Linear Quadratic Control for Sampled-data Systems with Stochastic Delays

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***Abstract*— We study optimal control for sampled-data systems with stochastic delays. Assuming that the delays can be modeled by a Markov chain and can be measured by controllers, we design a control law that minimizes an infinite-horizon continuous-time quadratic cost function. The resulting optimal control law can be efficiently computed offline by the iteration of a certain Riccati difference equation. We also obtain sufficient conditions in terms of linear matrix inequalities for stochastic stabilizability and detectability, which are used for the optimal controller design.**

I. Introduction

Time-varying delays often appear in networked communication and real-time implementation of control systems, as surveyed in [1], [2]. Since the variation of such delays can be large, it may not be suitable to use optimal controllers that are constructed for the case of constant delays, e.g., in [3, Chapter 6]. One solution to compensate for time-varying delays is to measure delays by time-stamped messages and exploit these measurements in the control algorithm. An example of this scenario is inter-area power systems [4]. Our goal is to design optimal control laws that employ real-time information about time-varying delays.

Delay-dependent controllers have been widely used for systems with time-varying delays, and the stabilization by such controllers has been studied in [4]–[8] and references therein. For systems with stochastic delays, the design of linear quadratic (LQ) controllers has also been developed for scenarios where delay measurements are available [9]–[12] and where they are not [13]–[17].

Although delay-dependent stabilizing controllers can be obtained offline, the LQ controllers in the earlier studies [9]–[11], [13]–[17] require online computation. Furthermore, those designs of LQ controllers are based on simplistic assumptions that the delays can take only finitely many values or follow independent and identically-distributed distribution, but networked-induced delays generally take a continuum of values in a given interval and are often modeled by a Markov chain (see, e.g., [18], [19] for the Markovian modeling of network-induced delays). A notable exception are found in [12], whose authors present a method for offline computation of LQ controllers for systems having continuous-valued Markovian delays. However, in the controller design of [12], we need to solve a nonlinear vector integral equation called the Riccati integral equation. Moreover, the LQ criterion employed in [12] is defined in the *discrete-time* domain, and therefore fails to incorporate the intersample behavior of the closed-loop system.

In this paper, we study LQ control for sampled-data linear systems and present a computationally efficient method for finding an optimal control law. We reduce our optimal control problem with a *continuous-time* LQ criterion to an LQ problem for a discrete-time Markov jump system whose jumps are modeled by a Markov chain taking values in a general Borel space. This reduction is the first contribution of this study and enables us to deal with Markovian delays that take arbitrary values in a given interval. Furthermore, we can obtain the LQ control law by the iteration of a Riccati difference equation based on the results of [20], and the derived controller is optimal including intersample behaviors.

The proposed method requires stochastic stabilizability and detectability of certain pairs of matrices constructed from a given sampled-data system. However, there has been relatively little work on the test of these properties. Our second contribution is to provide novel sufficient conditions for stochastic stabilizability and detectability in terms of linear matrix inequalities (LMIs). These conditions are inspired by the gridding methods for establishing the stability of networked control systems with aperiodic sampling and time-varying delays in [6], [21], [22].

This paper is organized as follows. In Section II, we introduce the closed-loop system and basic assumptions, and then formulate our LQ problem. In Section III, we reduce our LQ problem to an LQ problem for discrete-time Markov jump systems, which allows us to apply the general results in [20] to the reduced LQ problem in Section IV. Section V is devoted to the derivation of sufficient conditions for stochastic stabilizability and detectability. We illustrate the proposed method with a numerical simulation in Section VI and give concluding remarks in Section VII.

Due to space constraints, all proofs have been omitted and can be found in [23].

Notation: Let $\mathbb{Z}_+$ denote the set of nonnegative integers. For a complex matrix M, let us denote its complex conjugate by M^*. For simplicity, we write a partitioned Hermitian matrix $\begin{bmatrix} Q & W \\ W^* & R \end{bmatrix}$ as $\begin{bmatrix} Q & W \\ \star & R \end{bmatrix}$. For a Borel set $\mathcal{M}$ and a

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σ-finite measure μ on $\mathcal{M}$, we denote by $\mathbb{H}_1^{n,m}$ the space of matrix-valued functions $P(\bullet) : \mathcal{M} \to \mathbb{C}^{n,m}$ that are measurable and integrable in $\mathcal{M}$, and similarly, by $\mathbb{H}_{\sup}^{n,m}$ the space of matrix-valued functions $P(\bullet) : \mathcal{M} \to \mathbb{C}^{n,m}$ that are measurable and essentially bounded in $\mathcal{M}$. For simplicity, we will write $\mathbb{H}_1^n := \mathbb{H}_1^{n,n}$, $\mathbb{H}_{\sup}^n := \mathbb{H}_{\sup}^{n,n}$,

$$\mathbb{H}_{\sup}^{n+} := \{P \in \mathbb{H}_{\sup}^n : P(t) \geq 0,\ \mu\text{-almost everywhere in } \mathcal{M}\},$$

and

$$\widetilde{\mathbb{H}}_{\sup}^{n,m} := \{A \in \mathbb{H}_{\sup}^{n,m} : A(t) \text{ is real for all } t \in \mathcal{M}\}.$$

For a bounded linear operator T on a Banach space, let $r_\sigma(T)$ denote the spectral radius of T.

