Output Regulation for Non-square Multi-rate Systems

Duarte Antunes * **João P. Hespanha** ** **Carlos Silvestre** *

* *Dep. of Electrical Eng. and Computer Science, Instituto Superior Técnico, ISR, 1046-001 Lisboa, Portugal.* {dantunes,cjs}@isr.ist.utl.pt

** *Dept. of Electrical and Computer Eng., University of California, Santa Barbara, CA 93106-9560, USA.* hespanha@ece.ucsb.edu

Abstract:
We address the problem of regulating a subset of outputs of a linear time-invariant plant with multi-rate measurements so as to achieve asymptotic tracking of an exogenous signal generated by the free motion of a linear time-invariant system, denoted by exosystem. A solution to this problem is required to yield closed-loop stability and should be such that output regulation is achieved even in the presence of small plant uncertainties and exogenous disturbances also generated by the exosystem. Contrarily to previous works, we propose a solution to the general case where the plant may have more measured outputs than inputs. We show that this solution allows us to solve simultaneous stabilization and output regulation problems that are not possible to solve through the previous works. Besides incorporating an internal model of the exosystem, the key feature of our proposed controller is that it includes a system that blocks signals generated by the exosystem arriving to the controller from the non-regulated outputs.

Keywords: Multi-rate; Internal Model Principle; Periodic Blocking Zeros; Time-varying systems; Output regulation.

1. INTRODUCTION

One of the celebrated problems in automatic control, commonly known as output regulation, is that of controlling the output of a linear time-invariant system so as to achieve asymptotic tracking of an exogenous signal generated by the free motion of a linear time-invariant system, so-called exosystem, while guaranteeing closed loop stability. Several solutions were proposed in the seventies using different approaches (see, e.g., Davison (1972), Wonham and Pearson (1974), Francis and Wonham (1976), Francis (1977)), and all incorporate an internal model of the exosystem, as well as a controller that stabilizes the closed loop. The necessity of incorporating a model of the exosystem in the controller was proved in Francis (1977), and is thereafter known as the internal model principle.

Multi-rate systems appear naturally in multi-input multi-output control, due to the heterogeneity among sensors and possibly among actuators, which are likely to work at different rates. If these rates are sufficiently high, then sensors and actuators may be synchronized to match the rate of the slowest. However in many applications it is desirable, or even necessary, to take advantage of the full capacities of the available sensors and actuators. As an example, taken from Antunes et al. (2010), consider the control of unmanned vehicles, in which the linear position sensor, consisting of a Global Positioning System (GPS), is typically available at a slower rate than the remaining sensors, such as magnetometers and gyroscopes. It is often the case that delaying all the sensor rates to match the available GPS rate makes closed loop stability difficult to achieve, meaning that a multi-rate scheme must be employed.

The blossom of the research in multi-rate systems took place in the early nineties, where many standard problems for LTI systems have been extended to multi-rate systems. For example, structural properties for multi-rate systems including stabilizability and detectability notions are provided in Longhi (1994), a solution to the output pole placement problem can be found in Colaneri et al. (1990), and the linear quadratic gaussian problem for multi-rate systems is solved in Colaneri et al. (1992). The close relation between multi-rate and periodic systems (see, e.g., Bittanti and Colaneri (2000)), entails that these definitions and problem solutions are often based on the concept of lifting (see Meyer and Burrus (1975)), i.e., considering the LTI system obtained by writing the equations of the periodic system along a period.

The output regulation problem for multi-rate systems was addressed in Scattolini and Schiavoni (1993). See also Colaneri et al. (1991) for the special case where the exogenous references are constant signals. However, both in Scattolini and Schiavoni (1993) and Colaneri et al. (1991) the analysis is restricted to square systems, i.e., systems with the same number of inputs and measured outputs, which are also the outputs to be regulated. The solution in Scattolini and Schiavoni (1993) and Colaneri et al. (1991) consists of designing a stabilizing controller

[1] This work was partially supported by the NSF grant # CNS-0720842.

for the system obtained by adding an internal model of the exosystem in *series* with the input of the plant. The peculiarities of the multi-rate systems that prevented the authors from generalizing the solution to non-square systems, are discussed in Scattolini and Schiavoni (1993).

In the present paper, we address the output regulation problem for multi-rate systems for general non-square plants. For example, in the control of unmanned vehicles it is typically the case that some of the variables are required to track constant reference inputs (see Antunes et al. (2010)). The rotorcraft example given in Antunes et al. (2010) has only four inputs, while a twelve component vector comprising the linear and angular velocity vectors, linear position, and Euler angles is available for feedback. Naturally, it is desirable to take advantage of the information provided by the twelve measured components, instead of choosing only four in such a way that the system becomes square, and the solution in Scattolini and Schiavoni (1993) or Colaneri et al. (1991) can be applied. Moreover, a solution to the output regulation problem for non-square systems that achieves closed-loop stability is required when the non-regulated outputs are needed to guarantee the detectability of the plant. We illustrated this in the present paper through an example.

We propose a controller that achieves stability for the closed loop and output regulation for a number of regulated outputs equal to the number of inputs, while taking advantages of the remaining outputs for feedback. As in Scattolini and Schiavoni (1993) and Colaneri et al. (1991), the controller includes an internal model of the exosystem that is placed in series with the input of the plant. The key of our solution is to include a system that blocks signals generated by the exosystem that arrive to the controller from the non-regulated outputs. The concept of a system that blocks signals is made precise, by introducing the notion of blocking zero with respect to a matrix, both for LTI and periodic systems, which generalizes the standard notion of blocking zero for LTI systems (see, e.g., Zhou et al. (1995)). We show that there exists a stabilizing controller with the proposed structure, i.e., incorporating an internal model of the exosystem and a system that blocks signals generated by the exosystem, that achieves output regulation even in the presence of plant uncertainties and disturbances generated by the exosystem. The present paper may be viewed as a follow up work of Antunes et al. (2008), and Antunes et al. (2010), where the output regulation problem was considered for non-square multi-rate systems in the special case where the exogenous reference signals are constant signals. For constant reference signals, the proposed controller structure, is shown in Antunes et al. (2008), and Antunes et al. (2010) not only to be suited for output regulation but also that it can be exploited to implement nonlinear gain-scheduled controllers in such a way that a fundamental property known as linearization property is satisfied.

The remainder of the paper is organized as follows. In Section 2, we provide the problem setup and state the output regulation problem for multi-rate systems. In Section 3 we propose a solution to the problem statement summarized in our main result. The main result is proved in Section 4. An illustrative example showing that our solution allows to solve problems not possible to solve through previous works is provided in Section 5. Conclusions and directions for future work are given in Section 6. In the Appendix we introduce the notion of blocking zero with respect to a matrix, which is a key concept to prove the main results in Section 4.

Notation We denote by I_n and $0_{n\times m}$ the $n \times n$ identity and zero matrices, respectively. We drop the dimensions of these matrices when they are clear from the context. The notation $\mathrm{bdiag}(A_1, \dots, A_n)$ denotes a matrix with blocks $A_i \in \mathbb{R}^{n\times n}$ in the diagonal, i.e.,

$$\mathrm{bdiag}(A_1, \dots, A_q) := \begin{bmatrix} A_1 & 0 & \dots & 0 \\ 0 & A_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & A_q \end{bmatrix}.$$

For a matrix A, $A^\intercal$ denotes its transpose. For dimensionally compatible matrices A and B, we define $(A, B) := [A^\intercal \; B^\intercal]^\intercal$. An eigenvalue of a matrix A is denoted by $\lambda_i(A)$. The nomenclature unstable eigenvalues is used to denote the eigenvalues of a matrix A that have modulus greater than or equal to one. We denote the value at time $t \in \mathbb{R}_{\geq 0}$ of the continuous-time signals $x : \mathbb{R}_{\geq 0} \mapsto \mathbb{R}^n$ by $x(t)$, and the value at time $k \in \mathbb{N}$ of the discrete time signals $x : \mathbb{N} \mapsto \mathbb{R}^n$ by $x[k]$.

