

# Stochastic Networked Control Systems with Dynamic Protocols

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**Abstract**—We consider networked control systems in which sensors, controllers, and actuators communicate through a shared network that introduces stochastic intervals between transmissions, delays and packet drops. Access to the communication medium is mediated by a protocol that determines which node (one of the sensors, one of the actuators, or the controller) is allowed to transmit a message at each sampling/actuator-update time. We provide conditions for mean exponential stability of the networked closed loop in terms of matrix inequalities, both for investigating the stability of given protocols, such as static round-robin protocols and dynamic maximum error first-try once discard protocols, and to design new dynamic protocols. The main result entailed by these conditions is that if the networked closed loop is stable for a static protocol then we can provide a dynamic protocol for which the networked closed loop is also stable. The stability conditions also allow for obtaining an observer-protocol pair that reconstructs the state of an LTI plant in a mean exponential sense and less conservative stability results than other conditions that previously appeared in the literature.

## I. INTRODUCTION

The proliferation of network communication systems in recent years paved the way for important research in the area of networked control systems. This research area addresses control loops closed via a shared network that provides the medium for sensor, actuator and controller nodes to communicate.

Walsh and co-authors [1] made strides in the analysis of control systems closed via a local area network, such as the controller-area network, the ethernet and wireless 802.11 networks. The key assumptions in [1] are that there exists a bound on the interval between transmissions denoted by *maximum allowable transfer interval* (MATI), and that transmission delays and packet drops are negligible. In [1] an emulation set-up is considered in the sense that the controller for the networked control system is obtained from a previously designed stabilizing continuous-time controller. Two basic types of protocols have been proposed: static protocols, such as the round-robin (RR) protocol where nodes take turns transmitting data in a periodic fashion; and dynamic protocols such as the maximum error first-try once discard (MEF-TOD) protocol, where the node that has the top priority in using the communication medium is the one whose current value to transmit differs the most from the last transmitted value. Under this setup, one can

attempt to provide an upper bound on the MATI for which stability can be guaranteed. Since these protocols have been proposed in the papers referenced above, MATI bounds have been improved [2], [3], [4]. Although, [1] illustrate through simulations that using the MEF-TOD protocol allows for preserving stability of the networked closed loop for a larger MATI than that obtained when using the RR protocol, and similar conclusions are obtained in [2]- [4] from sufficient stability conditions, no analytical result has been established proving that this holds in general.

As mentioned in [1], the occurrence of transmission events on the network is often more appropriately modeled as a random process. This feature is taken into account in [5], which considers networked control systems with MEF-TOD and RR protocols, and independent and identically distributed (i.i.d.) intervals between transmissions. It is shown that stability can be guaranteed for distributions of the inter-transmission intervals that have a support larger than previously derived deterministic upper bounds for the MATI [1], [2], [3]. The conservativeness of these results for linear networked control systems using the RR protocol was eliminated in [6]. Recently, [7] addresses a model of networked control systems with i.i.d. intervals between transmissions and stochastic delays for a class of quadratic protocols that is more general than MEF-TOD. Through a convex over-approximation approach, sufficient conditions are given for mean exponential stability. In [8] a method is proposed to design an observer-protocol pair to asymptotically reconstruct the states of an LTI plant where the plants outputs are sent through a network with constant intervals between transmissions. The protocol to be designed can be viewed as a weighted version of the MEF-TOD.

In the present paper we follow this line of research considering that the network imposes i.i.d. intervals between transmissions. We also take into account stochastic delays modeled as in [7], and packet drops. We consider that access to the network is mediated by a dynamic protocol specified as follows. Associated to each node there is a set of quadratic state functions, which are evaluated at a given transmission time. The node allotted to transmit is the one corresponding to the least value of these quadratic state functions. These protocols are more general than quadratic protocols considered in [7] and thus more general than the MEF-TOD protocol.

We establish two stability results for the networked control system, both providing conditions in terms of linear matrix inequalities (LMIs) for investigating the stability in a mean exponential sense of given protocols, and conditions in terms of BMIs to design quadratic state functions, specifying the

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dynamic protocol, that yield the networked closed loop stable. The first stability result allows to prove that if the networked closed loop is stable for a static protocol then we can provide a dynamic protocol for which the networked closed loop is also stable. This is the main contribution of the paper and gives an analytical justification on why one should utilize dynamic protocols rather than static, while, e.g., in [1], this conclusion is only illustrated through simulation. The second stability result allows us to extend the work [8] to the case where transmission intervals are stochastic.

We illustrate through benchmark examples, that the conditions in this paper are significantly less conservative than other conditions that previously appeared in the literature.

