

Stability of Networked Control Systems with Asynchronous Renewal Links: An Impulsive Systems Approach ^{*}

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Abstract

We consider networked control systems in which sensors, actuators, and controller transmit through asynchronous communication links, each introducing independent and identically distributed intervals between transmissions. We model these scenarios through impulsive systems with several reset maps triggered by independent renewal processes, i.e., the intervals between jumps associated with a given reset map are identically distributed and independent of the other jump intervals. For linear dynamic and reset maps, we establish that mean exponential stability is equivalent to the spectral radius of an integral operator being less than one. We also prove that the origin of a non-linear impulsive system is (locally) stable with probability one if its local linearization about the zero equilibrium is mean exponentially stable, which justifies the importance of studying the linear case. The applicability of the results is illustrated through an example using a linearized model of a batch-reactor.

Key words: Networked Control Systems, Stability of Stochastic Systems, Asynchronous Systems, Impulsive Systems.

1 Introduction

Consider a networked control system in which a remote controller communicates with a plant through two independent communication links; e.g., the actuation data is sent from the controller to the plant through a shared wired network and the sensor data is sent from the plant to the controller through a shared wireless network. A reasonable assumption in control over networks utilizing CSMA-type protocols, such as the Ethernet or the Wireless 802.11, is to take the lengths of times needed to gain access to the shared network and to transmit data to be independent and identically distributed

(see, e.g. [5], [20], [18]). In the present paper, we show that this networked control system, and, more generally, networked control systems where several sensors, actuators, and controller, are linked through networks operating asynchronously, can be modeled by impulsive systems of the following form

$$\begin{aligned} \dot{x}(t) &= a(x(t)), \quad t \geq 0, \quad t \neq t_k^\ell, \quad x(0) = x_0, \\ x(t_k^\ell) &= j_\ell(x(t_k^{\ell-})), \quad k \geq 1, \quad \ell \in \mathcal{L} := \{1, \dots, n_\ell\}, \end{aligned} \quad (1)$$

where the duration of the intervals $\{h_k^\ell = t_{k+1}^\ell - t_k^\ell, k \geq 0\}$ between jumps times t_k associated with a given *reset map* j_ℓ , are independent and identically distributed and also independent of the jump intervals $\{h_k^j, j \neq \ell, j \in \mathcal{L}\}$. In (1), the notation $x(t_k^{\ell-})$ indicates the limit from the left of x at the point t_k^ℓ .

We provide stability results for (1), from which one can directly infer stability properties for the networked control systems just described. Our main result establishes that when the dynamic map a and the reset maps j_ℓ are linear, mean exponential stability is equivalent to the spectral radius of an integral operator being less than one, which can be efficiently tested numerically. To prove this result, we first derive conditions for mean exponential stability for (1) with general non-linear dy-

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dynamic and reset maps. When specialized to linear dynamic and reset maps, these stability conditions can be expressed in terms of the existence of a solution to an integro-differential equation, which, in turn, is related to the spectral radius of an integral operator. For the general nonlinear case, we show that the origin of (1) is (locally) stable with probability one if the linearization of (1) about zero equilibrium is mean exponentially stable, which justifies the importance of studying the linear case.

To illustrate the applicability of our results, we consider the linearized model of a batch-reactor that appeared in [2], where we assumed that the sensors transmit in a round-robin fashion through a single shared link. We can now test mean exponential stability in the case where the sensors transmit through two asynchronous links both introducing independent and identically distributed intervals between transmissions.

Several references to related work on networked control systems can be found in [12], including systems with asynchronous data transmissions, e.g., [22], and systems with independent and identically distributed intervals between transmission, e.g., [18]. See also [13], [20], [5], [11]. However, the networked control problem we consider here, and the associated class of impulsive systems, seem to have not been studied in the literature. Stability results for deterministic impulsive systems can be found in [1], [17]. The proof of our results builds upon results for piecewise deterministic systems [10] and the stochastic Lyapunov approach [16].

The remainder of the paper is organized as follows. The connection between (1) and networked control systems is given in Section 2. In Section 3 we state and discuss our main results. Our main result concerning stability of linear impulsive systems, is proved in Section 4; the results concerning mean exponential stability and stability with probability one of (1), are proved in the Appendix. An example is given in Section 5. Final conclusions are given in Section 6. A subset of the results in this paper was presented in the conference paper [3].

Notation: For vectors a_i , (a_1, \dots, a_n) denotes the column vector $[a_1^\top \dots a_n^\top]^\top$. The notation 1_n indicates a vector of n ones.

2 Modeling Networked Control Systems with Impulsive Systems

Consider a continuous-time plant and a controller described by

$$\text{Plant: } \dot{x}_P = f_P(x_P, \hat{u}), \quad y = g(x_P) \quad (2)$$

$$\text{Controller: } \dot{x}_C = f_C(x_C, \hat{y}), \quad u = h(x_C, \hat{y}). \quad (3)$$

The controller is assumed to yield the closed-loop stable when the plant and the controller are directly connected, i.e., $\hat{u}(t) = u(t)$, $\hat{y}(t) = y(t)$. However, sensors, actuators, and controller may be spatially distributed and linked via communication networks, in which case this ideal assumption is not valid. Suppose that there are \tilde{n}_y sensors, among which, n_y are linked to the controller via n_y communication networks, i.e., each sensor transmits through a different network. Then we can partition y as $y = (y_D, y_{\bar{D}}) := (g_D(x_P), g_{\bar{D}}(x_P)) = g(x_P)$ where

$$y_D = (y_1, \dots, y_{n_y}),$$

comprises the measurement signals $y_i \in \mathbb{R}^{s_i}$, $1 \leq i \leq n_y$ of the n_y sensors linked to the controller via a network, and $y_{\bar{D}}$ comprises the measurement signals of the sensors whose connection to the controller is ideal. Therefore, partitioning \hat{y} as $\hat{y} = (\hat{y}_1, \dots, \hat{y}_{n_y}, \hat{y}_{\bar{D}})$, $\hat{y}_i \in \mathbb{R}^{s_i}$, we have $y_{\bar{D}}(t) = \hat{y}_{\bar{D}}(t)$. Likewise, assuming that there are \tilde{n}_u actuators, among which, n_u communicate to the plant via a communication network, we can partition u as $u = (u_D, u_{\bar{D}}) := (h_D(x_C), h_{\bar{D}}(x_C)) = h(x_C)$, where

$$u_D = (u_1, \dots, u_{n_u}),$$

comprises the actuation signals $u_j \in \mathbb{R}^{r_j}$, $1 \leq j \leq n_u$ of the n_u actuators linked to the controller via a network, and $u_{\bar{D}}$ comprises the actuation signals of the actuators whose connection to the controller is ideal. Also here, partitioning \hat{u} as $\hat{u} = (\hat{u}_1, \dots, \hat{u}_{n_u}, \hat{u}_{\bar{D}})$, $\hat{u}_j \in \mathbb{R}^{r_j}$, we have $u_{\bar{D}}(t) = \hat{u}_{\bar{D}}(t)$. Let $n_\ell := n_y + n_u$, and for a given $\ell \in \{1, \dots, n_y\}$, let $\{t_k^\ell, k \geq 0\}$ denote the transmission times of the sensor y_ℓ and for a given $\ell \in \{n_y+1, \dots, n_\ell\}$, let $\{t_k^\ell, k \geq 0\}$ denote the transmission times of the actuator $u_{\ell-n_y}$. Between transmission times we assume that \hat{y}_i and \hat{u}_j remain constant

$$\hat{y}_\ell(t) = \hat{y}_\ell(t_k^\ell), \quad t \in [t_k^\ell, t_{k+1}^\ell), \quad 1 \leq \ell \leq n_y, \quad (4)$$

and

$$\hat{u}_{\ell-n_y}(t) = \hat{u}_{\ell-n_y}(t_k^\ell), \quad t \in [t_k^\ell, t_{k+1}^\ell), \quad 1 \leq \ell - n_y \leq n_u, \quad (5)$$

while at transmission times we have the following update equations

$$\hat{y}_\ell(t_k^\ell) = y_\ell(t_k^{\ell-}), \quad 1 \leq \ell \leq n_y, \quad (6)$$

and

$$\hat{u}_{\ell-n_y}(t_k^\ell) = u_{\ell-n_y}(t_k^{\ell-}), \quad 1 \leq \ell - n_y \leq n_u. \quad (7)$$

We assume that in each of the n_ℓ networks that connect sensors and actuators to the controller, the intervals between transmissions are independent and identically distributed, i.e., $\{h_k^\ell := t_{k+1}^\ell - t_k^\ell\}$ are independent and

identically distributed random variables, and also independent of the transmission intervals in the remaining networks. Defining,

$$e := (e_y, e_u) := (\hat{y}_D - y_D, \hat{u}_D - u_D), \quad (8)$$

and using the fact that we can write $\hat{y} = (e_y + g_D(x_P), g_{\bar{D}}(x_P))$, we can model the networked control system (2)-(8) as an impulsive system taking the form (1), where $x = (x_P, x_C, e)$ is the state;

$$a(x) = \begin{bmatrix} f_P(x_P, (e_u + h_D(x_C, \hat{y}), h_{\bar{D}}(x_C, \hat{y}))) \\ f_C(x_C, (e_y + g_D(x_P), g_{\bar{D}}(x_P))) \end{bmatrix} \quad (9)$$

models the plant, controller, and error dynamics; and

$$j_\ell(x) = (x_P, x_C, \hat{j}_\ell(e_1), \dots, \hat{j}_\ell(e_{n_\ell})), \quad (10)$$

models the transmissions at which the error associated with the transmitting sensor/actuator is reset to zero, i.e., $\hat{j}_\ell(e_i) = 0$, if $i = \ell$, and $\hat{j}_\ell(e_i) = e_i$, if $i \neq \ell$.

3 Main results

We start by providing in Subsection 3.1 a stability result for (1) with general non-linear dynamic and reset maps. Building upon this result, we are able to establish our main result, presented in Subsection 3.2, which provides necessary and sufficient stability conditions when the dynamic and reset maps in (1) are linear. In Subsection 3.3 we relate the stability of the non-linear impulsive system with that of its linearization.

