L^2-Gain Analysis of Systems with Clock Offsets

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***Abstract*— In this paper, we analyze the L^2-gain of sampled-data systems with asynchronous sensing and control. The closed-loop system has time-varying clock offsets and sampling intervals and can be modeled as a parameter-uncertain system. We use a lifting approach and a gridding and norm-bounded approach in order to analyze intersample behavior and deal with parameter uncertainty, respectively. An upper bound on the L^2-gain is derived via linear matrix inequalities. We illustrate the proposed method by applying it to an active suspension system.**

I. Introduction

Recent developments in communication technology allow us to control systems through networks. Networked control systems offer advantages in terms of flexibility and maintenance simplicity, but simultaneously introduce new challenges such as time-varying delays and sampling intervals. One control approach is to design controllers that are robust with respect to such time-domain uncertainties, which has been actively studies; see, e.g., [1], [2].

Another solution is time-stamping: The sensor sends the measurement together with a time-stamp of the sampling time, and the controller compensates variable delays and sampling periods by using the time-stamp [3], [4]. This time-stamp approach requires that the clocks of the sensor and the controller be synchronized. However, clock offsets are unavoidable [5].

The objective of this paper is to analyze the L^2-gain analysis of systems that have clock offsets between the sensor and the controller. For such systems, stability under input saturation [6] and limitations on clock offsets for stabilization [7], [8] have been studied. The approach in [6], [7] uses the discretization of the closed-loop system and that in [8] is based on the analysis developed for data rate limitations in quantized control [9].

However, these approaches cannot be employed for investigating how much disturbances affect the system performance between sampling times. We therefore employ the lifting technique proposed in [10], [11], which discretizes sampled-data systems without ignoring intersample behavior. Lifting has been first developed for the robust control of sampled-data systems with constant sampling rate (see, e.g., the book [12]), and has been recently exploited to the stability analysis of systems with non-uniform sampling [13] and to the L^2-gain analysis of hybrid systems [14] and stochastic systems [15].

A major difficulty in the L^2-gain analysis of the closed-loop system stems from the fact that sampling is not periodic and clock offsets are time varying. Many lifting techniques employ the periodicity of sampling. Moreover, although L^2-gain analysis for time-varying systems has been studied in [16], there is a drawback that the derived matrix inequality depends on time-varying parameters, which requires the computation of infinitely many LMIs.

For lifting, here we utilize the periodicity of the update of the zero-order hold (ZOH), instead of aperiodic sampling periods. To deal with two time-varying parameters arising from sampling times and clock offsets, we use a gridding and norm-bounded approach in [17]–[20]. We obtain an upper bound on the L^2-gain of the closed-loop system for a given offset bound via finitely many linear matrix inequalities (LMIs). The proposed method is illustrated with a numerical simulation of an active suspension system by showing the trade-off between L^2-gain performance and robustness against clock offsets.

L^2-gain analysis has been extensively studied for systems with aperiodic sampling and variable delays, e.g., in [21]–[23] and the references therein. These previous works are based on the input-delay approach introduced in [24], [25], which models sampled-data systems as continuous-time systems with a time-varying delayed control input. Stability analysis for systems with plant uncertainty and synchronization errors has also been studied by the input-delay approach in [26]. The difference between the proposed approach (the combination of lifting and gridding) and the input-delay approach is the type of Lyapunov functions. The proposed approach uses a *time-varying discrete-time* Lyapunov function, whereas the input-delay approach constructs a *continous-time* Lyapunov *functional*.

This paper is organized as follows. In Section II, we introduce the closed-loop system and basic assumptions, and then present the problem formulation. Section III is devoted to giving an upper bound on the L^2-gain of the networked control system. In Section IV, we apply the proposed method to an active suspension system. Concluding remarks are given in Section V.

Notation: Let $\mathbb{Z}_+$ be the set of nonnegative integers. We denote by $\mathrm{conv}(S)$ the convex hull of a set S. For a fixed $T \in (0,\infty]$, $L^2([0,T),\mathbb{R}^p)$ denotes the Hilbert space of vector-valued functions $f : [0,T) \to \mathbb{R}^p$ that are

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square (Lebesgue) integrable over the interval $[0,T)$. We denote the characteristic function of the interval $[a,b)$ by $\mathbf{1}_{[a,b)}$. For an interval $[a,b) \subset [0,h)$, we use the projection operator $\Pi_{[a,b)}: L^2([0,T),\mathbb{R}^p) \to L^2([0,T),\mathbb{R}^p)$ defined by $\Pi_{[a,b)} f := f\mathbf{1}_{[a,b)}$.

The Euclidean norm of $v \in \mathbb{R}^n$ is denoted by $\|v\|$. For a bounded linear operator $\mathbf{D}$, $\|\mathbf{D}\|$ denotes the operator norm. For a bounded self-adjoint linear operator $\mathbf{D}$ on a Hilbert space $\mathcal{E}$, the notation $\mathbf{D} \geq \mathbf{0}$ means that $(\mathbf{D}x, x) \geq 0$ for all $x \in \mathcal{E}$. We denote the L^2-norm of $z \in L^2([0,T),\mathbb{R}^p)$ by $\|z\|_{L^2[0,T)}$. For notational convenience, the L^2-norm of $z \in L^2([0,T),\mathbb{R}^p)$ is denoted by $\|z\|_{L^2}$.