II. Problem Statement

A. Plant

Consider the following linear continuous-time plant:

$$\Sigma_c : \ \dot{x}(t) = A_c x(t) + B_c u(t), \qquad x(0) = x_0 \tag{1}$$

where $x(t) \in \mathbb{R}^n$ and $u(t) \in \mathbb{R}^m$ are the state and the input of the plant. This plant is connected to a controller through a time-driven sampler with period $h > 0$ and an event-driven zero-order hold.

The state x is measured at each sampling time $t = kh$ ($k \in \mathbb{Z}_+$). The controller receives the sampled state $x(kh)$ with a sensor-to-controller delay τ_k and generates the control value u_k at time $t = kh + \tau_k$. We assume that the lengths of the delays are known to the controller at the time when the sample arrives. One way to measure the delays is to mark every output of the sampler with a time-stamp. The zero-order hold transforms the discrete-time control input u_k to the continuous-time control input

$$u(t) = \begin{cases} u_{-1} & 0 \leq t < \tau_0 \\ u_k & kh + \tau_k \leq t < (k+1)h + \tau_{k+1},\ k \in \mathbb{Z}_+. \end{cases} \tag{2}$$

where u_{-1} is a known deterministic initial condition for the zero-order hold. In (2), we have ignored controller-to-actuator delays. However, if the controller-to-actuator delays τ_{ca} are constant (see, e.g., [1, Section 2.3.2] for this situation) and if we consider the total delays $\tau_k + \tau_{ca}$ instead of the sensor-to-controller delays τ_k, then we can use the proposed method in the presence of the controller-to-actuator delays.

B. Problem Formulation

Throughout this paper, we fix the probability space $(\Omega, \mathcal{F}, P)$ and assume that the delays $\{\tau_k : k \in \mathbb{Z}_+\}$ is smaller than one sampling period and is modeled by a continuous-state Markov chain.

Assumption 2.1 (Delays modeled by Markov chains): *The delay sequence $\{\tau_k : k \in \mathbb{Z}_+\}$ is modeled by a time-homogeneous Markov chain taking values in an interval $\mathcal{M} := [\tau_{\min}, \tau_{\max}] \subset [0, h)$ and having transition probability kernel $\mathcal{G}(\bullet|\bullet)$ with a density $g(\bullet|\bullet)$ with respect to a σ-finite measure μ on $\mathcal{M}$, so that for every $k \in \mathbb{Z}_+$ and every Borel set $\mathcal{B}$ of $\mathcal{M}$, $\mathcal{G}(\tau_{k+1} \in \mathcal{B}|\tau_k = \tau) = \int_{\mathcal{B}} g(t|\tau)\mu(dt)$.*

We assume that the pair of the initial state and delay (x_0, τ_0) has a distribution $\widehat{\mu}$ on $\mathbb{R}^n \times \mathcal{M}$. Define $\widehat{\mu}_{\mathcal{M}}$ by $\widehat{\mu}_{\mathcal{M}}(\mathcal{B}) := \widehat{\mu}(\mathbb{R}^n, \mathcal{B})$ for all Borel sets $\mathcal{B}$ of $\mathcal{M}$. We place the following mild assumption on the initial distribution $\widehat{\mu}$:

Assumption 2.2 (Initial distribution): *The initial distribution $\widehat{\mu}$ of (x_0, τ_0) satisfies A1) $E(\|x_0\|^2) < \infty$ and A2) $\widehat{\mu}_{\mathcal{M}}$ is absolutely continuous with respect to μ.*

The assumption of absolute continuity guarantees the existence of the Radon-Nikodym derivative of $\widehat{\mu}_{\mathcal{M}}$.

Let $\{\mathcal{F}_k : k \in \mathbb{Z}_+\}$ denote a filtration, where $\mathcal{F}_k$ represents the σ-field generated by $\{x(0), \tau_0, \ldots, x(kh), \tau_k\}$. Set $\mathcal{U}_c$ as the class of control inputs $u = \{u_k : k \in \mathbb{Z}_+\}$ such that u_k is $\mathcal{F}_k$ measurable and the system (1) and (2) is mean square stable in the sense that $E(\|x(t)\|^2) \to 0$ as $t \to \infty$ and $E(\|u_k\|^2) \to 0$ as $k \to \infty$ for every initial distribution $\widehat{\mu}$ satisfying Assumption 2.2. For all $u \in \mathcal{U}_c$, we consider the infinite-horizon continuous-time quadratic cost function $\mathcal{J}_c$ defined by

$$\mathcal{J}_c(\widehat{\mu}, u) := E\left(\int_0^\infty x(t)^* Q_c x(t) + u(t)^* R_c u(t) dt\right), \tag{3}$$

where $Q_c \geq 0$ and $R_c > 0$ are weighting matrices. The optimal cost is defined as $\inf_{u \in \mathcal{U}_c} \mathcal{J}_c(\widehat{\mu}, u)$.