2. PROBLEM FORMULATION

We describe first the multi-rate set-up, and the exosystem. Then we state the output regulation problem.

2.1 Multi-Rate Set-up

We consider a continuous-time plant

$$\begin{bmatrix} \dot{x}_P(t) \\ y_P(t) \end{bmatrix} = \begin{bmatrix} A_P & B_P \\ C_P & 0 \end{bmatrix} \begin{bmatrix} x_P(t) \\ u_P(t) \end{bmatrix} + \begin{bmatrix} E_P \\ 0 \end{bmatrix} v_P(t), \quad t \geq 0, \tag{1}$$

where $x_P(t) \in \mathbb{R}^n$ is the state, $u_P(t) \in \mathbb{R}^m$ is the input, and $v_P(t) \in \mathbb{R}^{n_v}$ is a disturbance vector generated by the following system

$$\begin{aligned} \dot{w}_P(t) &= S_V w_P(t), \quad t \geq 0, \\ v_P(t) &= E_V w_P(t). \end{aligned} \tag{2}$$

The output vector $y_P(t) \in \mathbb{R}^p$ can be partitioned into

$$y_P(t) = (y_P^1(t), \dots, y_P^{n_y}(t))$$

where $y_P^i(t) \in \mathbb{R}^{p_i}$, is associated with sensor $i \in \{1, \dots, n_y\}$, and $\sum_{i=1}^{n_y} p_i = p$. The sensors are assumed to operate at different sampling rates, with periods that are rationally related. This model for the measurements can be described by

$$y[k] := (y^1[k], \dots, y^{n_y}[k]),$$

where

$$y^i[k] := \Gamma_k^i y_P^i(t_k), \quad k \geq 0, \quad 1 \leq i \leq n_y, \tag{3}$$

$t_k := kt_s$, for some $t_s > 0$, $\Gamma_k^i = \gamma_k^i I_{p_i}$, and

$$\gamma_k^i := \begin{cases} 1, \text{ if sensor } i \text{ is sampled at } t_k \\ 0, \text{ otherwise} \end{cases}. \tag{4}$$

Let

$$\Gamma_k := \mathrm{bdiag}(\Gamma_k^1, \dots, \Gamma_k^{n_y}).$$

Note that, due to our assumption that the periodicities of the sensors sampling are rationally related, Γ_k is a periodic function of k, i.e., there exists h such that $\Gamma_k = \Gamma_{k+h}, \forall_k$. We can assume that each diagonal entry of Γ_k is nonzero at least once in a period since otherwise a given sensor component would never be sampled and could be disregarded.

The actuator mechanism is assumed to be a standard sample and hold device, and it is assumed to be available for update at every sampling time t_k, i.e.,

$$u(t) = u[k], t \in [t_k, t_{k+1}), \tag{5}$$

where $u[k]$ is the actuation update at time t_k.

Denote the sampled state at times t_k by $x[k] := x(t_k)$. Then we can write (1), (3) and (5) at times t_k as

$$\mathsf{P} := \left\{ \quad \begin{bmatrix} x[k+1] \\ y[k] \end{bmatrix} = \begin{bmatrix} A & B \\ \Gamma_k C & 0 \end{bmatrix} \begin{bmatrix} x[k] \\ u[k] \end{bmatrix} + \begin{bmatrix} B_V \\ 0 \end{bmatrix} v[k] \right. \tag{6}$$

where $A = e^{A_P h}$, $B = \int_0^{t_s} e^{A_P s} ds B_P$, $C = C_P$, $B_V = \int_0^{t_s} e^{A(t_s - s)} E_P E_V e^{S_V s} ds$, and $v[k] = v_P(t_k)$ is generated by the discretization of (2), which is given by

$$\begin{aligned} w_V[k+1] &= e^{S_V t_s} w_V[k], \;\; k \geq 0, \\ v[k] &= E_V w_V[k]. \end{aligned} \tag{7}$$

2.2 Exosystem

Suppose that we further partition the output vector $y[k]$ according to

$$y[k] = \begin{bmatrix} y_m[k] \\ y_r[k] \end{bmatrix} = \begin{bmatrix} \Gamma_{mk} C_m \\ \Gamma_{rk} C_r \end{bmatrix} x[k] \tag{8}$$

where Γ_{mk} and Γ_{rk} are $n_m \times n_m$ and $m \times m$ matrices, respectively, such that $\Gamma_k = \text{bdiag}(\Gamma_{mk}, \Gamma_{rk})$. Note that $n_m = p - m$. The component $y_r[k] \in \mathbb{R}^m$ is a set of outputs that we wish to asymptotically track a reference signal $r[k]$, and $y_m[k] \in \mathbb{R}^{p_m}$ is an additional set of measurements available for feedback. Subsumed in this partition is that $p > m$. A solution to the output regulation problem in the case where $p \leq m$ can be found in Scattolini and Schiavoni (1993).

The reference signal $r[k] \in \mathbb{R}^m$ and the disturbance signal $v[k] \in \mathbb{R}^{n_v}$ are assumed to be generated by the following model, which we denote by *exosystem*,

$$\begin{aligned} w[k+1] &= S w[k] \\ r[k] &= C_R w[k] \\ v[k] &= C_V w[k], \end{aligned} \tag{9}$$

where $w[k] \in \mathbb{R}^{n_w}$. The matrices S and C_V must be compatible with (7), i.e., the same signal $v[k]$ should be generated by (9) and (7). Consider the Jordan canonical form of S i.e.,

$$S = V \text{bdiag}(S_1, \ldots, S_{n_s}) V^{-1} \tag{10}$$

where V is an invertible matrix and the matrices S_j take the form

$$S_j = \begin{bmatrix} \mu_j & 1 & 0 & \ldots \\ 0 & \mu_j & 1 & \ldots \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & \mu_j \end{bmatrix} \in \mathbb{C}^{\kappa_j \times \kappa_j}, \;\; 1 \leq j \leq n_s \tag{11}$$

where $n_s \leq n_w$, and $\sum_{j=1}^{n_s} \kappa_j = n_w$. We assume that:

(S1) $\|\mu_i\| \geq 1$, $\forall_{1 \leq i \leq n_s}$.

(S2) $\mu_i \neq \mu_j$ for $i \neq j$.

To see that (S1) and (S2) introduce no loss of generality, note first that the exosystem can generated signals $r[k]$ and $v[k]$ taking the form

$$\xi[k] = \sum_{j=1}^{n_s} \sum_{l=0}^{\kappa_j - 1} b_{jl} \binom{k}{l} \mu_j^{k-l}, \tag{12}$$

where b_{jl} can be made arbitrarily by properly choosing C_R, C_V, and $w[0]$. Since, as we shall see shortly, output regulation is an asymptotic property, if (S1) would not hold then the disturbance and reference terms in (12) corresponding to stable eigenvalues of S would play no role. If (S2) would not hold, the exosystem would still only be able to generate the same class of reference and disturbance signals (12). Another way of stating (S2) is to say that the characteristic polynomial of S coincides with the minimal polynomial of S, which is a statement more commonly seen in related works addressing the internal model principle (see, e.g., Byrnes and Isidori (2000)).

2.3 Problem Statement

Consider a linear controller for the system (6), i.e., a map $y[k] \mapsto u[k]$ with state $x_c[k] \in \mathbb{R}^{n_c}$. We say that the closed-loop is stable if $(x[k], x_c[k]) \to 0$ as $k \to \infty$, when $r[k] = 0$, $\forall_{k \geq 0}$. Moreover, we say that output regulation is achieved if $(C_r x[k] - r[k]) \to 0$ as $k \to \infty$.

The problem we are interested in this paper can be stated as follows.

Problem 1. Find a linear controller for the system (6) such that the closed loop is stable and output regulation is achieved.

3. MAIN RESULT

The structure of the controller that we propose to solve the Problem 1 is shown in Figure 1.

