L^2-Gain Analysis of Regenerative Switched Linear Systems with Sampled-Data State-Feedback Control

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***Abstract*— In this paper, we analyze the L^2-gain of a class of switched linear systems under sampled-data state-feedback control. We consider switched linear systems whose switching signal is a regenerative process. Using the lifting approach and piecewise-constant approximations, we derive a sequence whose limit inferior upper-bounds the L^2-gain of the closed-loop system. Each term of the sequence can be found by solving a linear matrix inequality. We illustrate the results by the L^2-gain analysis of a linear system with a failure-prone controller.**

I. Introduction

The sampled-data control of continuous-time switched linear systems is an important subject due to the ability of switched linear systems to model various systems in applications such as robotics [1], networked control [2], and epidemiology [3], [4], as well as the obvious practicality of the framework of sampled-data control [5]. However, the attention by researchers has been mostly put on a special class of switched linear systems called Markov jump linear systems [6], a class of switched linear systems whose switching signal is a (time-homogeneous) Markov process. For example, the authors in [7] study the output-feedback sampled-data stabilization of Markov jump linear systems. Also, the optimal state-feedback H^2 control problem for Markov jump linear systems is studied in the recent paper [8].

Though Markov jump linear systems are relatively easy to analyze due to the simple structure of Markov processes, they do not necessarily provide a rich class of dynamics for modeling dynamical systems in application. For example, Weibull distributions, which are often used for modeling failures of components, are known to be not well described by Markov processes. This fact has been motivating authors to consider switching signals belonging to more general classes of stochastic processes. For example, the authors in [9] and [10]–[12] analyze the stability of autonomous switched linear systems whose switching signals are regenerative processes [13] and semi-Markov processes [14], respectively. The former class of switched linear systems, called regenerative switched linear systems, in particular contains various dynamical systems including controlled systems under periodic maintenance [15] and inverted pendulums with stochastic fluctuations [16], neither of which can be modeled by Markov processes nor semi-Markov processes.

In this paper, we study the L^2-gain of regenerative switched linear systems under sampled-data state-feedback control law. By applying an extended version of the lifting operator [17] that is compatible with the underlying regenerative process, we reduce the analysis of the L^2-gain to the discrete-time domain where we can appropriately use a quadratic Lyapunov function. We then apply fast-sample fast-hold approximations [18] to the lifted system and obtain approximate upper bounds for the L^2-gain of the closed-loop system.

This paper is organized as follows. After giving necessary notations, in Section II, we briefly review the definition of regenerative switched linear systems. The lifting representation of those systems is introduced in Section III. Based on the representation, in Section IV, we present the proposed upper bound for the L^2-gain of regenerative switched linear systems subject to sampled-data feedback control. Section V is devoted to numerical examples.

Notations and Conventions

The Euclidean norm on $\mathbb{R}^n$ is denoted by $\|\cdot\|$. Let I and O denote the identity and the zero matrices, respectively. For a continuous operator A defined between Hilbert spaces, we denote its adjoint by A^*. When A is self-adjoint and defined on the Hilbert space X, we say that A is positive semi-definite, denoted by $A \geq 0$, if $(x, Ax) \geq 0$ for every $x \in X$. For another continuous and self-adjoint operator B on X, we write $A \geq B$ if $A - B \geq 0$. The notation $A \leq B$ defined in the same manner.

Let $(\Omega, \mathcal{M}, P)$ be a fixed probability space. For an integrable random variable X on Ω, its expected value is denoted by $E[X]$. The random variables that appear in this paper are assumed to be integrable. For positive integers p and q, we define $(L[0,\infty))^p$ ($(L[0,\infty))^{p\times q}$) as the space of $\mathbb{R}^p$-valued ($\mathbb{R}^{p\times q}$-valued, respectively) stochastic processes $f = \{f(t)\}_{t\geq 0}$ such that $f(t)$ is measurable with respect to $\mathcal{M}$ for every $t \geq 0$. The above defined spaces are denoted simply by $L[0,\infty)$ when there would arise no confusion. Then, we define $L^2[0,\infty)$ as the set of $f \in L[0,\infty)$ such that the integral $\int_0^\infty E[\|f(t)\|^2]\,dt$ is finite. For $f \in L^2[0,\infty)$, we define its norm by $\|f\|^2 = \int_0^\infty E[\|f(t)\|^2]\,dt$. For a positive constant T, the spaces $L[0,T)$ and $L^2[0,T)$ are introduced

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in the obvious manner. Similarly, for a set $\mathcal{K}$, let $\ell_{\mathcal{K}}$ denote the space of $\mathcal{K}$-valued stochastic processes $f = \{f_k\}_{k\geq 0}$ such that f_k is measurable for every $k \geq 0$. If $\mathcal{K}$ is a Hilbert space, then we define $\ell^2_{\mathcal{K}}$ as the set of $f \in \ell_{\mathcal{K}}$ such that the sum $\sum_{k=0}^{\infty} E[\|f_k\|^2]$ is finite. We define the norm of $f \in \ell^2_{\mathcal{K}}$ by $\|f\|^2 = \sum_{k=0}^{\infty} E[\|f_k\|^2]$.