The remainder of the paper is organized as follows. The networked control problem set up is given in Section II. The main results are stated in Section III. In Section IV we compare our results with previous works. Concluding remarks are given in Section V.

*Notation* We denote by  $I_n$  and  $O_n$  the  $n \times n$  identity and zero matrices, respectively, and by  $\text{diag}([A_1 \dots A_n])$  a block diagonal matrix with blocks  $A_i$ . For dimensionally compatible matrices  $A$  and  $B$ , we define  $(A, B) := [A^\top B^\top]^\top$ .

## II. PROBLEM FORMULATION

We start by introducing the networked control stability problem and then we show that it can be casted into analyzing the stability of an impulsive system.

### A. Networked Control Set-up

We consider a networked control system for which sensors, actuators, and a controller, are connected through a communication network, possibly shared with other users. The plant and controller are described by the following state-space model:

$$\text{Plant:} \quad \dot{x}_P = A_P x_P + B_P \hat{u}, \quad y = C_P x_P \quad (1)$$

$$\text{Controller:} \quad \dot{x}_C = A_C x_C + B_C \hat{y}, \quad u = C_C x_C + D_C \hat{y}. \quad (2)$$

Following an emulation approach, we assume that the controller has been designed to stabilize the closed loop, when the process and the controller are directly connected, i.e.,  $\hat{u}(t) = u(t)$ ,  $\hat{y}(t) = y(t)$ , and we are interested in analyzing the effects of the network on the stability of the closed loop. We denote the times at which a node transmits a message by  $\{t_k, k \in \mathbb{N}\}$ , and assume that,  $\hat{u}$  and  $\hat{y}$  are held constant, between transmission times, i.e.,

$$\hat{u}(t) = \hat{u}(t_k), \quad \hat{y}(t) = \hat{y}(t_k), \quad t \in [t_k, t_{k+1}), \quad k \in \mathbb{Z}_{\geq 0}, \quad (3)$$

We denote by  $e$  the error signal between process output and controller input ( $\hat{y} - y$ ) and between controller output and process input ( $\hat{u} - u$ ). In particular,

$$e(t) := (\hat{y}(t) - y(t), \hat{u}(t) - u(t)). \quad (4)$$

We assume that while  $m$  nodes compete for the network, only one of them is allowed to transmit at each given transmission time. However, in our terminology a single transmitting node could be associated with several entries of the process output  $y$  or with several entries of the controller output  $u$ . For

simplicity we assume that the sensors and actuators have been ordered in such a way that we can partition the error vector as  $e = (e_1, \dots, e_m)$ , where each  $e_i(t) \in \mathbb{R}^{s_i}$  is the error associated with node  $i \in \mathcal{M} := \{1, \dots, m\}$ . The state of the networked control system is thus defined by the vector  $x := (x_P, x_C, e)$ , where  $x_P \in \mathbb{R}^{n_P}$ ,  $x_C \in \mathbb{R}^{n_C}$ ,  $e \in \mathbb{R}^{n_e}$ , and  $x \in \mathbb{R}^{n_x}$ . We are interested in scenarios for which the following assumptions hold:

- (i) The time intervals  $\{h_k := t_{k+1} - t_k\}$  are i.i.d. described by a probability measure  $\mu$  with support on  $[0, \gamma]$ ,  $\gamma \in \mathbb{R}_{\geq 0} \cup \{+\infty\}$ , i.e.,  $\text{Prob}(a \leq h_k \leq b) = \int_a^b \mu(dr)$  for  $a, b \in [0, \gamma]$ .
- (ii) Corresponding to a transmission at time  $t_k$  there is a transmission delay  $d_k$  no greater than  $h_k = t_{k+1} - t_k$ ; A joint stationary probability density  $\chi$  describes  $(h_k, d_k)$ , in the sense that

$$\text{Prob}(a \leq h_k \leq b, c \leq d_k \leq d) = \int_a^b \int_c^d \chi(dr, ds) \quad (5)$$

where  $a, b \in [0, \gamma]$  and  $\text{Prob}(a \leq h_k \leq b, c \leq d_k \leq d) = 0$  if  $c > b$ . In view of (i) and (ii), we have that  $\mu([a, b]) = \chi([a, b], [0, b])$ .