In this paper, we study the following LQ problem:

Problem 2.3: *Consider the system (1) and (2), and let Assumptions 2.1 and 2.2 hold. Find an optimal control law $u^{\text{opt}} \in \mathcal{U}_c$ achieving $\mathcal{J}_c(\widehat{\mu}, u^{\text{opt}}) = \inf_{u \in \mathcal{U}_c} \mathcal{J}_c(\widehat{\mu}, u)$ for every distribution $\widehat{\mu}$ on (x_0, τ_0).*

III. Reduction to Discrete-time LQ Problem

In this section, we transform Problem 2.3 to the LQ problem of discrete-time Markov jump linear systems.

Define

$$\xi_k := \begin{bmatrix} x(kh) \\ u_{k-1} \end{bmatrix}. \tag{4}$$

This stochastic process can be viewed as the state of the following stochastic discrete-time linear system

$$\Sigma_d : \ \xi_{k+1} = A(\tau_k)\xi_k + B(\tau_k)u_k, \tag{5}$$

where

$$A(\tau) := \begin{bmatrix} A_d & B_d - \Gamma(\tau) \\ 0 & 0 \end{bmatrix}, \quad B(\tau) := \begin{bmatrix} \Gamma(\tau) \\ I \end{bmatrix} \tag{6}$$

$$A_d := e^{A_c h},\ B_d := \int_0^h e^{A_c s} B_c ds,\ \Gamma(\tau) := \int_0^{h-\tau} e^{A_c s} B_c ds.$$

This discretized system is widely used for the analysis of time-delay systems, e.g., in [6], [9]–[12], [22].

We denote by $\mathcal{U}_d$ the discrete-time counterpart of $\mathcal{U}_c$, defined as the class of control inputs $\{u_k : k \in \mathbb{Z}_+\}$ such that u_k is $\mathcal{F}_k$ measurable and $E(\|\xi_k\|^2) \to 0$ as $k \to \infty$ for every initial distribution $\widehat{\mu}$ satisfying Assumption 2.2. The following result establishes that $\mathcal{U}_d = \mathcal{U}_c$:

Lemma 3.1: Consider the plant Σ_c and its discretization Σ_d. We have that

$$\lim_{k\to\infty} E(\|\xi_k\|^2) = 0$$
$$\Leftrightarrow \quad \lim_{t\to\infty} E(\|x(t)\|^2) = 0 \quad \text{and} \quad \lim_{k\to\infty} E(\|u_k\|^2) = 0.$$

The cost $\mathcal{J}_c$ in (3) can be expressed as the following (discrete-time) summation $\mathcal{J}_c = \sum_{k=0}^{\infty} \mathcal{J}_k$, where

$$\mathcal{J}_k := \int_{kh}^{(k+1)h} x(t)^* Q_c x(t) + u(t)^* R_c u(t) dt.$$

The next lemma shows that each $\mathcal{J}_k$ is a quadratic form on the state and input of the discrete-time system Σ_d in (4).

Lemma 3.2: *For every $k \geq 0$, we have*

$$\mathcal{J}_k = \begin{bmatrix} \xi_k \\ u_k \end{bmatrix}^* \begin{bmatrix} Q(\tau_k) & W(\tau_k) \\ \star & R(\tau_k) \end{bmatrix} \begin{bmatrix} \xi_k \\ u_k \end{bmatrix}, \tag{7}$$

where the matrices Q, W, R are defined by

$$\begin{aligned}
Q(\tau) &:= \begin{bmatrix} Q_{11}(\tau) & Q_{12}(\tau) \\ \star & Q_{22}(\tau) \end{bmatrix} \\
W(\tau) &:= \int_{\tau}^{h} \begin{bmatrix} \alpha(\theta)^* Q_c \gamma(\tau,\theta) \\ \beta(\theta)^* Q_c \gamma(\tau,\theta) - \gamma(\tau,\theta)^* Q_c \gamma(\tau,\theta) \end{bmatrix} d\theta \\
R(\tau) &:= (h-\tau) R_c + \int_{\tau}^{h} \gamma(\tau,\theta)^* Q_c \gamma(\tau,\theta) d\theta \\
Q_{11}(\tau) &:= \int_0^h \alpha(\theta)^* Q_c \alpha(\theta) d\theta \\
Q_{12}(\tau) &:= \int_0^h \alpha(\theta)^* Q_c \beta(\theta) d\theta - \int_{\tau}^{h} \alpha(\theta)^* Q_c \gamma(\tau,\theta) d\theta \\
Q_{22}(\tau) &:= \tau R_c + \int_0^h \beta(\theta)^* Q_c \beta(\theta) d\theta \\
&\quad + \int_{\tau}^{h} \gamma(\tau,\theta)^* Q_c \gamma(\tau,\theta) d\theta \\
&\quad - \int_{\tau}^{h} \left(\beta(\theta)^* Q_c \gamma(\tau,\theta) + \gamma(\tau,\theta)^* Q_c \beta(\theta) \right) d\theta
\end{aligned}$$

and the functions α, β, γ are defined by

$$\alpha(\theta) := e^{A_c \theta}, \quad \beta(\theta) := \int_0^{\theta} e^{A_c s} B_c ds$$
$$\gamma(\tau,\theta) := \int_0^{\theta-\tau} e^{A_c s} B_c ds.$$