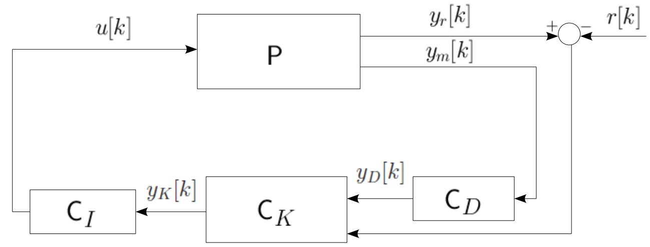

Fig. 1. Proposed controller structure to achieve output regulation; Plant:P; Controller: C_I-internal model, C_D blocking system, C_K stabilizer

The purpose of the systems C_I, C_D and C_K and the rationale behind this structure are briefly explained as follows. Since the controller must provide the adequate input value $u[k]$ such that the output $y_r[k]$ of the linear plant tracks the desired reference signal $r[k]$, the system C_I is such that it is capable of providing such an input to the plant, when the input to C_I is identically zero. Denote by *steady state* the state of the plant at which output regulation is achieved. At steady state, due to the linearity of the plant, the non-regulated output $y_m[k]$ consists of a signal with the same frequency content of the input to the plant. The system C_D has the purpose of yielding a zero output when the steady state signal $y_m[k]$ is applied to its input, while assuring that other signals of interest, do not yield a zero output and thus can still be utilized to control the plant.

Closed loop can be guaranteed if the following condition is met

(G1) The system "seen" by the controller, i.e., obtained by computing the series of C_I, P and C_D, with input $y_K[k]$ and output $(y_D[k], y_r[k] - r[k])$, is detectable and stabilizable.

In fact, from standard results for linear systems, if (G1) holds one can compute a stabilizing controller C_K. Note that, when output regulation is achieved, both the input to the controller and its output are zero.

Before we state our main result, we provide possible choices for C_I and C_D. We shall impose in Section 4 some requirements that C_I and C_D must meet to prove our main output regulation result. We shall see that the systems proposed below meet these requirements. However other choices may exist that satisfy these requirements.

3.1 System C_I

The system C_I is an LTI system described by

$$\mathsf{C}_I := \left\{ \begin{bmatrix} x_I[k+1] \\ y_I[k] \end{bmatrix} = \begin{bmatrix} A_I & B_I \\ C_I & 0 \end{bmatrix} \begin{bmatrix} x_I[k] \\ u_I[k] \end{bmatrix} \right. \tag{13}$$

where $x_I[k] \in \mathbb{R}^{n_I}$, $u_I[k] \in \mathbb{R}^m$, and $y_I[k] \in \mathbb{R}^m$. The system C_I should be capable of providing the adequate input to the plant such that output regulation is achieved. Such an input takes the general form (12). One realization for (13) that achieves this is given by

$$\begin{aligned} A_I &= \mathrm{bdiag}(S, \dots, S) \in \mathbb{R}^{(mn_w)\times(mn_w)}. \\ B_I &= \mathrm{bdiag}(B_J, \dots, B_J) \in \mathbb{R}^{(mn_w)\times m} \\ C_I &= \mathrm{bdiag}(C_J, \dots, C_J) \in \mathbb{R}^{m\times(mn_w)} \end{aligned} \tag{14}$$

where $B_J \in \mathbb{R}^{n_I \times 1}$ is such that (S, B_J) is stabilizable, and $C_J \in \mathbb{R}^{1\times n_w}$ is such that (C_J, S) is detectable. It is straightforward to verify that this implies that (A_I, B_I) is observable, and (C_I, A_I) is detectable, respectively (cf. Proposition 6 below). Note that, the system (13) with matrices (14) incorporates an $m-$fold reduplication of the exosystem (9), in the sense of Francis (1977).

3.2 System C_D

The system C_D , is a linear periodically time-varying system, described by

$$\mathsf{C}_D := \left\{ \begin{bmatrix} x_D[k+1] \\ y_D[k] \end{bmatrix} = \begin{bmatrix} A_{Dk} & B_{Dk} \\ C_{Dk} & D_{Dk} \end{bmatrix} \begin{bmatrix} x_D[k] \\ u_D[k] \end{bmatrix} \right. \tag{15}$$

where $x_D[k] \in \mathbb{R}^{n_D}$, $u_D[k] \in \mathbb{R}^{n_m}$, and $y_D[k] \in R^{n_m}$, and A_{Dk}, B_{Dk}, C_{Dk} and D_{Dk} are $h-$periodic, i.e., e.g., $A_{Dk} = A_{Dk+h}$, $\forall_{k\geq 0}$.

The system C_D has the purpose of blocking the signals that can be generated by the exosystem. By this we mean that the output of (15) is zero for every input generated by the exosystem, which takes the general form (12). The notion is made precise in the Appendix (cf. Definition 17).

One realization for (15) that achieves this is given by

$$\begin{aligned} A_{Dk} &= \mathrm{bdiag}(A_k^1, \dots, A_k^{n_m}) \in \mathbb{R}^{n_m n_w \times n_m n_w}. \\ B_{Dk} &= \mathrm{bdiag}(B_k^1, \dots, B_k^{n_m}) \in \mathbb{R}^{(n_m n_w)\times n_m} \\ C_{Dk} &= \mathrm{bdiag}(C_k^1, \dots, C_k^{n_m}) \in \mathbb{R}^{n_m \times (n_m n_w)} \\ D_{Dk} &= \mathrm{bdiag}(D_k^1, \dots, D_k^{n_m}) \in \mathbb{R}^{n_m \times n_m}, \quad k \geq 0 \end{aligned} \tag{16}$$

where, corresponding to each output $i \in \{1, \dots, \}$, the matrices are given by

$$A_k^i = \begin{bmatrix} 0 & 0 \\ I_{n_w-1} & 0 \end{bmatrix}, \; B_k^i = \begin{bmatrix} 1 \\ 0_{n_w-1} \end{bmatrix}, \; C_k^i = (c^{ik})^\intercal, \; D_k^i = 1, \tag{17}$$

if the output i is sampled at t_k, and

$$A_k^i = I_{n_w}, \; B_k^i = 0_{n_w\times 1}, \; C_k^i = 0_{1\times n_w} \; D_k^i = 0, \tag{18}$$

otherwise, where c^{ik} is a $h-$periodic vector, i.e., $c^{ik} = c^{i(k+h)}$ which is described shortly. The matrices A_k^i and B_k^i, which correspond to the ith component of the output vector $y_m[k]$, are such that the system C_D holds the last n_w sampled values of the output i when k is sufficiently large. A condition for k that assures this is $k \geq n_w h$. In fact, from (17) we see that if the output is sampled at t_k then the new measurement is introduced in the state while the least recent is dropped. From (18) we see that if the output is not sampled at t_k the system C_D holds the previous state. The matrices C_k^i and D_k^i are such that the output is zero when steady state is achieved, in which case $y_m[k]$ is a signal taking the form (12). This will be shown in Proposition 7. The n_w dimensional periodic vector c^{ik} can be determined as follows. Since the c^{ik} are $h-$periodic we need only to specify c^{ik} along a period, i.e., e.g., for $k \in \{1, \dots, h\}$, and for values for which $\gamma_k^i = 1$ (otherwise $C_k^i = 0$), where γ_k^i is given by (4). Let $\lfloor k \rfloor$ be the remainder of the division of k by h if $k \geq 1$, i.e., e.g., $\lfloor k+1 \rfloor = k+1$ if $1 \leq k \leq h-1$, and $\lfloor k+1 \rfloor = 1$ if $k = h$. Moreover if $k \leq 0$ use the same notation to denote $\lfloor k \rfloor := \lfloor k + rh \rfloor$ for some $r \in \mathbb{N}$ such that $k + rh \geq 1$. For each $k \in \{1, \dots, h\}$ such that $\gamma^{ik} = 1$, define a set of $n_w + 1$ indexes $\{\tau_l^{ik}, 0 \leq l \leq n_w\}$ by

$$\tau_l^{ik} = \begin{cases} 0, \text{ if } l = 0, \\ \tau_{l-1}^{ik} + \min\{k_1 > 0 : \gamma^i_{\lfloor k - \tau_{l-1}^{ik} - k_1 \rfloor} = 1\}, \\ \qquad\qquad\qquad \text{if } 1 \leq l \leq n_w. \end{cases}$$

