Distributed Output-Feedback Model Predictive Control for Multi-Agent Consensus

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Abstract

We propose a distributed output-feedback model predictive control approach for achieving consensus among multiple agents. Each agent computes a distributed control action based on an output-feedback measurement of a local neighborhood tracking error and communicates information only to its neighbors, according to a communication network modeled as a directed graph. Each agent computes its distributed control action by solving a local min-max optimization problem that simultaneously computes a local state estimate and control input under worst-case assumptions on unmeasured input disturbances and measurement noise. Under easily verified controllability and observability assumptions, this distributed output-feedback model predictive control approach provides an upper bound on the group consensus error, thereby ensuring practical consensus in the presence of unmeasured disturbances and noise. A numerical example with four agents connected in a directed graph is given to illustrate the results.

Keywords: distributed model predictive control, output-feedback, moving horizon estimation, multi-agent consensus

1. Introduction

With the increasing availability of devices with communication and computational capabilities, the number of applications for distributed control is growing dramatically. Centralized control schemes often confront intractable computational burdens, lack necessary system-wide knowledge, and raise privacy concerns. Distributed control approaches, on the other hand, distribute the computational burden, only require local/partial information of the system, can mitigate problems due to flaws in communication, and can maintain privacy. One distributed control problem, multi-agent consensus, has found many applications in computer science, sensor networks, and multi-vehicle coordination problems such as rendezvous, formation control, cooperative search, attitude alignment, and flocking [22, 25]. In these problems, there is a communication network with directed information sharing, and the objective is to achieve consensus among all of the agents by performing local control actions using only local information as feedback.

Many distributed control approaches have been studied, but in this work, we focus on distributed model predictive control (MPC). MPC is an optimal control approach that performs an open-loop optimization at each time step in order to find a sequence of future control actions and corresponding predictions of the future states [20, 24]. Distributed MPC involves multiple agents each solving a local MPC problem in order to determine a local action to take with the goal of achieving desired global behavior (see, e.g., [3, 4, 24, 26]). Distributed MPC has been used in numerous applications; several examples include chemical process control for a four-tank system [11, 21], power system load-frequency control [19] and automatic generation control [29], and vehicle platooning [32].

MPC schemes often assume full state feedback, but output-feedback approaches must be considered in many practical applications where the full state is not known or is not available for feedback. Several output-feedback approaches for multi-agent consensus exist [9, 14, 18, 31]. However, fewer results are available for distributed output-feedback MPC; in fact, the authors of [4] highlighted output-feedback MPC as a direction for future work. When considering linear systems, distributed MPC approaches can include an observer, such as a Kalman filter [28, 30] or Luenberger observer [10], for (decentralized) state estimation. In this work, we propose a distributed output-feedback MPC approach that employs moving horizon estimation (MHE) [23], explicitly handles input constraints, and is robust to worst-case disturbances and measurement noise.

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Specifically, in this work, we present a distributed output-feedback MPC approach for achieving multi-agent consensus for agents with linear discrete-time dynamics. We consider a strongly connected graph structure for the communication of the agents and specifically investigate leader-follower consensus [15]. We extend results for the estimation and control approach proposed in [5, 8], which simultaneously solves MHE and MPC problems as a single min-max optimization problem, to the distributed multi-agent consensus problem. In order to do this, we use the concept of local neighborhood error, as in [1, 13], and assume that only partial noisy measurements of this local error are available for feedback. In addition to these partial local noisy measurements, we only assume that each agent has knowledge of its own dynamics and the dynamics of its neighbors, and it communicates past and future computed inputs only to its neighbors. In this way, each agent only has access to local information to perform simultaneous state estimation and control computation at each time step. Communication of state estimates and full input trajectories is common in distributed MPC approaches, and depending on the algorithm used, this data may be communicated multiple times within each time step to iteratively converge to a solution [4].

Under appropriate local controllability and observability assumptions, we show that the local neighborhood tracking error for each agent converges to a small value in the presence of input disturbances and noise. We combine this result with results from [1], which shows that the group consensus error can be made arbitrarily small by decreasing the local neighborhood tracking errors, in order to prove that the group consensus error is bounded. Thus, we prove practical multi-agent consensus in the presence of input disturbances and measurement noise.

The remainder of the paper is organized as follows. Next we introduce some notation used throughout the paper. The problem is formulated in Section 2, the proposed estimation and control approach is described in Section 3, the main results are given in Section 4, a numerical example is provided in Section 5, and finally, conclusions are discussed in Section 6.

Notation. We denote the set of real numbers as $\mathbb{R}$, the set of non-negative integers as $\mathbb{Z}_{\geq 0}$, and by $\mathbb{Z}_a^b$ the set of consecutive integers $\{a, \dots, b\}$. Given a discrete-time signal $z : \mathbb{Z}_{\geq 0} \to \mathbb{R}^{n_z}$ and two times $t_0, t \in \mathbb{Z}_{\geq 0}$ with $t_0 \leq t$, we denote by $z_{t_0:t}$ the sequence $\{z_{t_0}, z_{t_0+1}, \dots, z_t\}$. With a slight abuse of notation, we write $z_{t_0:t} \in \mathcal{Z}$ to mean that each element of $z_{t_0:t}$ belongs to the set $\mathcal{Z}$. Finally, given a matrix $R \in \mathbb{R}^{n_z \times n_z}$ and denoting the transpose of z_t as $z_t^\top$, we denote by $\|z_t\|_R^2$ the operation $z_t^\top R z_t$.

2. Problem Formulation

We consider the problem of achieving consensus, or synchronization, of $N + 1$ agents. One agent is the leader, and N agents utilize only local information to compute a local control action in order to achieve consensus with the leader. We label the leader as agent 0 and model its dynamics as a linear discrete-time system without control given as

$$x_{t+1}^0 = A x_t^0,$$

where $x_t^0 \in \mathbb{R}^{n_x}$ denotes the leader's state at time t, and $A \in \mathbb{R}^{n_x \times n_x}$ is the system matrix that captures the leader's dynamics. In general, these dynamics can be stable or unstable and can generate a large class of useful command trajectories for multi-agent systems. We model every other agent $i \in \mathcal{I} := \{1, 2, \dots, N\}$ as a linear discrete-time system with dynamics given by

$$x_{t+1}^i = A x_t^i + B_i u_t^i + D_i d_t^i, \tag{1}$$

where $x_t^i \in \mathbb{R}^{n_x}$ denotes the *state* of agent i at time t, $u_t^i \in \mathbb{R}^{n_u}$ denotes the *control input* of agent i at time t, which is constrained to belong to the set $\mathcal{U}_i \subset \mathbb{R}^{n_u}$, and $d_t^i \in \mathbb{R}^{n_d}$ denotes the *unmeasured disturbance* that agent i is subjected to at time t, which belongs to the set $\mathcal{D}_i \subset \mathbb{R}^{n_d}$. All of the agents have the same system matrix $A \in \mathbb{R}^{n_x \times n_x}$, but each agent i may have different input matrices $B_i \in \mathbb{R}^{n_x \times n_u}$ and $D_i \in \mathbb{R}^{n_x \times n_d}$, which allows for inputs and disturbances of agents with different capabilities to affect their states in different ways.

The objective is to design distributed control inputs u_t^i, using only local information, in order to achieve consensus between every agent and the leader, that is, for all $i \in \mathcal{I}$, $\lim_{t\to\infty} \|x_t^i - x_t^0\| = 0$. In the presence of input disturbances and measurement noise, it may not be possible to drive this limit to 0, so, instead, we try to bound it by a small value β, i.e., $\lim_{t\to\infty} \|x_t^i - x_t^0\| \leq \beta$, thereby achieving practical consensus.