As in the linear time-invariant case [24, Section 3.4], we can remove the cross term in (7) by transforming the input u_k into $\bar{u}_k$ as

$$\bar{u}_k = u_k + R(\tau_k)^{-1} W(\tau_k)^* \xi_k.$$

Under this transformation, we can obtain the next lemma.

Lemma 3.3: *Assume that the weighting matrix in Lemma 3.2 satisfies*

$$\begin{bmatrix} Q(\tau) & W(\tau) \\ \star & R(\tau) \end{bmatrix} \geq 0, \quad \tau \in \mathcal{M}. \tag{8}$$

Define

$$\bar{A}(\tau) := A(\tau) - B(\tau) R(\tau)^{-1} W(\tau)^*, \tag{9}$$

and let $C(\tau)$ and $D(\tau)$ be obtained from the following Cholesky decompositions:

$$\begin{aligned}
C(\tau)^* C(\tau) &= Q(\tau) - W(\tau) R(\tau)^{-1} W(\tau)^* \\
D(\tau)^* D(\tau) &= R(\tau).
\end{aligned} \tag{10}$$

Consider the following discrete-time Markov jump system

$$\xi_{k+1} = \bar{A}(\tau_k)\xi_k + B(\tau_k)\bar{u}_k \tag{11}$$

and LQ cost

$$\mathcal{J}_d(\widehat{\mu}, \bar{u}) := \sum_{k=0}^{\infty} E\left(\|C(\tau_k)\xi_k\|^2 + \|D(\tau_k)\bar{u}_k\|^2 \right). \tag{12}$$

If $\bar{u}^{\mathrm{opt}} \in \mathcal{U}_d$ is the optimal control law achieving $\mathcal{J}_d(\widehat{\mu}, u^{\mathrm{opt}}) = \inf_{u \in \mathcal{U}_d} \mathcal{J}_d(\widehat{\mu}, u)$ for the discrete-time Markov jump system (11) and the discrete-time LQ cost (12), then the desired optimal input $u^{\mathrm{opt}} \in \mathcal{U}_c$ for the continuous-time system (1) and the continuous-time LQ cost (3) is given by

$$u_k^{\mathrm{opt}} = \bar{u}_k^{\mathrm{opt}} - R(\tau_k)^{-1} W(\tau_k)^* \xi_k \tag{13}$$

for every $k \in \mathbb{Z}_+$.

Remark 3.4: If, in (8), we have strict positive definiteness

$$\begin{bmatrix} Q(\tau) & W(\tau) \\ \star & R(\tau) \end{bmatrix} > 0, \quad \tau \in \mathcal{M} = [\tau_{\min}, \tau_{\max}] \tag{14}$$

instead of the semidefiniteness, then

$$Q(\tau) - W(\tau) R(\tau)^{-1} W(\tau)^* > 0$$

from the Schur complement formula. Hence $C(\tau)$ and $D(\tau)$, derived from the Cholesky decompositions (10), are continuous with respect to τ. Moreover, $C(\tau)$ and $D(\tau)$ are unique in the following sense: For all $\tau \in [\tau_{\min}, \tau_{\max}]$, there exist unique upper triangular matrices $C(\tau)$ and $D(\tau)$ with strictly positive diagonal entries such that (10) holds; see, e.g., [25] for Cholesky decompositions.

Remark 3.5: Although the authors in [9], [12] also designed LQ optimal controllers by transforming time-delays systems to discrete-time Markov jump systems, only discrete-time LQ criteria were considered.

Remark 3.6: When delays are larger than one sampling period h, we can discretize the system (1) and (2) as in [26]. Under the assumption that certain variables calculated from the delays can be modeled by a Markov chain, we can extend the proposed approach to the case with large delays.

IV. LQ Control for Discrete-time Markov Jump Systems

In this section, we recall the results of [20] for the LQ problem of discrete-time Markov Jump systems.

First we define stochastic stability for discrete-time Markov jump linear systems. Consider the following autonomous system

$$\xi_{k+1} = A(\tau_k)\xi_k, \tag{15}$$

where $A \in \widetilde{\mathbb{H}}^n_{\sup}$ and the sequence $\{\tau_k : k \in \mathbb{Z}_+\}$ is a time-homogeneous Markov chain in a Borel space $\mathcal{M}$. Throughout this section, we assume that the initial distribution $\widehat{\mu}$ of (ξ_0, τ_0) satisfies the following conditions as in Assumption 2.2: A1') $E(\|\xi_0\|^2) < \infty$ and A2') $\widehat{\mu}_{\mathcal{M}}(B) = \widehat{\mu}(\mathbb{R}^n, B)$ is absolutely continuous with respect to μ.