Define the following set of matrices

$$M_k^i = \begin{bmatrix} N_k^i(\mu_1) \\ \vdots \\ N_k^i(\mu_{n_s}) \end{bmatrix} \in \mathbb{C}^{n_w \times n_w}$$

where $k \in \{1, \dots, h\}$, and

$$N_k^i(\mu_j) := \begin{bmatrix} \mu_j^{-\tau_1^{ik}} & \mu_j^{-\tau_2^{ik}} & \dots & \mu_j^{-\tau_{n_w}^{ik}} \\ \tau_1^{ik} \mu_j^{-\tau_1^{ik}} & \tau_2^{ik} \mu_j^{-\tau_2^{ik}} & \dots & \tau_{n_w}^{ik} \mu_j^{-\tau_{n_w}^{ik}} \\ \vdots & \vdots & \vdots & \vdots \\ (\tau_1^{ik})^{\kappa_j} \mu_j^{-\tau_1^{ik}} & (\tau_2^{ik})^{\kappa_j} \mu_j^{-\tau_2^{ik}} & \dots & (\tau_{n_w}^{ik})^{\kappa_j} \mu_j^{-\tau_{n_w}^{ik}} \end{bmatrix}.$$

Define also the set of vectors

$$b_k^i = \begin{bmatrix} d_1 \\ \vdots \\ d_{n_s} \end{bmatrix} \in \mathbb{R}^{n_w \times 1}$$

where $k \in \{1, \ldots, h\}$ and

$$d_j := \begin{bmatrix} 1 & 0_{1\times(\kappa_j-1)} \end{bmatrix}^\intercal, 1 \leq j \leq n_s.$$

We make the following assumption:

$$M_k^i \text{ is invertible for every } k \in \{1, \ldots, h\},\ 1 \leq i \leq n_y. \tag{19}$$

Take $c^{ik} = [c_1^{ik} \ \ldots \ c_{n_w}^{ik}]$ as the solution to

$$M_k^i c^{ik} = -b_k, \tag{20}$$

which is unique due to (19) and it is real. To see that it is real note that if μ_i is a complex eigenvalue of S then so is its conjugate since S is real. Note also that to the eigenvalue μ_i and to its conjugate, correspond complex conjugate rows of M_k^i and therefore both c^{ik} and its conjugate satisfy (20).

The assumption holds in the special case where the sensors are sampled at a single-rate. In fact, in this case we have that $\tau_l^{ik} = l, \forall_{1\leq k\leq h, 1\leq i\leq n_y, 1\leq l\leq n_w}$, and the rows of M_i^k correspond to linear independent functions $l^r \mu_j^{-l}$ where $1 \leq r \leq \kappa_j$. This is also in general true in the multi-rate case where the τ_l^{ik} are in general not equal to l. However, the assumption may fail in some pathological cases as we illustrate in the next example.

Example 2. Suppose that $y_m[k]$ is one dimensional and corresponds to a sensor which is sampled once every five times in a period, i.e., $h = 5$, $\gamma_k^1 = 1$, if $k = 1$ and, $\gamma_k^1 = 0$ if $k \in \{2, 3, 4, 5\}$. If $n_w = n_s = 3$, then we can obtain that $\tau_1^{1k} = 5$, $\tau_2^{1k} = 10$, and $\tau_3^{3k} = 15$ if $k = 1$ and τ_l^{ik} do not need to be specified if $k \in \{2, 3, 4, 5\}$. Suppose that $\mu_1 = e^{i2\pi/5}$, $\mu_2 = e^{-i2\pi/5}$, $\mu_3 = 1$. Then the matrices M_k^1, which needs only be specified for $k = 1$, is given by

$$M_1^1 = \begin{bmatrix} (e^{i2\pi/5})^{-5} & (e^{i2\pi/5})^{-10} & (e^{i2\pi/5})^{-15} \\ (e^{-i2\pi/5})^{-5} & (e^{-i2\pi/5})^{-10} & (e^{-i2\pi/5})^{-15} \\ 1 & 1 & 1 \end{bmatrix},$$

which is singular and therefore the Assumption 19 does not hold.

3.3 Main Result

We make the following assumptions on the multi-rate discrete-time plant (6):

(P1) (A, B) is stabilizable and (C, A) is detectable.

(P2) There are no invariant zeros from the input of the plant to the regulated output that coincide with the eigenvalues of S, i.e.,

$$\begin{bmatrix} A - \mu_j I_n & B \\ C_r & 0 \end{bmatrix} \text{ is invertible } \forall_{1\leq j\leq n_s}.$$

(P3) Consider the following systems

$$\begin{aligned} \hat{x}[k+1] &= A\hat{x}[k] & \hat{w}[k+1] &= S\hat{w}[k] \\ \hat{y}_r[k] &= \Gamma_{rk} C_r \hat{x}[k] & \hat{r}[k] &= \Gamma_{mk} C_M \hat{w}[k] \\ \hat{y}_m[k] &= \Gamma_{mk} C_m \hat{x}[k] & & \end{aligned} \tag{21}$$

with initial conditions $x[0] = x_0$ and $w[0] = w_0$. Then, there does not exist x_0 different from the zero vector, such that there exists a w_0 and C_M for the free motion of (21), such that $\hat{y}_r[k] = 0, \forall_{k\geq 0}$ and $\hat{y}_m[k] = \hat{r}[k], \forall_{k\geq 0}$.

We assume (P1)-(P3) to obtain (G1). The assumptions (P1) and (P2) are typical in related problems (cf. Francis (1977)). The assumption (P3) is closely related to the following assumption, which is easier to test:

(P3') A and S do not share an eigenvalue, or if A and S share an eigenvalue $\mu_j, 1 \leq j \leq n_s$, then the corresponding eigenvector of A is not in the kernel of $\Gamma_{rk} C_r$ for every $k \in \{1, \ldots, h\}$.

While (P3) and (P3') are not equivalent, it is straightforward to show that (P3) implies (P3'). However if (P3') holds then (P3) may not hold in pathological cases.

Example 3. Suppose that $\Gamma_{rk} = 1, \forall_{k\geq 0}$ and $\Gamma_{mk} = 1$ if $k = 0$, $\Gamma_{mk} = 0$ if $k \in \{1, 2, 3, 4\}$ and $\Gamma_{mk} = \Gamma_{m(k+5)}$, $\forall_{k\geq 0}$. Let $S = 1$, $C_m = [1\,0\,0]$, $C_r = [0\,0\,1]$, and

$$A = \begin{bmatrix} \cos(\frac{2\pi}{5}) & -\sin(\frac{2\pi}{5}) & 0 \\ \sin(\frac{2\pi}{5}) & \cos(\frac{2\pi}{5}) & 0 \\ 0 & 0 & \frac{1}{2} \end{bmatrix}.$$

Although A does not have eigenvalues that coincide with the eigenvalues of S, and therefore (P3') holds, if we make $x_0 = [0\,1\,0]^\intercal$ and $w_0 = 1$ then we have that $y_r[k] = 0, \forall_{k\geq 0}$, and $\hat{r}[k] = y_m[k] = 1, \forall_{k\geq 0}$, which means that (P3) does not hold.

The following is the main result of the paper. We denote by plant uncertainties the fact that the matrices A, B, C in (6) might not be known exactly, i.e., although the controller of Fig. 1 is designed for the model (6), the actual plant is described by the matrices $\tilde{A}$, $\tilde{B}$, $\tilde{C}$ and is given by

$$\tilde{\mathsf{P}} := \left\{ \begin{bmatrix} x[k+1] \\ y[k] \end{bmatrix} = \begin{bmatrix} \tilde{A} & \tilde{B} \\ \Gamma_k \tilde{C} & 0 \end{bmatrix} \begin{bmatrix} x[k] \\ u[k] \end{bmatrix} + \begin{bmatrix} B_V \\ 0 \end{bmatrix} v[k] \right. \tag{22}$$

Since asymptotic stability is a robust property, if $\tilde{A}$, $\tilde{B}$, and $\tilde{C}$ are sufficiently close to A, B, and C, respectively, and if the controller of Fig. 1 designed for (23) asymptotic stabilizes the closed-loop, then asymptotic stability is preserved when P is replaced by the actual plant $\tilde{\mathsf{P}}$.