For a matrix M, let us denote by $M^\top$ its transpose. For a bounded linear operator $\mathbf{D}$, $\mathbf{D}^*$ stands for its adjoint. For simplicity, the symmetric term in the symmetric matrix is denoted by '$\star$'.

II. Problem Statement

A. Plant and information structure

Consider the following plant:

$$\dot{x}(t) = Ax(t) + B_1u(t) + B_2w(t), \tag{1}$$

where $x(t) \in \mathbb{R}^n$ and $u(t) \in \mathbb{R}^m$ are the state and the input of the plant and $w(t) \in \mathbb{R}^p$ is the disturbance. Let the performance signal $z(t) \in \mathbb{R}^q$ be

$$z(t) = Cx(t) + Du(t). \tag{2}$$

The first assumption we require is the invertibility of A, which simplifies the representation of the closed-loop system.

Assumption 2.1: *The matrix A is invertible.*

Let $s_0, s_1, \ldots$ be sampling instants. The sensor observes the state $x(s_k)$ and sends it to the controller together with a time-stamp. However, due to asynchronous sensors and controllers, the time-stamp typically includes an unknown offset with respect to the sampling time of the controller clock, and environmental changes make this offset variable. Therefore, for each $k \in \mathbb{Z}_+$, the time-stamp $\hat{s}_k$ reported from the sensor is given by

$$\hat{s}_k = s_k + \Delta_k$$

for some $\Delta_k \in \mathbb{R}$. The clock offset Δ_k is unknown to the sensor and the controller because of time-varying communication delay and is time-varying due to environmental changes such as temperature and humidity.

Let $h > 0$ be the update period of the ZOH. The control signal $u(t)$ is updated at each time $t_k = kh$ ($k \in \mathbb{Z}_+$) with controller outputs u_k: $u(t) = u_k$ for $t \in [t_k, t_{k+1})$. While the control-updating instants t_k are assumed to be periodic, the sampling instants s_k and the reported time stamps $\hat{s}_k$ may not be periodic. However, we do assume that s_k and $\hat{s}_k$ both take values in $[t_k, t_{k+1})$, and that the data $x(t_k)$ and $\hat{s}_k$ are available to the controller by time $t = t_{k+1}$. This assumption is formally stated as follows.

Assumption 2.2: *Let $h > 0$. For all $k \in \mathbb{Z}_+$,*

$$t_{k+1} - t_k = h, \quad s_k, \hat{s}_k \in [t_k, t_{k+1}).$$

The controller receives $x(t_k)$ and $\hat{s}_k$ from the sensor by the time $t = t_{k+1}$.

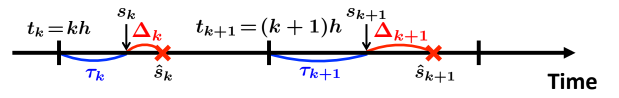

Fig. 1: Sampling instants s_k, reported time-stamps $\hat{s}_k$, and updating instants t_k of the zero-order hold.

Fig. 1 shows the timing diagram of the sampling instants s_k, the reported time-stamps $\hat{s}_k$, and updating instants t_k of the control inputs, where we define $\tau_k := s_k - t_k$.

B. Controller

As in the model-based control of networked control systems (see, e.g., [27]), the controller can estimate the state of the plant at time $t = t_{k+1}$ using the following dynamics:

$$\begin{aligned} \dot{\hat{x}}(t) &= A\hat{x}(t) + Bu(t) && (\hat{s}_k \leq t < t_{k+1}) \\ \hat{x}(\hat{s}_k) &= x(s_k) && (k \in \mathbb{Z}_+), \end{aligned} \tag{3}$$

where $\hat{x} \in \mathbb{R}^n$ is the estimated state. For a given feedback controller gain K, the controller output u_k is $K\hat{x}(t_k)$ for each $k \in \mathbb{Z}_+$, and hence the plant input $u(t)$ is given by

$$u(t) = K\hat{x}(t_k) \qquad (t_k \leq t < t_{k+1}). \tag{4}$$

C. Control objective

Before stating our control objective, we define exponential stability for the closed-loop system (1)–(4).

Definition 2.3 (Exponential stability): *Define*

$$\xi(t) := \begin{bmatrix} x(t) \\ \hat{x}(t) \end{bmatrix}.$$

The system (1)–(4) *is exponentially stable with decay rate $\theta > 0$ if there exists $\Omega \geq 1$ such that $\|\xi(t)\| \leq \Omega e^{-\theta t}\|\xi(0)\|$ for all $\xi(0) \in \mathbb{R}^{2n}$ and $t \in [0,\infty)$ in the case $w = 0$.*

Fix $\gamma > 0$. The objective of the present paper is to obtain a bound on the time-varying clock offset Δ_k under which the closed-loop system (1)–(4) is stable and satisfies the following L^2-gain condition:

$$z \in L^2([0,T),\mathbb{R}^q) \text{ and } \|z\|_{L^2} \leq \gamma\|w\|_{L^2} \text{ if } \xi(0) = 0 \tag{5}$$

for every $w \in L^2([0,\infty),\mathbb{R}^q)$ and every value of Δ_k in the corresponding range. To this end, in what follows we consider the problem below. Let $\mathcal{S}$ be a range of time-varying parameters Δ_k and $\tau_k = s_k - t_k$.