3.1 Non-linear dynamic and reset maps

The maps a and j_ℓ , $\ell \in \mathcal{L}$ are assumed to be differentiable and globally Lipschitz and the origin is an equilibrium point, i.e., $a(0) = 0$ and $j_\ell(0) = 0$, $\forall \ell \in \mathcal{L}$. We denote by n the dimension of the state $x \in \mathbb{R}^n$. We assume that the intervals between jump times $\{h_k^\ell = t_{k+1}^\ell - t_k^\ell, \ell \in \mathcal{L}\}$, are described by a probability density function $f_\ell(t) \geq 0$, with finite support in the interval $[0, \gamma_\ell]$, $\gamma_\ell > 0$. We assume that the $f_\ell(t)$ are differentiable¹ on $(0, \gamma_\ell)$ and we denote the survivor function by

$$r_\ell(s) := \text{Prob}[h_k^\ell > s] = \int_s^{\gamma_\ell} f_\ell(r) dr, k \geq 1, s \in [0, \gamma_\ell],$$

and the hazard rates by

$$\lambda_\ell(\tau_\ell) := \frac{f_\ell(\tau_\ell)}{r_\ell(\tau_\ell)}, \tau_\ell \in B_\ell. \quad (11)$$

¹ We assume differentiability on most functions of interest in the paper to avoid complicating the proofs of our main results.

The system (1) is started at $t = 0$, where it is subsumed that a time $\tau_\ell := -t_0^\ell$ has elapsed since the last jump associated with map ℓ . In other words, we consider that for each reset map $\ell \in \mathcal{L}$, the first jump times t_1^ℓ , satisfy

$$\text{Prob}([t_1^\ell > s]) = \frac{r_\ell(\tau_\ell + s)}{r_\ell(\tau_\ell)}, s \in [0, \gamma_\ell - \tau_\ell], \quad (12)$$

which is the probability that the next jump after $t = 0$ occurs after time s , given that at $t = 0$ a time τ_ℓ has elapsed since the map j was triggered.

We need to define the following auxiliary process

$$v(t) = (v_1(t), \dots, v_{n_\ell}(t)), v_\ell(t) := t - t_{k_\ell}^\ell, v(0) = \tau, \quad (13)$$

where $k_\ell := \max\{k \geq 0 : t_k^\ell \leq t\}$. The process $v(t)$ keeps track of the time elapsed since the last jump associated with each of the reset maps, and therefore at time $t = 0$, $v(0) = \tau = (\tau_1, \dots, \tau_{n_\ell}) = -(t_0^1, \dots, t_0^{n_\ell})$. Note that $v(t) \in B$, where

$$B := B_1 \times \dots \times B_{n_\ell}. \quad (14)$$

and $B_\ell := [0, \gamma_\ell]$, $\ell \in \mathcal{L}$. We also define

$$\mathbf{x}(t) := (x(t), v(t)), \mathbf{x}(0) := \mathbf{x} = (x_0, \tau), \quad (15)$$

which, as we shall see, is a Markov process, although, in general, $x(t)$ is not. In fact, (15) can be constructed as a piecewise deterministic process (cf. Theorem 14 in the Appendix), which allows to establish the following key result in what follows. The proof is given in the Appendix.

Theorem 1 *If V is a differentiable function, such that*

$$\mathbb{E}[\sum_{t_k^\ell \leq n} |V(\mathbf{x}(t_k^\ell)) - V(\mathbf{x}(t_k^{\ell-}))|] < \infty, \forall n \in \mathbb{N} \quad (16)$$

where $\ell \in \mathcal{L}$, then

$$\mathbb{E}[V(\mathbf{x}(t))] = V(\mathbf{x}) + \mathbb{E} \int_0^t \mathfrak{A}V(\mathbf{x}(s)) ds, \forall t \geq 0, \quad (17)$$

where

$$\begin{aligned} \mathfrak{A}V(\mathbf{x}) := & \frac{\partial}{\partial \tau} V(\mathbf{x}) + \\ & \mathfrak{X}_x V(\mathbf{x}) + \sum_{\ell=1}^{n_\ell} \lambda_\ell(\tau_\ell) [V((j_\ell(x), \pi_\ell^0(\tau))) - V(\mathbf{x})], \end{aligned} \quad (18)$$

for $\mathbf{x} = (x, \tau) \in \mathbb{R}^n \times B$, and $\mathfrak{X}_x V(\mathbf{x}) := \sum_{i=1}^n \frac{\partial V(\mathbf{x})}{\partial x_i} a_i(x)$.

□

We consider the following definition of stability for (1).

Definition 2 We say that (1) is mean exponentially stable (MES) if there exists constants $c > 0, \alpha > 0$ such that for any initial condition x_0 , the following holds

$$\mathbb{E}[x(t)^\top x(t)] \leq ce^{-\alpha t} x_0^\top x_0, \forall t \geq 0. \quad (19)$$

□

The following result establishes general conditions for (1) to be MES, providing a stochastic analog of a well known result for deterministic non-linear systems (cf. [14, Th.4.10]). The proof can be found in the Appendix.

Theorem 3 The system (1) is MES if and only if there exists a differentiable positive function $V(\mathbf{x})$ which equals zero at zero, and positive constants c_1, c_2, r such that for every $\mathbf{x} = (x, \tau) \in \mathbb{R}^n \times B$,

$$c_1 \|x\|^2 \leq V(\mathbf{x}) \leq c_2 \|x\|^2 \quad (20)$$

$$\mathfrak{A}V(\mathbf{x}) \leq -r \|x\|^2 \quad (21)$$

□

3.2 Linear dynamic and reset maps

In this subsection, we consider the following linear version of (1)

$$\begin{aligned} \dot{x}(t) &= Ax(t), \quad t \geq 0, \quad t \neq t_k^\ell, \quad x(0) = x_0, \\ x(t_k^\ell) &= J_\ell x(t_k^{\ell-}), \quad k \geq 1, \quad \ell \in \mathcal{L}, \end{aligned} \quad (22)$$

where $\{h_k^\ell = t_{k+1}^\ell - t_k^\ell, k \geq 1\}$ and t_0^ℓ are as described for (1).

Define a variable $\tau = (\tau_1, \dots, \tau_{n_\ell}) \in B$ where $\tau_j \in B_j$. Furthermore, let $\hat{B}_j = B_1 \times \dots \times B_{j-1} \times B_{j+1} \times \dots \times B_{n_\ell}$ and define the two following maps: π_ℓ that removes the component ℓ from the vector τ , i.e., $\pi_\ell : B \mapsto \hat{B}_\ell$, $\pi_\ell(\tau) = (\tau_1, \dots, \tau_{\ell-1}, \tau_{\ell+1}, \dots, \tau_{n_\ell})$; and π_ℓ^0 that sets the component ℓ of the vector τ to zero, i.e.,

$$\pi_\ell^0 : B \mapsto B, \pi_\ell^0(\tau) = (\tau_1, \dots, \tau_{\ell-1}, 0, \tau_{\ell+1}, \dots, \tau_{n_\ell}). \quad (23)$$

Let \mathcal{P}_a be the Cartesian product space of n_ℓ real symmetric matrix functions, each defined in \hat{B}_ℓ , i.e., if $P = (P_1(\hat{\tau}_1), \dots, P_{n_\ell}(\hat{\tau}_{n_\ell})) \in \mathcal{P}_a$, then $P_\ell(\hat{\tau}_\ell)^\top = P_\ell(\hat{\tau}_\ell), \forall \hat{\tau}_\ell \in \hat{B}_\ell$. Sum and multiplication by scalar in \mathcal{P}_a are defined in a natural way and we consider the usual inner product $\langle Q, R \rangle = \sum_{\ell=1}^{n_\ell} \int_{\hat{B}_\ell} Q_\ell(\hat{\tau}_\ell)^\top R_\ell(\hat{\tau}_\ell) d\hat{\tau}_\ell$ for $Q, R \in \mathcal{P}$. Then we define the space $\mathcal{P} \subset \mathcal{P}_a$ as the space of elements $P \in \mathcal{P}_a$ such that $\langle P, P \rangle < \infty$, which

can be shown to be a Hilbert space. Let $\mathfrak{L} : \mathcal{P} \mapsto \mathcal{P}$ be the following integral operator

$$(Q_1, \dots, Q_{n_\ell}) = \mathfrak{L}(P_1, \dots, P_{n_\ell}) \quad (24)$$

where $Q_\ell(\pi_\ell(\tau)) := R(\pi_\ell^0(\tau))$,

$$\begin{aligned} R(\tau) &:= \\ &\sum_{\ell=1}^{n_\ell} \int_0^{\bar{\gamma}} (J_\ell e^{As})^\top P_\ell(\pi_\ell(\tau) + s \mathbf{1}_{n_\ell-1}) J_\ell e^{As} \frac{\hat{r}_\ell(\tau, s)}{\bar{r}_\ell(\tau)} \frac{f_\ell(\tau_\ell + s)}{r_\ell(\tau_\ell)} ds, \end{aligned} \quad (25)$$

$\hat{r}_\ell(\tau, s) := \prod_{j=1, j \neq \ell}^{n_\ell} r_j(\tau_j + s)$, $\bar{r}_\ell(\tau) := \prod_{j=1, j \neq \ell}^{n_\ell} r_j(\tau_j)$, $\bar{\gamma} := \min\{\gamma_\ell - \tau_\ell, \ell \in \mathcal{L}\}$ and $\mathbf{1}_{n_\ell-1}$ is a vector with $n_\ell - 1$ components set to one.

For example for $n_\ell = 1$, (24), (25) take the form

$$Q_1 = \int_0^{\gamma_1} (J_1 e^{As})^\top P_1 J_1 e^{As} f_1(s) ds, \quad (26)$$

where P_1 and Q_1 are symmetric matrices, and in this special case \mathcal{P} is a finite dimensional space since its elements are matrices and not matrix-valued functions. For $n_\ell = 2$, we have

$$\begin{aligned} Q_1(\tau_2) &= \int_0^{\bar{\gamma}_1} (J_1 e^{As})^\top P_1(\tau_2 + s) J_1 e^{As} \frac{r_2(\tau_2 + s)}{r_2(\tau_2)} f_1(s) ds \\ &\quad + \int_0^{\bar{\gamma}_1} (J_2 e^{As})^\top P_2(s) J_2 e^{As} r_1(s) \frac{f_2(\tau_2 + s)}{r_2(\tau_2)} ds, \\ &\quad \tau_2 \in [0, \gamma_2], \\ Q_2(\tau_1) &= \int_0^{\bar{\gamma}_2} (J_2 e^{As})^\top P_2(\tau_1 + s) J_2 e^{As} \frac{r_1(\tau_1 + s)}{r_1(\tau_1)} f_2(s) ds \\ &\quad + \int_0^{\bar{\gamma}_2} (J_1 e^{As})^\top P_1(s) J_1 e^{As} r_2(s) \frac{f_1(\tau_1 + s)}{r_1(\tau_1)} ds. \\ &\quad \tau_1 \in [0, \gamma_1], \end{aligned} \quad (27)$$

where $\bar{\gamma}_1 = \min(\gamma_1, \gamma_2 - \tau_2)$, and $\bar{\gamma}_2 = \min(\gamma_1 - \tau_1, \gamma_2)$.