- (iii) At each transmission time there is a probability  $p_{\text{drop}}$  that a packet may not arrive at its destination or that it may arrive corrupted (packet drop).
- (iv) The nodes implement one of the two protocols:

*Dynamic protocol (DP):* This protocol is specified by  $m_D$  symmetric matrices  $\{R_i, i \in \mathcal{M}_D\}$ ,  $\mathcal{M}_D := \{1, \dots, m_D\}$ ,  $m_D \geq m$ . A subset of these matrices  $\{R_i, i \in I_j\}$  is associated with node  $j \in \mathcal{M}$  where  $I_j := \{i_1^j, i_2^j, \dots, i_{r_j}^j\}$  is an index subset of  $\mathcal{M}_D$ . These subsets are assumed to be nonempty, i.e.,  $r_j \geq 1$ , disjoint, and the  $r_j$  are such that  $\sum_{j=1}^m r_j = m_D$ . The node  $j$  allotted to transmit at  $t_k$  is determined by the map  $d: \mathbb{R}^{n_x} \mapsto \mathcal{M}$ ,

$$d(x(t_k^-)) = d_1 \circ d_2(x(t_k^-)), \quad (6)$$

where  $d_2: \mathbb{R}^{n_x} \mapsto \mathcal{M}_D$  is given by

$$d_2(x(t_k^-)) := \text{argmin}_{i \in \mathcal{M}_D} x(t_k^-)^\top R_i x(t_k^-), \quad (7)$$

and  $d_1: \mathcal{M}_D \mapsto \mathcal{M}$  is given by

$$d_1(i) := \{j : i \in I_j\}. \quad (8)$$

In case the minimum in (7) is achieved simultaneously for several value of the index  $i$ , stability of the networked control system should be guaranteed regardless of the specific choice for the argmin. In view of (6), the error  $e$  is updated at time  $t_k$  according to

$$e(t_k) = (I_{n_e} - \Lambda_{d(x(t_k^-))})e(t_k^-), \quad (9)$$

where  $\Lambda_j := \text{diag}([0_{\sum_{i=1}^{j-1} s_i} I_{s_j} 0_{\sum_{i=j+1}^m s_i}])$ ,  $j \in \mathcal{M}$ . That is, only the components of  $\hat{y}$  or  $\hat{u}$  associated with the node that transmits are updated by the corresponding components of  $y(t_k^-)$  or  $u(t_k^-)$ .

*Static Protocol (SP)*: The nodes transmit in a  $m_S$ -periodic sequence determined by a periodic function

$$s : \mathbb{N} \mapsto \mathcal{M} \quad (10)$$

with period  $m_S$ . In this case, the error  $e$  is updated at time  $t_k$  according to

$$e(t_k) = (I_{n_e} - \Lambda_{s(k)})e(t_k^-). \quad (11)$$

We assume that  $s$  is onto, i.e., each node transmits at least once in a period. When  $m_S = m$ , each node transmits exactly once in a period.

As mentioned in Section I, the assumptions (i)-(iii) are appropriate for networked control systems in which feedback loops are closed via local area networks (cf. [1], [9]). The class of dynamic protocols that we describe in (iv) allow a node to transmit if the state of the networked control system lies on a given region of the state space, partitioned according to quadratic restrictions. This class of protocols boils down to the quadratic protocols introduced in [7] when  $m_D = m$ . Thus, our definition allows for ampler partitions of the state-space than quadratic protocols, and as we shall see it also allows to obtain that dynamic protocols are in a sense better than static ones. If we make  $m_D = m$  and chose  $P > 0$  such that  $R_i = P - \text{diag}([0_{n_P+n_C} \ \Lambda_i]) > 0$ , then (6) becomes the usual MEF-TOD protocol, where the node that transmits is the one with the maximum norm of the error  $e_i(t)$  between its current value and its last transmitted value.

### B. Impulsive systems

Suppose that there are no delays, i.e.  $d_k = 0$ , and no packet drops, i.e.,  $p_{\text{drop}} = 0$ . Then we can write the networked control system (1), (2), (3), (4), in the form of the following impulsive system

$$\begin{aligned} \dot{x}(t) &= Ax(t), \quad x(0) = x_0 \\ x(t_k) &= J_{p(x(t_k^-), k)} x(t_k^-), \end{aligned} \quad (12)$$

where  $x \in \mathbb{R}^{n_x}$  and  $t_{k+1} - t_k$  are i.i.d. random variables characterized by the probability density  $\mu$ , and the map  $p$  takes the following form for dynamic and static protocols

$$\text{DP:} \quad p(x(t_k^-), k) = d(x(t_k^-)) \quad (13)$$

$$\text{SP:} \quad p(x(t_k^-), k) = s(k). \quad (14)$$

For example, the following expressions for  $A$  and  $\{J_i, i \in \mathcal{M}\}$ , correspond to the case in which the controller and plant are directly connected and only the outputs are transmitted through the network, i.e.,  $\hat{u}(t) = u(t)$ ,  $x = (x_P, x_C, \hat{y} - y)$ .

$$\begin{aligned} A &= \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \\ A_{11} &= \begin{bmatrix} A_P + B_P D_C C_P & B_P C_C \\ B_C C_P & A_C \end{bmatrix} \\ A_{12} &= \begin{bmatrix} B_P D_C \\ B_C \end{bmatrix} \\ A_{21} &= -[C_P \ 0] A_{11} \\ A_{22} &= -[C_P \ 0] A_{12} \\ J_i &= \text{diag}([I_{n_P+n_C} \ I_{n_e} - \Lambda_i]), \quad i \in \mathcal{M}. \end{aligned} \quad (15)$$

This case will be considered in Section IV. Expressions for the general case considered in Section II can be easily obtained (see, e.g., [10, p. 5]).