Problem 2.4: *Given a set $\mathcal{S} \subset (-h,h)\times[0,h)$, determine if for a given feedback gain K, the closed-loop system* (1)–(4) *is exponentially stable and satisfies the L^2-gain condition* (5) *for all $(\Delta_k, \tau_k) \in \mathcal{S}$.*

In order to analyze the L^2-gain of the system (1)–(4), we construct the following lifted system based on the lifting approach [10]–[12]:

$$\xi_{k+1} = A_{cl}(\Delta_k)\xi_k + \mathbf{B}(\Delta_k,\tau_k)w_k \tag{6}$$

$$z_k = \mathbf{C}\xi_k + \mathbf{D}w_k, \tag{7}$$

where

$$\xi_k := \begin{bmatrix} x(t_k) \\ \hat{x}(t_k) \end{bmatrix},\ w_k(t) := w(t_k+t),\ z_k(t) := z(t_k+t)$$

for $t \in [0,h)$ and the matrix A and the operators $\mathbf{B}, \mathbf{C}, \mathbf{D}$ are defined by

$$\begin{aligned}
A_{cl}(\Delta_k) &: \mathbb{R}^{2n} \to \mathbb{R}^{2n} \\
&: \xi_k \mapsto \begin{bmatrix} e^{Ah} & (e^{Ah}-I)A^{-1}B_1K \\ e^{A(h-\Delta_k)} & (e^{A(h-\Delta_k)}-I)A^{-1}B_1K \end{bmatrix} \xi_k \\
\mathbf{B}(\Delta_k,\tau_k) &: L^2([0,T),\mathbb{R}^p) \to \mathbb{R}^{2n} \\
&: w_k \mapsto \begin{bmatrix} I & 0 \\ 0 & e^{-A\Delta_k} \end{bmatrix} \begin{bmatrix} \int_0^h e^{A(h-t')}B_2 w_k(t')dt' \\ \int_0^{\tau_k} e^{A(h-t')}B_2 w_k(t')dt' \end{bmatrix} \\
\mathbf{C} &: \mathbb{R}^{2n} \to L^2([0,T),\mathbb{R}^q) \\
&: \xi_k \mapsto \begin{bmatrix} Ce^{At} & C(e^{At}-I)A^{-1}B_1K + DK \end{bmatrix} \xi_k \\
\mathbf{D} &: L^2([0,T),\mathbb{R}^p) \to L^2([0,T),\mathbb{R}^q) \\
&: w_k \mapsto C\int_0^t e^{A(t-t')}B_2 w_k(t')dt'.
\end{aligned}$$

The direct calculation gives (6) and (7). We therefore skip the derivation for brevity. Note that w_k in the integral of $\mathbf{B}$ and $\mathbf{D}$ is an independent variable.

III. Main Results

The lifted system (6) and (7) is described by the operators $\mathbf{B}, \mathbf{C}, \mathbf{D}$, which lead to an operator inequality for L^2-gain analysis. In Section III.A, to avoid solving an operator inequality, first we obtain a matrix inequality whose feasibility is sufficient for the closed-loop system to be stable and achieves the L^2-gain condition (5). Since some matrices in the derived inequality depends on the time-varying parameters Δ_k, τ_k, we next obtain a polytopic overapproximation of such matrices in Section III.B. Finally, in Section III.C, using this polytopic overapproximation, we derive the main result, a sufficient LMI condition for the closed-loop stability and L^2-gain performance.

A. Reduction to matrix inequality

Inspired by Theorem 2 in [16], we obtain a sufficient condition for the stability and L^2-gain condition to hold. This sufficient condition is described by an inequality of matrices dependent on Δ_k and τ_k.

Lemma 3.1: *Assume that $\gamma > \|\mathbf{D}\|$. Let $\mathcal{S} \subset (-h,h) \times [0,h)$ and $\rho \in (0,1)$. Define matrices $A_d(\Delta)$, $B_d(\Delta,\tau)$, and C_d such that*

$$\begin{aligned}
&A_d(\Delta) = A_{cl}(\Delta) + \mathbf{B}(\Delta,\tau)\mathbf{D}^*(\gamma^2 I - \mathbf{D}\mathbf{D}^*)^{-1}\mathbf{C} \\
&B_d(\Delta,\tau)B_d(\Delta,\tau)^\top = \gamma^2 \mathbf{B}(\Delta,\tau)(\gamma^2 I - \mathbf{D}^*\mathbf{D})^{-1}\mathbf{B}(\Delta,\tau)^* \\
&C_d^\top C_d = \gamma^2 \mathbf{C}^*(\gamma^2 I - \mathbf{D}\mathbf{D}^*)^{-1}\mathbf{C}.
\end{aligned} \tag{8}$$