Since \mathfrak{L} operates in a real space \mathcal{P} , to define its spectral radius we consider the complexification of \mathcal{P} (cf. [15, p. 77]), i.e., the space $\tilde{\mathcal{P}} := \{Q = P + iR : P, R \in \mathcal{P}\}$. For $Q = P + iR \in \tilde{\mathcal{P}}$, one defines $\mathfrak{L}(Q) := \mathfrak{L}(P) + i\mathfrak{L}(R)$. The spectral radius is defined as follows:

$$r_\sigma(\mathfrak{L}) := \max\{|\lambda| : \lambda \in \sigma(\mathfrak{L})\}, \quad (28)$$

where $\sigma(\mathfrak{L}) := \{\lambda : \mathfrak{L} - \lambda I \text{ is not invertible in } \tilde{\mathcal{P}}\}$ denotes the spectrum and I the identity. Note that, defining \mathcal{P} as a real space, and defining the spectral radius of \mathfrak{L} acting on \mathcal{P} as in (28) is generally different from considering \mathcal{P} to be a complex space, where the matrix components P_ℓ of $P = (P_1, \dots, P_{n_\ell}) \in \mathcal{P}$ are self-adjoint matrices, and defining the spectral radius of \mathfrak{L} as usual. We shall use

the first construction since this will allow us to readily use the results for positive operators given in [15] to prove our results in Section 4.

The following is the main result of the paper.

Theorem 4 *The system (22) is MES if and only if $r_\sigma(\mathfrak{L}) < 1$.*

□

The theorem is proved in Section 4. We discuss next how one can numerically compute $r_\sigma(\mathfrak{L})$, and some special cases of the impulsive system, for which one can provide alternative stability conditions to Theorem (4).

Computation of the spectral radius of \mathfrak{L}

One can show that \mathfrak{L} is a compact operator (using, e.g., [8, p. 165, Th. 4.1]) and therefore its spectrum consists either of a finite set of eigenvalues $\lambda : \mathfrak{L}P = \lambda P$ for some $P \in \mathcal{P}$ or a countable set of eigenvalues with no accumulation point other than zero (cf., e.g., [8, p. 117, Th. 2.34]). For simplicity, consider first the case $n_\ell = 2$, where \mathfrak{L} is described by (27). A numerical method to compute $r_\sigma(\mathfrak{L})$ is the following. Take a grid of points in the interval $\bar{\tau}_{1j} \in [0, \gamma_1]$, $1 \leq j \leq n_{d1}$, and $\bar{\tau}_{2j} \in [0, \gamma_2]$, $1 \leq j \leq n_{d2}$ and consider the map obtained by replacing $P(\tau_1), P(\tau_2)$ in (27) by interpolating piecewise polynomials at the points $\{P_1(\bar{\tau}_{2k}), P_2(\bar{\tau}_{1k})\}$, and evaluating the integrals (27) at the grid points to obtain $\{Q_1(\bar{\tau}_{2k}), Q_2(\bar{\tau}_{1k})\}$. This yields a finite rank operator, from the chosen space of piecewise polynomials in \mathcal{P} , to itself, and assuming $n_{d1} = n_{d2} = n_d$, its matrix representation has dimension $2n_d^2 n(n+1)/2$, since $P_i, Q_i, i \in \{2\}$ are symmetric. Denote by L_{n_d} , the finite rank operator obtained. The method just described is known as the collocation method, a special case of the projection method (cf. [8, p. 177]), and one can conclude from the results in [8] that $r_\sigma(L_{n_d}) \rightarrow r_\sigma(\mathfrak{L})$ as $n_d \rightarrow \infty$, for typical piecewise polynomial approximations, such as the trapezoidal approximation.² For general n_ℓ , this method involves computing the spectral radius of a $n_d^{n_\ell} n_\ell n(n+1)/2$ matrix, which means that computing $r_\sigma(\mathfrak{L})$ may require significant computational effort when the number of reset maps is large.

Special cases

A first special case is when there is only one reset map, i.e., $n_\ell = 1$. In this case \mathcal{P} is simply the finite dimensional space of symmetric matrices and \mathcal{L} is the linear

² In fact, from the spectral characterization of compact operators described above, one can conclude from [8, p.232, Th. 5.5 and p.250, Example 5.14]), that the eigenvalues of the compact operators L_{n_d} converge to the eigenvalues of the compact operator \mathfrak{L} which allows us to conclude that $r_\sigma(L_{n_d}) \rightarrow r_\sigma(\mathfrak{L})$.

map $P_1 \mapsto Q_1$ between two finite dimensional space defined in (26). In this case $r_\sigma(\mathfrak{L}) < 1$ reduces to testing if the spectral radius of a matrix is less than one. This condition can be shown to be equivalent to the stability condition given in [5], where the case $n_\ell = 1$ is analyzed using a different approach, based on Volterra equations, which does not appear to generalize to the problem considered in this paper.

A second special case is when the maps A, J_ℓ commute, i.e., $AJ_\ell = J_\ell A$, and $J_\ell J_r = J_r J_\ell, \forall \ell, r \in \mathcal{L}$. In this case, the following result, proved in the Appendix, provides alternative stability conditions to Theorem 4.

Theorem 5 *When the maps A and $J_\ell, \ell \in \mathcal{L}$ commute, the system (22) is MES if*

$$2\bar{\lambda} + \sum_{\ell=1}^{n_\ell} \alpha_\ell < 0, \quad (29)$$

where $\bar{\lambda}$ is the maximum real part of the eigenvalues of A and the α_ℓ are given by

$$\alpha_\ell = \begin{cases} -\infty, & \text{if } r_\sigma(J_\ell^\top \otimes J_\ell^\top) = 0 \\ a \in \mathbb{R} : \int_0^{\gamma_\ell} e^{-as} f_\ell(s) ds = \frac{1}{r_\sigma(J_\ell^\top \otimes J_\ell^\top)}, & \text{otherwise} \end{cases} \quad (30)$$

□

It is important to emphasize, that even for the commuting case, the condition (29) is sufficient but not necessary, as shown in the following example.

Example 6 *Suppose that $A = [0]_{2 \times 2}$, $n_\ell = 1$, $J_1 = \begin{bmatrix} a_1 & 0 \\ 0 & 0 \end{bmatrix}$, and $J_2 = \begin{bmatrix} 0 & 0 \\ 0 & a_2 \end{bmatrix}$, where $a_1 > 1, a_2 > 1$. Then, $\bar{\lambda} = 0$ and, from (30), we can conclude that $\alpha_1 > 0$ and $\alpha_2 > 0$. Thus (29) does not hold. However, since the state remains constant between jump times $t_k^\ell, \ell \in \{1, 2\}$, and component i is reset to zero when the reset map i is triggered, $i \in \{1, 2\}$, we conclude that $\mathbb{E}[x(t)^\top x(t)] = 0, t > \max(\gamma_1, \gamma_2)$ and therefore (22) is MES.*

□

A third special case is when the probability densities f_ℓ correspond to exponential distributions, i.e., $r_\ell(y) = e^{-\beta_\ell y}$. Since the support of the probability density functions is not bounded we assume the following

$$2\bar{\lambda} < \sum_{\ell=1}^{n_\ell} \beta_\ell, \quad (31)$$

where $\bar{\lambda}$ is the maximum real part of the eigenvalues of A . This assumption can be shown to assure that the

expected value of a quadratic function of the state of the system does not go unbounded between jump times. This assertion can be obtained using a similar reasoning to [5, Th. 3]. The next theorem states that, in this case, we can provide stability conditions in the form of LMIs. The result is proved in the appendix.

Theorem 7 *Suppose that $r_\ell(y) = e^{-\beta_\ell y}$ and that (31) holds. Then the system (22) is MES if and only if*

$$\exists P_{>0} : A^\top P + PA + \sum_{\ell=1}^{n_\ell} \beta_\ell (J_\ell^\top P J_\ell - P) < 0 \quad (32)$$

□

3.3 Stability with probability one

The following definition is adapted from [16].

Definition 8 *We say that the origin of the system (1) is (locally) stable with probability one if for every $\rho > 0$ and $\epsilon > 0$ there is a $\delta(\rho, \epsilon) > 0$ such that, if $\|x_0\| < \delta(\rho, \epsilon)$ then*

$$\text{Prob}\left\{ \sup_{\infty > t \geq 0} \|x(t)\| \geq \epsilon \right\} \leq \rho. \quad (33)$$

The following result shows that one can assert stability with probability one of the origin of (1), by establishing mean exponential stability for its linearization, which can be tested by Theorem 4. The proof is provided in the Appendix.

Theorem 9 *If (22) is MES with $A = \frac{\partial}{\partial x} a(x)|_{x=0}$ and $J_\ell = \frac{\partial}{\partial x} j_\ell(x)|_{x=0}$, $\ell \in \mathcal{L}$ where a and j_ℓ are the non-linear maps in (1), then the origin of (1) is stable with probability one.*

□

This theorem allow us to conclude a property analogous to the one proved in [17] stating that a standard periodic sampled-data connection of a non-linear plant and a non-linear controller is locally stable if the sampled-data connection of the linearization of the plant and of the linearization of the controller is stable. In fact, from Theorem 9, we can conclude that in the setup of Section 2, the non-linear networked control system described by (9), (10) is stable with probability one, if the networked control system obtained by replacing f_P , g , f_C , and h , by their local linearizations about the zero equilibrium is mean exponentially stable, which can be tested by Theorem 4.

4 Proof of the Theorem 4

We prove the Theorem 4 through three steps: (i) we show that specializing the stability conditions of Theorem 3 to

the system (22), yields mean square stability conditions for (22) in terms of the existence of a solution to an integro-differential equation; (ii) we establish that these conditions are equivalent to the existence of a solution to a Fredholm equation; (iii) we prove that (ii) is equivalent to the spectral radius of the integral operator of the Fredholm equation being less than one.

(i) *MES for Linear impulsive systems*

The Theorem 3 can be specialized to (22) as follows.

Theorem 10 *The system (22) is MES if and only if for every differentiable functions $Y(\tau)$ and $Z(\tau)$ such that $a_1 I < Y(\tau) < a_2 I$, $\forall \tau \in B$ and $b_1 I < Z(\tau) < b_2 I$, $\forall \tau \in B$ there exists a differentiable function $X(\tau)$, such that $c_1 I < X(\tau) < c_2 I$, $\forall \tau \in B$, and for every $\tau \in B$,*

$$\begin{aligned} \sum_{\ell=1}^{n_\ell} \frac{\partial}{\partial \tau_\ell} X(\tau) + A^\top X(\tau) + X(\tau) A + \\ \sum_{\ell=1}^{n_\ell} \lambda_\ell(\tau_\ell) [J_\ell^\top X(\pi_\ell^0(\tau)) J_\ell - X(\tau) + Z(\tau)] + Y(\tau) = 0, \end{aligned} \quad (34)$$

where π_ℓ^0 is defined by (23), and $a_i, b_i, c_i, i \in \{1, 2\}$ are positive constants.