To take into account delays and packet drops modeled as described in Section II, we consider the following impulsive system

$$\begin{aligned} \dot{x}(t) &= Ax(t), \quad x(0) = x_0. \\ x(t_k) &= K_{p(x(t_k^-), k)}^{q_k} x(t_k^-), \\ x(s_k) &= Lx(s_k^-), \quad t_k \leq s_k \leq t_{k+1}, \end{aligned} \quad (16)$$

where  $p(x_k, k)$  is defined as in (13) for dynamic protocols and as in (14) for static protocols. The random variables  $t_k$  and  $s_k$  are completely defined by the inter-sampling times  $h_k := t_{k+1} - t_k$  and by the delays  $d_k := s_k - t_k$ . The  $(h_k, d_k)$  are i.i.d., and are as described by (5). The  $q_k \in \{1, \dots, n_q\}$  are i.i.d., and such that  $\text{Prob}[q_k = j] = w_j \forall j \in \{1, \dots, n_q\}, k \geq 0$ . We provide below expressions for  $A$ ,  $L$ ,  $w_i$  and  $K_i^j$ ,  $i \in \mathcal{M}$ ,  $j \in \{1, \dots, n_q\}$  which model the case where the controller and the plant are directly connected and only the plant outputs are transmitted through the network, i.e.,  $\hat{u}(t) = u(t)$ . The state is now considered to be  $x = (x_P, x_C, \hat{y}, v) \in \mathbb{R}^{n_x}$  where  $v \in \mathbb{R}^{n_e}$  is an auxiliary vector  $(v_1, \dots, v_m)$  that is updated with the sampled value  $v_j = y_j(t_k)$  at each sampling time  $t_k$  at which node  $j$  is allowed to transmit. However, the update only takes place if a packet sent at  $t_k$  is not dropped and the sampled value  $v_j$  is only used to update the value of  $\hat{y}_j$  after a transmission delay  $d_k$ , at the time  $s_k = t_k + d_k$ .

$$\begin{aligned} A &= \begin{bmatrix} A_{11} & A_{12} & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \\ A_{11} &= \begin{bmatrix} A_P & B_P C_C \\ 0 & A_C \end{bmatrix} \\ A_{12} &= \begin{bmatrix} B_P D_C \\ B_C \end{bmatrix} \\ n_q &= 2, \quad w_1 = 1 - p_{\text{drop}}, \quad w_2 = p_{\text{drop}} \\ K_i^1 &= \begin{bmatrix} I_{n_P} & 0 & 0 & 0 \\ 0 & I_{n_C} & 0 & 0 \\ 0 & 0 & I_{n_e} & 0 \\ \Lambda_i C_P & 0 & 0 & I_{n_e} - \Lambda_i \end{bmatrix}, \\ K_i^2 &= I_{n_P+n_C+2n_e}, \quad i \in \mathcal{M} \\ L &= \begin{bmatrix} I_{n_P+n_C} & 0 & 0 \\ 0 & 0 & I_{n_e} \\ 0 & 0 & I_{n_e} \end{bmatrix}. \end{aligned} \quad (17)$$

Again, the expressions for the general case considered in Section II can be easily obtained. It is also important to mention that there are other ways to model the setup with delays and packet drops described in Section II. For example one can find a similar model to (16) but introduce the dependency on the variable  $q_k$  modeling the packet drops in the matrix  $L$ .

We say that (12) is *mean exponentially stable (MES)* if there exists constants  $c > 0$  and  $0 < \alpha < 1$  such that for

any initial condition  $x_0$ , we have that

$$\mathbb{E}[x(t_k)^\top x(t_k)] \leq c\alpha^k x_0^\top x_0, \forall k \geq 0. \quad (18)$$

The same definition of MES is used for the system (16). We assume that the following condition holds

$$e^{2\bar{\lambda}(A)t} r(t) < ce^{-\alpha_1 t} \text{ for some } c > 0, \alpha_1 > 0. \quad (19)$$

where  $\bar{\lambda}(A)$  is the real part of the eigenvalues of  $A$  with largest real part and  $r(t) := \mu((t, \gamma))$  denotes the survivor function. Assuming (19), we were able to prove in [6], considering only static protocols, that (18) is equivalent to the more usual notion of mean exponential stability in continuous-time where one requires  $\mathbb{E}[x(t)^\top x(t)]$  to decrease exponentially. In the present paper we make no such assertion, although assuming (19) is still useful (e.g., (19) guarantees that (21) is bounded).