2.1. Graphs, error dynamics, and consensus

Let $\mathcal{G}$ be a directed graph (digraph) defined as the pair $\mathcal{G} = (V, \mathcal{E})$ with a nonempty finite set of N vertices $V = \{v_1, v_2, \dots, v_N\}$, one for each non-leader agent, and a set of edges $\mathcal{E} \subseteq V \times V$, denoting communication links. We assume that $\mathcal{G}$ is simple, i.e., it contains no repeated edges and $(v_i, v_i) \neq \mathcal{E}, \forall i$ (no self-loops). The connectivity matrix E is defined such that $E = [e_{ij}]$ with $e_{ij} > 0$ if $(v_j, v_i) \in \mathcal{E}$ and $e_{ij} = 0$ otherwise. The set of neighbors of agent i is $N_i = \{v_j : (v_j, v_i) \in \mathcal{E}\}$. A directed path is a sequence of vertices, e.g., $v_1, v_2, \dots, v_r$, such that $(v_i, v_{i+1}) \in \mathcal{E}$, $i \in \{1, 2, \dots, r-1\}$. A directed tree is a connected digraph in which every vertex except one, called the root, has in-degree equal to one. A digraph is said to contain a spanning tree if a subset of the edges form

a directed tree. A digraph is said to be strongly connected if every vertex is the root of a spanning tree. Lastly, consider a vertex for the leader agent v_0 that is connected to a small percentage of the vertices in $\mathcal{G}$.

To study the consensus problem on graphs, we define a local neighborhood tracking error [1, 13], denoted as $\varepsilon_t^i \in \mathbb{R}^{n_\varepsilon}$ for each agent i at time t, as

$$\varepsilon_t^i = \sum_{j \in N_i} e_{ij}(x_t^j - x_t^i) + g_i(x_t^0 - x_t^i), \tag{2}$$

where $g_i \geq 0$ is the pinning gain of agent i, which is nonzero only if the vertex v_i is connected to the leader's vertex v_0. The dynamics of the local neighborhood tracking error, for every agent $i \in \mathcal{I}$, are given as follows:

$$\varepsilon_{t+1}^i = A\varepsilon_t^i - (w_i + g_i)(B_i u_t^i + D_i d_t^i) + \sum_{j \in N_i} e_{ij}(B_j u_t^j + D_j d_t^j) \tag{3a}$$

$$y_t^i = C_i \varepsilon_t^i + n_t^i, \tag{3b}$$

where $w_i = \sum_{j \in N_i} e_{ij}$ is the weighted in-degree of vertex v_i, and $y_t^i \in \mathbb{R}^{n_y}$ denotes the *measured output* of agent i at time t subjected to the *measurement noise* n_t^i, which belongs to the set $\mathcal{N}_i \subset \mathbb{R}^{n_y}$. This measured output depends on the states of agent i and its in-neighbors and can be computed either by having each agent take a local measurement of its own state and sending that measurement to its out-neighbors at every time t or by taking local measurements of the difference between the states of neighboring agents.

If we define the weighted in-degree matrix as $W = \text{diag}\{w_i\}$, then the Laplacian matrix $\mathcal{L}$ corresponding to the digraph $\mathcal{G}$ is defined as $\mathcal{L} = W - E$. If we also define $G = \text{diag}\{g_i\}$, then the vector of local neighborhood tracking errors for all agents at time t is given by

$$\varepsilon_t = -((\mathcal{L} + G) \otimes I_{n_x})x_t + ((\mathcal{L} + G) \otimes I_{n_x})\underline{x}_t^0, \tag{4}$$

where the global state vector at time t is $x_t = [x_t^{1\top}, x_t^{2\top}, \ldots, x_t^{N\top}]^\top$, the vector of local neighborhood tracking errors at time t is $\varepsilon_t = [\varepsilon_t^{1\top}, \varepsilon_t^{2\top}, \ldots, \varepsilon_t^{N\top}]^\top$, $\underline{x}_t^0 = (\mathbf{1} \otimes I_{n_x})x_t^0$, and $\mathbf{1}$ is the N-vector of ones. The symbol $\otimes$ denotes the Kronecker product, and I_{n_x} denotes the identity matrix with dimensions $n_x \times n_x$.

Defining the global disagreement (group consensus error) vector [22, 1] at time t as

$$\eta_t = (x_t - \underline{x}_t^0) \in \mathbb{R}^{n_x N}, \tag{5}$$

we can re-write (4) as $\varepsilon_t = -((\mathcal{L} + G) \otimes I_{n_x})\eta_t$. The following result from [1] provides a bound for the size of the group consensus error η_t.

Lemma 1. *[1] If $(\mathcal{L} + G)$ is nonsingular, then*

$$\|\eta_t\| \leq \|\varepsilon_t\| / \underline{\sigma}(\mathcal{L} + G), \tag{6}$$

where $\underline{\sigma}(\mathcal{L} + G)$ denotes the smallest singular value of $(\mathcal{L} + G)$.

Proof. Since $(\mathcal{L} + G)$ is nonsingular, $\underline{\sigma}(\mathcal{L} + G) \neq 0$, and (4) and (5) imply (6). ■

As stated in [1], if the digraph $\mathcal{G}$ contains a spanning tree, and $g_i \neq 0$ for at least one root vertex i, then $(\mathcal{L} + G)$ is nonsingular. This result implies that the group consensus error can be made arbitrarily small by driving the local neighborhood tracking errors to a small value.

3. Estimation and Control Approach

With the main objective being practical consensus of the group of agents, the control objective for each agent is to minimize its local neighborhood tracking error given actuation constraints and only output-feedback information. Therefore, each agent must estimate its local neighborhood tracking error and compute control signals that minimize future predicted errors. To do this, we propose an approach that formulates and solves MHE and MPC optimization problems as a single min-max optimization problem, which was proposed for centralized problems in [5, 8].

Specifically, each agent's *control objective* is to select the sequence of future control signals $u_{t:t+T-1}^i \in \mathcal{U}_i$, so as to minimize a finite-horizon criterion of the form

$$J_t^i(\varepsilon_{t-L}^i, u_{t-L:t+T-1}^i, d_{t-L:t+T-1}^i, y_{t-L:t}^i) := \sum_{s=t}^{t+T-1} \|\varepsilon_s^i\|_{Q^i}^2 + \|\varepsilon_{t+T}^i\|_{\overline{Q}^i}^2 + \sum_{s=t}^{t+T-1} \|u_s^i\|_{R_u^i}^2 - \sum_{s=t-L}^{t} \|n_s^i\|_{R_n^i}^2 - \sum_{s=t-L}^{t+T-1} \|d_s^i\|_{R_d^i}^2 \tag{7}$$

under worst-case assumptions on the unknown initial local neighborhood tracking error ε_{t-L}^i, unmeasured disturbances d_s^i, and measurement noise n_s, subject to constraints on individual variables and those imposed by the system dynamics and measurements y_s collected up to the current time t. Therefore, the local performance criterion (7) depends on agent i's unknown initial local neighborhood tracking error ε_{t-L}^i, unknown disturbance input sequence $d_{t-L:t+T-1}^i$, measured output sequence $y_{t-L:t}^i$, (known) past control inputs $u_{t-L:t-1}^i$ that have already been applied, and future control inputs $u_{t:t+T-1}^i$ that still need to be selected. The parameter $T \in \mathbb{Z}_{>0}$ is the forward prediction and control horizon, $L \in \mathbb{Z}_{>0}$ is the backward estimation horizon, $Q^i \in \mathbb{R}^{n_\varepsilon \times n_\varepsilon}$ is a positive semidefinite matrix weighting the local neighborhood tracking error, $\overline{Q}^i \in \mathbb{R}^{n_\varepsilon \times n_\varepsilon}$ is a positive semidefinite matrix weighting the error at the end of the prediction horizon, and $R_u^i \in \mathbb{R}^{n_u \times n_u}$, $R_n^i \in \mathbb{R}^{n_y \times n_y}$, and $R_d^i \in \mathbb{R}^{n_d \times n_d}$ are positive definite matrices that weight the control input, measurement noise, and input disturbance, respectively, for

each agent $i \in \mathcal{I}$. The last two terms on the right-hand-side of (7) are included to penalize the maximizer for choosing large (unlikely) values for n_s and d_s, respectively.

