the difference between the ideal admittance (8) and the real admittance (9):

$$\left\| W(z)\Big(\frac{G(z)}{1+G(z)C(z)} - G(z)\Big) \right\|_\infty^2, \tag{10}$$

where $\|\cdot\|_\infty$ denotes the H_∞-norm and $W(z)$ a function that allows us to weight the admittance error differently at different frequencies. In practice, we are mostly interested in preserving transparency in the frequency range, where humans are most sensitive. The choice of a proper weight function would be determined experimentally based on humans sensitivity over the range of frequencies. A controller that achieves transparency should minimize the criteria (10). This allow us to reformulate Problem 1 as selecting the controller $C(z)$ that minimizes the transparency criteria (10) among all that satisfy the stability condition (5). Hence Problem 1 can be written in mathematical form as follows:

Problem 2: For a given frequency-weighting function $W(z)$, find the controller $C(z)$ that

$$\begin{aligned} &\text{minimizes } \gamma \\ &\text{subject to } \left\| W(z)\Big(\frac{G(z)}{1+G(z)C(z)} - G(z)\Big) \right\|_\infty^2 < \gamma \quad (11) \\ &\qquad \frac{z-1}{Tz}\Big(C(z) - \frac{T}{2b}\Big) \in DSPR. \quad (12) \end{aligned}$$

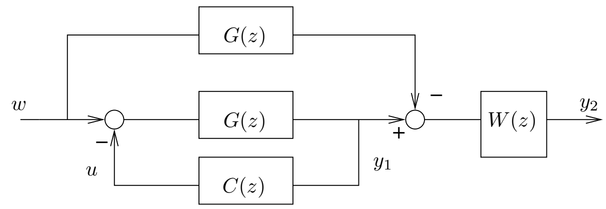

Fig. 2. Block diagram representation of $W(z)(\frac{G(z)}{1+G(z)C(z)} - G(z))$

This problem falls in the class of (closed-loop) H_∞ minimization subject to an (open loop) controller constraint. Such problems are typically hard to solve because they result in a non-convex matrix inequalities (specially the condition on the controller). However, we show that it can be solved with the recently introduced cone complementarity algorithm.

A. Matrix inequality derivation

Fig. 2 shows the transfer function in (11) and the goal is to minimize the H_∞ norm from input w to output y_2 with the constraint (12). We restrict our search among the controllers of the form

$$C(z) = \bar{C}(z) + \frac{dz}{z-1} + D_c, \tag{13}$$

where $\bar{C}(z)$ is a strictly proper transfer function and D_c is a scalar (for simplicity we pulled the direct feed-through out).

Remark 1: The condition (12) with controller (13) can be written as

$$\frac{z-1}{Tz}\big(\bar{C}(z) - \frac{T}{2b} + D_c\big) + \frac{d}{T} \in DSPR.$$

For any choice of $\bar{C}(z)$ and D_c this condition would not be satisfied unless $d > 0$ because at $z = 1$ the condition should also hold. This observation shows that restricting the search of controllers over the type (13) does not produce any conservativeness and in fact this choice is necessary to get good numerical results.

The realization of the controller $\bar{C}(z)$ can be written as:

$$\begin{aligned} x_c(k) &= A_c x_c(k) + B_c y(k), \\ u(k) &= C_c x_c(k). \end{aligned} \tag{14}$$

Then the minimal realization matrices of the transfer function $\frac{z-1}{Tz}\big(C(z) - \frac{T}{2b}\big)$ are

$$\bar{A}_c := \begin{bmatrix} A_c & B_c \\ 0 & 0 \end{bmatrix}, \qquad \bar{B}_c := \begin{bmatrix} \frac{1}{T}B_c \\ -\frac{1}{T} \end{bmatrix},$$

$$\bar{C}_c := \begin{bmatrix} C_c & D_c - \frac{T}{2b} \end{bmatrix}, \qquad \bar{D}_c := \frac{d}{T} - \frac{1}{2b} + \frac{D_c}{T}.$$