II. Regenerative Switched Linear Systems

In this section, we review the definition of the class of regenerative switched linear systems [9]. We start with recalling the definition of regenerative processes [19].

Definition 2.1 ([19]): Let N be a positive integer. A stochastic process $\sigma = \{\sigma(t)\}_{t\geq 0}$ taking values in $[N]$ is called a *regenerative process* if there exists a real random variable $r_1 > 0$, called a regeneration epoch, such that 1) $\{\sigma(t+r_1)\}_{t\geq 0}$ is independent of $\{\{\sigma(t)\}_{t<r_1}, r_1\}$ and 2) $\{\sigma(t+r_1)\}_{t\geq 0}$ is stochastically equivalent to $\{\sigma(t)\}_{t\geq 0}$.

In the following, we quote from [19] some consequences of the above definition. By repeatedly applying the definition, one can obtain a sequence of independent and identically distributed random variables $\{r_k\}_{k\geq 1}$, called *cycle lengths*, which can be used to break σ into independent and identically distributed *cycles* $\{\sigma(t)\}_{0\leq t<r_1}$, $\{\sigma(t)\}_{r_1\leq t<r_1+r_2}$, $\ldots$. Notice that, since r_1 is finite with probability one, there exist infinitely many cycles. Then, the stochastic process $\{z_k\}_{k\geq 1}$ defined by $z_k = r_1 + \cdots + r_k$ is called the *embedded renewal process* of σ. Throughout this paper, for the sake of convenience, we set $z_0 = 0$ and call $\{z_k\}_{k\geq 0}$ the embedded renewal process of σ.

We then introduce the class of switched linear systems studied in this paper. Let n, m, p, q, and N be positive integers. For each $i \in [N]$, let $A_i \in \mathbb{R}^{n\times n}$, $B_i \in \mathbb{R}^{n\times m}$, $C_i \in \mathbb{R}^{q\times n}$, $D_i \in \mathbb{R}^{q\times m}$, and $E_i \in \mathbb{R}^{n\times p}$. Consider the system of stochastic differential equations:

$$\begin{cases} \dfrac{dx}{dt} = A_{\sigma(t)}x(t) + B_{\sigma(t)}u(t) + E_{\sigma(t)}w(t), \\ \quad y = C_{\sigma(t)}x(t) + D_{\sigma(t)}u(t). \end{cases} \tag{1}$$

The variables x, u, w, and y represent the state, control input, disturbance input, and the measured output of the system. We assume that both $x(0) = x_0$ and w are *deterministic*. Following [9], we call the differential equations (1) a *regenerative switched linear system*. By the generality of regenerative processes, the class of regenerative switched linear systems contains various dynamical systems including controlled system under periodic maintenance [9] and inverted pendulums with stochastic fluctuations [16], neither of which can be represented as Markov jump linear systems. We also remark that, as a particular and trivial case, linear time-invariant systems are regenerative switched linear systems. This is because, if σ is a constant function, then we can regard σ as a regenerative process with regeneration epoch $r_1 = T$ for a constant $T > 0$.

The objective of this paper is to analyze the performance of the closed-loop system Σ consisting of the regenerative switched linear system (1) and the following sampled-data state-feedback control law

$$u(t) = Kx(z_k),\ z_k \leq t < z_{k+1},\ k \geq 0, \tag{2}$$

where $K \in \mathbb{R}^{m\times n}$. Notice that the sampling of the state variable is performed at the times z_k when a regeneration of σ occurs. We introduce the following two performance measures for Σ:

Definition 2.2 ([9]):

1) We say that Σ is *internally mean square stable* [9] (*mean square stable* for short) if there exist $\alpha > 0$ and $\beta > 0$ such that $E[\|x(t)\|^2] \leq \alpha e^{-\beta t}\|x_0\|^2$ for all $x(0) = x_0$ and $t \geq 0$ provided $w \equiv 0$.
2) We say that Σ is L^2*-stable* if there exists $\gamma > 0$ such that, for every function $w \in L^2[0,\infty)$, we have that $y \in L^2[0,\infty)$ and $\|y\| \leq \gamma\|w\|$ provided $x_0 = 0$. If Σ is L^2-stable, then we define the L^2*-gain* of Σ by

$$\|\Sigma\| = \inf\{\gamma \geq 0 : \|y\| \leq \gamma\|w\| \text{ for all } w \in L^2[0,\infty)\}.$$

III. Lifting

In this section, we introduce a discrete-time representation of the system Σ based the lifting of stochastic processes, which extends the one originally introduced [17] for sampled-data control of linear time-invariant systems. The representation to be obtained will be used to analyze the L^2-gain of Σ in Section IV. We will also introduce piecewise-constant approximations of stochastic processes.

Throughout the paper, we place the following assumption on regenerative processes:

Assumption 3.1: There exists a constant $T > 0$ such that $r_1 \leq T$ with probability one.