Definition 4.1 (Stochastic stability, [27]): *The autonomous Markov jump linear system (15) is said*

to be stochastically stable if $\sum_{k=0}^{\infty} E(\|\xi_k\|^2) < \infty$ *for any initial distribution* $\widehat{\mu}$ *satisfying A1') and A2').*

Let $g(\bullet|\bullet)$ be the density function with respect to a σ-finite measure μ on $\mathcal{M}$ for the transition of the Markov chain $\{\tau_k : k \in \mathbb{Z}_+\}$. For every $A \in \widetilde{\mathbb{H}}^n_{\sup}$, define the operator $\mathcal{L}_A : \mathbb{H}^n_1 \to \mathbb{H}^n_1$ by

$$\mathcal{L}_A(V)(\bullet) := \int_{\mathcal{M}} g(\bullet|t)A(t)V(t)A(t)^*\mu(dt) \tag{16}$$

We recall a relationship among stochastic stability, the spectral radius $r_\sigma(\mathcal{L}_A)$, and a Lyapunov inequality condition.

Theorem 4.2 ([27]): *The following assertions are equivalent:*

1) *System (15) is stochastically stable.*
2) *The spectral radius* $r_\sigma(\mathcal{L}_A) < 1$*, where* $\mathcal{L}_A$ *is as in (16) above.*
3) *There exists* $S \in \mathbb{H}^{n+}_{\sup}$ *and* $\epsilon > 0$ *such that the following Lyapunov inequality holds for* μ*-almost every* $\tau \in \mathcal{M}$*:*

$$S(\tau) - A(\tau)^* \left(\int_{\mathcal{M}} g(t|\tau)S(t)\mu(dt) \right) A(\tau) \geq \epsilon I. \tag{17}$$

We next provide the definition of stochastic stabilizability and stochastic detectability.

Definition 4.3 (Stochastical stabilizability, [20]): *Let* $A \in \widetilde{\mathbb{H}}^n_{\sup}$ *and* $B \in \widetilde{\mathbb{H}}^{n,m}_{\sup}$*. We say that* (A, B) *is stochastically stabilizable if there exists* $F \in \widetilde{\mathbb{H}}^{m,n}_{\sup}$ *such that* $r_\sigma(\mathcal{L}_{A+BF}) < 1$*, where* $\mathcal{L}_{A+BF}$ *is as in (16) above. In this case,* F *is said to stochastically stabilize* (A, B)*.*

Definition 4.4 (Stochastic detectability, [20]): *Let* $A \in \widetilde{\mathbb{H}}^n_{\sup}$ *and* $C \in \widetilde{\mathbb{H}}^{r,n}_{\sup}$*. We say that* (C, A) *is stochastically detectable if there exists* $L \in \widetilde{\mathbb{H}}^{n,r}_{\sup}$ *such that* $r_\sigma(\mathcal{L}_{A+LC}) < 1$*, where* $\mathcal{L}_{A+LC}$ *is as in (16) above.*

Define the operators $\mathcal{E} : \mathbb{H}^{n+}_{\sup} \to \mathbb{H}^{n+}_{\sup}$, $\mathcal{V} : \mathbb{H}^{n+}_{\sup} \to \mathbb{H}^{m+}_{\sup}$, and $\mathcal{R} : \mathbb{H}^{n+}_{\sup} \to \mathbb{H}^{n+}_{\sup}$ as follows:

$$\begin{aligned}
\mathcal{E}(Z)(\bullet) &:= \int_{\mathcal{M}} Z(t)g(t|\bullet)\mu(dt) \\
\mathcal{V}(Z) &:= D^*D + B^*\mathcal{E}(Z)B \\
\mathcal{R}(Z) &:= C^*C + \bar{A}^*(\mathcal{E}(Z) - \mathcal{E}(Z)B\mathcal{V}(Z)^{-1}B^*\mathcal{E}(Z))\bar{A}.
\end{aligned}$$

Using the above operators, we can obtain the optimal control law for discrete-time Markov jump linear systems from the iteration of a Riccati difference equation.