If there exist a matrix function $P(\Delta,\tau)$ and constants $\beta_1, \beta_2 > 0$ such that $\beta_1 I \le P(\Delta_k,\tau_k) \le \beta_2 I$ and the matrix inequality (A) *holds for all (Δ_k,τ_k), $(\Delta_{k+1},\tau_{k+1}) \in \mathcal{S}$, then the system* (1)–(4) *is exponentially stable with decay rate $\theta \ge 1/(2h)\cdot\log(1/\rho)$ and satisfies the L^2-gain condition* (5).

The proof follows the same line as in that of Theorem 2 in [16]. Hence we omit it due to space constraints.

Remark 3.2: We can check $\gamma > \|\mathbf{D}\|$ by LMIs; see, e.g., [28].

In order to use the matrix inequality in Lemma 3.1, we have to calculate the matrices A_d, B_d, and C_d in (8). Lemma 3.3 below gives key equations for this calculation. Using these equations, we can compute the matrices A_d, B_d, and C_d in (A) after the Cholesky factorizations of $B_d B_d^\top$ and $C_d^\top C_d$.

Lemma 3.3: *The operators in* (8) *satisfy*

$$\mathbf{B}(\Delta,\tau)\mathbf{D}^*(\gamma^2 I - \mathbf{D}\mathbf{D}^*)^{-1}\mathbf{C} = \begin{bmatrix} I & 0 \\ 0 & e^{-A\Delta} \end{bmatrix} T \tag{9}$$

$$\begin{aligned}
&\mathbf{B}(\Delta,\tau)(\gamma^2 I - \mathbf{D}^*\mathbf{D})^{-1}\mathbf{B}(\Delta,\tau)^* \\
&\quad = \begin{bmatrix} I & 0 \\ 0 & e^{-A\Delta} \end{bmatrix} Y \begin{bmatrix} I & 0 \\ 0 & e^{-A\Delta} \end{bmatrix}^\top
\end{aligned} \tag{10}$$

$$\begin{aligned}
&\mathbf{C}^*(\gamma^2 I - \mathbf{D}\mathbf{D}^*)^{-1}\mathbf{C} \\
&\quad = \frac{1}{\gamma^2}\left(R^\top M \begin{bmatrix} Q_{11}^{-1} & 0 \\ 0 & 0 \end{bmatrix} N - R^\top L + J \right),
\end{aligned} \tag{11}$$

where T and Y are defined by (B) *in the next page and*

$$\underline{A} := \begin{bmatrix} A & B_1K \\ 0 & 0 \end{bmatrix},\quad E := \begin{bmatrix} -A^\top & -C^\top C \\ B_2B_2^\top/\gamma^2 & A \end{bmatrix}$$

$$X_1 := \begin{bmatrix} C & DK \end{bmatrix}^\top \begin{bmatrix} 0 & C \end{bmatrix},$$

$$X_2 := \begin{bmatrix} C & 0 \end{bmatrix}^\top \begin{bmatrix} C & DK \end{bmatrix}$$

$$X_3 := \begin{bmatrix} C & DK \end{bmatrix}^\top \begin{bmatrix} C & DK \end{bmatrix}$$

$$\begin{bmatrix} P & M & L \\ 0 & Q & N \\ 0 & 0 & R \end{bmatrix} := \exp\left(h \begin{bmatrix} -\underline{A}^\top & X_1 & 0 \\ 0 & E & X_2 \\ 0 & 0 & \underline{A} \end{bmatrix} \right)$$

$$\begin{bmatrix} Q_{11}(t) & Q_{12}(t) \\ Q_{21}(t) & Q_{22}(t) \end{bmatrix} := Q(t) := e^{Et}$$

$$\begin{bmatrix} Q_{11} & Q_{12} \\ Q_{21} & Q_{22} \end{bmatrix} := Q := Q(h) = e^{Eh}$$

$$\Gamma := Q_{21}Q_{11}^{-1}Q_{12} - Q_{22}$$

$$\begin{bmatrix} * & \bar{M}(\tau) \\ 0 & * \end{bmatrix} := \exp\left((h-\tau) \begin{bmatrix} -\underline{A}^\top & X_1 \\ 0 & E \end{bmatrix} \right)$$

$$\begin{bmatrix} * & \bar{J}_{12} \\ 0 & \bar{J}_{22} \end{bmatrix} := \exp\left(h \begin{bmatrix} -\underline{A}^\top & X_3 \\ 0 & \underline{A} \end{bmatrix} \right)$$

$$J := \bar{J}_{22}^\top \bar{J}_{12}.$$

We can obtain this result from a calculation similar to the case without clock offsets in [10] and [12, Sec. 13.6]. The proof is therefore omitted for brevity.