Remark 1 (MHE "arrival cost"). *In the MHE literature (see, e.g., [23]), it is common to include an additional term in the optimization criterion, often called the "arrival cost," that can be used to account for the measurements collected before time $t - L$. Our results do not need such a term, but instead they require the past horizon L to be "sufficiently large" so that Assumption 2 holds.* □

3.1. Finite-Horizon Online Optimization

We formulate the combined MPC/MHE problem for each agent $i \in \mathcal{I}$ as a finite-horizon min-max optimization problem, to be solved at each time t, of the form

$$\min_{\hat{\boldsymbol{u}}^i} \quad \max_{\hat{\boldsymbol{\varepsilon}}^i, \hat{\boldsymbol{d}}^i} \quad J_t^i(\hat{\boldsymbol{\varepsilon}}^i, \boldsymbol{u}^i, \hat{\boldsymbol{u}}^i, \hat{\boldsymbol{d}}^i, \boldsymbol{y}^i) \tag{8}$$

with cost function $J_t^i(\cdot)$ as defined in (7) and subject to

$$\begin{aligned}
\hat{\varepsilon}^i_{s+1|t} &= A\hat{\varepsilon}^i_{s|t} - (w_i + g_i)(B_i u^i_s + D_i \hat{d}^i_{s|t}) \\
&\quad + \sum_{j \in N_i} e_{ij}(B_j u^j_s + D_j \hat{d}^{j*}_{s|t}), \quad \forall s \in \mathbb{Z}^{t-1}_{t-L} && \text{(9a)} \\
\hat{\varepsilon}^i_{s+1|t} &= A\hat{\varepsilon}^i_{s|t} - (w_i + g_i)(B_i \hat{u}^i_{s|t} + D_i \hat{d}^i_{s|t}) \\
&\quad + \sum_{j \in N_i} e_{ij}(B_j \hat{u}^{j*}_{s|t} + D_j \hat{d}^{j*}_{s|t}), \quad \forall s \in \mathbb{Z}^{t+T-1}_{t} && \text{(9b)} \\
\hat{n}^i_{s|t} &= C_i \hat{\varepsilon}^i_{s|t} - y^i_s \in \mathcal{N}_i, && \forall s \in \mathbb{Z}^{t}_{t-L} && \text{(9c)} \\
\hat{d}^i_{s|t} &\in \mathcal{D}_i, && \forall s \in \mathbb{Z}^{t+T-1}_{t-L} && \text{(9d)} \\
\hat{u}^i_{s|t} &\in \mathcal{U}_i, && \forall s \in \mathbb{Z}^{t+T-1}_{t}, && \text{(9e)}
\end{aligned}$$

where the following shorthand notation has been introduced: $\hat{\boldsymbol{u}}^i := \hat{u}^i_{t:t+T-1|t}$, $\hat{\boldsymbol{d}}^i := \hat{d}^i_{t-L:t+T-1|t}$, $\hat{\boldsymbol{\varepsilon}}^i := \hat{\varepsilon}^i_{t-L|t}$, $\boldsymbol{u}^i := u^i_{t-L:t-1}$, $\boldsymbol{y}^i := y^i_{t-L:t}$. The subscript $_{\cdot|t}$ denotes that the variable is computed by solving the MPC/MHE optimization at time t. The sequence of control inputs that minimizes (8) is defined as $\hat{\boldsymbol{u}}^{i*} := \hat{u}^{i*}_{t:t+T-1|t} \in \mathcal{U}_i$. Similarly, we define the variables that maximize (8) as $\hat{\boldsymbol{d}}^{i*} := \hat{d}^{i*}_{t-L:t+T-1|t} \in \mathcal{D}_i$ and $\hat{\boldsymbol{\varepsilon}}^{i*} := \hat{\varepsilon}^{i*}_{t-L|t}$. These can be thought of as "worst-case" estimates of the past and future disturbance variables and initial local neighborhood tracking error, respectively.

The constraints (9a)-(9b) ensure that the local neighborhood error dynamics in (3a) are satisfied. These constraints depend on the solutions of agent i's neighbors, i.e., $\hat{\boldsymbol{u}}^{j*}$ and $\hat{\boldsymbol{d}}^{j*}$ for all $j \in N_i$. Constraint (9c) makes sure the output equation in (3b) and constraint on the measurement noise are satisfied, and (9d) and (9e) enforce constraints on the disturbance and control inputs, respectively.

The optimization (8) is repeated at each time step t, and, for every agent, we use as the control input the first element of the sequence $\hat{\boldsymbol{u}}^{i*}$ that minimizes (8), leading to the following control law:

$$u^i_t = \hat{u}^{i*}_{t|t}, \quad \forall t \in \mathbb{Z}_{\geq 0}. \tag{10}$$

More details on this combined MPC/MHE approach can be found in [5, 8], and its application for adaptation and learning of systems with parametric uncertainty is discussed in [6].

3.2. Distributed Implementation

In this approach, a cooperative, communication-based distributed algorithm, similar to those considered in, e.g., [12], may be used to iteratively find a solution to (8). In this way, at each time step t, each agent i computes a solution and communicates that solution to its neighbors multiple times in order to converge to a solution to (8). Specifically, within each time step, for each iteration p, every agent $i \in \mathcal{I}$ has knowledge of its neighbors' solutions $\hat{d}^{j*(p-1)}_{t-L:t+T-1|t} \in \mathcal{D}_j$ and $\hat{u}^{j*(p-1)}_{t:t+T-1|t} \in \mathcal{U}_j$ from iteration $p-1$ and solves the following optimization:

$$\min_{\hat{\boldsymbol{u}}^{i(p)}} \quad \max_{\hat{\boldsymbol{\varepsilon}}^{i(p)}, \hat{\boldsymbol{d}}^{i(p)}} \quad \bar{J}_t^i(\hat{\boldsymbol{\varepsilon}}^{i(p)}, \boldsymbol{u}^i, \hat{\boldsymbol{u}}^{i(p)}, \hat{\boldsymbol{d}}^{i(p)}, \boldsymbol{y}^i) \tag{11}$$

with the new local cost function $\bar{J}_t^i(\cdot)$, which should be related to (7), and subject to

$$\begin{aligned}
\hat{\varepsilon}^{i(p)}_{s+1|t} &= A\hat{\varepsilon}^{i(p)}_{s|t} - (w_i + g_i)(B_i u^i_s + D_i \hat{d}^{i(p)}_{s|t}) \\
&\quad + \sum_{j \in N_i} e_{ij}(B_j u^j_s + D_j \hat{d}^{j*(p-1)}_{s|t}), \quad \forall s \in \mathbb{Z}^{t-1}_{t-L} && \text{(12a)} \\
\hat{\varepsilon}^{i(p)}_{s+1|t} &= A\hat{\varepsilon}^{i(p)}_{s|t} - (w_i + g_i)(B_i \hat{u}^{i(p)}_{s|t} + D_i \hat{d}^{i(p)}_{s|t}) \\
&\quad + \sum_{j \in N_i} e_{ij}(B_j \hat{u}^{j*(p-1)}_{s|t} + D_j \hat{d}^{j*(p-1)}_{s|t}), \quad \forall s \in \mathbb{Z}^{t+T-1}_{t} && \text{(12b)} \\
\hat{n}^{i(p)}_{s|t} &= C_i \hat{\varepsilon}^{i(p)}_{s|t} - y^i_s \in \mathcal{N}_i, && \forall s \in \mathbb{Z}^{t}_{t-L} && \text{(12c)} \\
\hat{d}^{i(p)}_{s|t} &\in \mathcal{D}_i, && \forall s \in \mathbb{Z}^{t+T-1}_{t-L} && \text{(12d)} \\
\hat{u}^{i(p)}_{s|t} &\in \mathcal{U}_i, && \forall s \in \mathbb{Z}^{t+T-1}_{t} && \text{(12e)}
\end{aligned}$$

with the new notation $\hat{\boldsymbol{u}}^{i(p)} := \hat{u}^{i(p)}_{t:t+T-1|t}$, $\hat{\boldsymbol{d}}^{i(p)} := \hat{d}^{i(p)}_{t-L:t+T-1|t}$, $\hat{\boldsymbol{\varepsilon}}^{i(p)} := \hat{\varepsilon}^{i(p)}_{t-L|t}$. We define the sequence of control inputs that minimizes (11) as $\hat{\boldsymbol{u}}^{i*(p)} := \hat{u}^{i*(p)}_{t:t+T-1|t} \in \mathcal{U}_i$ and the variables that maximize (11) as $\hat{\boldsymbol{d}}^{i*(p)} := \hat{d}^{i*(p)}_{t-L:t+T-1|t} \in \mathcal{D}_i$ and $\hat{\boldsymbol{\varepsilon}}^{i*(p)} := \hat{\varepsilon}^{i*(p)}_{t-L|t}$.