Theorem 4.5 ([20]): *Consider the Markov jump system (11) with LQ cost* $\mathcal{J}_d$ *in (12). If* $(\bar{A}, B)$ *is stochastically stabilizable and* $(C, \bar{A})$ *is stochastically detectable, then there exists a unique function* $S \in \widetilde{\mathbb{H}}^{n+}_{\sup}$ *such that* $S = \mathcal{R}(S)$ *and* $K := -\mathcal{V}(S)^{-1}B^*\mathcal{E}(S)\bar{A}$ *stochastically stabilizes* $(\bar{A}, B)$*. The optimal control input* $\bar{u}^{\rm opt}_k \in \mathcal{U}_d$ *is given by* $\bar{u}^{\rm opt}_k := K(\tau_k)\xi_k$ *and achieves*

$$\mathcal{J}_d(\widehat{\mu}, u^{\rm opt}) = \inf_{u \in \mathcal{U}_d} \mathcal{J}_d(\widehat{\mu}, u) = E(\xi_0^* S(\tau_0)\xi_0)$$

for every initial distribution $\widehat{\mu}$ *satisfying A1') and A2'). Moreover, we can compute the solution* S *in the following way: For any* $\Xi \in \widetilde{\mathbb{H}}^{n+}_{\sup}$ *and the sequence* $Y^\eta_k = \mathcal{R}(Y^\eta_{k+1})$*,* $Y^\eta_\eta = \Xi$*, we have that* $Y^\eta_0(\tau) \to S(\tau)$ *as* $\eta \to \infty$ μ*-almost everywhere in* $\mathcal{M}$*.*

If the weighting matrix of $\mathcal{J}_d$ is positive definite, that is, if (14) holds, then we do not have to compute the Cholesky decompositions in (10) for the design of the optimal controller. This follows from the definition of the operators $\mathcal{V}$, $\mathcal{R}$ and Proposition 4.6 below.

Proposition 4.6: *Define* $\bar{A}$ *and* C *as in (9) and (10), respectively. The pair* (A, B) *is stochastically stabilizable if and only if* $(\bar{A}, B)$ *is stochastically stabilizable. Moreover, under the positive definiteness of the weighting matrix in (14),* $(C, \bar{A})$ *is stochastically detectable if and only if* $(C^*C, \bar{A})$ *is stochastically detectable.*

V. Sufficient Conditions for Stochastic Stabilizability and Detectability

From the results of Sections 3 and 4, we can obtain the optimal controller under the assumption that (A, B) in (6) is stochastically stabilizable and $(C, \bar{A})$ in Lemmas 3.2 and 3.3 is stochastically detectable. This assumption does not hold in general (and hence the solution of the Riccati difference equation may diverge) even if (A_c, B_c) is controllable and (Q_c, A_c) is observable. The major difficulty in the controller design is to check the stochastic stabilizability and detectability. In this section, we provide sufficient conditions for these properties in terms of LMIs.

For simplicity, we assume that $\{\tau_k : k \in \mathbb{Z}_+\}$ is a time-homogeneous Markov chain taking values in the interval $\mathcal{M} = [\tau_{\min}, \tau_{\max}]$. We can easily extend the results here to more general Borel spaces $\mathcal{M}$ such as bounded sets in $\mathbb{R}^\ell$. Let $g(\bullet|\bullet)$ be the density function with respect to a σ-finite measure μ on $\mathcal{M}$ for the transition of the Markov chain $\{\tau_k : k \in \mathbb{Z}_+\}$. All matrices in this section are complex-valued, but similar results hold for real-valued matrices.

A. Stochastic Stabilizability

We first study the stochastic stabilizability of the pair $A \in \widetilde{\mathbb{H}}^n_{\sup}$ and $B \in \widetilde{\mathbb{H}}^{n,m}_{\sup}$. In our LQ problem, we need to check the stochastic stabilizability of the pair (A, B) defined by (6).

Consider the piecewise-constant feedback gain $K \in \widetilde{\mathbb{H}}^{m,n}_{\sup}$ defined by

$$F(\tau) := F_i \in \mathbb{C}^{m,n}, \qquad \tau \in [s_i, s_{i+1}), \tag{18}$$

where

$$\tau_{\min} =: s_1 < s_2 < \cdots < s_{N+1} := \tau_{\max}. \tag{19}$$

We provide a sufficient condition for the feedback gain K in (18) to stochastically stabilize (A, B), inspired by the gridding approach developed, e.g., in [6], [21], [22].

Theorem 5.1: *For each* $i = 1, \ldots, N$*, define*

$$\beta_i(\tau) := \int_{[s_i, s_{i+1})} g(t|\tau)dt \geq 0, \quad \tau \in [\tau_{\min}, \tau_{\max}],$$

$$s_i^{\rm m} := \frac{s_i + s_{i+1}}{2}, \tag{20}$$

$$\begin{bmatrix} U_i+U_i^* & 0 & U_i\Upsilon_{A,i}+\bar{L}_i\Upsilon_{C,i} & \kappa_{A,i}\begin{bmatrix}U_i & \bar{L}_i\end{bmatrix} & 0 & \Upsilon_{\beta,i}\boldsymbol{S} & \rho_i I \\ \star & \lambda_i I & \lambda_i I & 0 & 0 & 0 & 0 \\ \star & \star & S_i & 0 & 0 & 0 & 0 \\ \star & \star & \star & \lambda_i I & 0 & 0 & 0 \\ \star & \star & \star & \star & \rho_i I & \kappa_{\beta,i}\boldsymbol{S} & 0 \\ \star & \star & \star & \star & \star & \boldsymbol{S} & 0 \\ \star & \star & \star & \star & \star & \star & \rho_i I \end{bmatrix} > 0, \quad \text{where } \boldsymbol{S} := \mathrm{diag}(S_1,\dots,S_N). \tag{A}$$