□

Proof To prove sufficiency we use Theorem 3 and consider the function $V(\mathbf{x}(t)) = x^\top(t) X(v(t)) x(t)$ where $\mathbf{x} = (x(t), v(t))$ and $X(\tau), \tau \in B$ satisfies (34) and the remaining conditions of the theorem. Then from (18) we have that

$$\begin{aligned} \mathfrak{A}(x^\top X(v) x) = x^\top \left[\sum_{\ell=1}^{n_\ell} \frac{\partial}{\partial \tau_\ell} X(v) + A^\top X(v) + X(v) A + \right. \\ \left. + \sum_{\ell=1}^{n_\ell} \lambda_\ell(v_\ell) [J_\ell^\top X(\pi_\ell^0(v)) J_\ell - X(v)] \right] x \end{aligned} \quad (35)$$

Using (34) we obtain

$$\mathfrak{A}(x^\top X(v) x) = -x^\top \left[\left(\sum_{\ell=1}^{n_\ell} \lambda_\ell(v_\ell) Z(v) + Y(v) \right) \right] x \leq -a_1 \|x\|^2,$$

and since $c_1 I \leq X(v) \leq c_2 I$, we have that $c_1 \|x\|^2 \leq V(\mathbf{x}) \leq c_2 \|x\|^2$. Using Theorem 3, applied to (22), we conclude that (22) is MES.

Necessity follows by using the same arguments as in the proof of Theorem 3 and noticing that the function (69) takes the form $V(\mathbf{x}) = x_0^\top X(\tau) x_0$, for $\mathbf{x} = (x_0, \tau) \in \mathbb{R}^n \times B$,

$$X(\tau) = X_1(\tau) + X_2(\tau), \quad (36)$$

where

$$X_1(\tau) = \int_0^{+\infty} \mathbb{E}_\tau[\Phi(t)^\top Y(v(t))\Phi(t)]dt, \quad (37)$$

$$X_2(\tau) = \sum_{k>0, \ell \in \mathcal{L}} \mathbb{E}_\tau[\Phi(t_k^\ell)^\top Z(v(t_k^\ell))\Phi(t_k^\ell)],$$

$\Phi(t)$ is the transition matrix of the system (22), i.e.,

$$\Phi(t) = e^{A(t-t_r^\ell)} J_{\ell_{r-1}} \dots J_{\ell_1} e^{Ah_{1}^{\ell_1}} J_{\ell_0} e^{Ah_0^{\ell_0}}, \quad (38)$$

where $\{\ell_j \in \mathcal{L}, j \geq 0\}$ is the triggered sequence of reset maps, $r = \max\{k : t_k \leq t\}$ and \mathbb{E}_τ emphasizes that expectation subsumes that the process $\Phi(t)$ depends on the initial conditions τ of the process $v(t)$. Since from Theorem 3, $c_1\|x_0\|^2 \leq V(x) \leq c_2\|x_0\|^2$ it follows that $c_1I \leq X(\tau) \leq c_2I$. From [10, p.92, Th.(32.2)] it follows that

$$\mathfrak{A}(x_0^\top X_1(\tau)x_0) = -x_0^\top Y(\tau)x_0$$

and from [10, p.90,91] we have that

$$\mathbb{E}[x_0^\top X_2(\tau)x_0] = \mathbb{E}\left[\int_0^{+\infty} \lambda_T(v(t))x(t)^\top Z(v(t))x(t)\right],$$

where

$$\lambda_T(\tau) := \sum_{j=1}^{n_\ell} \lambda_j(\tau_j), \quad (39)$$

from which one can conclude again from [10, p.92, Th. (32.2)] that

$$\mathfrak{A}(x_0^\top X_2(\tau)x_0) = -\lambda_T(\tau)x_0^\top Z(\tau)x_0.$$

Thus, for every (x_0, τ) , we have

$$\mathfrak{A}(x_0^\top X(\tau)x_0) = -x_0^\top (Y(\tau) + \lambda_T(\tau)Z(\tau))x_0 \quad (40)$$

Computing $\mathfrak{A}(x^\top X(v)x)$ from (18) we obtain (35) which must be equal to (40) when $x = (x_0, \tau)$ is replaced by $x = (x, v)$, from which we conclude (34). \blacksquare

(ii) Fredholm equation (4)

Let \mathcal{U} be the space of elements $(U_1(\hat{\tau}_1), \dots, U_{n_\ell}(\hat{\tau}_{n_\ell})) \in \mathcal{P}$ for which $U_\ell(\hat{\tau}_\ell) \geq 0, \forall \ell \in \mathcal{L}, \forall \hat{\tau}_\ell \in B_\ell$. The space $\mathcal{V} \subset \mathcal{U}$ is defined similarly but requiring $U_\ell(\hat{\tau}_\ell) > 0, \forall \ell \in \mathcal{L}, \forall \hat{\tau}_\ell \in B_\ell$.

Theorem 11 *The system (1) is MES if and only if for every differentiable functions $Y(\tau)$ and $Z(\tau)$ such that $a_1I < Y(\tau) < a_2I, \forall \tau \in B$ and $b_1I < Z(\tau) < b_2I, \forall \tau \in B$, there exists a solution $P \in \mathcal{V}$ to the Fredholm equation*

$$P = \mathfrak{L}(P) + U, \quad (41)$$

where $U = (U_1, \dots, U_{n_\ell})$,

$$U_\ell(\pi_\ell(\tau)) := W(\pi_\ell^0(\tau)), \quad (42)$$

and

$$W(\tau) := \sum_{\ell=1}^{n_\ell} \int_0^{\bar{\gamma}} e^{A^\top s} Z(\tau + s\mathbf{1}_{n_\ell}) e^{As} \frac{\hat{r}_\ell(\tau, s)}{\bar{r}_\ell(\tau)} \frac{f_\ell(\tau_\ell + s)}{r_\ell(\tau_\ell)} ds + \int_0^{\bar{\gamma}} e^{A^\top s} Y(\tau + s\mathbf{1}_{n_\ell}) e^{As} \prod_{\ell=1}^{n_\ell} \frac{r_\ell(\tau_\ell + s)}{r_\ell(\tau_\ell)} ds$$

□

Proof Suppose that (1) is MES and therefore there exists a solution X to (34) given by (36). Let

$$P(\pi_\ell(\tau)) := X(\pi_\ell^0(\tau)), \quad \pi_\ell(\tau) \in B_\ell. \quad (43)$$

We prove next that $X(\tau)$ satisfies

$$X(\tau) = W(\tau) + \sum_{\ell=1}^{n_\ell} \int_0^{\gamma_\ell} (J_\ell e^{As})^\top X(\pi_\ell^0(\tau + s\mathbf{1}_{n_\ell})) J_\ell e^{As} \frac{\hat{r}_\ell(\tau, s)}{\bar{r}_\ell(\tau)} \frac{f_\ell(\tau_\ell + s)}{r_\ell(\tau_\ell)} ds. \quad (44)$$

Then sufficiency follows by directly using (44) in (43), and noticing that the fact that $X(\tau)$, given by (36), satisfies $X(\tau) > c_1I, \forall \tau \in B$, implies that $P \in \mathcal{V}$.

To this effect, we start by conditioning the integrand in (37) on the time of the first jump $t_1 := \min\{t_1^\ell, \ell \in \mathcal{L}\}$,

$$\begin{aligned} \mathbb{E}_\tau[\Phi(t)^\top Y(v(t))\Phi(t)] &= \mathbb{E}_\tau[(\Phi(t)^\top Y(v(t))\Phi(t))\mathbf{1}_{[t_1 > t]}] \\ &+ \sum_{\ell=1}^{n_\ell} \mathbb{E}_\tau[(\Phi(t)^\top Y(v(t))\Phi(t))\mathbf{1}_{C_\ell(t)}] \end{aligned} \quad (45)$$

where $C_\ell(t) = [\min\{t_1^j, j \in \mathcal{L}\} = t_1^\ell = t_1 \leq t] \wedge [t_1^j > t_1, j \neq \ell]$, is the event that jump ℓ is the first to trigger, given that a trigger has occurred at time t . Using (12) the first term on the right hand side of (45) is given by $e^{A^\top t} Y(\tau + t\mathbf{1}_{n_\ell}) e^{At} \prod_{\ell=1}^{n_\ell} \frac{r_\ell(\tau_\ell + t)}{r_\ell(\tau_\ell)}$. Note that for a function $G(t_1)$,

$$\begin{aligned} \mathbb{E}_\tau[G(t_1)\mathbf{1}_{C_\ell(t)}] &= \int_0^t \mathbb{E}[G(s)\mathbf{1}_{[t_1^j > s, j \neq \ell]} | t_1^\ell = s] \frac{f_\ell(\tau_\ell + s)}{r_\ell(\tau_\ell)} ds \\ &= \int_0^t G(s) \frac{\hat{r}_\ell(\tau, s)}{\bar{r}_\ell(\tau)} \frac{f_\ell(\tau_\ell + s)}{r_\ell(\tau_\ell)} ds \end{aligned}$$

and that $\Phi(t) = \hat{\Phi}_\ell(t - t_1)(J_\ell e^{At_1})$ when the transition $\ell \in \mathcal{L}$ is first triggered, where $\hat{\Phi}_\ell(t - t_1)$ is the transition matrix from t_1 to t starting the process at $\pi_\ell^0(\tau + s\mathbf{1}_{n_\ell})$

where π_ℓ^0 is defined by (23). Thus

$$\begin{aligned} \mathbb{E}_\tau[\hat{\Phi}(t)^\top Y(v(t))\hat{\Phi}(t)\mathbf{1}_{C_\ell(t)}] &= \int_0^t (J_\ell e^{As})^\top \dots \\ \mathbb{E}_{\pi_\ell^0(\tau+s1_{n_\ell})}[\hat{\Phi}(t-s)^\top Y(v(t-s))\hat{\Phi}(t-s)] & J_\ell e^{As} \hat{\alpha}_\ell(\tau, s) ds, \end{aligned} \quad (46)$$

where $\alpha_\ell(\tau, s) := \frac{\hat{r}_\ell(\tau, s)}{\bar{r}_\ell(\tau)} \frac{f_\ell(\tau_\ell + s)}{r_\ell(\tau_\ell)}$. By construction of the process

$$\begin{aligned} \mathbb{E}_{\pi_\ell^0(\tau+s1_{n_\ell})}[(\hat{\Phi}_\ell(t-s)^\top Y(v(t-s))\hat{\Phi}_\ell(t-s)] &= \\ \mathbb{E}_{\pi_\ell^0(\tau+s1_{n_\ell})}[\hat{\Phi}(t-s)^\top Y(v(t-s))\hat{\Phi}(t-s)]. \end{aligned} \quad (47)$$