Definition 1 (Convergent distributed algorithm). *We say that a distributed algorithm for this approach is convergent if the following holds: Given $\rho_i := \|\hat{\boldsymbol{u}}^{i*(p)} - \hat{\boldsymbol{u}}^{i*(p-1)}\|_2$ and a scalar tolerance δ, if each time step t is long enough for the agents to iterate to convergence such that $\rho_i < \delta$, for all agents $i \in \mathcal{I}$ and a small δ, then in the limit as $p \to \infty$, $\rho_i \to 0$, and the solution to* (11) *converges to that of* (8). □

Any distributed algorithm that satisfies Definition 1 can be implemented, and the results in the remainder of the paper will hold.

Remark 2 (Communication and computation). *With simultaneous state estimation and control computations, more information may be required to be shared among agents than, e.g., in an approach using state feedback. However, depending on the algorithm used, this approach may require more or less information sharing and computation than other approaches. For instance, in the cooperative distributed MPC with output-feedback approaches in [28, 30], information is shared multiple times, and both distributed state estimation and MPC are iteratively solved, within a single time step. Also, in other sequential or iterative distributed MPC approaches (see, e.g., [4, 17]), full input trajectories are shared among multiple neighbors several times within a time step. Furthermore, we consider a directed communication graph, so less data is communicated compared to an undirected graph. When communication or computation capabilities of the agents are limited, rather than iteratively solving* (11) *and implementing control law* (10) *at each time step t, multiple elements of the sequence* $\hat{\boldsymbol{u}}^{i*}$ *may be implemented while iteratively solving* (11) *to find solutions for a future time step.* □

4. Main Results

In this section, we show that practical group consensus of all agents $i \in \mathcal{I}$ can be achieved using the proposed distributed MPC/MHE approach. First, we present necessary assumptions.

4.1. Assumptions

Assumption 1. *For all* $i \in \mathcal{I}$*, there exists a saddle-point solution to the min-max optimization* (8)*. Specifically, for every time* $t \in \mathbb{Z}_{\geq 0}$*, past control input sequence* $\boldsymbol{u}^i \in \mathcal{U}_i$*, and past measured output sequence* $\boldsymbol{y}^i$*, there exist a finite scalar* $J_t^{i*} \in \mathbb{R}$*, a future control input sequence* $\boldsymbol{u}^{i*} \in \mathcal{U}_i$*, a disturbance sequence* $\boldsymbol{d}^{i*} \in \mathcal{D}_i$*, and a noise sequence* $\boldsymbol{n}^{i*} \in \mathcal{N}_i$*, such that*

$$J_t^{i*} = J_t^i(\hat{\boldsymbol{\varepsilon}}^{i*}, \boldsymbol{u}^i, \hat{\boldsymbol{u}}^{i*}, \hat{\boldsymbol{d}}^{i*}, \boldsymbol{y}^i) \tag{13a}$$

$$= \max_{\hat{\boldsymbol{\varepsilon}}^i, \hat{\boldsymbol{d}}^i} J_t^i(\hat{\boldsymbol{\varepsilon}}^i, \boldsymbol{u}^i, \hat{\boldsymbol{u}}^{i*}, \hat{\boldsymbol{d}}^i, \boldsymbol{y}^i) \tag{13b}$$

$$= \min_{\hat{\boldsymbol{u}}^i} J_t^i(\hat{\boldsymbol{\varepsilon}}^{i*}, \boldsymbol{u}^i, \hat{\boldsymbol{u}}^i, \hat{\boldsymbol{d}}^{i*}, \boldsymbol{y}^i). \tag{13c}$$

□

If we define $\overline{A} := A$, $\overline{B} := -(w_i + g_i)B_i$, and $\overline{D}_i := -(w_i + g_i)D_i$, then the constraints (9a)-(9b) can be rewritten as

$$\hat{\varepsilon}^i_{s+1|t} = \overline{A}\hat{\varepsilon}^i_{s|t} + \overline{B}_i u^i_s + \overline{D}_i \hat{d}^i_{s|t} + h^i_s, \quad \forall s \in \mathbb{Z}^{t-1}_{t-L} \tag{14a}$$

$$\hat{\varepsilon}^i_{s+1|t} = \overline{A}\hat{\varepsilon}^i_{s|t} + \overline{B}_i \hat{u}^i_{s|t} + \overline{D}_i \hat{d}^i_{s|t} + h^i_s, \quad \forall s \in \mathbb{Z}^{t+T-1}_{t} \tag{14b}$$

where $h^i_s = \sum_{j \in N_i} e_{ij}(B_j u^j_s + D_j d^j_s)$. Then the optimization in (8) looks just like a classic affine-quadratic zero-sum dynamic game [2] with forward and backward horizons. Thus, since we consider linear systems and quadratic costs, Assumption 1 is satisfied if the system is observable and the weighting matrices in (7) are chosen such that $Q^i \geq 0$, $\overline{Q}^i \geq 0$, $R^i_u > 0$, and $R^i_n > 0$ and $R^i_d > 0$ are sufficiently large [7].

Assumption 2 (Observability). *For all* $i \in \mathcal{I}$*, there exists a bounded set* $\mathcal{N}_{pre} \subset \mathbb{R}^n$ *such that, for every time* $t \in \mathbb{Z}_{\geq 0}$*, every error sequence* $\hat{\varepsilon}^i_{t-L:t}$*, every disturbance sequence* $\hat{d}^i_{t-L:t} \in \mathcal{D}_i$*, and every noise sequence* $\hat{n}^i_{t-L:t} \in \mathcal{N}_i$ *that are compatible with the applied control input* u^i_s*,* $s \in \mathbb{Z}_{\geq 0}$*, and the measured output* y^i_s*,* $s \in \mathbb{Z}_{\geq 0}$*, in the sense that, for all* $s \in \{t-L, t-L+1, \ldots, t-1\}$*,*

$$\hat{\varepsilon}^i_{s+1} = \overline{A}\hat{\varepsilon}^i_s + \overline{B}_i u^i_s + \overline{D}_i \hat{d}^i_s + h^i_s, \tag{15a}$$

$$y^i_s = C_i \hat{\varepsilon}^i_s + \hat{n}^i_s, \tag{15b}$$

there exist a "predecessor" error estimate $\hat{\varepsilon}^i_{t-L-1}$*, disturbance estimate* $\hat{d}^i_{t-L-1} \in \mathcal{D}_i$*, and noise estimate* $\hat{n}_{t-L-1} \in \mathcal{N}_{pre}$ *such that* (15a)-(15b) *also hold for time* $s = t-L-1$*.* □

This assumption is satisfied if the error dynamics (3) are observable and the set $\mathcal{N}_{\text{pre}}$ is chosen sufficiently large [8].

Assumption 3 (ISS-control Lyapunov function). *The terminal cost* $\|\varepsilon^i_t\|^2_{\overline{Q}^i}$ *acts as an input-to-state-stability (ISS) control Lyapunov function in the sense that, for all* $i \in \mathcal{I}$ *and every* $t \in \mathbb{Z}_{\geq 0}$*,* ε^i_t*, there exist a control* $u^i_t \in \mathcal{U}_i$ *and weighting matrices* $\overline{Q}^i$*,* Q^i*,* R^i_u*, and* R^i_d *such that, for all* $d^i_t \in \mathcal{D}_i$*,*

$$\|\varepsilon^i_{t+1}\|^2_{\overline{Q}^i} - \|\varepsilon^i_t\|^2_{\overline{Q}^i} \leq -\|\varepsilon^i_t\|^2_{Q^i} - \|u^i_t\|^2_{R^i_u} + \|d^i_t\|^2_{R^i_d}. \tag{16}$$

□

Assumption 3 requires the terminal cost to be a control Lyapunov function for the closed-loop [20], which is a common assumption in MPC. Without d^i_t, the terminal cost could be viewed as a control Lyapunov function that decreases along system trajectories for an appropriate control input u^i_t [27]. With d^i_t, the terminal cost should be viewed as an ISS-control Lyapunov function that satisfies an ISS condition for the disturbance input d^i_t and an appropriate control input u^i_t [16].