and

$$\begin{bmatrix}\Gamma_{A,i} & \Gamma_{B,i}\end{bmatrix} := \begin{bmatrix} \sqrt{\beta_1(s_i^{\mathrm{m}})}A(s_i^{\mathrm{m}}) & \sqrt{\beta_1(s_i^{\mathrm{m}})}B(s_i^{\mathrm{m}}) \\ \vdots & \vdots \\ \sqrt{\beta_N(s_i^{\mathrm{m}})}A(s_i^{\mathrm{m}}) & \sqrt{\beta_N(s_i^{\mathrm{m}})}B(s_i^{\mathrm{m}}) \end{bmatrix}.$$

Take a scalar $\kappa_i > 0$ such that for all $\tau \in [s_i, s_{i+1})$,

$$\left\| \begin{bmatrix} \sqrt{\beta_1(\tau)}A(\tau) & \sqrt{\beta_1(\tau)}B(\tau) \\ \vdots & \vdots \\ \sqrt{\beta_N(\tau)}A(\tau) & \sqrt{\beta_N(\tau)}B(\tau) \end{bmatrix} - \begin{bmatrix}\Gamma_{A,i} & \Gamma_{B,i}\end{bmatrix} \right\| \le \kappa_i. \tag{21}$$

If there exist positive definite matrices $R_i > 0$, (not necessarily symmetric) matrices $U_i, \bar{F}_i$, and scalars $\lambda_i > 0$ such that the following LMIs are feasible for all $i = 1\dots,N$:

$$\begin{bmatrix} U_i+U_i^*-R_i & 0 & U_i^*\Gamma_{A,i}^*+\bar{K}_i^*\Gamma_{B,i}^* & \kappa_i\begin{bmatrix}U_i^* & \bar{F}_i^*\end{bmatrix} \\ \star & \lambda_i I & \lambda_i I & 0 \\ \star & \star & \boldsymbol{R} & 0 \\ \star & \star & \star & \lambda_i I \end{bmatrix} > 0, \tag{22}$$

where $\boldsymbol{R} := \mathrm{diag}(R_1,\dots,R_N)$, the pair (A,B) is stochastically stabilizable and the controller (18) with $F_i := \bar{F}_i U_i^{-1}$ stochastically stabilizes (A,B).

B. Stochastic Detectability

Next we study the stochastic detectability of the pair $A \in \widetilde{\mathbb{H}}^n_{\sup}$ and $C \in \widetilde{\mathbb{H}}^{r,n}_{\sup}$. In our LQ problem, we need to check the stochastic detectability of $(C, \bar{A})$ or $(Q - WR^{-1}W^*, \bar{A})$ in Lemmas 3.2 and 3.3.

Define the observer gain $L \in \widetilde{\mathbb{H}}^{n,r}_{\sup}$ as the piecewise-constant function:

$$L(\tau) := L_i \in \mathbb{C}^{n,r} \qquad \tau \in [s_i, s_{i+1}), \tag{23}$$

where the interval $[s_i, s_{i+1})$ is defined as in (19) for each $i = 1,\dots,N$. Note that the positions of the variables K, L are different between $A + BK$ (stabilizability) and $A + LC$ (detectability). Moreover, unlike the case of countable state Markov chains (see, e.g., [28]), the duality of stochastic stabilizability and stochastic detectability is not proved yet for the case of continuous-state Markov chains. Hence we cannot use Theorem 5.1 directly, but the gridding method also provides a sufficient condition for stochastic detectability in terms of LMIs.

Theorem 5.2: *For each $i = 1,\dots,N$, define β_i, s_i^{m} as in Theorem 5.1 and*

$$\begin{bmatrix}\Upsilon_{A,i} \\ \Upsilon_{C,i}\end{bmatrix} := \begin{bmatrix}A(s_i^{\mathrm{m}}) \\ C(s_i^{\mathrm{m}})\end{bmatrix}, \quad \Upsilon_{\beta,i} := \begin{bmatrix}\sqrt{\beta_1(s_i^{\mathrm{m}})}I & \cdots & \sqrt{\beta_N(s_i^{\mathrm{m}})}I\end{bmatrix}.$$

Take scalars $\kappa_{A,i}, \kappa_{\beta,i} > 0$ such that for all $\tau \in [s_i, s_{i+1})$,

$$\begin{aligned} &\left\| \begin{bmatrix}A(\tau) \\ C(\tau)\end{bmatrix} - \begin{bmatrix}\Upsilon_{A,i} \\ \Upsilon_{C,i}\end{bmatrix} \right\| \le \kappa_{A,i} \\ &\left\| \begin{bmatrix}\sqrt{\beta_1(\tau)}I & \cdots & \sqrt{\beta_N(\tau)}I\end{bmatrix} - \Upsilon_{\beta,i} \right\| \le \kappa_{\beta,i}. \end{aligned} \tag{24}$$

If there exists positive definite matrices $S_i > 0$, (not necessarily symmetric) matrices $U_i, \bar{L}_i$, and scalars $\lambda_i, \rho_i > 0$ such that the LMIs in (A) are feasible for all $i = 1,\dots,N$, then (C, A) is stochastically detectable, and the desired observer gain L is given by (23) with $L_i := U_i^{-1}\bar{L}_i$.