Notice that this assumption implies $r_k \leq T$ with probability one for every $k \geq 1$ because $\{r_k\}_{k\geq 1}$ are identically distributed. Under this assumption, we introduce a lifting representation of stochastic processes as follows:

Definition 3.2: Given a regenerative process σ satisfying Assumption 3.1, the *lifting operator* on $L[0,\infty)$ is defined as the mapping $\mathcal{L}\colon L[0,\infty) \to \ell_{L[0,T)}\colon \phi \mapsto \{\phi_k\}_{k\geq 0}$ given by

$$\phi_k(\theta) = \begin{cases} \phi(z_k + \theta), & \text{if } 0 \leq \theta < r_{k+1}, \\ 0, & \text{if } r_{k+1} \leq \theta < T, \end{cases}$$

for every $k \geq 0$ (see Fig. 1 for an illustration).

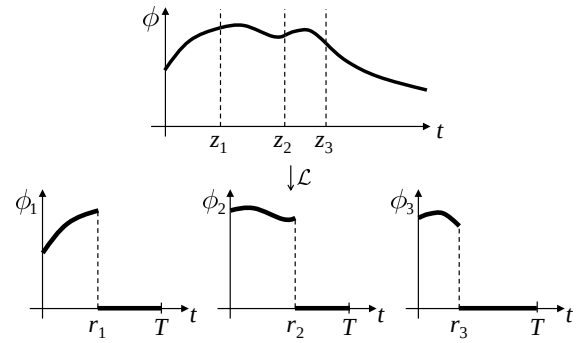

Fig. 1. Lifting

Remark 3.3: Unlike in the standard problem setting where the sampling period is a constant, the sampled-data controller (2) in Σ has random and time-varying sampling intervals r_k, $k = 1, 2, \ldots$. Therefore, simply decomposing a signal, say, ϕ, into pieces $\{\phi_k\}_{k\geq 0}$ given by $\phi_k(\theta) = \phi(kh + \theta)$ for a constant period $h > 0$ does not respect the random and time-varying sampling intervals and hence will not allow the discussion given below.

The following lemma readily follows from the definition of the lifting operator:

Lemma 3.4: $\mathcal{L}$ gives an isomorphism from $L^2[0,\infty)$ to $\ell^2_{L^2[0,T)}$.

Proof: Take an arbitrary $\phi \in L^2[0,\infty)$ and let $\mathcal{L}\phi = \{\phi_k\}_{k\geq 0}$. By the definition of $\mathcal{L}$, we have

$$\begin{aligned}\|\phi_k\|^2 &= \int_0^{r_{k+1}} E[\|\phi(z_k+\theta)\|^2]\,d\theta \\ &= \int_{z_k}^{z_{k+1}} E[\|\phi(t)\|^2]\,dt.\end{aligned}$$

Therefore, $\|\mathcal{L}\phi\|^2 = \int_0^\infty E[\|\phi(t)\|^2]\,dt = \|\phi\|^2$, as desired. ■

Using the lifting operator, we can then derive a lifting representation of Σ. In order to state the representation, we need to introduce the following notations. For every $t \geq 0$ and $\tau \leq t$, let $\Phi(t,\tau)$ denote the solution of the stochastic differential equation $\partial\Phi/\partial t = A_{\sigma(t)}\Phi(t,\tau)$ with initial condition $\Phi(\tau,\tau) = I$. This Φ gives the transition matrix of the autonomous system $dx/dt = A_{\sigma(t)}x$. Then, for all $k \geq 0$, $t \geq 0$, and $\tau \leq t$, define $\Phi_k(t,\tau) = \Phi(t+z_k, \tau+z_k)$. Also, we will use the lifting of system matrices denoted by $\{B_k\}_{k\geq 0} = \mathcal{L}(B_{\sigma(\cdot)})$, $\{C_k\}_{k\geq 0} = \mathcal{L}(C_{\sigma(\cdot)})$, $\{D_k\}_{k\geq 0} = \mathcal{L}(D_{\sigma(\cdot)})$, and $\{E_k\}_{k\geq 0} = \mathcal{L}(E_{\sigma(\cdot)})$. We can now state the next proposition:

Proposition 3.5: For each $k \geq 0$, define the random operator

$$\mathcal{M}_k = \begin{bmatrix} A_{d,k} & \mathcal{B}_k \\ \mathcal{C}_k & \mathcal{D}_k \end{bmatrix} : \mathbb{R}^n \oplus (L^2[0,T))^p \to \mathbb{R}^n \oplus L^2[0,T)^q$$

by

$$\begin{aligned}A_{d,k} &= \Phi_k(r_{k+1},0) + \int_0^{r_{k+1}} \Phi_k(r_{k+1},\tau)B_k(\tau)\,d\tau\,K, \\ \mathcal{B}_k(w) &= \int_0^{r_{k+1}} \Phi_k(r_{k+1},\tau)E_k(\tau)w(\tau)\,d\tau, \\ \mathcal{C}_k(\xi)(t) &= \big(C_k(t)\Phi_k(t,0) + \\ &\qquad C_k(t)\int_0^t \Phi_k(t,\tau)B_k(\tau)K\,d\tau + D_k(t)K\big)\xi, \\ \mathcal{D}_k(w)(t) &= C_k(t)\int_0^t \Phi_k(t,\tau)E_k(\tau)w(\tau)\,d\tau.\end{aligned}$$