$$\begin{bmatrix} A_d(\Delta_k)^\top P(\Delta_{k+1},\tau_{k+1})A_d(\Delta_k) - \rho P(\Delta_k,\tau_k) + C_d^\top C_d & A_d(\Delta_k)^\top P(\Delta_{k+1},\tau_{k+1})B_d(\Delta_k,\tau_k) \\ \star & B_d(\Delta_k,\tau_k)^\top P(\Delta_{k+1},\tau_{k+1})B_d(\Delta_k,\tau_k) - \gamma^2 I \end{bmatrix} < 0 \tag{A}$$

$$T := \begin{bmatrix} \begin{bmatrix}(Q_{11}^{-1})^\top & 0\end{bmatrix} M^\top \\ e^{A(h-\tau)}\left(\begin{bmatrix}Q_{11}(-\tau)^\top + (Q_{11}^{-1}Q_{12}Q_{21}(-\tau))^\top & 0\end{bmatrix} M^\top + \begin{bmatrix}I & 0\end{bmatrix}\bar{M}(\tau)^\top\right)\end{bmatrix} R$$
$$Y := \begin{bmatrix} Q_{21}Q_{11}^{-1} & \Gamma Q_{21}(-\tau)e^{A^\top(h-\tau)} \\ e^{A(h-\tau)}Q_{21}(\tau)Q_{11}^{-1} & e^{A(h-\tau)}Q_{21}(\tau)(Q_{11}(-\tau)+Q_{11}^{-1}Q_{12}Q_{21}(-\tau))e^{A^\top(h-\tau)} \end{bmatrix} \tag{B}$$

B. Polytopic overapproximation

The matrices A_d and B_d obtained from Lemmas 3.1 and 3.3 depend on the time-varying parameters Δ_k and τ_k in a nonlinear way. Based on an overapproximation technique in [17]–[19], the original model can be embedded into a larger class of polytopic models with a norm-bounded additive uncertainty. The polytopic model is used in the next subsection in order to obtain a sufficient LMI condition for the closed-loop system to be stable and achieves the L^2-gain condition (5). This overapproximation procedure is stated in the following lemma:

Lemma 3.4: *Define triangles*

$$\mathcal{S}^{[l]} := \mathrm{conv}(\{(\Delta_1^{[l]},\tau_1^{[l]}),(\Delta_2^{[l]},\tau_2^{[l]}),(\Delta_3^{[l]},\tau_3^{[l]})\}) \subset \mathcal{S} \tag{12}$$

such that they are disjoint and $\sum_l \mathcal{S}^{[l]} = \mathcal{S}$. Let us denote the vertices of the triangles $\mathcal{S}^{[l]}$ by $\{(\Delta_i,\tau_i)\}_{i=1}^N$ without duplication. Define scalars κ by

$$\kappa = \max_l \kappa_l,$$

where we take $\kappa_l > 0$ so that

$$\left\|\begin{bmatrix}A_d(\Delta) & B_d(\Delta,\tau)\end{bmatrix} - \sum_{i=1}^{3}\alpha_i\begin{bmatrix}A_d(\Delta_i^{[l]}) & B_d(\Delta_i^{[l]},\tau_i^{[l]})\end{bmatrix}\right\| \le \kappa_l \tag{13}$$

for all $(\Delta,\tau) \in \mathcal{S}^{[l]}$. Then

$$\begin{aligned}&\left\{\begin{bmatrix}A_d(\Delta) & B_d(\Delta,\tau)\end{bmatrix} : (\Delta,\tau)\in\mathcal{S}\right\} \\ &\subset \left\{\sum_{i=1}^N \alpha_i\begin{bmatrix}A_d(\Delta_i) & B_d(\Delta_i,\tau_i)\end{bmatrix} + \kappa\Phi : \begin{bmatrix}\alpha_1\\ \vdots \\ \alpha_N\end{bmatrix} \in \mathcal{A},\ \Phi \in \mathbf{\Phi}\right\},\end{aligned} \tag{14}$$

where $\mathcal{A}$ and $\mathbf{\Phi}$ are given by

$$\mathcal{A} := \left\{\alpha \in \mathbb{R}^N : \sum_{i=1}^N \alpha_i = 1,\ \alpha_i \ge 0 \text{ for } i = 1,\dots,N\right\}$$
$$\mathbf{\Phi} := \left\{\begin{bmatrix}\Phi_A & \Phi_B\end{bmatrix} :\ \Phi_A \in \mathbb{R}^{2n\times 2n},\ \Phi_B \in \mathbb{R}^{2n\times\bar{p}},\ \left\|\begin{bmatrix}\Phi_A & \Phi_B\end{bmatrix}\right\| \le 1\right\}.$$

and $\bar{p}$ is the number of columns of B_d.