Given these three assumptions, we are now ready to present the main results.

4.2. Practical Stability/Consensus

Theorem 1 (Local error bound). *Suppose that Assumptions 1, 2, and 3 hold. Along any trajectory of the closed-loop system defined by the error dynamics* (3) *and the control law* (10)*, we have that*

$$\|\varepsilon_t^i\|_{Q^i}^2 \leq J_L^{i*} - \|u_t^i\|_{R_u^i}^2 + \sum_{s=0}^{t-L-1} \|\tilde{d}_s^i\|_{R_d^i}^2 + \sum_{s=0}^{t-L-1} \|\tilde{n}_s^i\|_{R_n^i}^2 + \sum_{s=t-L}^{t} \|n_s^i\|_{R_n^i}^2 + \sum_{s=t-L}^{t} \|d_s^i\|_{R_d^i}^2, \quad \forall t \in \mathbb{Z}_{\geq L}, \tag{17}$$

for appropriate sequences $\tilde{d}_{0:t-L-1}^i \in \mathcal{D}_i$, $\tilde{n}_{0:t-L-1}^i \in \mathcal{N}_{\text{pre}}$.

Proof. The proof of Theorem 1 follows the proof of Theorem 1 in [8] with the particular cost function (7), the local dynamics (14), Assumptions 1, 2, and 3, and the result in Lemma 2. The full proof is given in the Appendix. ■

This bound depends on the value of the local cost function J_L^{i*} at time L, the actual past noise and disturbance sequences $n_{t-L:t}^i$ and $d_{t-L:t}^i$, respectively, and the terms $\sum_{s=0}^{t-L-1} \|\tilde{d}_s^i\|_{R_d^i}^2$ and $\sum_{s=0}^{t-L-1} \|\tilde{n}_s^i\|_{R_n^i}^2$ that can be thought of as the "arrival cost" that appears in the MHE literature to capture the quality of the estimate at the beginning of the current estimation window [23].

The bound in (17) will be finite in the case of "vanishing" future noise and disturbance signals, in the sense that

$$\sum_{s=t-L}^{\infty} \|n_s^i\|_{R_n^i}^2 < \infty, \qquad \sum_{s=t-L}^{\infty} \|d_s^i\|_{R_d^i}^2 < \infty, \tag{18}$$

or when exponentially time-weighted functions of the noise and disturbance signals are employed in the criterion (7). For more details, see [8].

Remark 3. *The bound on the local neighborhood tracking error in* (17) *can be used to find a bound on the size of the group consensus error. Denoting the right-hand-side of* (17) *as* α_t^i*, and choosing* $Q^i \geq I_{n_\varepsilon}$*, we get the inequality*

$$\|\varepsilon_t^i\| \leq \|\varepsilon_t^i\|_{Q^i} \leq \left(\alpha_t^i\right)^{1/2}.$$

Then, utilizing the triangle inequality, we can derive a bound on the size of the vector of local neighborhood tracking errors as

$$\|\varepsilon_t\| \leq \sum_{i\in\mathcal{I}} \|\varepsilon_t^i\| \leq \sum_{i\in\mathcal{I}} \left(\alpha_t^i\right)^{1/2}. \tag{19}$$

□

Corollary 1 (Group Practical Stability/Consensus). *Suppose that Assumptions 1, 2, and 3 hold. Then, given the state-space model* (1)*, the local neighborhood error dynamics* (3)*, and the control law* (10)*, for all* $i \in \mathcal{I}$*, the size of the group consensus error at time* $t \in \mathbb{Z}_{\geq 0}$ *is bounded as follows:*

$$\|\eta_t\| \leq \frac{\sum_{i\in\mathcal{I}} \left(\alpha_t^i\right)^{1/2}}{\underline{\sigma}(\mathcal{L}+G)}. \tag{20}$$

Proof. The proof follows directly from the result in (6) and the bound in (19). ■

Therefore, the group consensus error is bounded, and the agents achieve practical consensus in the presence of unmeasured disturbances and measurement noise when using the distributed control law (10).

5. Numerical Example

Figure 1 depicts the strongly connected graph considered in this example. The objective is for agents 1, 2, and 3 to achieve consensus with the leader agent 0 by using the distributed estimation and control approach proposed in Section 3, while being subjected to unmeasured input disturbances and measurement noise.

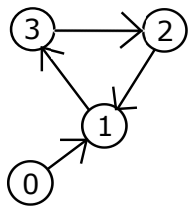

Figure 1: Graph communication structure with four agents. Agent 0 is the leader.

The matrices and weights defining each agent's dynamics, as in (1), and local neighborhood tracking error dynamics, as in (3), are given as

$$A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix},\ B_1 = \begin{bmatrix} 2 & 1 \end{bmatrix}^\top,$$
$$B_2 = \begin{bmatrix} 1 & -1 \end{bmatrix}^\top,\ B_3 = \begin{bmatrix} 2 & -2 \end{bmatrix}^\top,$$
$$D_0 = B_0,\ D_1 = B_1,\ D_2 = B_2,\ D_3 = B_3,$$
$$C_1 = C_3 = \begin{bmatrix} 0 & 1 \end{bmatrix},\ C_2 = \begin{bmatrix} 1 & 0 \end{bmatrix},$$

$g_1 = 2$, $g_2 = g_3 = 0$, $w_i = 1$ for all i, and $e_{ij} = 1$ for all i and j. Therefore, the agents' dynamics are unstable, and only partial, noisy measurements of the local neighborhood tracking errors are available for feedback. The actual noise and disturbances were normally distributed random variables given as $n_t^i \sim \mathcal{N}(0.05, 0.1^2)$ and $d_t^i \sim \mathcal{N}(0, 0.03^2)$, respectively, for all $i \in \mathcal{I}$ and all $t \in \mathbb{Z}_{\geq 0}$.

A distributed implementation, as discussed in section 3.2, was used with $\bar{J}_t^i = J_t^i$ as defined in (7). The forward and backward horizons were chosen as $T = 5$ and $L = 5$, respectively. The other parameters and constraint sets included in optimization (11) were chosen as $Q^i = 1$, $\overline{Q}^i = 1$, $R_u^i = 1$, $R_d^i = 100$, $R_n^i = 1000$, $\mathcal{U}_i := \{u_t^i \in \mathbb{R} : -0.5 \leq u_t^i \leq 0.5\}$, $\mathcal{D}_i := \{d_t^i \in \mathbb{R} : -0.1 \leq d_t^i \leq 0.1\}$, and $\mathcal{N}_i := \mathbb{R}$ for all $i \in \mathcal{I}$. The tolerance described in Definition 1 was chosen as $\delta = 10^{-6}$.

Figures 2, 3, and 4 show the results. Figure 2 shows that the states of all agents converge close to those of the leader agent 0; therefore practical consensus is achieved. Figure 3 shows the distributed control actions, implemented according to (10), that each agent applied, as well as the actual unmeasured disturbances and noise that each agent was subjected to. Finally, Figure 4 shows the local neighborhood tracking errors for each agent; these errors converge to a small value, so the agents achieve practical consensus.

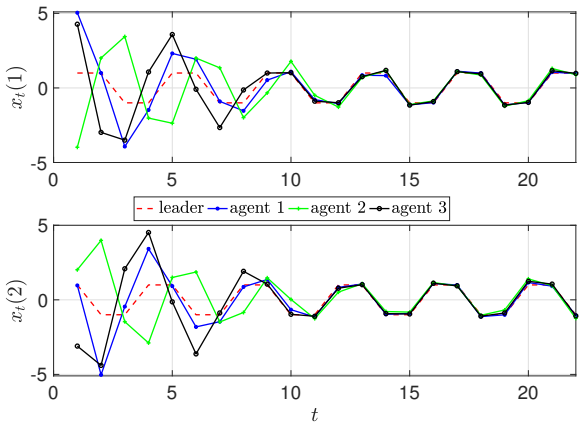

Figure 2: The states $x_t(1)$ and $x_t(2)$ of all agents converge close to those of the leader, thereby achieving practical consensus.