### III. MAIN RESULTS

For simplicity, we assume in Subsection III-A and III-B, that there are no delays, i.e.,  $d_k = 0, \forall k$ , and no packet drops, i.e.,  $p_{\text{drop}} = 0$ , and in Subsection III-C we consider the general case.

#### A. Stability Result I and dynamic vs. static protocols

The following is our first stability result for (12).

*Theorem 1:* The system (12) with dynamic protocol (13) is MES if there exist scalars  $\{0 \leq p_{ji} \leq 1, j, i \in \mathcal{M}_D\}$ , with  $\sum_{j=1}^{m_D} p_{ji} = 1, \forall i \in \mathcal{M}_D$ , and  $n_x \times n_x$  symmetric matrices  $\{R_i > 0, i \in \mathcal{M}_D\}$  such that

$$J_{d_1(i)}^\top \left( \sum_{j=1}^{m_D} p_{ji} E(R_j) \right) J_{d_1(i)} - R_i < 0, \forall i \in \mathcal{M}_D, \quad (20)$$

where

$$E(R_j) := \int_0^\gamma e^{A^\top h} R_j e^{Ah} \mu(dh). \quad (21)$$

□

This result can be used to analyze if a given protocol yields the networked control system stable or to synthesize a protocol that achieves this.

*Analysis:* Note first that a given dynamic protocol specified by  $R_i > 0, i \in \mathcal{M}_D$ , is equivalent to a dynamic protocol specified by

$$\tilde{R}_i = P + R_i > 0, i \in \mathcal{M}_D, \quad (22)$$

where  $P$  can be any symmetric matrix such that  $P + R_i > 0$ . If we replace in (20) the matrices  $R_i$  by  $\tilde{R}_i$ , given by (22), we obtain that (20), (22) are LMIs in the variables  $P$  and  $p_{ji}$  (using the fact that  $\sum_{j=1}^m p_{ji} = 1, \forall i \in \mathcal{M}_D$ ).

*Synthesis:* If we allow  $R_i$  to be variables in (20), then (20) are in general BMIs. In fact, if we chose a basis  $B_l$  for the linear space of symmetric matrices, we have  $R_i = \sum_{l=1}^{n_s} b_{il} B_l$  and (20) depends on the products  $p_{ji} b_{il}$ . In this case the dynamic protocol, determined by the matrices  $R_i$  comes out from the solution to (20).

To state the next theorem, we need the following result which can be found in [6]. Let  $[i] := i$  if  $i \in \{1, \dots, m_S - 1\}$  and  $[i] = 1$  if  $i = m_S$ . Let  $\mathcal{M}_S := \{1, \dots, m_S\}$ .

*Theorem 2:* The system (12), with static protocol (14) is MES if and only if there exists  $n_x \times n_x$  symmetric matrices  $\{R_i > 0, i \in \mathcal{M}_S\}$  such that

$$J_{s(i)}^\top E(R_{[i+1]}) J_{s(i)} - R_i < 0, \forall i \in \mathcal{M}_S \quad (23)$$

where  $E(R_{[i+1]})$  is given as in (21).

From Theorems 1 and 2 we can conclude the following result.

*Theorem 3:* If the networked control system is MES for a static protocol with period  $m_S$  then there exists a dynamic protocol taking the form (6), with  $m_D = m_S$ , that yields the networked control system MES.

*Proof:* Since the stability conditions of Theorem 2 are necessary and sufficient, there exists a static protocol with period  $m_S$  that yields the networked control system MES if and only if there exists  $\{R_i, i \in \mathcal{M}_S\}$  such that (23) holds for (12) with matrices defined by (15). This implies that if we consider a dynamic protocol with  $m_D = m_S$ ,  $I_j = \{k \in \mathcal{M}_S : s(k) = j\}, j \in \mathcal{M}$ , then  $d_1(i) = s(i)$ , for  $i \in \mathcal{M}_S$  and (20) holds with

$$p_{ji} = \begin{cases} 1, & \text{if } i < m_D \text{ and } j = i + 1, \\ 1, & \text{if } i = m_D \text{ and } j = 1, \\ 0 & \text{otherwise} \end{cases}$$

and with  $\{R_i, i \in \mathcal{M}_S = \mathcal{M}_D\}$  taken to be the solution to (23). ■

From the proof of Theorem 3 we see that the matrices  $\{R_i, i \in \mathcal{M}_D\}$  that characterize the dynamic protocol mentioned in its statement can be taken to be the solution to (23). Note that in the special case where  $m_D = m = m_S$ , the Theorem 3 states that if there exists a round-robin protocol with period  $m_S = m$ , i.e., each node only transmits exactly once in a period, that yields the networked control system MES, then one can find a quadratic protocols as introduced in [7] that also yields the networked control system MES.