Proof: Since $\|A_d\|$ and $\|B_d\|$ are bounded for all $(\Delta,\tau) \in \mathcal{S}^{[l]}$, it follows that κ_l satisfying (13) always exists for every $(\Delta,\tau) \in \mathcal{S}$. By the definition of κ, we have that for all $(\Delta,\tau) \in \mathcal{S}$,

$$\begin{bmatrix}A_d(\Delta) & B_d(\Delta,\tau)\end{bmatrix} - \sum_{i=1}^N \alpha_i\begin{bmatrix}A_d(\Delta_i) & B_d(\Delta_i,\tau_i)\end{bmatrix} = \kappa\Phi$$

for some $\begin{bmatrix}\alpha_1 & \cdots \alpha_N\end{bmatrix}^\top \in \mathcal{A}$ and $\Phi \in \mathbf{\Phi}$. Thus (14) holds. ■

Remark 3.5: Suppose that the time-stamps $\hat{s}_k$ are periodic, namely, $\hat{s}_k = t_k + q = kh + q$ for some constant $q \ge 0$, and that the actual sampling instants s_k are not equal to $\hat{s}_k$. This means that the sensor acts periodically with respect to its clock but not with respect to the controller clock. In this case, since τ_k and Δ_k satisfy $\Delta_k = q - \tau_k$, the set $\mathcal{S}$ is one dimensional. Hence it is enough to consider the intervals $I^{[l]} := [\Delta^{[l]}, \Delta^{[l+1]}]$ instead of the triangles $\mathcal{S}^{[l]}$ in (12).

C. L^2-gain analysis via LMIs

Combining the polytopic overapproximation technique and Lemmas 3.1 and 3.3, we obtain the main result, which gives a sufficient LMI condition for the closed-loop system to achieve the desired exponential stability and L^2-gain performance.

Theorem 3.6: *Assume that $\gamma > \|\mathbf{D}\|$ and fix $\rho \in (0,1]$. Define $\{(\Delta_i,\tau_i)\}_{i=1}^N$ and κ as in Section III. B. If there exist matrices $P_i > 0$ and scalars $\sigma_{ij} > 0$ for $i,j = 1,\dots,N$ such that*

$$\begin{bmatrix} \rho P_i - C_d^\top C_d - \sigma_{ij} I & 0 & 0 & A_d(\Delta_i)^\top P_j \\ \star & (\gamma^2 - \sigma_{ij})I & 0 & B_d(\Delta_i,\tau_i)^\top P_j \\ \star & \star & \sigma_{ij} I & \kappa P_j \\ \star & \star & \star & P_j \end{bmatrix} > 0 \tag{15}$$

for all $i,j = 1,\dots,N$, then the system (1)–(4) *is exponentially stable with decay rate $\theta > 1/(2h)\cdot\log(1/\rho)$ and satisfies the L^2-gain condition* (5).

Proof: Since the eigenvalues of a matrix are continuous functions of the entries of the matrix, if (15) holds with $\rho = \rho_0$, then there exists $\delta > 0$ such that for all $\epsilon \in [0,\delta)$, (15) holds with $\rho = \rho_0 - \epsilon$. Hence we can use $\rho \in (0,1]$ in contrast with $\rho \in (0,1)$ in Lemma 3.1 and achieve a decay rate $\theta > 1/(2h)\cdot\log(1/\rho)$, not $\theta \ge 1/(2h)\cdot\log(1/\rho)$.

Since $(\Delta_k,\tau_k) \in \mathcal{S}$, it follows from (14) that

$$\begin{aligned}&\begin{bmatrix}A_d(\Delta_k) & B_d(\Delta_k,\tau_k)\end{bmatrix} \\ &\quad = \sum_{i=1}^N \alpha_{k,i}\begin{bmatrix}A_d(\Delta_i) & B_d(\Delta_i,\tau_i)\end{bmatrix} + \kappa\Phi_k\end{aligned} \tag{16}$$

with $[\alpha_{k,1},\dots,\alpha_{k,N}] \in \mathcal{A}$ and $\Phi_k \in \mathbf{\Phi}$. Let $P_i > 0$ and define a matrix function $P(\Delta_k,\tau_k)$ by

$$P(\Delta_k,\tau_k) = \sum_{i=1}^N \alpha_{k,i}P_i \tag{17}$$

for all $(\Delta_k,\tau_k) \in \mathcal{S}$, where $\{\alpha_{k,i}\}_{i=1}^N$ is consistent to that in (16). Since $P_i > 0$, it follows that for all $i = 1,\dots,N$, there exist constants $\beta_{1,i}, \beta_{2,i} > 0$ such that $\beta_{1,i} I \leq P_i \leq \beta_{2,i} I$. We therefore have

$$\min_{i=1,\dots,N} \beta_{1,i} I \leq P(\Delta_k, \tau_k) \leq \max_{i=1,\dots,N} \beta_{2,i} I,$$

and from Lemma 3.1, it suffices to show that the LMI (15) implies the matrix inequality (A).