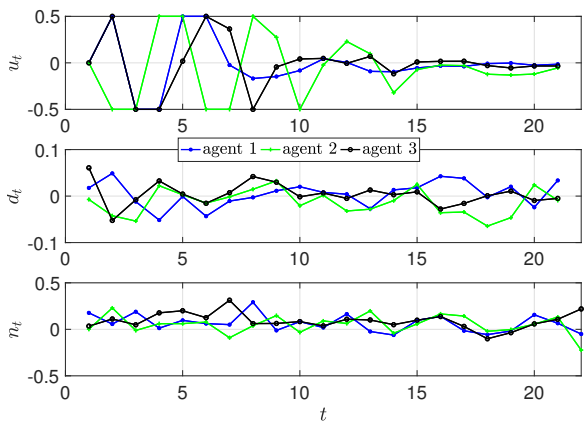

Figure 3: u_t applied, actual d_t, and actual n_t for all three agents.

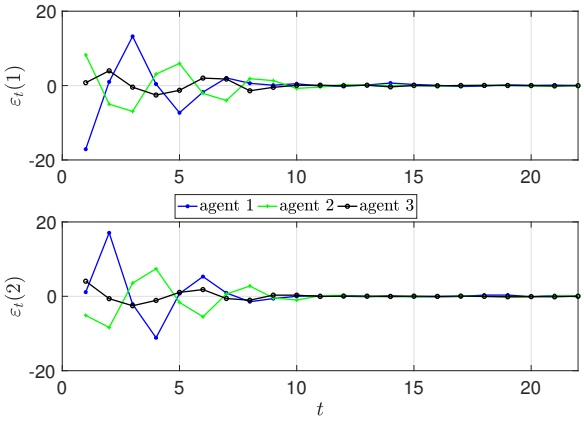

Figure 4: Local neighborhood tracking errors, $\varepsilon_t(1)$ and $\varepsilon_t(2)$, for all three agents.

6. Conclusions and Future Work

We presented a distributed output-feedback model predictive control approach for achieving consensus in multi-agent systems. The agents only have knowledge of local information determined by a communication network modeled as a directed graph. Each agent computes a local control input by solving a min-max optimization based on the output-feedback measurement of its neighborhood tracking error. Using this approach, we were able to prove a bound on the size of the group consensus error, thereby ensuring practical consensus in the presence of unmeasured input disturbances and measurement noise. A numerical example was given showcasing these results.

In future work, an example of a provably convergent distributed algorithm will be developed. Furthermore, the results could be extended for the case of nonlinear dynamics, or a model-free approach could be considered that, for instance, includes a reinforcement learning algorithm.

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Conflict of interest

The authors confirm that there are no known conflicts of interest associated with this publication, and there has been no significant financial support for this work that could have influenced its outcome.

Appendix

The proof of Theorem 1 follows very closely the proof of Theorem 1 in [8].

Before proving Theorem 1 (practical stability of the error dynamics (3)), we introduce a key technical lemma that establishes a monotonicity-like property of the sequence $\{J_t^{i*} : t \in \mathbb{Z}_{\geq 0}\}$ computed along solutions to the closed loop.

Lemma 2. *Suppose that Assumptions 1, 2, and 3 hold. Along any trajectory of the closed-loop system defined by the error dynamics* (3) *and the control law* (10)*, the sequence* $\{J_t^{i*} : t \in \mathbb{Z}_{\geq 0}\}$*, whose existence is guaranteed by Assumption 1, satisfies*

$$J_{t+1}^{i*} - J_t^{i*} \leq \|\tilde{n}_{t-L}^i\|_{R_n^i}^2 + \|\tilde{d}_{t-L}^i\|_{R_d^i}^2, \quad \forall t \in \mathbb{Z}_{\geq L} \tag{21}$$

for appropriate sequences $\tilde{d}_{0:t-L-1}^i \in \mathcal{D}_i$, $\tilde{n}_{0:t-L-1}^i \in \mathcal{N}_{\text{pre}}$. □

The following notation will be used in the remainder of the proof to denote the solution to process (3): given a past control input sequence, as previously denoted $\boldsymbol{u}^i$, and a past disturbance input sequence $d_{t-L:t-1}^i$, we denote by

$$\varphi_t^i(t-L, \varepsilon_{t-L}^i, \boldsymbol{u}^i, d_{t-L:t-1}^i)$$

the solution ε_t^i of the system (3) at time t for the given inputs and initial condition ε_{t-L}^i.

Proof of Lemma 2. From (13c) in Assumption 1 at time $t+1$, we conclude that there exist an initial condition $\hat{\varepsilon}_{t-L+1|t+1}^{i*}$ and sequences $\hat{d}_{t-L+1:t+T|t+1}^{i*} \in \mathcal{D}_i$, $\hat{n}_{t-L+1:t+1|t+1}^{i*} \in \mathcal{N}_i$ such that

$$J_{t+1}^{i*} = \min_{\hat{u}_{t+1:t+T|t+1}^i} \; J_{t+1}^i(\hat{\varepsilon}_{t-L+1|t+1}^{i*}, u_{t-L+1:t}^i, \hat{u}_{t+1:t+T|t+1}^i, \hat{d}_{t-L+1:t+T|t+1}^{i*}, y_{t-L+1:t+1}^i). \tag{22}$$

From Assumption 3 at time $t+T$, with $d_{t+T}^i = \hat{d}_{t+T|t+1}^{i*}$ and

$$\varepsilon_{t+T}^i = \hat{\varepsilon}_{t+T|t+1}^{i*} := \varphi_{t+T}^i(t-L+1, \hat{\varepsilon}_{t-L+1|t+1}^{i*}, u_{t-L+1:t}^i, \hat{u}_{t+1:t+T|t+1}^{i*}, \hat{d}_{t-L+1:t+T|t+1}^{i*}),$$

we conclude that there exists a control $\tilde{u}_{t+T}^i \in \mathcal{U}_i$ such that

$$\|\hat{\varepsilon}_{t+T+1|t+1}^{i*}\|_{\overline{Q}^i}^2 - \|\hat{\varepsilon}_{t+T|t+1}^{i*}\|_{\overline{Q}^i}^2 + \|\hat{\varepsilon}_{t+T|t+1}^{i*}\|_{Q^i}^2 + \|\tilde{u}_{t+T}^i\|_{R_u^i}^2 - \|\hat{d}_{t+T|t+1}^{i*}\|_{R_d^i}^2 \leq 0. \tag{23}$$

Moreover, we conclude from Assumption 2, that there exist vectors $\tilde{\varepsilon}_{t-L}^i, \tilde{d}_{t-L}^i \in \mathcal{D}_i$, $\tilde{n}_{t-L}^i \in \mathcal{N}_{\text{pre}}$ such that

$$\begin{aligned} \hat{\varepsilon}_{t-L+1|t+1}^{i*} &= \overline{A}\tilde{\varepsilon}_{t-L}^i + \overline{B}_i u_{t-L}^i + \overline{D}_i \tilde{d}_{t-L}^i + h_{t-L}^i, \\ y_{t-L}^i &= C_i \tilde{\varepsilon}_{t-L}^i + \tilde{n}_{t-L}^i, \end{aligned} \tag{24}$$

Using now (13b) in Assumption 1 at time t, we conclude that there also exist a finite scalar $J_t^{i*} \in \mathbb{R}$ and a sequence $\hat{\boldsymbol{u}}^{i*} \in \mathcal{U}_i$ such that

$$J_t^{i*} = \max_{\hat{\boldsymbol{\varepsilon}}^i, \hat{\boldsymbol{d}}^i} J_t^i(\hat{\boldsymbol{\varepsilon}}^i, \boldsymbol{u}^i, \hat{\boldsymbol{u}}^{i*}, \hat{\boldsymbol{d}}^i, \boldsymbol{y}^i). \tag{25}$$