Then, for each $k \geq 0$, the random variables $\xi_k = x(z_k)$, $\{w_k\}_{k\geq 0} = \mathcal{L}w$, and $\{y_k\}_{k\geq 0} = \mathcal{L}y$ satisfy the stochastic difference equation

$$\Sigma_d : \begin{bmatrix} \xi_{k+1} \\ y_k \end{bmatrix} = \mathcal{M}_k \begin{bmatrix} \xi_k \\ w_k \end{bmatrix}. \tag{3}$$

Moreover, the random operators $\{\mathcal{M}_k\}_{k\geq 0}$ are independent and identically distributed.[1]

Proof: The derivation of (3) is almost the same as that for linear time-invariant systems. We refer the readers to [5, Chapter 10]. Also, the latter claim directly follows from the definition of regenerative processes. We omit the details due to limitations of space. ■

Before closing this section, let us also introduce piecewise-constant approximations of stochastic processes, which will be used to state the main result of this paper. Let N be a positive integer and write $h = T/N$. let X_N be the space of $\mathbb{R}^p$-valued step functions that are defined on $[0,T)$ and are constant on the intervals $[ih,(i+1)h)$ $(i = 0,1,\ldots,N-1)$. Then, the orthogonal projection of $f \in (L^2[0,T))^p$ onto X_N equals $h^{-1}\sum_{i=1}^N f_i^N \chi_i^N$, where $f_i^N = \int_{(i-1)h}^{ih} f(t)\,dt$, and $\chi_i^N = \chi_{[(i-1)h,ih)} : \mathbb{R} \to \{0,1\}$ is the indicator function of the interval $[(i-1)h, ih)$. This orthogonal projection will give us a finite-dimensional approximation of functions in the infinite-dimensional space $L^2[0,T)$.

In the sequel, we choose to identify X_N with the Euclidean space $\mathbb{R}^{Np}$ via the isomorphism

$$X_N \to \mathbb{R}^{Np} : \sum_{i=1}^N v_i\chi_i^N \mapsto \sqrt{h}\,\mathrm{col}(v_1,\ldots,v_N),$$

where $\mathrm{col}(v_1,\ldots,v_N) = [v_1^\top \ \cdots \ v_N^\top]^\top$. Composed with this isomorphism, the orthogonal projection from $(L^2[0,T))^p$ to X_N yields the mapping $\mathcal{S}_N : (L^2[0,T))^p \to \mathbb{R}^{Np}$ given by $\mathcal{S}_N(f) = \mathrm{col}(f_1^N,\ldots,f_N^N)/\sqrt{h}$, which has the adjoint $\mathcal{S}_N^* : \mathrm{col}(v_1,\ldots,v_N) \mapsto \sum_{i=1}^N v_i\chi_i^N/\sqrt{h}$. We will need the following lemma to prove our main result. Its proof is omitted due to limitations of space.

Lemma 3.6: Let $f \in L^2[0,T)$ be arbitrary. Then, it holds that $\lim_{N\to\infty}(\mathcal{S}_N^*\mathcal{S}_N f) = f$ and $\lim_{N\to\infty}\|\mathcal{S}_N f\| = \|f\|$.

IV. L^2-Gain Analysis

In this section, we analyze the L^2-gain of the closed-loop system Σ based on its lifting representation Σ_d and the piecewise-constant approximations introduced in the last section. Specifically, we show that the L^2-gain is upper-bounded by the limit inferior of a sequence, each term of which can be found by solving a linear matrix inequality. In the rest of this paper, when k is omitted from the operator $\mathcal{M}_k$, we understand that $k = 0$, i.e., we write $\mathcal{M} = \mathcal{M}_0$. The same notation is applied to the operators $A_{d,k}$, $\mathcal{B}_k$, $\mathcal{C}_k$, and $\mathcal{D}_k$.

Also, we state the following assumption:

Assumption 4.1: For every nonzero $x \in \mathbb{R}^n$, there exists $v \in (L^2[0,T))^p$ such that $(\mathcal{B}v, x)$ is not the zero random variable.

Remark 4.2: Assume that Σ is time-invariant (i.e., σ is a constant function) and let us regard σ as a regenerative process with the regeneration epoch $T > 0$. If Assumption 4.1 is true, then letting $v = \mathcal{B}^*x$ shows that $\mathcal{B}\mathcal{B}^*$ is positive definite. This shows that the pair (A,E) is controllable because $\mathcal{B}\mathcal{B}^*$

[1] Basic notions about real random variables such as independence and expectations can be naturally extended to random operators, that is, random elements taking the values in the space of continuous operators between Hilbert spaces. We refer the readers to the monograph [20] for the details.

equals the controllability Gramian of the pair. We can in fact show that, in this time-invariant case, Assumption 4.1 is equivalent to the controllability of (A,E).