Theorem 4. Suppose that (P1)-(P3) hold for the plant P, and that C_I is given by (13), and C_D is given by (16), (17), and (18). Then there exists matrices B_J, C_J for C_I, and C_K such that the closed-loop in Figure 1 is stable. Moreover, output regulation for $y_m[k]$ is achieved even in the presence of plant uncertainties that do not destroy closed-loop stability.

4. PROOF OF THE MAIN RESULT

We start by reviewing some general definitions for periodically time-varying systems. Then we state the assumptions that we make on C_I, C_D that lead to establishing that there exists a system C_K that yields the closed-loop of Figure 1 stable. After establishing the existence of such a system C_K we prove the main result. In the Appendix we introduce the notion of blocking zero with respect to a matrix, which is key to understand the assumptions on the block C_D, and to prove some of the results in the present Section.

4.1 Periodic Systems

Consider a discrete-time linear periodic system

$$\mathsf{R} = \begin{cases} x[k+1] = A_k x[k] + B_k u[k] \\ \quad\; y[k] = C_k x[k] + D_k u[k], \;\; k \geq 0, \end{cases} \tag{23}$$

where $x[k] \in \mathbb{R}^{n_A}$, $u[k] \in \mathbb{R}^{n_B}$, $y[k] \in \mathbb{R}^{n_C}$, and A_k, B_k, C_k, D_k are h-periodic matrices, e.g., $A_k = A_{k+h}$. Many system analytical notions for (23) are defined by considering the *lifted* time-invariant system $\bar{\mathsf{R}}$ associated with R, which is defined as

$$\bar{\mathsf{R}} = \begin{cases} \bar{x}[l+1] = \bar{A}\bar{x}[l] + \bar{B}\bar{u}[l] \\ \quad\; \bar{y}[l] = \bar{C}\bar{x}[l] + \bar{D}\bar{u}[l], \;\; l \geq 0, \end{cases} \tag{24}$$

where $\bar{x}[l] = x[lT]$,

$$\bar{u}[l] := (u[lh], u[lh+1], \ldots, u[lh+h-1]),$$
$$\bar{y}[l] := (y[lT], y[lT+1], \ldots, y[lT+h-1]),$$

and the system matrices in (24) are given by $\bar{A} := \Phi(h,0)$,

$$\bar{B} := [\Phi(h,1)B_0 \;\; \Phi(h,2)B_1 \ldots B_{h-1}],$$
$$\bar{C} := (C_0, C_1\Phi(1,0), \ldots, C_{h-1}\Phi(h-1,0)),$$
$$\bar{D} := \begin{bmatrix} E_{11} & \ldots & E_{1T} \\ \vdots & \vdots & \vdots \\ E_{T1} & \ldots & E_{Th} \end{bmatrix},$$
$$E_{ij} := \begin{cases} C_{i-1}\Phi(i-1,j)B_{j-1} & i > j \\ D_i & i = j \\ 0 & i < j \end{cases},$$

where $\Phi(i,j) := A_{i-1}A_{i-2}\ldots A_j$, for $i > j$ and $\Phi(i,i) := I$.

The system R is stable, stabilizable and detectable if and only if $\bar{\mathsf{R}}$ is stable, stabilizable or detectable, respectively. Equivalently, stability of (23) is characterized by all the eigenvalues of the matrix $\bar{A}$ having norm less that one, i.e., $\|\lambda_i(\bar{A})\| < 1, \;\; \forall i$, stabilizability of (23), denoted by (A_k, B_k) is stabilizable, is characterized by there exists a set of periodic matrices F_k, $F_k = F_{k+h}$, such that $x[k+1] = (A_k + B_kF_k)x[k]$ is stable, and detectability of (23), denoted by (C_k, A_k) is detectable, is characterized by there exists a set of periodic matrices G_k, $G_k = G_{k+h}$, such that $x[k+1] = (A_k + G_kC_k)x[k]$ is stable (cf. Bittanti and Bolzern (1985)).

4.2 Assumptions on C_I:

The system C_I, described by (13), must be such that:

(I1) (A_I, B_I) is stabilizable and (C_I, A_I) is detectable,

(I2) C_I does not have invariant zeros at the unstable eigenvalues of the plant P, i.e.,

$$\begin{bmatrix} A_I - \rho I_{n_I} & B_I \\ C_I & 0 \end{bmatrix} \text{ is invertible} \tag{25}$$

for $\rho \in \{\lambda_i(A) : \|\lambda_i(A)\| \geq 1\}$.

(I3) For every $Z \in \mathbb{R}^{m \times n_w}$, the following equation

$$\begin{aligned} A_I \Pi_I &= \Pi_I S \\ C_I \Pi_I &= Z \end{aligned} \tag{26}$$

has a solution $\Pi_I \in \mathbb{R}^{n_I \times n_w}$. Moreover such solution is unique.

(I4) The eigenvalues of A_I belong to the set of eigenvalues of S.

We assume (I1)-(I2) to obtain (G1). The assumption (I4) is required to limit the set of possible systems C_I. As stated in the next proposition the assumption (I3) is closely related to the purpose of the system C_I, i.e., to provide the adequate input to the plant so that it tracks the reference signal. Note that due to the linearity of the plant, an adequate input takes the same form of the reference signal one wishes to follow, i.e., it is generated by

$$\begin{aligned} w[k+1] &= Sw[k], \;\; k \geq 0, \\ u[k] &= C_U w[k]. \end{aligned} \tag{27}$$

Proposition 5. If (I3) holds then for any signal $u[k]$ generated by (27) there exists an initial condition x_0 such that the free motion of

$$\begin{aligned} x_I[k+1] &= A_I x_I[k], \;\; x_I[0] = x_0 \\ y_I[k] &= C_I x_I[k] \end{aligned}$$

is such that $y_I[k] = u[k]$.

□

Proof (of proposition 5) If (I3) holds then for an initial condition $x_I[0] = \Pi_I w[0]$, we have that

$$y_I[k] = C_I A_I^k \Pi_I w[0] = C_I \Pi_I S^k w[0] = Z S^k w[0].$$

The result follows by making $Z = C_U$.

■

Due to the following proposition it is always possible to find B_J and C_J such that (I1)-(I3) hold for the system (13) with matrices (14).

Proposition 6. The set of matrices $(B_J, C_J) \in \mathbb{R}^{n_w \times 1} \times \mathbb{R}^{1 \times n_w}$ for which (I1)-(I3) do not hold for the system (13) with matrices (14) is a set of measure zero in $\mathbb{R}^{n_w \times 1} \times \mathbb{R}^{1 \times n_w}$.

□

Proof Since the union of sets of measure zero has measure zero it suffices to prove that for each assumption (I1), (I2), and (I3) the set of matrices $(B_J, C_J) \in \mathbb{R}^{n_w \times 1} \times \mathbb{R}^{1 \times n_w}$ which do not satisfy each of these assumption has measure zero. We start by showing this for (I1). If (S, B_J) is not stabilizable, there exists a left eigenvalue of S, say w_i, for some $1 \leq i \leq n_s$, such that $w_i^\intercal B_J = 0$. Since the set $\{B_J : w_i^\intercal B_J = 0, \text{ for every } 1 \leq i \leq n_s\}$ where (S, B_J) is not stabilizable has measure zero, we establish that (S, B_J) is stabilizable except in a set of measure zero in $B_J \in \mathbb{R}^{n_w \times 1}$. A similar reasoning allows to conclude that (C_J, S) is detectable except in a set of measure zero in $C_J \in \mathbb{R}^{1 \times n_w}$. Since stabilizability of (S, B_J) implies that of (A_I, B_I) and detectability of (C_J, S) implies that of (C_I, A_I), we see that (I1) holds almost everywhere for the matrices (14). Next we consider (I2). Note that due to the structure of the matrices (14), the condition (25) is equivalent to the following condition

$$\begin{bmatrix} S - \rho I_{n_w} & B_J \\ C_J & 0 \end{bmatrix} \text{ is invertible }, \tag{28}$$

for $\rho \in \{\lambda_i(A) : \|\lambda_i(A)\| \geq 1\}$. For each $\rho \in \{\lambda_i(A) : \|\lambda_i(A)\| \geq 1\}$, the set

$$\{(B_J, C_J) : \det(\begin{bmatrix} S - \rho I_{n_w} & B_J \\ C_J & 0 \end{bmatrix}) = 0\}$$

is at most a manifold of dimension one in $\in \mathbb{R}^{n_w \times 1} \times \mathbb{R}^{1 \times n_w}$, from which we conclude that (28) holds except in a set of measure zero.