Going back to (22), we then conclude that

$$J_{t+1}^{i*} \leq J_{t+1}^i(\hat{\varepsilon}_{t-L+1|t+1}^{i*}, u_{t-L+1:t}^i, \hat{u}_{t+1:t+T-1|t}^{i*}, \tilde{u}_{t+T}^i, \hat{d}_{t-L+1:t+T|t+1}^{i*}, y_{t-L+1:t+1}^i) \tag{26}$$

because the minimization in (22) with respect to $\hat{u}_{t+1:t+T|t+1}^i \in \mathcal{U}_i$ must lead to a value no larger than what would be obtained by setting $\hat{u}_{t+1:t+T-1|t+1}^i = \hat{u}_{t+1:t+T-1|t}^{i*}$ and $\hat{u}_{t+T|t+1}^i = \tilde{u}_{t+T}^i$.
Similarly, we can conclude from (25) that

$$\begin{aligned} J_t^{i*} &\geq J_t^i(\tilde{\varepsilon}_{t-L}^i, \boldsymbol{u}^i, \hat{\boldsymbol{u}}^{i*}, \tilde{d}_{t-L}^i, \hat{d}_{t-L+1:t+T-1|t+1}^{i*}, \boldsymbol{y}^i) \\ &= J_t^i(\tilde{\varepsilon}_{t-L}^i, u_{t-L:t}^i, \hat{u}_{t+1:t+T-1|t}^{i*}, \tilde{d}_{t-L}^i, \hat{d}_{t-L+1:t+T-1|t+1}^{i*}, \boldsymbol{y}^i), \end{aligned} \tag{27}$$

because the maximization in (25) with respect to $\hat{\boldsymbol{\varepsilon}}^i$ and $\hat{\boldsymbol{d}}^i$ must lead to a value no smaller than what would be obtained by setting $\hat{\boldsymbol{\varepsilon}}^i = \tilde{\varepsilon}_{t-L}^i$, $\hat{d}_{t-L|t}^i = \tilde{d}_{t-L}^i$ and $\hat{d}_{t-L+1:t+T-1|t}^i = \hat{d}_{t-L+1:t+T-1|t+1}^{i*}$. The last equality in (27) is obtained by applying the control law (10).
Combining (26), (27), and (24) leads to

$$\begin{aligned} J_{t+1}^{i*} - J_t^{i*} \leq\; & J_{t+1}^i\Big(\overline{A}\tilde{\varepsilon}_{t-L}^i + \overline{B}_i u_{t-L}^i + \overline{D}_i \tilde{d}_{t-L}^i + h_{t-L}^i, \\ & u_{t-L+1:t}^i, \hat{u}_{t+1:t+T-1|t}^{i*}, \tilde{u}_{t+T}^i, \hat{d}_{t-L+1:t+T|t+1}^{i*}, y_{t-L+1:t+1}^i\Big) \\ & - J_t^i\Big(\tilde{\varepsilon}_{t-L}^i, u_{t-L:t}^i, \hat{u}_{t+1:t+T-1|t}^{i*}, \tilde{d}_{t-L}^i, \hat{d}_{t-L+1:t+T-1|t+1}^{i*}, y_{t-L:t}^i\Big). \end{aligned} \tag{28}$$

A crucial observation behind this inequality is that *both* terms $J_{t+1}^i(\cdot)$ and $J_t^i(\cdot)$ on the right-hand side of (28) are computed along a trajectory initialized at time $t-L$ with the *same initial state* $\tilde{\varepsilon}_{t-L}^i$ and share the *same control input* sequence $u_{t-L:t}^i, \hat{u}_{t+1:t+T-1|t}^{i*}$ and the *same disturbance input* sequence $\tilde{d}_{t-L}^i, \hat{d}_{t-L+1:t+T-1|t+1}^{i*}$. We shall denote this common state trajectory by $\tilde{\varepsilon}_s^i$, $s \in \{t-L, \ldots, t+T\}$, and the shared control and disturbance sequences by

$$\tilde{d}_s^i := \hat{d}_{s|t+1}^{i*}, \quad \forall s \in \{t-L+1, \ldots, t+T-1\},$$

$$\tilde{u}_s^i := \begin{cases} u_s^i & s \in \{t-L, \ldots, t\} \\ \hat{u}_{s|t}^{i*} & s \in \{t+1, \ldots, t+T-1\}. \end{cases}$$

The vectors $\tilde{u}^i_{t+T}$ and $\tilde{d}^i_{t-L}$ have been previously defined, but we now also define $\tilde{d}^i_{t+T} := \hat{d}^{i*}_{t+T|t+1}$, $\tilde{\varepsilon}^i_{t+T+1} := \overline{A}\tilde{\varepsilon}^i_{t+T} + \overline{B}_i\tilde{u}^i_{t+T} + \overline{D}_i\tilde{d}^i_{t+T} + h^i_{t+T}$, and $\tilde{n}^i_s := y^i_s - C_i\tilde{\varepsilon}^i_s$, $s \in \{t-L, \dots, t\}$. All of these definitions enable us to express both terms $J^i_{t+1}(\cdot)$ and $J^i_t(\cdot)$ on the right-hand side of (28) as follows:

$$
\begin{aligned}
J^{i*}_{t+1} - J^{i*}_t \leq & \sum_{s=t+1}^{t+T} (\|\tilde{\varepsilon}^i_s\|^2_{Q^i} + \|\tilde{u}^i_s\|^2_{R^i_u}) + \|\tilde{\varepsilon}^i_{t+T+1}\|^2_{\overline{Q}^i} \\
& - \sum_{s=t-L+1}^{t+1} \|\tilde{n}^i_s\|^2_{R^i_n} - \sum_{s=t-L+1}^{t+T} \|\tilde{d}^i_s\|^2_{R^i_d} \\
& - \sum_{s=t}^{t+T-1} (\|\tilde{\varepsilon}^i_s\|^2_{Q^i} + \|\tilde{u}^i_s\|^2_{R^i_u}) - \|\tilde{\varepsilon}^i_{t+T}\|^2_{\overline{Q}^i} \\
& + \sum_{s=t-L}^{t} \|\tilde{n}^i_s\|^2_{R^i_n} + \sum_{s=t-L}^{t+T-1} \|\tilde{d}^i_s\|^2_{R^i_d} \\
= & \|\tilde{\varepsilon}^i_{t+T}\|^2_{Q^i} + \|\tilde{u}^i_{t+T}\|^2_{R^i_u} + \|\tilde{\varepsilon}^i_{t+T+1}\|^2_{\overline{Q}^i} \\
& - \|\tilde{\varepsilon}^i_{t+T}\|^2_{\overline{Q}^i} - \|\tilde{d}^i_{t+T}\|^2_{R^i_d} + \|\tilde{n}^i_{t-L}\|^2_{R^i_n} \\
& + \|\tilde{d}^i_{t-L}\|^2_{R^i_d} - \|\tilde{\varepsilon}^i_t\|^2_{Q^i} - \|\tilde{u}^i_t\|^2_{R^i_u} - \|\tilde{n}^i_{t+1}\|^2_{R^i_n}.
\end{aligned}
$$

Equation (21) follows from this, (23), and the fact that $\|\tilde{\varepsilon}^i_t\|^2_{Q^i}$, $\|\tilde{u}^i_t\|^2_{R^i_u}$ and $\|\tilde{n}^i_{t+1}\|^2_{R^i_n}$ are all non-negative. ■

Now we are ready to prove Theorem 1.