Fig. 3 shows the transfer function in (11) with the controller $\bar{C}(z)$ where

$$G_2(z) := \frac{G(z)}{1 + G(z)\big(\frac{dz}{z-1} + D_c\big)}$$

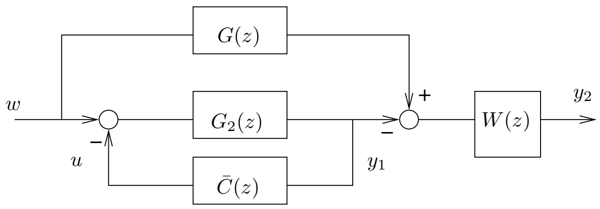

Fig. 3. Block diagram representation of $W(z)(\frac{G(z)}{1+G_2(z)\bar{C}(z)} - G(z))$

The realization of the four port process in Fig. 3 with inputs w, u and outputs y_1, y_2 is

$$\begin{aligned} x(k+1) &= Ax(k) + B_w w(k) + B_u u(k), \\ y_1(k) &= C_{y_1} x(k) + D_{y_1 w} w(k) + D_{y_1 u} u(k), \\ y_2(k) &= C_{y_2} x(k) + D_{y_2 w} w(k) + D_{y_2 u} u(k), \end{aligned} \qquad (15)$$

where all the matrices are known and functions of the realization matrices of $W(z)$, $G(z)$ and $G_2(z)$. The closed-loop system matrices with the process (15) and the controller (14) are

$$\mathcal{A} := \begin{bmatrix} A & B_u C_c \\ B_c C_{y_1} & A_c + B_c D_{y_1 u} C_c \end{bmatrix}, \quad \mathcal{B} := \begin{bmatrix} B_w \\ B_c D_{y_1 w} \end{bmatrix},$$
$$\mathcal{C} := \begin{bmatrix} C_{y_2} & D_{y_2 u} C_c \end{bmatrix}, \qquad \mathcal{D} := D_{y_2 w}. \qquad (16)$$

We consider a controllable canonical form for the controller

$$A_c = \begin{bmatrix} -\alpha_1 & -\alpha_2 & \dots & -\alpha_{n_c-1} & -\alpha_{n_c} \\ 1 & 0 & \dots & 0 & 0 \\ 0 & 1 & \dots & 0 & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \dots & 1 & 0 \end{bmatrix}, \quad B_c = \begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix},$$
$$C_c = \begin{bmatrix} c_1 & c_2 \dots c_{n_c-1} & c_{n_c} \end{bmatrix},$$

where the matrix C_c does not have any particular structure and n_c is the size of the controller. Defining

$$\chi := (\alpha_1, \alpha_2, \dots, \alpha_{n_c}, c_1, c_2, \dots c_{n_c}),$$

the closed-loop system can be written as

$$\begin{aligned} x_{cl}(k+1) &= \mathcal{A}_\nu(\chi) x_{cl}(k) + \mathcal{B}_\nu(\chi) w(k), \\ z(k) &= \mathcal{C}_\nu(\chi) x_{cl}(k) + \mathcal{D}_\nu(\chi) w(k), \end{aligned}$$

where $\nu := (d, D_c)$. Note that for each fixed value of ν, the matrices $\mathcal{A}_\nu(\chi), \mathcal{B}_\nu(\chi), \mathcal{C}_\nu(\chi), \mathcal{D}_\nu(\chi)$ are affine functions of the unknown controller parameters χ. The constraints (11) and (12) can be written in the matrix inequality form