Replacing (47) in (46) and (46) and (45) we obtain ■

$$X_1(\tau) = \sum_{\ell=1}^{n_\ell} X_1^\ell(\tau) + \int_0^{\bar{\gamma}} e^{A^\top t} Y(\tau+t1_{n_\ell}) e^{At} \prod_{\ell=1}^{n_\ell} \frac{r_\ell(\tau_\ell + t)}{r_\ell(\tau_\ell)} dt \quad (48)$$

where

$$X_1^\ell(\tau) = \int_0^\infty \int_0^t \mathbb{E}_{\pi_\ell^0(\tau+s1_{n_\ell})}[(J_\ell e^{As})^\top (\hat{\Phi}(t-s)^\top Y(v(t-s)) \dots \hat{\Phi}(t-s)(J_\ell e^{As})\alpha(\tau, s))] ds dt$$

Changing the order of integration in this last expression we have that (48) can be written as

$$\begin{aligned} X_1(\tau) &= \int_0^{\bar{\gamma}} e^{A^\top t} Y(\tau + s1_{n_\ell}) e^{At} \prod_{\ell=1}^{n_\ell} \frac{r_\ell(\tau_\ell + t)}{r_\ell(\tau_\ell)} dt + \\ &\sum_{\ell=1}^{n_\ell} \int_0^{\bar{\gamma}} (J_\ell e^{As})^\top X_1(\pi_\ell^0(\tau + s1_{n_\ell})) J_\ell e^{As} \frac{\hat{r}_\ell(\tau, s)}{\bar{r}_\ell(\tau)} \frac{f_\ell(\tau_\ell + s)}{r_\ell(\tau_\ell)} ds \end{aligned}$$

With similar computations one can conclude that

$$\begin{aligned} X_2(\tau) &= \sum_{\ell=1}^{n_\ell} \int_0^{\bar{\gamma}} e^{A^\top s} Z(\tau + s1_{n_\ell}) e^{As} \frac{\hat{r}_\ell(\tau, s)}{\bar{r}_\ell(\tau)} \frac{f_\ell(\tau_\ell + s)}{r_\ell(\tau_\ell)} ds + \\ &\sum_{\ell=1}^{n_\ell} \int_0^{\bar{\gamma}} (J_\ell e^{As})^\top X_2(\pi_\ell^0(\tau + s1_{n_\ell})) J_\ell e^{As} \frac{\hat{r}_\ell(\tau, s)}{\bar{r}_\ell(\tau)} \frac{f_\ell(\tau_\ell + s)}{r_\ell(\tau_\ell)} ds \end{aligned}$$

Since $X(\tau) = X_1(\tau) + X_2(\tau)$ adding $X_1(\tau)$ and $X_2(\tau)$ we obtain (44).

Conversely, suppose that there exists a solution $P \in \mathcal{V}$ to (41). Then one can verify that

$$\begin{aligned} X(\tau) &= W(\tau) + \\ &\sum_{\ell=1}^{n_\ell} \int_0^{\bar{\gamma}} (J_\ell e^{As})^\top P_\ell(\hat{\pi}_\ell(\tau) + s1_{n_\ell-1}) J_\ell e^{As} \frac{\hat{r}_\ell(\tau, s)}{\bar{r}_\ell(\tau)} \frac{f_\ell(\tau_\ell + s)}{r_\ell(\tau_\ell)} ds \end{aligned} \quad (49)$$

satisfies all the assumptions of the function $X(\tau)$ of Theorem 10, and therefore (1) is MES. In fact, if there exists

a solution $P \in \mathcal{V}$ to (41) one can obtain an explicit expression for the solution to (41) (cf. Theorem 12), which is given by

$$P = \sum_{i=0}^{\infty} \mathfrak{L}^i(U), \quad (50)$$

where \mathfrak{L}^i denotes the composite operator obtained by applying i times \mathfrak{L} , e.g., $\mathfrak{L}^2(P) = \mathfrak{L}(\mathfrak{L}(P))$ and $\mathfrak{L}^0(P) := P$. From (50) we can conclude that P is bounded and differentiable with respect to τ , since we assume that the f_ℓ are differentiable. Then, it is clear that $X(\tau)$ is bounded, $X(\tau) \geq W(\tau) > c_1 I, \forall \tau \in B$, for some $c_1 > 0$ and (34) can be obtained by direct computation.

(iii) Positive solution of the Fredholm equation

As a prelude to the next result, we note that \mathcal{U} is a cone in the Hilbert space (and hence Banach space, with the usual norm inherit by the inner product) \mathcal{P} , in the sense of [15] since (i) it is closed; (ii) if $U, W \in \mathcal{U}$ then $\alpha_1 U + \alpha_2 W \in \mathcal{U}$ for $\alpha_1 \geq 0$ and $\alpha_2 \geq 0$; and (iii) the set $^3 -\mathcal{U} := \{-P : P \in \mathcal{U}\}$ intersects \mathcal{U} only at the zero vector. Moreover, this cone is reproducing in \mathcal{P} , i.e., if $Z \in \mathcal{P}$, then there exists $U, W \in \mathcal{U}$ such that $Z = U - W$ (take for example, $U_i(\tau) = Z_i(\tau) + \epsilon I$ and $W_i(\tau) = \epsilon I$ for sufficiently large $\epsilon > 0$ such that $P_i(\tau) + \epsilon I > 0$ for all $i \in \{1, \dots, n_\ell\}, \tau \in B$). The operator \mathfrak{L} is a positive operator with respect to \mathcal{U} , i.e., $\mathfrak{L}(U) \in \mathcal{U}$ if $U \in \mathcal{U}$.

Theorem 12 *The equation (41) has a solution $P \in \mathcal{V}$ if and only if $r_\sigma(\mathfrak{L}) < 1$.* □

Note that the main result, Theorem 4, can be concluded from Theorems 11, 12.

Proof Sufficiency follows from the fact that if $r_\sigma(\mathfrak{L}) < 1$ then $P = \sum_{i=0}^{\infty} \mathfrak{L}^i(U)$ exists which is the solution to $P = \mathfrak{L}(P) + U$. Since \mathfrak{L} is a positive operator with respect to \mathcal{U} , P is a summation of $U \in \mathcal{V}$ plus elements in \mathcal{U} . Thus, taking into account the definitions of \mathcal{U} and \mathcal{V} , we conclude that P belongs to \mathcal{V} .

To prove necessity, we start by noticing that it is possible to prove that the dual cone (cf. [7, Sec. 2.6]⁴) of \mathcal{U} can be identified with itself, i.e., using the nomenclature of [7, Sec. 2.6], \mathcal{U} is self-dual. The proof follows similar arguments used to prove that the cone of positive semi-definite matrices is self-dual (cf. [7, p.52]), and is therefore omitted. From [15, p.22, Th. 2.5], we conclude that

³ Recall that addition and multiplications by scalar are defined in a natural way in \mathcal{P} , e.g. if $P = (P_1, \dots, P_{n_\ell}) \in \mathcal{P}$ then $-P := (-P_1, \dots, -P_{n_\ell})$

⁴ The nomenclature used in [15, Ch. 2] is adjoint cone instead of dual cone

the adjoint operator \mathfrak{L}^* is also a positive operator with respect to \mathcal{U} , and using [15, Th. 9.2] which states that a completely continuous positive operator with respect to a reproducing cone has an eigenvalue that equals the spectral radius and an eigenvector that belongs to the solid cone, we conclude that there exists $W \in \mathcal{U}$ (other than the zero element) such that

$$\mathfrak{L}^*(W) = r_\sigma(\mathfrak{L}^*)W. \quad (51)$$

In fact, \mathfrak{L}^* is a compact operator, since \mathfrak{L} is a compact operator (cf. [9, p.178]), which can be concluded from [8, p. 165, Th. 4.1], and a compact operator in a Banach space is completely continuous (cf. [9, p.177]). Suppose that $r_\sigma(\mathfrak{L}) \geq 1$ and (41) has a solution $P \in \mathcal{V}$. Then $r_\sigma(\mathfrak{L}^*) = r_\sigma(\mathfrak{L}) \geq 1$. Taking the inner product on both sides of (41) with $W \in \mathcal{U}$, such that (51) holds, yields

$$\begin{aligned} \langle W, P \rangle &= \langle W, \mathfrak{L}(P) \rangle + \langle W, U \rangle \Leftrightarrow \\ \langle W, P \rangle &= \langle \mathfrak{L}^*(W), P \rangle + \langle W, U \rangle \Leftrightarrow \\ \langle W, P \rangle (1 - r_\sigma(\mathfrak{L}^*)) &= \langle W, U \rangle \end{aligned} \quad (52)$$

Now $\langle W, P \rangle \geq 0$, since $W, P \in \mathcal{U}$. Moreover, one can conclude that $\langle W, U \rangle > 0$, since W is different from the zero element in \mathcal{U} and one can conclude from (42) that $U \in \mathcal{V}$. Thus, from (52) we conclude that $r_\sigma(\mathfrak{L}^*) = r_\sigma(\mathfrak{L}) \geq 1$ leads to a contradiction. ■

5 Batch Reactor

This example considers the control of a linearized model of an open loop unstable two-input two-output batch reactor, controlled by a PI controller. It is a widely used example in networked control (see, e.g., [21], [13]). The plant and controller take the form (2), and (3), with $f_P(x_P, \hat{u}) = A_P x_P + B_P \hat{u}$, $g(x_P) = C_P x_P$, and $f_C(x_C, \hat{y}) = A_C x_C + B_C \hat{y}$, $h(x_C, \hat{y}) = C_C x_C + D_C \hat{y}$. The expressions for (A_P, B_P, C_P) and (A_C, B_C, C_C, D_C) can be found in [2]. The actuator is directly connect $\hat{u}(t) = u(t)$. However, the sensors are linked to the plant through communications networks.

In [2], it is assumed that the outputs are sent in a round-robin fashion through a single shared communication network. When the distribution of the intervals between consecutive transmissions is assumed to be, e.g., uniform with a support γ , we can use the results in [2] to study the stability of this system.