In order to state the main result, we introduce the following operators. For $\gamma > 0$ and an $n \times n$ symmetric matrix P, define the random operator $\mathcal{R}(P)$ on $\mathbb{R}^n \oplus (L^2[0,T))^p$ by

$$\mathcal{R}(P) = \mathcal{M}^* \begin{bmatrix} P & O \\ O & I \end{bmatrix} \mathcal{M} - \begin{bmatrix} P & O \\ O & \gamma I \end{bmatrix}. \tag{4}$$

Also, we introduce the operator

$$\tilde{\mathcal{S}}_N = I \oplus \mathcal{S}_N \colon \mathbb{R}^N \oplus (L^2[0,T))^p \to \mathbb{R}^N \oplus \mathbb{R}^{Np}.$$

The next theorem, which is the main result of this paper, gives a real sequence for upper-bounding the L^2-gain:

Theorem 4.3: Suppose that Assumption 4.1 holds true. For $\varepsilon > 0$ and a positive integer N, consider the inequality

$$\tilde{\mathcal{S}}_N E[\mathcal{R}(P_N)] \tilde{\mathcal{S}}_N^* \leq -\varepsilon I. \tag{5}$$

Let δ be a positive constant. Define

$$\gamma^\star_{N,\varepsilon,\delta} = \min\{\gamma \geq 0 : \text{there exists } P_N \geq \delta \text{ such that (5) holds}\}$$

and let $\gamma^\star_{\varepsilon,\delta} = \liminf_{N\to\infty} \gamma^\star_{N,\varepsilon,\delta}$. Then,

$$\|\Sigma\|^2 \leq \gamma^\star_{\varepsilon,\delta}.$$

Moreover, $\gamma^\star_{\varepsilon,\delta}$ is increasing with respect to both ε and δ.

Before presenting the proof of this theorem in Subsection IV-A, we here briefly discuss how to solve the inequality (5) for P_N and γ. For this purpose, we temporarily fix and omit writing N. By the definition (4) of the operator $\mathcal{R}$, the inequality (5) can be written as

$$\begin{bmatrix} Q_{11} & Q_{12} \\ \star & Q_{22} \end{bmatrix} \leq -\varepsilon I, \tag{6}$$

where

$$\begin{aligned} Q_{11} &= E[A_d^\top P A_d] + E[\mathcal{C}^*\mathcal{C}] - P, \\ Q_{12} &= E[A_d^\top P \mathcal{B}\mathcal{S}^*] + E[\mathcal{C}^*\mathcal{D}]\mathcal{S}^*, \\ Q_{22} &= E[\mathcal{S}\mathcal{B}^* P \mathcal{B}\mathcal{S}^*] + \mathcal{S}E[\mathcal{D}^*\mathcal{D}]\mathcal{S}^* - \gamma \mathcal{S}\mathcal{S}^*. \end{aligned}$$

For $i,j \in \{1,\ldots,n\}$, let $U_{ij} \in \mathbb{R}^{n\times n}$ denote the $\{0,1\}$-matrix whose entries are all zero except its (i,j)- and (j,i)-elements being one. Then, we can express P as $P = \sum_{(i,j):i\leq j} p_{ij} U_{ij}$ using real and independent variables p_{ij} $(1 \leq i \leq j \leq n)$. With this expression, we have

$$\begin{aligned} Q_{11} &= \sum_{i\leq j} E[A_d^\top U_{ij} A_d] p_{ij} + E[\mathcal{C}^*\mathcal{C}] - P, \\ Q_{12} &= \sum_{i\leq j} E[A_d^\top U_{ij} \mathcal{B}\mathcal{S}^*] p_{ij} + E[\mathcal{C}^*\mathcal{D}]\mathcal{S}^*, \\ Q_{22} &= \sum_{i\leq j} E[\mathcal{S}\mathcal{B}^* U_{ij} \mathcal{B}\mathcal{S}^*] p_{ij} + E[\mathcal{S}\mathcal{D}^*\mathcal{D}\mathcal{S}^*] - \gamma I, \end{aligned} \tag{7}$$

which makes (6) into standard linear matrix inequalities with variables γ and p_{ij} $(1 \leq i \leq j \leq n)$. In Section V, we will demonstrate how to approximately compute the expectations appearing in (7).

A. Proof of Theorem 4.3

We prove Theorem 4.3 in this subsection. We start with the following definition on the stability of the lifted system Σ_d. We say that Σ_d is *mean square stable* if there exist $\lambda \in [0,1)$ and $\alpha \geq 0$ such that, if $w_k = 0$ for every $k \geq 0$, then $E[\|\xi_k\|^2] \leq \alpha\lambda^k \|\xi_0\|^2$ for every $\xi_0 \in \mathbb{R}^n$. The next proposition shows that the stability of Σ and Σ_d are closely related:

Lemma 4.4: The following statements are equivalent:

1) Σ is mean square stable;
2) Σ_d is mean square stable;
3) There exists an $n \times n$ positive definite matrix P such that $E[A_{d,k}^\top P A_{d,k}] - P < 0$.

Proof: Since the random matrices $\{A_{d,k}\}_{k\geq 0}$ are independent and identically distributed by Proposition 3.5, we can apply [21, Theorem 5.5] to prove the equivalence [2 $\Leftrightarrow$ 3]. Also, we can prove the equivalence [1 $\Leftrightarrow$ 2] in the same way as the proof of [9, Theorem 12]. The details are omitted due to limitations of space. ■

Then, extending [22, Theorem 5.3] given for a class of finite-dimensional discrete-time systems having independent and identically distributed coefficient matrices, the next lemma provides an upper bound on the L^2-gain via a linear *operator* inequality:

Lemma 4.5: Let $\gamma > 0$ be arbitrary. If there exists $\varepsilon > 0$ and a positive definite matrix $P \in \mathbb{R}^{n\times n}$ such that

$$E[\mathcal{R}(P)] \leq -\varepsilon I, \tag{8}$$

then Σ is mean square and L^2-stable. Moreover, $\|\Sigma\|^2 \leq \gamma$.