$$\begin{bmatrix} \mathcal{A}'_\nu(\chi) P_1 \mathcal{A}_\nu(\chi) - P_1 & \mathcal{A}_\nu(\chi)' P_1 \mathcal{B}_\nu(\chi) \\ * & \mathcal{B}_\nu(\chi)' P_1 \mathcal{B}_\nu(\chi) \end{bmatrix} + \qquad (17)$$
$$\begin{bmatrix} \mathcal{C}'_\nu(\chi) \\ \mathcal{D}'_\nu(\chi) \end{bmatrix} \begin{bmatrix} \mathcal{C}_\nu(\chi) & \mathcal{D}_\nu(\chi) \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & -\gamma I \end{bmatrix} < 0,$$
$$\begin{bmatrix} \bar{A}'_c P_2 \bar{A}_c - P_2 & \bar{A}'_c P_2 \bar{B}_c \\ * & \bar{B}'_c P_2 \bar{B}_c \end{bmatrix} + \begin{bmatrix} 0 & -\bar{C}'_c \\ * & -\bar{D}_c - \bar{D}'_c \end{bmatrix} < 0, \quad (18)$$

respectively, where $P_1 > 0$ and $P_2 > 0$. The inequalities (17) and (18) are bilinear matrix inequalities (BMIs) in the unknowns P_1, P_2, and χ. The next lemma taken from [12], is required to develop a numerically tractable procedure.

Lemma 1: Assume that $Q(M)$ is a symmetric matrix and matrix variables M and N are independent. There exists a symmetric matrix $N > 0$ such that

$$S(M)' N S(M) - U' N U + Q(M) + R'(M) R(M) < 0, \qquad (19)$$

if and only if there exist symmetric matrices X and Y such that $X = Y^{-1}$ and

$$\begin{bmatrix} U' X U - Q(M) & S(M)' & R(M)' \\ * & Y & 0 \\ * & * & I \end{bmatrix} > 0. \qquad (20)$$

The proof is obtained by Schur lemma and the equation $X = Y^{-1} = N$. Lemma 1 changes the BMI (19) in M and N into the LMI (20) in $X, M,$ and Y with a non-convex constraint that $X = Y^{-1}$. Applying Lemma 1 to (17) and (18) we obtain the following formulation of Problem 2:

minimizes γ

subject to:

$$\begin{bmatrix} X_1 & 0 & \mathcal{A}'_\nu(\chi) & \mathcal{C}'_\nu(\chi) \\ * & \gamma I & \mathcal{B}'_\nu(\chi) & \mathcal{D}'_\nu(\chi) \\ * & * & Y_1 & 0 \\ * & * & * & I \end{bmatrix} > 0, \qquad (21)$$
$$\begin{bmatrix} X_2 & \bar{C}'_c & \bar{A}'_c \\ * & \bar{D}_c + \bar{D}'_c & \bar{B}'_c \\ * & * & Y_2 \end{bmatrix} > 0, \qquad (22)$$
$$X_1 = Y_1^{-1}, \qquad X_2 = Y_2^{-1}. \qquad (23)$$

The inequalities (21) and (22) are LMIs with the variables X_1, X_2, Y_1, Y_2, χ but the condition (23) is non-convex. Next we introduce a numerical procedure to solve this problem.

B. Numerical procedure and example

The Cone complementarity linearization algorithm [13] reformulates the optimization (21)-(23) with non-convex constraints into the following convex minimization problems:

1) *Choose* $\gamma > 0$ (a big number).
2) *Choose* D_c, d (d is positive, larger than $T/(2b) - D_c$).
3) *Find feasible points* X^0, Y^0 *for the set of LMIs (21)-(23) and*

$$\begin{bmatrix} X & I \\ I & Y \end{bmatrix} \geq 0, \qquad (24)$$

where $X = \begin{bmatrix} X_1 & 0 \\ 0 & X_2 \end{bmatrix}, Y = \begin{bmatrix} Y_1 & 0 \\ 0 & Y2 \end{bmatrix}$. IF no feasible point can be found, changed d and again do step 3 until the LMIs are feasible.