*Remark 4:* The fact that the stability conditions of Theorem 2 are necessary and sufficient is key to obtain Theorem 3. In the work [4] a similar reasoning to Theorem 3 can be used to prove that if the stability conditions provided there for quadratic protocols (cf. [4, Th. 3]) hold then so do the stability conditions for a static protocol in the special case where each node transmits only once in a period (cf. [7, Th. 3]). However, since the conditions provided in [4] are only sufficient for the RR protocol, it does not allow to conclude that if a stabilizing static protocol exists then so does a dynamic protocol, as stated in Theorem 3. Although [7] does not explicitly presents stability conditions for a static protocol, the same remarks should apply, since convex over-approximations introduce conservativeness.

#### B. Stability Result II and observer-protocol design

The following is our second stability result for (12).

*Theorem 5:* The system (12) with dynamic protocol (13) is MES if there exists an  $n_x \times n_x$  symmetric matrix  $W > 0$ , scalars  $\{c_{ij} > 0, i, j \in \mathcal{M}_D, i \neq j\}$  and  $n_x \times n_x$  matrices

$R_i, i \in \mathcal{M}_D$  such that

$$J_{d(i)}^\top E(W) J_{d(i)} + \sum_{j=1, j \neq i}^{m_D} c_{ij} (R_j - R_i) - W < 0, \forall i \in \mathcal{M}_D \quad (24)$$

where  $E(W) := \int_0^\gamma (e^{Ah})^\top W e^{Ah} \mu(dh)$ .

□

Given a quadratic protocol, i.e., specific values for the matrices  $R_i$ , testing if (24) holds is an LMI feasibility problem. To design a protocol for which mean exponential stability of the networked control system is guaranteed, we can take the  $\{R_i, i \in \mathcal{M}_D\}$  as additional unknowns and (24) should now be viewed as a BMI feasibility problem.

We show next that Theorem 5 allows to extend the observer-protocol design proposed in [8].

1) *Observer Design:* Suppose that we wish to estimate the state of the following plant

$$\dot{x}_P(t) = A_P x_P(t), \quad y(t) = C_P x_P(t), \quad x_P(0) = x_{P0}$$

where the  $m$  outputs  $y(t) = (y_1, \dots, y_m)$ ,  $y_i \in \mathbb{R}^{s_i}$  are sent through a network that imposes i.i.d. intervals between transmissions to a remote observer. As in Section II, we denote by  $\mu$  the measure that defines the inter-transmissions times  $h_k := t_{k+1} - t_k$ . Let  $\mathcal{M} := \{1, \dots, m\}$  and  $\Psi_j := \text{diag}\{0_{s_1}, \dots, I_{s_j}, \dots, 0_{s_m}\}$ , for  $j \in \mathcal{M}$ . A natural linear remote observer for this system is defined by

$$\dot{\hat{x}}(t) = A_P \hat{x}(t) + L_k \Psi_{c(x_e(t_k^-))} (C_P \hat{x}(t_k^-) - y(t_k^-)), \quad (25)$$

where the observer gains  $L_k$  to be designed are allowed to depend on the index  $k$  and the map

$$c(x_e(t_k^-)) := \text{argmin}_{j \in \mathcal{M}} x_e(t_k^-)^\top C_P^\top S_j C_P x_e(t_k^-) \quad (26)$$

determines which node transmits at  $t_k$  based on the estimation error  $x_e(t_k^-) := \hat{x}_P(t_k^-) - x_P(t_k^-)$ , where  $\{S_j, j \in \mathcal{M}\}$  is a set of  $m$  matrices. As argued in [8], the sensors should run a replica of the remote observer to access  $\hat{x}(t)$ , which allows each node to encode in the message arbitration field  $x_e(t_k^-)^\top C_P^\top S_j C_P x_e(t_k^-)$ , where  $C_P x_e(t_k^-) = C_P \hat{x}(t) - y_j(t)$ ,  $j \in \mathcal{M}$ .