Proof: Assume that an $n \times n$ positive definite matrix P satisfies (8) and define the positive definite function $V(x) = (x, Px)$ for $x \in \mathbb{R}^n$. The (1,1)-block of the operator in the left hand side of (8) shows $E[A_{d,k}^\top P A_{d,k}] - P \leq -\varepsilon I$. Therefore, Σ is mean square stable by Lemma 4.4. Let us then show that Σ is L^2-stable and $\|\Sigma\|^2 \leq \gamma$. Take an arbitrary function $w \in L^2[0,\infty)$. For each $k \geq 0$, define the random operator

$$\mathcal{R}_k = \mathcal{M}_k^* \begin{bmatrix} P & O \\ O & I \end{bmatrix} \mathcal{M}_k - \begin{bmatrix} P & O \\ O & \gamma I \end{bmatrix}.$$

Then, it is straightforward to show that

$$E\left[\left(\mathcal{R}_k \begin{bmatrix} \xi_k \\ w_k \end{bmatrix}, \begin{bmatrix} \xi_k \\ w_k \end{bmatrix}\right)\right] = E[V(\xi_{k+1})] - E[V(\xi_k)] + \|y_k\|^2 - \gamma\|w_k\|^2. \tag{9}$$

On the other hand, since ξ_k and w_k are measurable with respect the σ-algebra Θ_k generated by the random variables $\{\sigma(t)\}_{0\leq t<z_k}$, the tower rule for conditional expectations yields that

$$\begin{aligned} E\left[\left(\mathcal{R}_k \begin{bmatrix} \xi_k \\ w_k \end{bmatrix}, \begin{bmatrix} \xi_k \\ w_k \end{bmatrix}\right)\right] &= E\left[E\left[\left(\mathcal{R}_k \begin{bmatrix} \xi_k \\ w_k \end{bmatrix}, \begin{bmatrix} \xi_k \\ w_k \end{bmatrix}\right) \;\middle|\; \Theta_k\right]\right] \\ &= E\left[\left(E[\mathcal{R}_k \mid \Theta_k] \begin{bmatrix} \xi_k \\ w_k \end{bmatrix}, \begin{bmatrix} \xi_k \\ w_k \end{bmatrix}\right)\right] \\ &= E\left[\left(E[\mathcal{R}_k] \begin{bmatrix} \xi_k \\ w_k \end{bmatrix}, \begin{bmatrix} \xi_k \\ w_k \end{bmatrix}\right)\right] \\ &\leq 0, \end{aligned} \tag{10}$$

where (8) is used to obtain the last inequality. From (9) and (10), we obtain $\|y_k\|^2 - \gamma\|w_k\|^2 \le E[V(\xi_k)] - E[V(\xi_{k+1})]$. Since $\xi_0 = 0$ and hence $E[V(\xi_0)] = 0$, for every $M \ge 0$ we have $\sum_{k=0}^{M}\|y_k\|^2 - \gamma\sum_{k=0}^{M}\|w_k\|^2 \le -E[V(\xi_{M+1})] \le 0$. Taking the limit $M \to \infty$ in this inequality, we obtain $\sum_{k=0}^{\infty}E[\|y_k\|^2] - \gamma\sum_{k=0}^{\infty}E[\|w_k\|^2] \le 0$ and therefore $\|y\|^2 \le \gamma\|w\|^2$ since $\mathcal{L}$ gives an isomorphism by Lemma 3.4. This completes the proof. ■

In practice, it is not easy to solve the operator inequality (8) due to the infinite-dimensionality of the operator $\mathcal{M}$. The next proposition shows that, by using the piecewise-constant approximations introduced in the last section, we can reduce the linear operator inequality to a sequence of linear matrix inequalities:

Proposition 4.6: Suppose that Assumption 4.1 holds true. Let $\varepsilon > 0$ and $\delta > 0$ be arbitrary. Assume that, for infinitely many positive integers N, there exists a positive definite matrix $P_N \in \mathbb{R}^{n\times n}$ satisfying $P_N \ge \delta I$ and (5). Then, $\|\Sigma\|^2 \le \gamma$.

Proof: Assume that a sequence of positive definite matrices $\{P_N\}_{N=1}^{\infty} \subset \mathbb{R}^{n\times n}$ satisfy (5) for infinitely many values of N. In this proof, we consider only the special case where the inequality (5) is satisfied for every N. The proof for the general case follows easily from the proof below.