4) *Set* $X^j = X^{j-1}$, $Y^j = Y^{j-1}$, *and find* X^{j+1}, Y^{j+1} *that solves the LMI problem*

$$\Sigma_j : \textit{min tr}(X^j Y + X Y^j)$$
$$\textit{subject to } (21), (22), (24).$$

5) *If the stopping criterion is satisfied, decrease* γ *and go to 2. Otherwise set* $j = j + 1$ *and go to step 4 if* $j < c$ *(a preset number) or exit if* $j \geq c$.

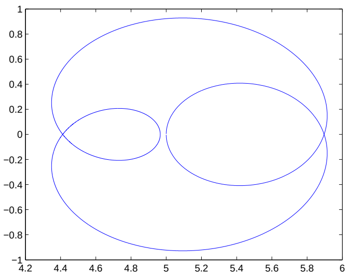

Fig. 4. Nyquist diagram of $\frac{z-1}{Tz}(C(z) - \frac{T}{2b})$.

If the solution of the minimization is equal to $2n_1 + 2n_2$ (n_1 and n_2 are the size of X_1 and X_2 respectively) then (21)-(22) are satisfied. It is numerically very difficult to obtain the minimum such that the $\mathrm{tr}(X_j Y + XY_j)$ equals $2n_1 + 2n_2$. Instead we choose (17) and (18) with $P_1 = X_1$ and $P_2 = X_2$ as the stopping criterion. The last value of γ is the minimum and the controller $C(z)$ with the state space (13) stabilizes the haptic system and maximizes transparency.

Example 1: We use the following values of parameters

$$T = 0.02,\ b = 10,\ K = 1000,\ B = 20,\ W(z) = \frac{0.1z + 0.03}{100z - 99}.$$

The condition (3) is not satisfied with this choice of parameters and we need to design a virtual coupling controller to guarantee the stability of the overall system. We obtained the following controller

$$C(z) = \frac{0.1z^3 - 0.0752z^2 + .0016z - 0.0101}{z^3 - 1.8407z^2 + 0.8444z - 0.0037}$$

using the numerical procedure mentioned above and searching among the controllers of the form (13) with the degree 3. It is important to note that $G_2(z)$ has a pole on the unit circle and the H_∞ design for the plants with poles on the unit circle (or on the jw axis for the continuous-time case) needs extra caution. To avoid this difficulty we moved the pole by $\epsilon = 0.0001$ placing it inside the unit circle for the controller design. Fig. 4 shows that the condition (12) on the virtual coupling controller is satisfied and therefore the overall haptic system is stable and oscillation free. The smallest γ is equal to 23.45 and shows that the admittance mismatch is small, leading to transparent virtual coupling in spite of a very slow sampling rate ($50Hz$). We may reduce γ by considering higher order controllers.

IV. Conclusions and Possible Extensions

We addressed the problem of designing controllers that stabilize the haptic system and maximize transparency. This approach is also amenable to incorporating other design requirements such robustness with respect to parameter uncertainty (e.g., in the mass m) or noise/disturbances rejection.

A direction for future research is the extension of these results to more general virtual environments as well as considering the effect of computational delay in the haptic system. Miller et al. [4] derived stability conditions for environments with nonlinearities and computational delay, for which the results here can be easily generalized. Addressing more general nonlinearities is still an open problem and may require different tools. Another avenue for future research consists of compensating the dynamics of the haptic interface to increase the fidelity of the system. The design procedure presented here may also be of use to this problem, but the sample-data effects need to be further investigated.

Appendix

Lemma 2: Let $H(z)$ be a square discrete-time real-rational transfer function such that $H(z)$ is discrete-time strictly positive-real (DSPR). If

$$x_{k+1} = Ax_k + Bu_k, \quad y_k = Cx_k + Du_k, \quad k \geq 0, \quad (25)$$

is a minimal realization of $H(z)$, there exists a positive-definite function V and strictly positive constants μ, φ, ρ such that along solutions to (25)

$$\Delta V(k) \leq -\mu V(x_k) - \varphi \|u_k\|^2 - \rho \|y_k\|^2 + u_k' y_k. \quad (26)$$

Proof can be found in the extend version [11].

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