The resulting estimation error  $x_e := \hat{x} - x_P$  evolves according to

$$\dot{x}_e(t) = A_P x_e(t) + L_k \Psi_{c(x_e(t_k^-))} C_P x_e(t_k). \quad (27)$$

We can cast this problem into the framework of (12) with dynamic protocol (13) by adding an auxiliary variable  $v$  that holds the value of  $x_e(t_k)$  between transmission times, considering  $x = (x_e, v)$  and

$$\begin{aligned} A &= \begin{bmatrix} A_P & I_{n_P} \\ 0_{n_P} & 0_{n_P} \end{bmatrix}, \\ J_i &= \begin{bmatrix} I_{n_P} & 0_{n_P} \\ 0_{n_P} & 0_{n_P} \end{bmatrix} + \begin{bmatrix} 0 \\ L_k \end{bmatrix} [\Psi_i C_P \quad 0], \\ R_i &= \begin{bmatrix} C_P^\top S_j C_P & 0 \\ 0 & 0_{n_P} \end{bmatrix} \end{aligned} \quad (28)$$

In the following theorem, we propose a method to obtain observer gains  $L_k$  that yield the networked control system MES. To state the result we need the following assumption:

$$H(s) := \int_0^s e^{A_P r} dr \text{ is invertible for every } s \in [0, \gamma] \quad (29)$$

While this assumption holds generically, it is possible to construct examples where it does not, as in the case where  $\gamma > s = 2\pi$  and  $A_P = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ , in which case  $H(s) = 0$ .

*Theorem 6:* Suppose that (29) hold. If there exists a  $n_P \times n_P$  symmetric matrix  $P > 0$ , a  $m \times m$  matrix  $Y$ , a  $n_P \times m$  matrix  $M$ ,  $m \times m$  matrices  $\{S_i, i \in \mathcal{M}\}$ , and scalars  $\{c_{ij} > 0, i, j \in \mathcal{M}\}$ , such that

$$\begin{aligned} F(P) + DM\Psi_i C_P + (DM\Psi_i C_P)^\top + C_P^\top Y C_P + \\ \sum_{j=1, j \neq i}^m c_{ij} (C_P^\top S_j C_P - C_P^\top S_i C_P) - P < 0, \forall i \in \mathcal{M} \\ \begin{bmatrix} P & M \\ M^\top & Y \end{bmatrix} > 0, \end{aligned} \quad (30)$$

where  $F(P) := \int_0^\gamma e^{A_P^\top r} P e^{A_P r} \mu(dr)$  and  $D := \int_0^\gamma e^{A_P r} \mu(dr)$ , then we have that the observer gain  $L_k = H(h_k)^{-1} P^{-1} M$  yields (12) with matrices (28) MES.

□

Note that our proposed observer gain  $L_k$  depends on the length  $h_k$  of the time interval  $\{t_{k+1} - t_k\}$ , which is not known at time  $t_k \leq t \leq t_{k+1}$  (25). In practice this results in a delay in constructing the state estimate that never needs to exceed  $h_k$  since the state of the remote observer (25) can only be updated with the measurement  $y(t_k)$  at the time  $t_{k+1}$  at which  $h_k$  can be computed.

Similarly to the Theorem 5, the conditions of the Theorem 6 can be used to investigate the stability of a given protocol determined by matrices  $R_j$ , in which case the problem reduces to an LMI feasibility problem, or they can be used to design a protocol, in which case one needs to solve a BMI feasibility problem.

*Remark 7:* When the intervals between transmission are constant, one can show that the stability conditions (30) and (31) are equivalent to the ones given in [8], where such an assumption is made. In this case, the matrices  $L_k$  do not depend on  $k$ , and can therefore be computed off-line.

### C. Extensions to handle delays and packet drops

Theorems 5 and 1 can be extended to the case where the network introduces packet drops and delays modeled by (16) with matrices (17). We state these extensions next.

*Theorem 8:* The system (16) with dynamic protocol (13) is MES if there exist scalars  $\{0 \leq p_{ji} \leq 1, j, i \in \mathcal{M}_D\}$ , with  $\sum_{j=1}^{m_D} p_{ji} = 1, \forall i \in \mathcal{M}_D$ , and  $n_x \times n_x$  symmetric matrices  $\{R_i > 0, i \in \mathcal{M}_D\}$  such that

$$\sum_{l=1}^{n_q} w_l (K_{d_1(i)}^l)^\top \left( \sum_{j=1}^{m_D} p_{ji} E(R_j) \right) K_{d_1(i)}^l - R_i < 0, \forall i \in \mathcal{M}_D,$$

where

$$E(R_j) := \int_0^\gamma \int_0^h (e^{Ah-s} L e^{As})^\top R_j e^{Ah-s} L e^{As} \chi(dh, ds). \quad (32)$$

□

**Theorem 9:** The system (16) with dynamic protocol (13) is MES if there exists an  $n_x \times n_x$  symmetric matrix  $W > 0$ , scalars  $\{c_{ij} > 0, i, j \in \mathcal{M}_D, i \neq j\}$  and  $n_x \times n_x$  matrices  $R_i, i \in \mathcal{M}_D$  such that

$$\sum_{l=1}^{n_q} w_l (K_{d_1(i)}^l)^\top E(W) K_{d_1(i)}^l + \sum_{j=1, j \neq i}^m c_{ij} (R_j - R_i) - W < 0,$$

$\forall i \in \mathcal{M}_D$ , where  $E(W)$  is defined as in (32).