Let us temporarily assume that the sequence $\{P_N\}_{N=1}^{\infty}$ has a bounded subsequence. By the compactness of the unit ball in $\mathbb{R}^{n\times n}$, the subsequence contains a converging sequence $\{P_{N_\ell}\}_{\ell=1}^{\infty}$ with a nonzero limit $P \ge \delta I$. Therefore, by Lemma 4.5, it is sufficient to prove $E[\mathcal{R}(P)] \le -\varepsilon I$ in order to show $\|\Sigma\|^2 \le \gamma$. Observe that, as $\ell \to \infty$, we have

$$\begin{aligned}&E[\mathcal{R}(P)] - E[\mathcal{R}(P_{N_\ell})]\\ =&E\left[\mathcal{M}^*\begin{bmatrix}P-P_{N_\ell} & O\\ O & O\end{bmatrix}\mathcal{M}\right] - \begin{bmatrix}P-P_{N_\ell} & O\\ O & O\end{bmatrix} \to O.\end{aligned}$$

Therefore, from Lemma 3.6, for every $f \in \mathbb{R}^n \oplus (L^2[0,T])^p$ it follows that

$$\begin{aligned}(f, E[\mathcal{R}(P)]f) &= \lim_{\ell\to\infty}\left(\tilde{\mathcal{S}}_{N_\ell}^*\tilde{\mathcal{S}}_{N_\ell}f, E[\mathcal{R}(P_{N_\ell})]\tilde{\mathcal{S}}_{N_\ell}^*\tilde{\mathcal{S}}_{N_\ell}f\right)\\ &= \lim_{\ell\to\infty}\left(\tilde{\mathcal{S}}_{N_\ell}f, \left(\tilde{\mathcal{S}}_{N_\ell}E[\mathcal{R}(P_{N_\ell})]\tilde{\mathcal{S}}_{N_\ell}^*\right)\tilde{\mathcal{S}}_{N_\ell}f\right)\\ &\le -\varepsilon\lim_{\ell\to\infty}\|\tilde{\mathcal{S}}_{N_\ell}f\|^2\\ &= -\varepsilon\|f\|^2\end{aligned}$$

and hence $E[\mathcal{R}(P)] \le -\varepsilon I$, as desired.

We omit the proof of the the existence of a bounded subsequence in $\{P_N\}_{N=1}^{\infty}$ due to limitations of space. ■

Now we have the elements to prove Theorem 4.3:

Proof of Theorem 4.3: Let $\varepsilon > 0$ and $\delta > 0$ be arbitrary. By the definition of $\gamma^\star_{\varepsilon,\delta}$, for any $\gamma > \gamma^\star_{\varepsilon,\delta}$, there exists infinitely many values of N such that (5) is feasible for some $P_N \ge \delta I$. Therefore, Proposition 4.6 implies that $\|\Sigma\|^2 \le \gamma$. The conclusion then follows immediately because $\gamma > \gamma^\star_{\varepsilon,\delta}$ was arbitrary. Also, since $\gamma^\star_{N,\varepsilon,\delta}$ is increasing with respect to ε and δ, so is $\gamma^\star_{\varepsilon,\delta}$. ■

V. Example

The aim of this section is to illustrate Theorem 4.3 by numerical examples. Let the switching signal σ be $\{1,2\}$-valued and consider the system given by $dx/dt = Ax(t) + B_{\sigma(t)}u(t) + w(t)$ and $y(t) = x(t)$, where $A \in \mathbb{R}^{n\times n}$ and $(B_1, B_2) = (B, O)$ for some $B \in \mathbb{R}^{n\times m}$. The above equations model a system with a failure-prone actuator; the actuator is operating normally when $\sigma(t) = 1$ and experiencing a failure when $\sigma(t) = 2$. Assume that the system is under the sampled-data feedback control $u(kT+\theta) = Kx(kT)$ for all $0 \le \theta < T$ and $k \ge 0$.

Instead of assuming that the transition of the mode signal σ can be described by a time-homogeneous Markov process [23], let us consider a more practical scenario when the controlled system is under a periodic maintenance [15], as studied in [9]. We in particular consider the situation where the maintenance action is performed whenever the control input is updated. Mathematically speaking, we assume that $\sigma_{kT} = 1$ with probability one for every $k \ge 0$ whatever value $\lim_{t\nearrow kT}\sigma(t)$ is. On the other hand, as for the occurrence of a failure, we assume that the probability of the occurrence of a failure equals $\lambda h + o(h)$ for a constant $\lambda \ge 0$ on any interval with length h, where the function $o(h)$ satisfies $\lim_{h\to 0} o(h)/h$. It is shown in [9] that σ is a regenerative process with a regeneration epoch $r_1 = T$, but neither a Markov process nor a semi-Markov process.

In order to apply Theorem 4.3, we need to compute the various expectations of random matrices appearing in (7). Define t_f as the minimum of T and the first time a failure occurs. Then, we have

$$\begin{aligned}A_d &= e^{AT} + \int_0^{t_f} e^{A(T-\tau)}\,d\tau BK,\\ \mathcal{B}(w) &= \int_0^T e^{A(T-\tau)}w(\tau)\,d\tau,\\ \mathcal{C}(\xi)(t) &= \left(e^{At} + \int_0^{\min(t,t_f)} e^{A(t-\tau)}\,d\tau BK\right)\xi,\\ \mathcal{D}(w)(t) &= \int_0^t e^{A(t-\tau)}w(\tau)\,d\tau.\end{aligned}$$

Also, the adjoints $\mathcal{B}$, $\mathcal{C}$, and $\mathcal{D}$ are given as

$$\begin{aligned}\mathcal{B}^*(\xi)(t) &= e^{A^\top(T-t)}\xi,\\ \mathcal{C}^*(y) &= \int_0^T \left(e^{At} + \int_0^{\min(t,t_f)} e^{A(t-\tau)}\,d\tau BK\right)^\top y(t)\,dt,\\ \mathcal{D}^*(y)(t) &= \int_t^T e^{A^\top(\tau-t)}y(\tau)\,d\tau.\end{aligned}$$

We can then express the expectations of random variables appearing in (6) as single, double, or triple integrals by a straightforward manipulation. The details are omitted due to limitations of space.