Finally, note that (I3) holds for the system (13) if and only if

$$\begin{aligned} S\Theta &= \Theta S \\ C_J \Theta &= Y \end{aligned} \tag{29}$$

has a solution $\Theta \in \mathbb{R}^{n_w \times n_w}$ for every $Y \in \mathbb{R}^{1 \times n_w}$. Considering the Jordan canonical decomposition of $S = VS_JV^{-1}$, where $S_J = \text{bdiag}(S_1, \dots, S_{n_w})$, and making $\Theta = V^{-1}[\Pi_1, \dots, \Pi_{n_s}]V$ we can conclude that (29) is equivalent to existing $\Pi_i \in \mathbb{R}^{n_w \times \kappa_i}$, $1 \le i \le n_s$ such that for every $W_i \in \mathbb{R}^{1 \times \kappa_i}$ we have that

$$\begin{aligned} S_J \Pi_i &= \Pi_i S_i \\ C_J \Pi_i &= W_i. \end{aligned} \tag{30}$$

It suffices to consider (30) for $i = 1$, i.e., prove that the set of C_J such that (30) does not hold for $i = 1$ has measure zero.

We can decompose Π_1 into $\Pi_1 = [v_1\, v_2\, \dots\, v_{\kappa_1}]$ where v_i are such that there exists α_i such that

$$S_J(\alpha_1 v_1) = \mu_1(\alpha_1 v_1)$$

and, for $i = 2, \dots, \kappa_1$,

$$v_i = w_i + \alpha_i v_1$$

for arbitrary α_i, where w_i are such that

$$(S_J - \mu_1 I)(w_i + \alpha_i v_1) = v_{i-1}.$$

Then

$$C_J \Pi_1 = [\alpha_1 C_J v_1 \;\; C_J w_2 + \alpha_2 C_J v_1 \;\; \dots C_J w_{\kappa_1} + \alpha_{\kappa_1} C_J v_1]$$

If $C_J v_1 \neq 0$ then we can choose $\alpha_1, \dots, \alpha_{\kappa_1}$ such that $C_J \Theta = W_1$ for arbitrary W_1. Since, the set of C_J such that $C_J v_1 = 0$ is a set of measure zero, we have that (I3) does not hold except in a set of measure zero.

■

4.3 Assumptions on C_D:

The system C_D must be such that:

(D1) (A_{Dk}, B_{Dk}) is stabilizable and (C_{Dk}, A_{Dk}) is detectable.

(D2) The following equations

$$\begin{aligned} A_{Dr}\Pi_{D\lfloor r+1 \rfloor} + B_{Dr}\Gamma_{mr}Y &= \Pi_{Dr}S \\ C_{Dr}\Pi_{Dr} + D_{Dr}\Gamma_{mr}Y &= 0, \qquad 1 \le r \le h \end{aligned} \tag{31}$$

have a solution $\Pi_{Dr} \in \mathbb{R}^{n_D \times n_w}$, $1 \le r \le h$ such that (31) holds for every $Y \in \mathbb{R}^{n_m \times n_w}$.

(D3) Consider the system

$$\begin{aligned} \hat{w}[k] &= \hat{S}\hat{w}[k], \;\; \hat{w}[0] = \hat{w}_0 \\ \hat{r}[k] &= C_R \hat{w}[k] \end{aligned} \tag{32}$$

and suppose that $\hat{S}$ is such that there exists $\hat{w}_0$ such that $\hat{r}[k] \neq r[k]$ for any arbitrary w_0, C_R in (9). Then (D2) does not hold when S is replaced by $\hat{S}$.

We assume (D1)-(D3) to obtain (G1). From the Definition 17 of a time-invariant blocking zero given in the Appendix, we see that (D2) is equivalent to the system obtained by computing the series between C_D and Γ_m having a time-invariant blocking zero with respect to S. From the interpretation of blocking zeros given in Proposition 18, we see that the assumption (D2) is closely related to the purpose of the system C_D, i.e., to block the signals that are generated from the exosystem (9). Moreover, using again the same interpretation of the Proposition 18, the assumption (D3) states that C_D does not block any signal other than the ones generated by the exosystem.

Proposition 7. The system (15) with matrices (17) satisfies (D1)-(D3).

□

Proof Note that (15) with matrices (17) is a stable system. In fact, one can check that all the eigenvalues of $\hat{A}_D = A_{D(h-1)} \dots A_{D2}A_{D1}A_{D0}$ are equal to zero, and therefore not only it is stable, but also the corresponding lifted system is a deadbeat system. Thus, its lifted system is stabilizable and detectable, and therefore so is (15) with matrices (17).

To prove (D2) it suffices, from the Proposition 18 in the Appendix, to prove that there exists an initial condition for C_D such that C_D has zero output for every signal generated by the following system

$$\begin{aligned} w[k+1] &= Sw[k] \\ r[k] &= \Gamma_{mk} C_R w[k] \end{aligned} \tag{33}$$

Denote the n_w basis functions that generate (12) by

$$f_\iota[k] = \binom{k}{l} \mu_j^{k-l}, \text{ if } \iota \in (\sum_{q=1}^{j-1} \kappa_q, \sum_{q=1}^{j} \kappa_q], \tag{34}$$

where $l = \iota - \sum_{q=1}^{j-1} \kappa_q - 1$, $1 \le \iota \le n_w$. Recall that at a given time $k \ge hn_w$ the state of C_D holds the last n_w sampled values corresponding to each output component of $y_r[k]$. Then, also by construction of the C_{Dr} and D_{Dr} of the system C_D, proving that for any input signal taking the form (12), there exists and initial condition $x_D[0]$, such that the output of C_D is zero, is equivalent to proving that the following holds

$$\begin{bmatrix} f_1[k] \\ \vdots \\ f_{n_w}[k] \end{bmatrix} + \begin{bmatrix} f_1[k - \tau_1^{ik}] & \dots & f_1[k - \tau_{n_w}^{ik}] \\ \vdots & \vdots & \vdots \\ f_{n_w}[k - \tau_1^{ik}] & \dots & f_{n_w}[k - \tau_{n_w}^{ik}] \end{bmatrix} \begin{bmatrix} c_1^{ik} \\ \vdots \\ c_{n_w}^{ik} \end{bmatrix} = 0, \tag{35}$$

for every index i corresponding to sensor i, $1 \le i \le n_s$, and for every $k \ge 0$, where (35) only needs to be verified for $k \in \{1, \dots, h\}$ such that sensor i is sampled at t_k. The c^{ik} are obtained from (20). We establish this by proving that each row of (35) imposes the same restriction as each row of (20). It is easy to see that to do so, it suffices to consider, without loss of generality, a single sensor ($i = n_s = 1$) and a single set of κ_1 rows corresponding to the eigenvalue μ_1, i.e., prove that (35) with $n_w = \kappa_1$, imposes the same restrictions on c^{1k} as the following set of equations

$$\begin{bmatrix} \mu_1^{-\tau_1^{1k}} & \mu_1^{-\tau_2^{1k}} & \dots & \mu_1^{-\tau_{n_w}^{1k}} \\ \tau_1^{1k}\mu_1^{-\tau_1^{1k}} & \tau_2^{1k}\mu_1^{-\tau_2^{1k}} & \dots & \tau_{n_w}^{1k}\mu_1^{-\tau_{n_w}^{1k}} \\ \dots & \dots & \dots & \dots \\ (\tau_1^{1k})^{\kappa_1}\mu_1^{-\tau_1^{1k}} & (\tau_2^{1k})^{\kappa_1}\mu_1^{-\tau_2^{1k}} & \dots & (\tau_{n_w}^{1k})^{\kappa_1}\mu_1^{-\tau_{n_w}^{1k}} \end{bmatrix} \begin{bmatrix} c_1^{1k} \\ \vdots \\ c_{n_w}^{1k} \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ \dots \\ 0 \end{bmatrix} \tag{36}$$