Using the Schur complement formula, we show that the matrix inequality (A) is equivalent to

$$\begin{bmatrix} \rho P^{[k]} - C_d^\top C_d & 0 & A_d(\Delta_k)^\top P^{[k+1]} \\ \star & \gamma^2 I & B_d(\Delta_k, \tau_k)^\top P^{[k+1]} \\ \star & \star & P^{[k+1]} \end{bmatrix} > 0, \quad (18)$$

where $P^{[k]} := P(\Delta_k, \tau_k)$. Define $\Phi_{k,A} \in \mathbb{R}^{2n \times 2n}$ and $\Phi_{k,B} \in \mathbb{R}^{2n,p}$ by $\begin{bmatrix} \Phi_{k,A} & \Phi_{k,B} \end{bmatrix} := \Phi_k$. Then, using (16) and (17), we can rewrite (18) as

$$\sum_{i=1}^N \alpha_{k,i} \sum_{j=1}^N \alpha_{k+1,j} \\ \times \begin{bmatrix} \rho P_i - C_d^\top C_d & 0 & (A_d(\Delta_i) + \kappa \Phi_{k,A})^\top P_j \\ \star & \gamma^2 I & (B_d(\Delta_i, \tau_i) + \kappa \Phi_{k,B})^\top P_j \\ \star & \star & P_j \end{bmatrix} > 0.$$

The above matrix inequality is feasible for all $\{\alpha_{k,i}\}_{i=1}^N, \{\alpha_{k+1,j}\}_{j=1}^N \in \mathcal{A}$ if and only if

$$\begin{bmatrix} \rho P_i - C_d^\top C_d & 0 & (A_d(\Delta_i) + \kappa \Phi_{k,A})^\top P_j \\ \star & \gamma^2 I & (B_d(\Delta_i, \tau_i) + \kappa \Phi_{k,B})^\top P_j \\ \star & \star & P_j \end{bmatrix} > 0 \quad (19)$$

for every $i, j = 1, \dots, N$. Since

$$\sigma_{ij} \left(I - \begin{bmatrix} \Phi_A^\top \\ \Phi_B^\top \end{bmatrix} \begin{bmatrix} \Phi_A & \Phi_B \end{bmatrix} \right) \geq 0$$

for all $\sigma_{ij} > 0$ and $\Phi = [\Phi_A \ \ \Phi_B] \in \mathbf{\Phi}$, it follows that (19) holds if

$$\begin{bmatrix} \rho P_i - C_d^\top C_d & 0 & (A_d(\Delta_i) + \kappa \Phi_{k,A})^\top P_j \\ \star & \gamma^2 I & (B_d(\Delta_i, \tau_i) + \kappa \Phi_{k,B})^\top P_j \\ \star & \star & P_j \end{bmatrix} \\ - \begin{bmatrix} \sigma_{ij} \left(I - \begin{bmatrix} \Phi_{k,A}^\top \\ \Phi_{k,B}^\top \end{bmatrix} \begin{bmatrix} \Phi_{k,A} & \Phi_{k,B} \end{bmatrix} \right) & 0 \\ 0 & 0 \end{bmatrix} > 0. \quad (20)$$

If we define the matrix on the left side of LMI (15) by Ω_{ij}, then the matrix on the left side of (20) equals $W^\top \Omega_{ij} W$, where W is defined by

$$W := \begin{bmatrix} I & 0 & 0 \\ 0 & I & 0 \\ \Phi_{k,A} & \Phi_{k,B} & 0 \\ 0 & 0 & I \end{bmatrix}.$$

Since W is full column rank for every $[\Phi_{k,A} \ \ \Phi_{k,B}] = \Phi_k \in \mathbf{\Phi}$, it follows that (15) implies (20) and hence (A). ■

IV. Numerical Example

Consider the active suspension system given in [22]. The system is described by (1) and (2) with

$$A = \begin{bmatrix} 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1 \\ -\frac{k_s}{m_s} & 0 & -\frac{c_s}{m_s} & \frac{c_s}{m_s} \\ \frac{k_s}{m_u} & -\frac{k_t}{m_u} & \frac{c_s}{m_u} & \frac{c_s + c_t}{m_u} \end{bmatrix}$$

$$B_1 = \begin{bmatrix} 0 & 0 & \frac{1}{m_s} & -\frac{1}{m_u} \end{bmatrix}^\top, \quad B_2 = \begin{bmatrix} 0 & -1 & 0 & -\frac{c_t}{m_u} \end{bmatrix}^\top$$

$$C = \begin{bmatrix} -\frac{k_s}{m_s} & 0 & -\frac{c_s}{m_s} - \frac{c_s}{m_s} \end{bmatrix}, \quad D = \frac{1}{m_s},$$

and $[x_1 \ \ x_2 \ \ x_3 \ \ x_4]^\top$ is the state x, where x_1 is the suspension deflection, x_2 is the tire deflection, x_3 is the sprung mass speed, and x_4 is the unsprung mass speed. The input u is the active input of the suspension system, and the disturbance w represents the vertical ground velocity of the road profile.