We let

$$A = \begin{bmatrix}-0.7 & 0.9\\ -0.6 & -0.7\end{bmatrix},\ B = \begin{bmatrix}0\\ 0.3\end{bmatrix},\ K = \begin{bmatrix}-0.7250 & -0.9521\end{bmatrix}.$$

Since the pair (A, I) is controllable, Assumption 4.1 is satisfied by the same reason as in Remark 4.2. We also note that the above feedback gain K is obtained by solving a linear-quadratic regulator problem for the failure-free case. First, we fix the values of T and λ as $T = 5$ and $\lambda = 0.1$ and observe the behavior of $\sqrt{\gamma^\star_{N,\varepsilon,\delta}}$ for $N = 1,\ldots,15$ with

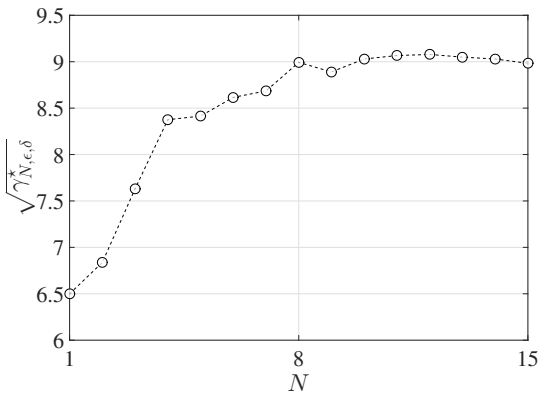

Fig. 2. Sequence $\sqrt{\gamma^\star_{N,\varepsilon,\delta}}$ for $T = 5$ and $\lambda = 0.1$

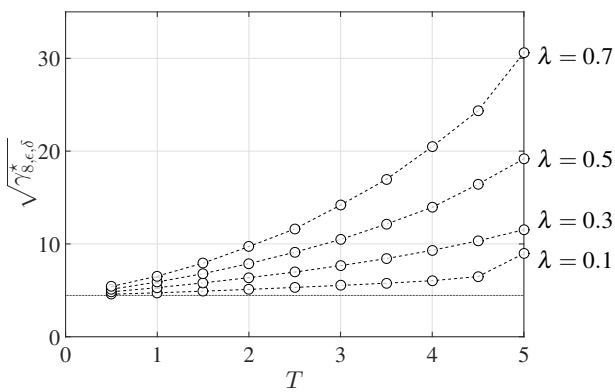

Fig. 3. Upper bounds on $\|\Sigma\|$ versus period T and failure rate λ. The horizontal line indicates the L^2-gain of the system without a failure ($\lambda = 0$).

$\varepsilon = \delta = 10^{-8}$. To compute the expectations of random matrices in (7), we use Monte Carlo integration with sample size 10000. Fig. 2 shows the sequence $\sqrt{\gamma^\star_{N,\varepsilon,\delta}}$ for $N = 1, \ldots, 15$. From the sequence and Theorem 4.3, we would conclude that the L^2-gain $\|\Sigma\|$ is less than 9.1.

Next, we vary the values of T and λ to see their effect on the L^2-gain. In this case, we for simplicity use the value of $\sqrt{\gamma^\star_{8,\varepsilon,\delta}}$ as the upper bound on $\|\Sigma\|$. Fig. 3 shows $\sqrt{\gamma^\star_{8,\varepsilon,\delta}}$ for various values of T and λ. We see that, as T or λ increase, the upper bound on the L^2-gain also increases as expected. We also observe that the upper bounds approach to the H^∞ norm of the failure-free system as T or λ decreases to zero, as expected.

VI. Conclusion and Discussion

In this paper, we analyzed the L^2-gain of regenerative switched linear systems under sampled-data feedback control. We first introduced a lifting operator to transform the closed-loop system into a discrete-time one with independent and identically distributed system parameters. Then, we presented a sequence whose limit inferior upper-bounds the L^2-gain of the system. Each term of the sequence can be computed by solving a linear matrix inequality. We illustrated the obtained result by presenting numerical examples.

The obtained results do not provide lower-bounds on the L^2-gain. In order to obtain the lower-bounds, we would need to show that, when $\varepsilon \to 0$, the feasibility of inequality (8) in Lemma 4.5 is also necessary for $\|\Sigma\|^2 \leq \gamma$. This fact is shown in [22] for finite-dimensional systems under the weak controllability of systems. Future work will be extending this result to the infinite-dimensional problem setting studied in this paper.

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