We argue by induction. The first row of (36) imposes the restriction

$$\left[\mu_1^{-\tau_1^{1k}} \; \mu_1^{-\tau_2^{1k}} \; \dots \; \mu_1^{-\tau_{n_w}^{1k}}\right] c^{1k} = 1 \tag{37}$$

while the first row of (35) imposes the restriction

$$\left[\mu_1^{k-\tau_1^{1k}} \; \mu_1^{k-\tau_2^{1k}} \; \dots \; \mu_1^{k-\tau_{n_w}^{1k}}\right] c^{1k} = \mu_1^k \tag{38}$$

which are obviously equivalent since $\mu_1 \neq 0$. It is also insightful to see that the second row of (36), given by,

$$\left[\tau_1^{1k}\mu_1^{-\tau_1^{1k}} \; \tau_2^{1k}\mu_1^{-\tau_2^{1k}} \; \dots \; \tau_{n_w}^{1k}\mu_1^{-\tau_{n_w}^{1k}}\right] c^{1k} = 0 \tag{39}$$

imposes the same restriction as the second row of (35), given by

$$\begin{aligned} &\left[(k-\tau_1^{1k})\mu_1^{k-\tau_1^{1k}} \; \dots \; (k-\tau_{n_w}^{1k})\mu_1^{k-\tau_{n_w}^{1k}}\right] c^{1k} = k\mu_1^k \\ \Leftrightarrow &\left[-\tau_1^{1k}\mu_1^{-\tau_1^{1k}} \; \dots \; -\tau_{n_w}^{1k}\mu_1^{-\tau_{n_w}^{1k}}\right] c^{1k} = 0 \end{aligned} \tag{40}$$

where we used (37).

Now assuming that the first $1 \leq r-1 < n_w-1$ rows of (35) impose the same restriction as the first $1 \leq r-1 < n_w-1$ rows (36), we prove that the same is true for the row r, i.e.,

$$\left[f_r[k-\tau_1^{1k}] \; f_r[k-\tau_2^{1k}] \dots \; f_r[k-\tau_{nw}^{1k}]\right] c^{1k} = f_r[k] \tag{41}$$

To this effect, note that we can write (34) as

$$\begin{aligned} f_r[k-\tau] &= \binom{k-\tau}{r}\mu_1^{k-r-\tau} \\ &= (\sum_{m=0}^{r} a_m k^{(r-m)}\tau^m)\mu_1^{k-r-\tau} \\ &= (\sum_{m=0}^{r-1} a_m k^{(r-m)}\tau^m + a_r\tau^r)\mu_1^{k-r-\tau} \end{aligned} \tag{42}$$

for some coefficients a_m implicitly defined from

$$\begin{aligned} \binom{k-\tau}{r} &= (k-\tau)(k-\tau-1)\dots(k-\tau-r) \\ &= \sum_{m=0}^{r} a_m k^{(r-m)}\tau^m. \end{aligned}$$

Note that $a_r = (-1)^{r+1}$. If we replace (42) in (41) and use the fact that the first $r-1$ restrictions of (36) hold, we obtain

$$a_r \left[(\tau_1^{1k})^r \mu_j^{k-r-\tau_1^{1k}} \; \dots \; (\tau_{n_w}^{1k})^r \mu_j^{k-r-\tau_{n_w}^{1k}}\right] c_k^1 = 0$$

which is equivalent to the restriction associated with row r of (36) since $\mu_1 \neq 0$ and $a_r \neq 0$.

To prove that (15) with matrices (17) satisfies (D3), note that using the interpretation of a blocking zero given in the Proposition 18, the assumption (D3) states that C_D does not block any other signal other than the ones generated by the exosystem. To see that this is true, suppose that there exists a signal $g[k]$ such that there exists an initial condition to the system C_D such that the output of the system C_D is identically zero when $g[k]$ is applied to its input. Then, by construction of C_D, i.e., by the fact that for $k \geq n_w h$, $x_D[k]$ will hold the value of the last n_w samples of every sensor $1 \leq i \leq n_s$, $g[k]$ must satisfy

$$g[k] + \left[g[k-\tau_1^{ik}] \; g[k-\tau_2^{ik}] \; \dots \; g[k-\tau_{n_w}^{ik}]\right] c^{ik} = 0$$

for every, $1 \leq i \leq n_s$ and $k \in \{1, \dots, h\}$ such that sensor i is sampled at t_k. From (35) and uniqueness of the c^{ik}(cf. (19), (20)) we conclude that $g[k]$ must be a linear combination of the $f_\iota[k]$, $1 \leq \iota \leq n_w$ and therefore $g[k]$ can be generated by the exosystem.

■

4.4 System C_K:

The system C_K takes the form

$$\mathsf{C}_K := \begin{cases} x_K[k+1] = A_{Kk}x_K[k] + B_{Kk}u_K[k] \\ y_K[k] = C_{Kk}x_K[k] \end{cases}. \tag{43}$$

where the matrices A_{Kk}, B_{Kk}, C_{Kk}, and D_{Kk} are $h-$periodic, i.e., e.g., $A_{Kk} = A_{K(k+h)}$ and is such that the closed loop of Figure 1 is stable. The next Lemma shows that such controller always exists.

Lemma 8. Suppose that P, C_I, C_D are such that (P1)-(P5), (I1)-(I4), (D1)-(D4) hold. Then there exists a stabilizing controller C_K for the closed loop system of the Fig. 1 taking the form (43).

□

To prove the Lemma 8 we need the following two results. For two dimensionally compatible LTI systems described by

$$\mathsf{C}_1 := \left\{ \begin{bmatrix} x_1[k+1] \\ y_1[k] \end{bmatrix} = \begin{bmatrix} A_1 & B_1 \\ C_1 & 0 \end{bmatrix} \begin{bmatrix} x_1[k] \\ u_1[k] \end{bmatrix} \right. \tag{44}$$

and

$$\mathsf{C}_2 := \left\{ \begin{bmatrix} x_2[k+1] \\ y_2[k] \end{bmatrix} = \begin{bmatrix} A_2 & B_2 \\ C_2 & D_2 \end{bmatrix} \begin{bmatrix} x_2[k] \\ u_2[k] \end{bmatrix} \right. \tag{45}$$

The series of the system C_2 and C_1 obtained by making $u_1[k] = y_2[k]$, is defined by:

$$\mathsf{C}_3 := \left\{ \begin{bmatrix} x_3[k+1] \\ y_1[k] \end{bmatrix} = \begin{bmatrix} A_3 & B_3 \\ C_3 & 0 \end{bmatrix} \begin{bmatrix} x_3[k] \\ u_2[k] \end{bmatrix} \right. \tag{46}$$

where

$$A_3 = \begin{bmatrix} A_1 & B_1C_2 \\ 0 & A_2 \end{bmatrix}, B_3 = \begin{bmatrix} 0 \\ B_2 \end{bmatrix}, C_3 = [C_1 \; D_1C_2].$$

Recall that for an LTI system, say (44), an eigenvalue λ_1 of A_1, i.e., there exists v_1 and w_1 such that $A_1v_1 = \lambda v_1$ and $w_1^\intercal A_1 = \lambda w_1^\intercal$, is observable if $C_1v_1 \neq 0$, and controlladble if $w_1^\intercal B_1 \neq 0$. The pair (A_1, C_1) is detectable if all the unstable eigenvalues of A_1 are observable, and the pair (A_1, B_1) is stabilizable if all the unstable eigenvalues of A_1 are controllable. One can find these definitions in Zhou et al. (1995) for continuous-time systems, which have an obvious counterpart for discrete-time systems. Denote the set of eigenvalues of A by

$$\Lambda_A := \{\lambda : Av = \lambda v, \text{ for some } v\},$$

where A can be replaced by A_1, A_2, A_3. Also denote the set of eigenvalues of A which are not eigenvalues of B by

$$\Lambda_{A/B} := \{\lambda : \lambda \in \Lambda_A \text{ and } \lambda \notin \Lambda_B\},$$

where A, B can be replace by A_1 and A_2.