Suppose now that, instead of transmitting the two measurements in a round robin fashion through the same communication network, the two sensors transmit data through two independent communication links. We assume that both links are shared with other users and that the intervals between consecutive transmissions can

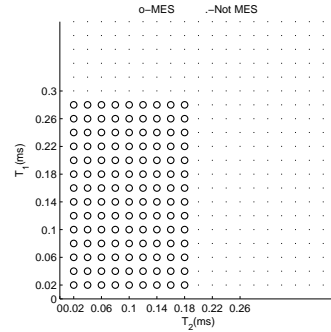


Fig. 1. MES for various values of the support of a uniform distributions of the transmission intervals of two independent links.

be modeled by independent processes with support in the interval $[0, \gamma_\ell]$ for the link associated with the output y_ℓ , $\ell \in \{1, 2\}$. We can cast this system in the framework of Section 2, and use the techniques developed here to study the stability in this latter case.

When two links are used to transmit the measurements of the two sensors, we can use Theorem 4 to investigate the stability of the system as a function of the distributions for the intersampling times on each network. The results obtained are summarized in Figure 1 for the case of uniform distributions with different supports.

For the case of two independent links we take a grid of values of γ_1 and γ_2 , i.e., the support of the distributions between transmissions, and test mean exponential stability of the closed-loop using the numerical method described in Section 3 to compute the spectral radius of the operator defined in the Theorem 4. The results obtained are summarized in the Figure 1. If the distributions of the two links have the same support then stability is preserved for every $\gamma_1, \gamma_2 \in [0, \bar{\gamma}]$, with $\bar{\gamma} = 0.18$. It is interesting to compare this with the case of a round-robin single-link protocol where it was shown in [2] that the maximum support of a uniform distribution for which stability could be guaranteed was $\gamma = 0.11$. With a round-robin protocol, this would lead to a distribution between consecutive samples for the same sensor that is triangular with support 0.22. However, note that in this case the duration of the intervals between transmissions of the two outputs are not independent, and a different approach must be used to assert stability (see [2]). If the two links have different supports one can conclude from the Figure 1 that the mean exponential stability of the closed-loop is lost for a lower value of the support of the distributions associated with the output y_2 than the value of the support associated with the output y_1 .

6 Conclusions

We provided several stability results to impulsive systems with several reset maps triggered at independent and identically distributed spaced times, motivated by their applications to networked control. The main result

establishes that in the case of linear dynamic and reset maps, stability is equivalent to the spectral radius of an integral operator being less than one. A topic for future work is to consider possible atom points and infinite support for the distributions for the intervals between resets. \square

7 Appendix

Proof of Theorems 1) and 3

We start by describing a construction for the process \mathbf{x} , described by (15), which mimics the construction of a piecewise deterministic process, as described in [10, p. 59]. Let $\Omega := \{u_k^1, u_k^2, k \geq 0\}$ where $\{u_k^1\}$ and $\{u_k^2\}$ are mutually independent and identically distributed random variables uniformly distributed in the interval $[0, 1]$. Let also $\phi_x(s, x(t_k))$, $\phi_v(s, v(t_k))$ be the flows at time s of the systems described by $\dot{x}(t) = a(x(t))$ and $\dot{v}(t) = 1$ with initial condition $x(t_k)$ and $v(t_k)$, respectively. Note that $\phi_v(s, v(t_k)) = v(t_k) + s1_{n_\ell}$. Set $k = 0$ and $t_0 = 0$, $\mathbf{x}(t_0) = (x_0, \tau)$, and consider the process $\mathbf{x}(t) = (x(t), v(t))$ obtained by iteratively repeating:

(I) Obtain h_k from

$$h_k = \inf\{t : e^{-\int_0^t \lambda_T(\phi_v(s, v(t_k))) ds} \leq u_k^1\}. \quad (53)$$

where λ_T is described by (39). Set $t_{k+1} = t_k + h_k$, and for $t \in [t_k, t_{k+1})$ make

$$\mathbf{x}(t) = (\phi_x(t - t_k, x(t_k)), \phi_v(t - t_k, v(t_k))) \quad (54)$$

(II) Make $\mathbf{x}(t_{k+1}) = \psi(u_k^2, \mathbf{x}(t_{k+1}^-))$, where

$$\psi(w, (x, \tau)) = (j_\ell(x), \pi_\ell^0(\tau)) \chi_{w \in (\sum_{j=1}^{\ell-1} \frac{\lambda_j(\tau_j)}{\lambda_T(\tau)}, \sum_{j=1}^{\ell} \frac{\lambda_j(\tau_j)}{\lambda_T(\tau)}]}, \quad (55)$$

and $\chi_{x \in A}$ denotes the characteristic function, i.e.,

$$\chi_{w \in A} = \begin{cases} 1 & \text{if } w \in A \\ 0 & \text{if } w \notin A \end{cases}. \quad (56)$$

\square

Remark 13 Note that (53) simply states that

$$\text{Prob}[h_k > s | \mathbf{x}(t_k)] = \prod_{j=1}^{n_\ell} r_j(s + v_j(t_k)), \quad \forall k \geq 0$$

and (55) simply states that

$$\text{Prob}[\mathbf{x}(t_k) = (j_\ell(x(t_k^-)), \pi_\ell^0(v(t_k^-))) | \mathbf{x}(t_k^-)] = \frac{\lambda_\ell(\tau_j(t_k^-))}{\lambda_T(\tau(t_k^-))}.$$

We choose to use the description (I) and (II) to mimic the piecewise deterministic process construction in [10, p. 59], which allows us to use the results from [10].

The next Theorem establishes the connection between (1), (13), and piecewise deterministic processes.

Theorem 14 *The stochastic process $(x(t), v(t))$, described by (1) and (13), can be realized in the probability space Ω and constructed as the piecewise deterministic process defined by the steps (I) and (II).*

\square

Proof For the process $(x(t), v(t))$, described by (1), (13), define $\{t_k \geq 0\}$ with $t_k < t_{k+1}, \forall k \geq 0$ as a set containing the union of all the jump times in (1), i.e., $\{t_k \geq 0\} = \cup_{\ell=1}^{n_\ell} \{t_{r_\ell}^\ell, r_\ell \geq 0\}$, let $\{h_k := t_{k+1} - t_k, k \geq 0\}, h_{-1} := 0$, and consider the following discrete-time process

$$z_k := (h_{k-1}, x(t_k), v(t_k)). \quad (57)$$

There exist a one to one relation between z_k and $\mathbf{x}(t)$, described by (1), and (13). In fact, given a sample path $(x(t), v(t))$ one can identify the jump times t_k by the times at which $v_\ell(t_k) = 0$ for some ℓ , and from these construct h_k and hence z_k . Conversely, from z_k we can obtain h_k and hence t_k , and construct $(x(t), \tau(t))$ from $(x(t_k), v(t_k))$ as

$$(x(t), v(t)) = (\phi_x(t - t_k, x(t_k)), \phi_v(t - t_k, v(t_k))), \quad t_k \leq t < t_{k+1}. \quad (58)$$

Moreover, z_k is a discrete-time Markov process. To see this, it suffices to prove that

$$\text{Prob}(z_{k+1} \in D | z_r, 0 \leq r \leq k) = \text{Prob}(z_{k+1} \in D | z_k), \quad \forall k \geq 0, \quad (59)$$

where D is a measurable set. Since at t_k , a time $\tau_\ell(t_k)$ has elapsed since the last jump associated with the reset map $\ell \in \mathcal{L}$, the time t_{k+1} equals $t_{k+1} = t_k + h_k$, where

$$h_k := \min_{\ell \in \{1, \dots, n_\ell\}} \{w_k^\ell\},$$

and w_k^ℓ are random variables such that $\text{Prob}(w_k^\ell > s) = \frac{r_\ell(v_\ell(t_k) + s)}{r_\ell(v_\ell(t_k))}$. Thus,

$$\text{Prob}(h_{k+1} > s | z_r, 0 \leq r \leq k) = \prod_{\ell=1}^{n_\ell} \frac{r_\ell(v_\ell(t_k) + s)}{r_\ell(v_\ell(t_k))}, \quad (60)$$

Let ξ_{k+1} denote which reset map triggers at t_{k+1} , i.e.,

$$\xi_{k+1} := \text{argmin}_{\ell \in \{1, \dots, n_\ell\}} \{w_k^\ell\}.$$

Then,

$$\begin{aligned} \text{Prob}[\xi_{k+1} = \ell | h_k \in [s, s + \epsilon) \wedge z_r, 0 \leq r \leq k] &= \\ \frac{\text{Prob}[h_k \in [s, s + \epsilon) \wedge \xi_{k+1} = \ell | z_r, 0 \leq r \leq k]}{\sum_{j=1}^{n_\ell} \text{Prob}[h_k \in [s, s + \epsilon) \wedge \xi_{k+1} = j | z_r, 0 \leq r \leq k]} &. \end{aligned} \quad (61)$$

Now,

$$\begin{aligned} \text{Prob}[h_k \in [s, s + \epsilon) \wedge \xi_{k+1} = \ell | z_r, 0 \leq r \leq k] &= \\ \text{Prob}[w_k^\ell \in [s, s + \epsilon) \wedge w_k^j > w_k^\ell, \forall j \neq \ell | z_r, 0 \leq r \leq k] &= \\ \int_s^{s+\epsilon} \prod_{j=1, j \neq \ell}^{n_\ell} \frac{r_j(v_j(t_k) + q)}{r_j(v_j(t_k))} \frac{f_\ell(v_\ell(t_k) + q)}{r_\ell(v_\ell(t_k))} dq & \end{aligned} \quad (62)$$

Replacing (62) in (61), taking the limit as $\epsilon \rightarrow 0$, and dividing the numerator and denominator of the right hand side of (61) by $\prod_{j=1}^{n_\ell} r_j(v_j(t_k + s))$, we obtain

$$\begin{aligned} \text{Prob}[\xi_{k+1} = \ell | h_k = s \wedge z_r, 0 \leq r \leq k] &= \\ \frac{\lambda_\ell(v_\ell(t_k) + s)}{\lambda_T(v(t_k) + s1_{n_\ell})} = \frac{\lambda_\ell(v_\ell(t_{k+1}^-))}{\lambda_T(v(t_{k+1}^-))} &. \end{aligned}$$

where $\lambda_T(v(t_{k+1}^-)) := \sum_{\ell=1}^{n_\ell} \lambda_\ell(v_\ell(t_{k+1}^-))$. Thus, we conclude that

$$\begin{aligned} \text{Prob}[(x(t_k), v(t_k)) = (j_\ell(x(t_k^-)), \pi_\ell^0(v(t_k^-))) | h_k \wedge z_r, 0 \leq r \leq k] &= \\ \frac{\lambda_\ell(v_\ell(t_k^-))}{\lambda_T(v(t_k^-))} & \end{aligned} \quad (63)$$

From (63), and (60), we conclude that

$$\begin{aligned} \text{Prob}[h_k \in [c, d], x(t_{k+1}) \in E_x, v(t_{k+1}) \in E_v | z_r, 0 \leq r \leq k] &= \\ \sum_{\ell=1}^{n_\ell} \left[\int_c^d \chi_{j_\ell(\phi_x(s, \mathbf{x}(t_k))) \in E_x \wedge \phi_v(s, \mathbf{x}(t_k)) \in E_v} \dots \right. & \\ \left. \left(\prod_{j=1, j \neq \ell}^{n_\ell} \frac{r_j(v_j(t_k) + s)}{r_j(v_j(t_k))} \right) \frac{f_\ell(v_\ell(t_k) + s)}{r_\ell(v_\ell(t_k))} \right] ds, & \end{aligned} \quad (64)$$

where $\chi_{(\dots)}$ denotes the characteristic function (56). Note that (64) depends only on $z_k = (h_{k-1}, x(t_k), v(t_k))$ and thus the Markov property (59) holds.