Proof of Theorem 1. Using (13b) in Assumption 1, we conclude that

$$
\begin{aligned}
J^{i*}_t &= \max_{\hat{\boldsymbol{\varepsilon}}^i, \hat{\boldsymbol{d}}^i} \; J^i_t(\hat{\boldsymbol{\varepsilon}}^i, \boldsymbol{u}^i, \hat{\boldsymbol{u}}^{i*}, \hat{\boldsymbol{d}}^i, \boldsymbol{y}^i) \\
&\geq J^i_t(\varepsilon^i_{t-L}, \boldsymbol{u}^i, \hat{\boldsymbol{u}}^{i*}, d^i_{t-L:t}, 0_{t+1:t+T-1}, \boldsymbol{y}^i) \\
&= J^i_t(\varepsilon^i_{t-L}, u^i_{t-L:t}, \hat{u}^{i*}_{t+1:t+T-1|t}, d^i_{t-L:t}, 0_{t+1:t+T-1}, \boldsymbol{y}^i).
\end{aligned}
$$

The first inequality results from the maximum leading to a value no smaller than what would have been obtained by setting $\hat{\boldsymbol{\varepsilon}}^i$ equal to the true state ε^i_{t-L}, setting $\hat{d}^i_{t-L:t}$ equal to the true (past) disturbances $d^i_{t-L:t}$ and setting $\hat{d}^i_{t+1:t+T-1}$ equal to zero. The final equality results from applying the control law (10).
To proceed, we replace $J^i_t(\cdot)$ by its definition in (7), while dropping all "future" positive terms in $\|\varepsilon^i_s\|^2_{Q^i}$, $\|u^i_s\|^2_{R^i_u}$, for $s > t$, and $\|\varepsilon^i_{t+T}\|^2_{\overline{Q}^i}$. This leads to

$$
J^{i*}_t \geq \|\varepsilon^i_t\|^2_{Q^i} + \|u^i_t\|^2_{R^i_u} - \sum_{s=t-L}^{t} \|n^i_s\|^2_{R^i_n} - \sum_{s=t-L}^{t} \|d^i_s\|^2_{R^i_d}. \quad (29)
$$

The future controls $\hat{u}^{i*}_{t+1:t+T-1|t}$ disappeared because we dropped all the (positive) terms involving the value of the state for times $s > t$, and the summation over future disturbances also disappeared since we set the sequence $\hat{d}^i_{t+1:t+T-1}$ to zero.

Adding both sides of (21) in Lemma 2 from time L to time $t-1$, leads to

$$
J^{i*}_t \leq J^{i*}_L + \sum_{s=0}^{t-L-1} \|\tilde{d}^i_s\|^2_{R^i_d} + \sum_{s=0}^{t-L-1} \|\tilde{n}^i_s\|^2_{R^i_n}, \; \forall t \in \mathbb{Z}_{\geq L}. \quad (30)
$$

The bound in (17) follows directly from (29) and (30). ■

References

[1] Abouheaf, M. I., Lewis, F. L., Vamvoudakis, K. G., Haesaert, S., Babuska, R., 2014. Multi-agent discrete-time graphical games and reinforcement learning solutions. Automatica 50, 3038–3053.

[2] Başar, T., Olsder, G. J., 1995. Dynamic Noncooperative Game Theory. Academic Press, London.

[3] Camponogara, E., Jia, D., Krogh, B. H., Talukdar, S., 2002. Distributed model predictive control. IEEE Control Systems 22 (1), 44–52.

[4] Christofides, P. D., Scattolini, R., Muñoz de la Peña, D., Liu, J., 2013. Distributed model predictive control: A tutorial review and future research directions. Computers and Chemical Engineering 51, 21–41.

[5] Copp, D. A., Hespanha, J. P., December 2014. Nonlinear output-feedback model predictive control with moving horizon estimation. In: IEEE Conference on Decision and Control. pp. 3511–3517.

[6] Copp, D. A., Hespanha, J. P., 2016. Addressing adaptation and learning in the context of model predictive control with moving horizon estimation. In: Control of Complex Systems: Theory and Applications. Elsevier, Ch. 6, pp. 187–209.

[7] Copp, D. A., Hespanha, J. P., 2016. Conditions for saddle-point equilibria in output-feedback MPC with MHE. In: American Control Conference (ACC). IEEE, pp. 13–19.

[8] Copp, D. A., Hespanha, J. P., March 2017. Simultaneous nonlinear model predictive control and state estimation. Automatica 77, 143–154.

[9] Ding, Z., 2013. Consensus output regulation of a class of heterogeneous nonlinear systems. IEEE Transactions on Automatic Control 58 (10), 2648–2653.

[10] Farina, M., Scattolini, R., 2011. An output feedback distributed predictive control algorithm. In: 2011 50th IEEE Conference on Decision and Control and European Control Conference. IEEE, pp. 8139–8144.

[11] Ferramosca, A., Limon, D., Alvarado, I., Camacho, E. F., April 2013. Cooperative distributed MPC for tracking. Automatica 49 (4), 906–914.

[12] Ferraz, H., Hespanha, J. P., 2019. Iterative algorithms for distributed leader-follower model predictive control. In: Submitted.

[13] Khoo, S., Xie, L., Man, Z., April 2009. Robust finite-time consensus tracking algorithm for multirobot systems. IEEE/ASME Transactions on Mechatronics 14 (2).

[14] Kim, H., Shim, H., Seo, J. H., January 2011. Output consensus of heterogeneous uncertain linear multi-agent systems. IEEE Transactions on Automatic Control 56 (1), 200 – 206.

[15] Li, Z., Wen, G., Duan, Z., Ren, W., April 2015. Designing fully distributed consensus protocols for linear multi-agent systems with directed graphs. IEEE Transactions on Automatic Control 60 (4), 1152 – 1157.

[16] Liberzon, D., Sontag, E. D., Wang, Y., 2002. Universal construction of feedback laws achieving ISS and integral-ISS disturbance attenuation. Systems & Control Letters 46 (2), 111–127.

[17] Liu, J., Chen, X., Muñoz de la Peña, D., Christofides, P. D., 2010. Sequential and iterative architectures for distributed model predictive control of nonlinear process systems. AIChE Journal 56 (8), 2137–2149.

[18] Liu, Q., Wang, Z., He, X., Zhou, D. H., September 2015. Event-based H∞ consensus control of multi-agent systems with relative output feedback: The finite-horizon case. IEEE Transactions on Automatic Control 60 (9), 2553–2558.

[19] Ma, M., Chen, H., Liu, X., Allgöwer, F., 2014. Distributed model predictive load frequency control of multi-area interconnected power system. International Journal of Electrical Power & Energy Systems 62, 289–298.

[20] Mayne, D. Q., Rawlings, J. B., Rao, C. V., Scokaert, P. O. M., 2000. Constrained model predictive control: Stability and optimality. Automatica 36 (6), 789–814.

[21] Mercangöz, M., Doyle, F. J., 2007. Distributed model predictive control of an experimental four-tank system. Journal of Process Control 17 (3), 297–308.

[22] Olfati-Saber, R., Murray, R. M., 2004. Consensus problems in networks of agents with switching topology and time-delays. IEEE Transactions on Automatic Control 49 (9), 1520–1533.

[23] Rao, C. V., Rawlings, J. B., Mayne, D. Q., 2003. Constrained state estimation for nonlinear discrete-time systems: Stability and moving horizon approximations. Automatic Control, IEEE Transactions on 48 (2), 246–258.

[24] Rawlings, J. B., Mayne, D. Q., 2009. Model Predictive Control: Theory and Design. Nob Hill Publishing.

[25] Ren, W., Beard, R. W., Atkins, E. M., 2007. Information consensus in multivehicle cooperative control. IEEE Control Systems 27 (2), 71–82.

[26] Scattolini, R., 2009. Architectures for distributed and hierarchical model predictive control–a review. Journal of process control 19 (5), 723–731.

[27] Sontag, E. D., 1999. Control-lyapunov functions. In: Open problems in mathematical systems and control theory. Springer, pp. 211–216.

[28] Stewart, B. T., Venkat, A. N., Rawlings, J. B., Wright, S. J., Pannocchia, G., August 2010. Cooperative distributed model predictive control. Systems & Control Letters 59 (8), 460–469.

[29] Venkat, A. N., Hiskens, I. A., Rawlings, J. B., Wright, S. J., 2008. Distributed mpc strategies with application to power system automatic generation control. IEEE transactions on control systems technology 16 (6), 1192–1206.

[30] Venkat, A. N., Rawlings, J. B., Wright, S. J., 2005. Stability and optimality of distributed model predictive control. In: 44th IEEE Conference on Decision and Control, 2005 and 2005 European Control Conference. IEEE, pp. 6680–6685.

[31] Zhang, H., Lewis, F. L., Das, A., August 2011. Optimal design for synchronization of cooperative systems: State feedback, observer and output feedback. IEEE Transactions on Automatic Control 56 (8), 1948 – 1952.

[32] Zheng, Y., Li, S. E., Li, K., Borrelli, F., Hedrick, J. K., 2017. Distributed model predictive control for heterogeneous vehicle platoons under unidirectional topologies. IEEE Transactions on Control Systems Technology 25 (3), 899–910.