#### IV. NETWORKED CONTROL RESULTS

In this section we show that Theorems 5 and 9 reduce the conservatism of the results in [5], [9] and [7]. These three works use the same benchmark problem for the control of a batch reactor, where the plant (1) and controller (2) matrices are given by

$$A_P = \begin{bmatrix} 1.38 & -0.2077 & 6.715 & -5.676 \\ -0.5814 & -4.29 & 0 & 0.675 \\ 1.067 & 4.273 & -6.654 & 5.893 \\ 0.048 & 4.273 & 1.343 & -2.104 \end{bmatrix},$$

$$B_P = \begin{bmatrix} 0 & 0 \\ 5.679 & 0 \\ 1.136 & -3.146 \\ 1.136 & 0 \end{bmatrix}, \quad C_P = \begin{bmatrix} 1 & 0 & 1 & -1 \\ 0 & 1 & 0 & 0 \end{bmatrix}.$$

$$A_C = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} B_C = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} C_C = \begin{bmatrix} -2 & 0 \\ 0 & 8 \end{bmatrix} D_C = \begin{bmatrix} 0 & -2 \\ 5 & 0 \end{bmatrix}.$$

Only the two outputs are sent through the network, i.e.,  $u(t) = \hat{u}(t)$ . The network imposes i.i.d. intervals between transmissions, possibly packet drops and no delays. The networked control closed loop can be written as in (12), (15) in the absence of drops and as in (16)-(17) when drops occur. Thus, the stability of the networked control system can be tested by Theorems 1, 5, and 8, 9. The results are shown in the Table I, considering two distributions  $\mu$  for the inter-transmissions intervals  $h_k$ : uniform in the interval  $[0, \gamma]$ , and exponential with expected value  $1/\lambda_{\text{exp}}$ .

TABLE I

STABILITY RESULTS FOR THE BATCH REACTOR EXAMPLE-MEF-TOD AND ROUND ROBIN PROTOCOL. NA STANDS FOR NOT AVAILABLE

	Dynamic Protocol		Static Protocol	
	no drops	$p = 0.5$	no drops	$p = 0.5$
Max. $\gamma : h_k \sim \text{Uni}(\gamma)$				
Results from [3]	NA	NA	NA	NA
Results from [5]	0.0372	0.0170	0.0517	0.0199
Results from [7]	0.11	NA	NA	NA
Ths. 5 and 9	0.0550	0.024	NA	NA
Ths. 1	0.111	NA	NA	NA
Th. 2	NA	NA	0.112	0.0385
Max. $1/\lambda_{\text{exp}} : h_k \sim \text{Exp}(\lambda_{\text{exp}})$				
Results from [3]	0.0095	0.0046	NA	NA
Results from [5]	0.0158	0.00795	0.0217	0.00924
Results from [7]	NA	NA	NA	NA
Ths. 5 and 9	0.0226	0.01124	NA	NA
Ths. 1	0.0357	NA	NA	NA
Th. 2	NA	NA	0.0417	0.0188

From Table I we can conclude that our results allow to significantly reduce the conservatism of the conditions in [5] and [3] for the same benchmark examples. The results in [7] are very close to the ones obtained with Theorem 1 and both outperform the results obtained with Theorem 5.

In Table II, we show the results obtained by allowing  $R_i$  in Theorem 1 to be additional variables, i.e., the protocol is to be designed. Note that Theorem 3 assures that the values obtained with Theorem 1 for the maximum support of a uniform distribution that preserves stability when a dynamic protocol (obtained from solving (20)) is utilized, are larger than the ones obtained with the necessary and sufficient conditions provided by Theorem 2 for the static protocol, which matches well with the results in Table II.

TABLE II

STABILITY RESULTS FOR THE BATCH REACTOR EXAMPLE-PROTOCOL DESIGN, NO PACKET DROPS

	Dynamic Prot. Design (Th.1)	Static Prot. (Th.2)
Max. $\gamma : h_k \sim \text{Uni}(\gamma)$	0.140	0.112

#### V. CONCLUSIONS

We provided stability results for networked control systems with stochastic intervals between transmissions, delays, and packet drops. Our main result, is to show that one can analytically prove that dynamic protocols preserve stability for larger sampling intervals between transmission than static protocols, and therefore less communication and control computations are required for such protocols in networked control systems.

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