Consider now the piecewise deterministic process defined by steps (I) and (II) and let

$$y_k := (h_{k-1}, x(t_k), v(t_k))$$

where $h_{-1} = 0$, and $h_{k-1}, t_k, x(t_k), v(t_k)$ are now the variables defined in steps (I) and (II). Then, by construction of the process and Remark 13, we immediately obtain that (60), and (63), also hold for this process, and this implies that (64) also holds for this process. Thus y_k is a Markov process with the same transition probability function, i.e., an alternative realization to z_k . Since there

is a one-to-one relation between z_k and the process defined by (1), (13), and there is a one-to-one relation between y_k and the process defined by the piecewise deterministic process construction described by steps (I) and (II), and both processes are completed in the same way between jump times (see (54), (58)), we conclude that the process (1), (13) can be constructed as the piecewise deterministic process specified by steps (I) and (II). \blacksquare

Proof (of Theorem 1) Theorem 14 allows us to apply the results available in [10]. In particular, Theorem 1 follows directly from [10, p.33, (14.17)], [10, p.66, Th. (26.14)] and [10, p. 70, Rem. (26.16)], provided that we can prove that the assumption in [10, p.60, (24.4)] that the expected value of the number of jumps up to a given time t is bounded, which when specialized to the stochastic process $(x(t), v(t))$, described by (1), is equivalent to saying that $\mathbb{E}[\sum_{\ell=1}^{n_\ell} N_\ell(t)] < \infty$, where

$$N_\ell(t) := \max\{k \in \mathbb{N} : t_k^\ell \leq t\}. \quad (65)$$

This is in fact true, since each $N_\ell(t)$ is a renewal process [19] with intervals between renewal times following a probability density function f_ℓ with no atom points, and therefore $\mathbb{E}[N_\ell(t)] < \infty$ (cf. [19, p. 186]). \blacksquare

Proof (of Theorem 3) To prove sufficiency, we use Theorem 1 applied to the function

$$W(\mathbf{x}(t), t) := e^{r_1 t} V(\mathbf{x})$$

where $V(\mathbf{x})$ satisfies (20) and (21), r_1 is a positive constant such that $r_1 > \frac{r}{c_2}$, and it is implicit that the process $(\mathbf{x}(t), t)$ is a piecewise deterministic process if $\mathbf{x}(t)$ is a piecewise deterministic process (cf. [10, p.84]). We need to show that W satisfies (16). Since we assume that a and j_ℓ are globally Lipschitz, we have $\|a(x)\| \leq L_1 \|x\|, \forall x \in \mathbb{R}^n$ and for every $\ell \in \mathcal{L}$, we have $\|j_\ell(x)\| \leq L_2 \|x\|, \forall x \in \mathbb{R}^n$. Thus, between jump times,

$$\|x(t)\|^2 \leq e^{2L_1(t-t_k^\ell)} \|x(t_k)\|^2, t \in [t_k^\ell, t_{k+1}^\ell),$$

(cf. [14, p.107, Exercise 3.17]) and at jump times, $\|x(t_k^\ell)\|^2 \leq (L_2)^2 \|x(t_k^{\ell-})\|^2$. Thus,

$$\mathbb{E}[\|x(t)\|^2] \leq \mathbb{E}[e^{2L_1 t} \prod_{\ell=1}^{n_\ell} (L_2)^{2N_\ell(t)}] \|x_0\|^2 \quad (66)$$

where $N_\ell(t)$ is the number of jumps associated with the reset map ℓ up to the time t , described by (65). We also

have

$$\begin{aligned} & \mathbb{E}\left[\sum_{t_k^\ell \leq n} |W(\mathbf{x}(t_k^\ell)) - W(\mathbf{x}(t_k^{\ell-}))|\right] \\ & \leq \mathbb{E}\left[\sum_{t_k^\ell \leq n} c_2 e^{r_1 n} \|x(t_k^\ell)\|^2\right] \\ & \leq e^{2L_1 n} e^{r_1 n} \|x_0\|^2 \sum_{j=1}^{n_\ell} \sum_{k=0}^{\infty} c_3 (L_2)^{2k} \mathbb{E}[\chi_{t_k^\ell \leq n}] \end{aligned} \quad (67)$$

where we used (20), and (66), and $\chi(\dots)$ denotes the characteristic function (56). Note that $\mathbb{E}[\chi_{t_k^\ell \leq n}] = \text{Prob}[N_\ell(n) \geq k]$. The fact that the right-hand side of (67) is bounded is a direct application of [19, p.186, Th. 3.3.1], and therefore W satisfies (16).

From Theorem 1

$$\mathbb{E}(W(\mathbf{x}(t), t)) = W(\mathbf{x}, 0) + \mathbb{E}\left[\int_0^t \mathfrak{A}W(\mathbf{x}(s), s) ds\right]$$

for an initial condition $\mathbf{x} = (x_0, \tau)$. From [10, p. 84], we can conclude that

$$\mathfrak{A}W(\mathbf{x}(s), s) = r_1 W(\mathbf{x}(s), s) + e^{r_1 s} \mathfrak{A}V(\mathbf{x}(s))$$

and using (21) we obtain

$$\begin{aligned} \mathbb{E}(W(\mathbf{x}(t), t)) & \leq W(\mathbf{x}, 0) + \\ & \mathbb{E}\left[\int_0^t r_1 W(\mathbf{x}(s), s) - r e^{r_1 s} \|x(s)\| ds\right] \end{aligned}$$

Using (20) and interchanging expectation with integral operations, we obtain

$$\mathbb{E}(W(\mathbf{x}(t), t)) \leq W(\mathbf{x}, 0) + (r_1 - \frac{r}{c_2}) \int_0^t \mathbb{E}[W(\mathbf{x}(s), s)] ds.$$

which implies, from the integral form of the Gronwall's inequality [6, Lemma 1], that

$$\begin{aligned} \mathbb{E}[V(\mathbf{x}(t))e^{r_1 t}] & = \mathbb{E}[W(\mathbf{x}(t), t)] \\ & \leq \mathbb{E}[W(\mathbf{x}, 0)]e^{(r_1 - \frac{r}{c_2})t} = V(\mathbf{x})e^{(r_1 - \frac{r}{c_2})t}. \end{aligned} \quad (68)$$

Note that we can apply the Gronwall's inequality since $r_1 - \frac{r}{c_2} > 0$, and this is the reason to work with W , instead of directly using V . From (68), we conclude that

$$\mathbb{E}[V(\mathbf{x}(t))] \leq V(\mathbf{x})e^{-\frac{r}{c_2}t}$$

and using (20), we conclude that:

$$\begin{aligned} \mathbb{E}[\|x(t)\|^2] & \leq \frac{1}{c_1} \mathbb{E}[V(\mathbf{x}(t))] \leq \frac{1}{c_1} V(\mathbf{x})e^{-\frac{r}{c_2}t} \\ & \leq \frac{c_2}{c_1} \|x_0\|^2 e^{-\frac{r}{c_2}t}. \end{aligned}$$

Necessity is obtained by proving that

$$V(\mathbf{x}) := V_1(\mathbf{x}) + V_2(\mathbf{x}) \quad (69)$$

satisfies (20) and (21) where

$$\begin{aligned} V_1(\mathbf{x}) & := \int_0^{+\infty} \mathbb{E}_x[x(s)^\top Y(v(s))x(s)] ds \\ V_2(\mathbf{x}) & := \sum_{\ell=1}^{n_\ell} \sum_{k>0} \mathbb{E}_x[x(t_k^{\ell-})^\top Z(v(t_k^{\ell-}))x(t_k^{\ell-})] ds, \end{aligned}$$

\mathbb{E}_x emphasizes that expectation subsumes that the process starts at an initial condition $\mathbf{x} = (x_0, \tau)$, and $a_1 I < Y(\tau) < a_2 I$, $\forall \tau \in B$, and $b_1 I < Z(\tau) < b_2 I$, $\forall \tau \in B$, are differentiable functions.

To see that $V(\mathbf{x}) \geq c_1 \|x_0\|^2$ denote by t_1 the time of the first jump, which can be from any of the n_ℓ reset maps, and note that $V(\mathbf{x}) \geq a_1 \int_0^{t_1} \mathbb{E}_x[x(s)^\top x(s)] ds + b_1 \mathbb{E}_x[x(t_1^-)^\top x(t_1^-)]$ for any $t_1 \geq 0$. Let L_1 be a Lipchitz constant for a , $\|a(x)\| \leq L_1 \|x\|$, which exists since we assume a is differentiable. Then we have $x(t)^\top x(t) \geq x_0^\top x_0 e^{-2L_1 t}$, $\forall t \geq 0$ (cf. [14, p.107, Exercise 3.17]). Thus $V(\mathbf{x}) \geq c_1 x_0^\top x_0$ where $c_1 = a_1 \int_0^{t_1} e^{-2L_1 t} dt + b_1 e^{-2L_1 t_1} > 0$, $\forall t_1 > 0$.

To see that $V(\mathbf{x}) \leq c_2 \|x_0\|^2$, note that since (1) is MES, we have that $\mathbb{E}_x[x(t)^\top x(t)] \leq a_2 c e^{-\alpha t} x_0^\top x_0$ for some constant $c > 0$. Thus $V_1(\mathbf{x}) \leq a_2 \frac{c}{\alpha} x_0^\top x_0$, and $V_2(\mathbf{x}) \leq k_2 x_0^\top x_0$ where $k_2 := \sum_{\ell=1}^{n_\ell} \sum_{k=1}^{\infty} \mathbb{E}[b_2 c e^{-\alpha t_k^\ell}]$. Note that $\mathbb{E}[e^{-\alpha t_k^\ell}] = \eta_0 \eta^{k-1}$, where $\eta_0 = \mathbb{E}[e^{-\alpha h_k^\ell}]$, $\eta = \mathbb{E}[e^{-\alpha h_{k_1}^\ell}] < 1$, for some $1 \leq k_1 < k$, and therefore $k_2 < \infty$. Thus, $V(\mathbf{x}) \leq c_2 \|x_0\|^2$ where $c_2 = a_2 \frac{c}{\alpha} + k_2$.