VI. Numerical Example

Consider the unstable batch reactor studied in [29], where the system matrices A_c and B_c in (1) are given by

$$A_c := \begin{bmatrix} 1.38 & -0.2077 & 6.715 & -5.676 \\ -0.5814 & -4.29 & 0 & 0.675 \\ 1.067 & 4.273 & -6.654 & 5.893 \\ 0.048 & 4.273 & 1.343 & -2.104 \end{bmatrix}$$

$$B_c := \begin{bmatrix} 0 & 0 \\ 5.679 & 0 \\ 1.136 & -3.146 \\ 1.136 & 0 \end{bmatrix}.$$

This model is widely used as a benchmark example, and for reasons of commercial security, the data were transformed by a change of basis and of time scale [29]. We take the sampling period $h = 0.3$ and the delay interval $[\tau_{\min}, \tau_{\max}] = [0, 0.4h]$.

The delay sequence $\{\tau_k\}$ is modeled by a Markov chain with transition probability kernel $\mathcal{G}$ satisfying

$$\mathcal{G}(\tau_{k+1} \in [s_1, s_2] | \tau_k = s) = \frac{\Phi_s(s_2) - \Phi_s(s_1)}{\Phi_s(\tau_{\max}) - \Phi_s(\tau_{\min})}$$

for every closed interval $[s_1, s_2] \subset [\tau_{\min}, \tau_{\max}]$, where $\Phi_s(x)$ is the probability distribution function of the normal distribution with mean s and standard deviation $1/150$.

The weighting matrices Q_c, R_c for the state and the input in (3) are the identity matrices with compatible dimensions. From Theorem 5.1, we see that (A, B) in (6) is stochastically stabilizable. Additionally, from Theorem 5.2, $(Q - WR^{-1}W^*, \bar{A})$ in Lemmas 3.2 and 3.3 is stochastically detectable. Hence it follows from Theorem 4.5 that we can derive the optimal controller u^{opt} from the iteration of a Riccati difference equation.

Time responses are computed for the initial state $x(0) = [-1 \;\; 1 \;\; -2 \;\; 3]^\top$ and the initial input $u_{-1} = [0 \;\; 0]^\top$. Fig. 1 depicts 10 sample paths of $\|x(t)\|^2$ with initial delay

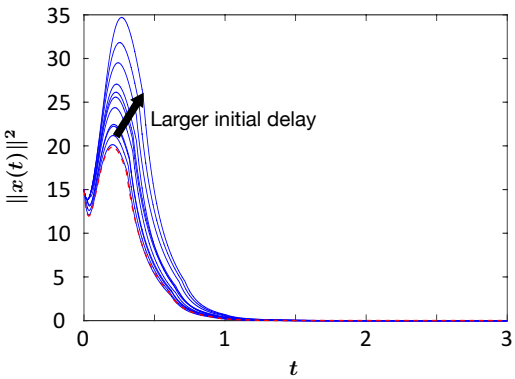

Fig. 1: 10 sample paths of $\|x(t)\|^2$: The blue lines are the time responses with initial state $x_0 = [-1\ 1\ -2\ 3]^\top$ and initial delay uniformly distributed in the interval $[\tau_{\min}, \tau_{\max}] = [0, 0.4h]$. The dotted red line is the time response with no delays, for which we used the (conventional) discrete-time LQ regulator computed with same weighting matrices.

uniformly distributed in the interval $[\tau_{\min}, \tau_{\max}] = [0, 0.4h]$. We observe that the time response with a small initial delay is similar to the response with no delays by the conventional discrete-time LQ regulator with same weighting matrices. Since $u_{-1} = 0$, the system is not controlled until the controller receives the initial state $x(0)$ at time $t = \tau_0$, which affects the time response as the initial delay τ_0 is larger. However, after receiving the sampled state, the optimal controller achieves almost the same decrease rate of $\|x(t)\|^2$ for every initial delay.

VII. Concluding Remarks

We presented an efficient method to obtain optimal controllers for sampled-data systems whose delays can be measured and be modeled by a continuous-state Markov chain. The proposed controller minimizes an infinite-horizon continuous-time quadratic cost function and can be efficiently computed offline by the iteration of a Riccati difference equation. To exploit the results on LQ control for Markov jump systems, we also derived sufficient conditions for stochastic stabilizability and detectability in terms of LMIs. Future work will focus on addressing more general systems by incorporating packet losses, output feedback, and delays with memory of more than one unit.

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