We denote by m_s the sprung mass representing the car chassis and by m_u the unsprung mass representing the wheel assembly, respectively; c_s and k_s are damping and stiffness of the passive suspension system, respectively; c_t and k_t stand for the damping and compressibility of the pneumatic tire, respectively. These parameters have the following values: $m_s = 972.2$ kg, $m_u = 113.6$ kg, $c_s = 1096$ Ns/m, $c_t = 14.6$ Ns/m, $k_s = 42719.6$ N/m, and $k_t = 101115$ N/m. As in [29], we took the sampling period to be $h = 10$ ms and the feedback control $u = Kx$ with

$$K = 10^3 \times \begin{bmatrix} 0.7646 & 3.6362 & -5.3292 & -0.0046 \end{bmatrix}.$$

As in Remark 3.5, we assume that the time-stamps $\hat{s}_k$ are periodic, that is, $\hat{s}_k = t_k + q$ for some constant $q \geq 0$. Hence the parameter set $\mathcal{S}$ is an interval.

First we took $\underline{\Delta} = -\alpha h$, $\overline{\Delta} = \alpha h$ and varied α from 0 to 0.4. We segmented $\mathcal{S}$ into 40 equal intervals. For $q = 0.5h$, which means that, in the absence of clock offsets, the control is updated in the mid-point between sensor sampling times. Fig. 2 illustrates an upper bound on the L^2-gain derived from Theorem 3.6 with $\rho = 1$. We observe that as the length $\overline{\Delta} - \underline{\Delta}$ of the offset interval becomes large, the upper bound on the L^2-gain increases exponentially. This is due to the uncertainty on Δ_k and potentially to the conservatism of the polytopic overapproximation.

Next we fixed the offset interval $[\underline{\Delta}, \overline{\Delta}] = [-0.1h, 0.1h]$ (or $[\underline{\Delta}, \overline{\Delta}] = [-0.2h, 0.2h]$), and changed the design parameter $q = \hat{s}_k - t_k = \hat{s}_k - kh$ from 0.2 to 0.8 (from 0.3 to 0.7). We divided $\mathcal{S}$ into 40 equal intervals. Fig. 3 shows an upper bound on the L^2-gain obtained from Theorem 3.6 with $\rho = 1$. The exact L^2-gain in the no-offset case is also given in Fig. 3. Since we have a better state estimate as q is closer to h, the exact L^2-gain in the case $\underline{\Delta} = \overline{\Delta} = 0$ decreases with q. However, if there are clock offsets, the local minimum of the upper bounds on the L^2-gain is around $[0.4h, 0.5h]$. This means that, in the presence of clock offsets, it is advantageous to update the control signal roughly in the mid-point between sensor sampling instants, as we had considered in Fig. 2. Furthermore, we see from Figs. 2 and

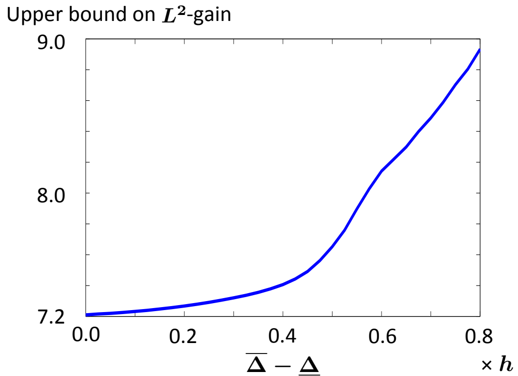

Fig. 2: $\overline{\Delta} - \underline{\Delta}$ versus an upper bound on the L^2-gain.

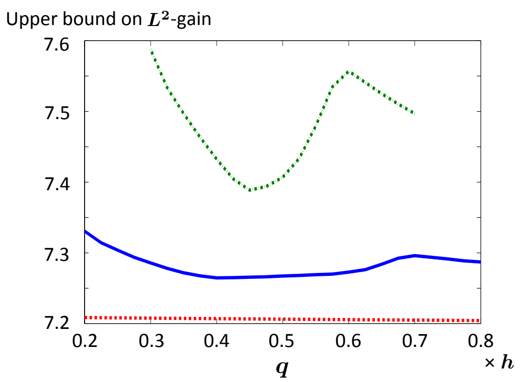

Fig. 3: $q = \hat{s}_k - t_k$ versus L^2-gain: The solid (dashed-dotted) line is an upper bound on the L^2-gain in the case $[\underline{\Delta}, \overline{\Delta}] = [-0.1h, 0.1h]$ ($[\underline{\Delta}, \overline{\Delta}] = [-0.2h, 0.2h]$). The dashed-line is the exact L^2-gain in the case $\underline{\Delta} = \overline{\Delta} = 0$.

3 that if clock offsets are small, then an upper bound on the L^2-gain derived from Theorem 3.6 is close to the exact L^2-gain in the no-offset case.

V. Concluding Remarks

We considered the L^2-gain analysis of sampled-data systems with asynchronous sensors and controllers. An upper bound on the L^2-gain of the closed-loop system was obtained via LMIs. First we gave the lifted representation of the closed-loop system and obtained a sufficient condition for the closed-loop system to be stable and achieve the desired L^2-gain performance. Since the derived condition contains matrices with parametric uncertainty, we next employed a polytopic overapproximation approach. Finally an active suspension system was considered to illustrate the proposed method. Future work involves analyzing robust stability for systems with model uncertainty in addition to clock offsets.

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