It follows from [10, p. 92, Cor. (32.6)] that $V_1(\mathbf{x})$ is differentiable since, as required in [10, p. 92, Cor. (32.6)] $\lambda_T(\tau)$, is continuous. From [10, p.92, Th. (32.2)]

$$\mathfrak{A}V_1(\mathbf{x}) = -x_0^\top Y(\tau)x_0$$

and from [10, p.90,91] we have that

$$\mathbb{E}[V_2(\mathbf{x})] = \mathbb{E}_x\left[\int_0^\infty \lambda_T(v(t))x(t)^\top Z(v(t))x(t) dt\right],$$

from which one can conclude again from [10, p.92, Th. (32.2)] that

$$\mathfrak{A}V_2(\mathbf{x}) = -\lambda_T(\tau)x_0^\top Z(\tau)x_0.$$

and that $V_2(\mathbf{x})$ is differentiable (again by [10, Cor.(32.6)]). Thus $V(\mathbf{x})$ is differentiable and $\mathfrak{A}V(\mathbf{x}) \leq -r x_0^\top x_0$ for $r = a_1$. ■

Proof of Theorems 5) and 7

Proof (of the Theorem 5) From the explicit solution to (22), described by (38), and the commuting property, we obtain that

$$\mathbb{E}[x(t)^\top x(t)] = x_0^\top e^{A^\top t} e^{At} \prod_{\ell=1}^{n_\ell} \mathbb{E}[(J_\ell^\top)^{N_\ell(t)} J_\ell^{N_\ell(t)}] x_0, \quad (70)$$

where $N_\ell(t)$ is described by (65). From [5, Th. 4], we can conclude for some symmetric matrix C and for α_ℓ described by (30), we have that $\mathbb{E}[(J_\ell^\top)^{N_\ell(t)} J_\ell^{N_\ell(t)}] \leq C e^{\alpha_\ell t}$, if $r_\sigma(J_\ell^\top \otimes J_\ell^\top) \neq 0$. If $r_\sigma(J_\ell^\top \otimes J_\ell^\top) = 0$, the state is reset to zero after a finite number of jumps, from which the result follows. \blacksquare

Proof (of the Theorem 7) Due to the memoryless property of the exponential distribution it is ready to establish that in this special case, the stochastic process defined by (22) can be constructed as a stochastic hybrid system with renewal transitions in the sense of [4] with one discrete mode and n_ℓ reset maps characterized by matrices J_ℓ and probability density functions f_ℓ . In [4] we provided necessary and sufficient stability conditions for the stability of stochastic hybrid systems with renewal transitions. Although we restricted ourselves to the case where the transition probability density functions have bounded support, the results provided in [4] can be generalized to the case of unbounded support using the same arguments provided in [4] for the special case where the stochastic hybrid system has only one discrete mode and only one reset map. Then, under the assumption (31), mean exponential stability can be shown to be equivalent to (special case of [4, Th.9.D])

$$\forall_{R>0} \exists_{P>0} : \sum_{\ell=1}^{n_\ell} Q_\ell(P) - P = -R \quad (71)$$

where the solution $P > 0$ is unique,

$$Q_\ell(P) := \int_0^\infty (J_\ell^\top e^{As})^\top P J_\ell^\top e^{As} e^{-\beta_T s} \beta_\ell ds,$$

and $\beta_T := \sum_{j=1}^{n_\ell} \beta_j$. Note that $Q_\ell(P)$ is guaranteed to converge due to (31). It suffices then to prove that (32) is equivalent to (71). To this effect, we start by noticing that

$$(A - \frac{\beta_T}{2} I)^\top Q_\ell(P) + Q_\ell(P) (A - \frac{\beta_T}{2} I) = -\beta_\ell J_\ell^\top P J_\ell \quad (72)$$

from which we conclude

$$\begin{aligned} (A - \frac{\beta_T}{2} I)^\top (\sum_{\ell=1}^{n_\ell} Q_\ell(P)) + (\sum_{\ell=1}^{n_\ell} Q_\ell(P)) (A - \frac{\beta_T}{2} I) \\ = - \sum_{\ell=1}^{n_\ell} \beta_\ell J_\ell^\top P J_\ell. \end{aligned} \quad (73)$$

Due to (31), $A - \frac{\beta_T}{2} I$ has all its eigenvalues in the left-half plane from which we conclude that for every $N > 0$ there exists a unique $M > 0$ such that

$$(A - \frac{\beta_T}{2} I)^\top M + M (A - \frac{\beta_T}{2} I) = -N. \quad (74)$$

To prove that (71) implies (32), choose $R = M$ in (71), where M satisfies (74), and replace $\sum_{\ell=1}^{n_\ell} Q_\ell(P) = P - M$ in (73). This yields

$$A^\top P + P A + \sum_{\ell=1}^{n_\ell} \beta_\ell (J_\ell^\top P J_\ell - P) = -N$$

which implies (32).

To prove that (32) implies (71), note that (32) can be written as

$$\exists_{P>0} : (A^\top - \frac{\lambda_T}{2}) P + P (A - \frac{\lambda_T}{2}) = - \sum_{\ell=1}^{n_\ell} \beta_\ell J_\ell^\top P J_\ell - H \quad (75)$$

for some matrix $H > 0$. Subtracting (75) and (73) one obtains

$$\begin{aligned} (A^\top - \frac{\lambda_T}{2} I) (P - \sum_{\ell=1}^{n_\ell} Q_\ell(P)) + \\ (P - \sum_{\ell=1}^{n_\ell} Q_\ell(P)) (A - \frac{\lambda_T}{2} I) = -H. \end{aligned}$$

From (74) with N replaced by H , we obtain that $P - \sum_{\ell=1}^{n_\ell} Q_\ell(P) = M > 0$ from which we conclude (71). \blacksquare

Proof of Theorem 9

Proof (of the Theorem 9) We rewrite the dynamic and reset maps in (1) as

$$a(x) = Ax + f_e(x), \quad j_\ell(x) = J_\ell x + g_{e\ell}(x), \quad (76)$$

where $f_e(x) := a(x) - Ax$, and $g_{e\ell}(x) := j_\ell(x) - J_\ell x$ are differentiable functions such that

$$\frac{\|f_e(x)\|}{\|x\|} \rightarrow 0, \quad \text{and} \quad \frac{\|g_{e\ell}(x)\|}{\|x\|} \rightarrow 0, \quad \text{as} \quad \|x\| \rightarrow 0 \quad (77)$$

(cf. [14, p.138]). Let $V(\mathbf{x}(t)) = x(t)^\top X(v(t))x(t)$, where $\mathbf{x}(t)$ is described by (15) and $X(\tau), \tau \in B$ satisfies (34) and $c_1 I < X(\tau) < c_2 I, \forall \tau \in B$. Then, there exists $\nu > 0$ such that

$$\mathfrak{A}V(\mathbf{x}) = -r(\mathbf{x}) \leq -d_1 \|\mathbf{x}\|^2, \forall_{\mathbf{x}: \|\mathbf{x}\| \leq \nu} \quad (78)$$

for some $d_1 > 0$, where $\mathfrak{A}V(\mathbf{x})$ is given by (18). This expression (78) can be obtained by directly replacing (76) in (18), and using (77) in a similar fashion to the proof of an analogous result for deterministic non-linear systems (cf. [14, p.139, Th. 4.7]).

Using similar arguments to [16, Th. 1, Ch. 2], we consider the stopped process $\mathbf{x}_S(t) := \mathbf{x}(t \wedge \tau_m)$, where $t \wedge \tau_m := \min(t, \tau_m)$; $\tau_m = \inf\{t : \mathbf{x}(t) \notin B_m\}$ is the first exit time from the set $B_m := \{\mathbf{x} : V(\mathbf{x}) < m\}$; and $m \leq \frac{\nu^2}{c_1}$ is such that $B_m \subseteq \{\mathbf{x} : \|\mathbf{x}\| \leq \nu\}$ and thus, from (78). It is easy to see that $\mathbf{x}(t \wedge \tau_m)$ is a piecewise deterministic process and from (18) and (78),

$$\mathfrak{A}V(\mathbf{x}(t \wedge \tau_m)) = \begin{cases} -r(\mathbf{x}) & \text{if } x \in B_m \\ 0 & \text{otherwise.} \end{cases} \quad (79)$$

Considering (17) for the process \mathbf{x}_S and using (79), we obtain that $\mathbb{E}_{\mathbf{x}}[V(\mathbf{x}(t \wedge \tau_m))] \leq V(\mathbf{x})$, i.e., $V(\mathbf{x}(t \wedge \tau_m))$ is a super-martingale. From this latter fact, and using the fact that $\lim_{t \rightarrow 0} \mathbb{E}_{\mathbf{x}}[V(\mathbf{x}(t))] = V(\mathbf{x})$, (cf. [10, p.77, Th. (27.6)]), we can apply the super-martingale theorem [16, p.26, Eq. 7.4], and conclude that

$$\text{Prob}_{\mathbf{x}}[\sup_{\infty > t \geq 0} V(\mathbf{x}(t \wedge \tau_m)) \geq m] \leq \frac{V(\mathbf{x})}{m} \quad (80)$$

where $\text{Prob}_{\mathbf{x}}$ denotes probability with respect to the Markov process \mathbf{x} started at initial condition $\mathbf{x} = (x_0, \tau)$. Given ϵ, ρ , choose $m = \frac{\min(\nu, \epsilon)^2}{c_1}$, and $\delta = \sqrt{\frac{\rho m}{c_2}}$. Then, for any $\|x_0\| \leq \delta$,

$$\begin{aligned} & \text{Prob}_{\mathbf{x}}[\sup_{\infty > t \geq 0} \|\mathbf{x}(t \wedge \tau_m)\| \geq \epsilon] \leq \\ & \text{Prob}_{\mathbf{x}}[\sup_{\infty > t \geq 0} V(\mathbf{x}(t \wedge \tau_m)) \geq m] \leq \frac{V(\mathbf{x})}{m} \leq \frac{c_2 \|x_0\|^2}{m} \leq \rho \end{aligned} \quad (81)$$

i.e., the origin of (1) is stable with probability one.

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