$\blacksquare$

Proposition 9. Consider C_1, C_2 and the series C_3. Then

(i) $\Lambda_{A_3} = \Lambda_{A_1} \cup \Lambda_{A_2}$
(ii) If $\lambda \in \Lambda_{A_1/A_2}$ then λ is an observable eigenvalue of C_3 if λ is an observable eigenvalue of C_1.
(iii) If $\lambda \in \Lambda_{A_2}$ then λ is an observable eigenvalue of C_3 if λ is an observable eigenvalue of C_2 and C_1 has no invariant zeros at λ.
(iv) If $\lambda \in \Lambda_{A_2/A_1}$ then λ is a controllable eigenvalue of C_3 if λ is a controllable eigenvalue of C_2.
(v) If $\lambda \in \Lambda_{A_1}$ then λ is a controllable eigenvalue of C_3 if λ is a controllable eigenvalue of C_1 and C_2 has no invariant zeros at λ.

$\square$

Proof (i) We can conclude from $\det(\lambda I - A_3) = \det(\lambda I - A_1)\det(\lambda I - A_2)$, which holds due to the structure of A_3. We prove only (ii) and (iii) since (iv) and (v) can be obtained from the fact that detectability and stabilizability are dual notions. To this effect consider the following equation for the eigenvalues of A_3

$$\begin{bmatrix} A_1 & B_1C_2 \\ 0 & A_2 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \lambda \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$$

If $\lambda \in \Lambda_{A_1/A_2}$ then $v_1 : A_1 v_1 = \lambda v_1$ and $v_2 = 0$. Thus $C_3[v_1^\intercal\, v_2^\intercal]^\intercal = C_1 v_1 \neq 0$, which means that λ is an observable eigenvalue of C_3 if it is an observable eigenvalue of C_1, which is (ii).

If $\lambda \in \Lambda_{A_2}$ then $v_2 : A_2 v_2 = \lambda v_2$ and $v_1 : A_1 v_1 + B_1 v_2 = \lambda v_1$. If λ was not an observable eigenvalue of C_3 then we would have $\mathsf{C}_3[v_1^\intercal\, v_2^\intercal]^\intercal = 0$ which would imply that

$$\begin{bmatrix} A_1 - \lambda_1 I & B_1 \\ C_1 & D_1 \end{bmatrix} \begin{bmatrix} v_1 \\ C_2 v_2 \end{bmatrix} = 0 \tag{47}$$

where $C_2 v_2 \neq 0$ since λ is an observable eigenvalue of C_2. The equation (47) contradicts the assumption that C_1 has no invariant zeros at λ.

$\blacksquare$

The series of two periodically time-varying systems is defined similarly to the series of two LTI systems.

Proposition 10. The lift of the series of two periodic systems is the series of the LTI lift systems of each periodic system.

$\square$

Proof Obtained by direct replacement.

$\blacksquare$

We shall need the following proposition.

Proposition 11. Consider the system

$$\begin{aligned} x[k+1] &= Ax[k] \\ y[k] &= \mathrm{bdiag}(\gamma_k^1, \ldots, \gamma_k^{n_y})Cx[k] \end{aligned} \tag{48}$$

where $y[k] \in \mathbb{R}^{n_y}$. The $\gamma_k^i \in \{0,1\}, 1 \leq i \leq n_y$ are $h-$periodic, i.e., $\gamma_k^i = \gamma_{k+h}^i$, and for each i are equal to 1 at least once in a period. The (48) is detectable if and only if the pair (A, C) is detectable.

$\square$

Proof See Colaneri et al. (1990)

$\blacksquare$

We shall also need to take into account that if we apply the lift for an LTI system we obtain the following eigenvalues of the lifted system.

$$\Lambda_{A^h} := \{\lambda^h, \lambda \in \Lambda_A\} \tag{49}$$

The Lemma 8 is proved next.

Proof In Colaneri (1991), it is shown that a periodic system R taking the form (23) is detectable and stabilizable if and only if there exists a periodic linear controller, taking the form (43) such that the closed loop system is asymptotically stable. Thus it suffices to prove the stabilizability and detectability of the periodic system obtained by computing the series of C_I, P, and C_D, which is a periodic system that we shall term C_A. This can proved by establishing stabilizability and detectability of the lifted LTI system, denoted by $\bar{\mathsf{C}}_A$. From Proposition 10 the lift of C_A is the series of the lift of each individual system C_I, P, and C_D, which are denoted by $\bar{\mathsf{C}}_I$, $\bar{\mathsf{P}}$, and $\bar{\mathsf{C}}_D$, respectively. Thus, we only have to prove the observability and controllability of the unstable eigenvalues of $\bar{\mathsf{C}}_A$, corresponding to the unstable eigenvalues of $\bar{\mathsf{C}}_I$ and $\bar{\mathsf{P}}$, since $\bar{\mathsf{C}}_D$ is a deadbeat system and thus stable.

We start by establishing stabilizability. From Proposition 9(iv) the controllability of the unstable eigenvalues of $\bar{\mathsf{C}}_A$ associated with $\bar{\mathsf{P}}$ or with $\bar{\mathsf{C}}_I$, do not depend of $\bar{\mathsf{C}}_D$ (which has only stable eigenvalues), and since C_I, P are LTI systems, this amounts to an LTI test. In fact, for an LTI system, considered to be a special case of a periodic system with period h, stabilizability of the lift is equivalent to stabilizability of the original LTI system, which can be concluded from the definition of stabilizability for periodic systems. Thus, it suffices to prove that the LTI series of C_I, P, which we denote by C_{IP}, is stabilizable. The eigenvalues of C_{IP} belonging to $\Lambda_{\mathsf{C}_I/\mathsf{C}_P}$ are controllable due to (I1) and Proposition 9(iv). The eigenvalues of C_{IP} belonging to Λ_{C_P} are controllable due to (P1), (I2), and Proposition 9(v).

We prove next detectability. To prove that observability of the unstable eigenvalues of $\bar{\mathsf{C}}_A$ belonging to $\Lambda_{\bar{\mathsf{C}}_I}$, which in turn correspond to unstable eigenvalues of C_I, due to (49), it suffices to prove that these eigenvalues are detectable from the plant output $y_r[k]$, i.e., if we consider the system obtained by the series of C_I and P_r where P_r is the system obtained by considering the plant with $y_r[k]$ has the only output. Again this amounts to an LTI test, since the series of C_I and P_r is an LTI system with multi-rate measurements as in Proposition 11, where we state that for such system detectability can be proved by an LTI test. The desired conclusion follows then from Assumptions (I1), (I4), (P2), and Proposition 9(iii).

To see the observability of the unstable eigenvalues of $\bar{\mathsf{C}}_A$ that correspond to those of P it suffices to consider those not observable through $y_r[k]$. By this we mean, the eigenvalues of the plant P that are not an observable eigenvalue of the pair (A, C_r). The existence of such eigenvalue, is equivalent to existing an initial condition for the plant $x[0]$ such that

$$\begin{aligned} x[k+1] &= Ax[k] \\ y_r[k] &= \Gamma_r C_r x[k] \end{aligned} \tag{50}$$

has zero output, i.e., $y_r[k] = 0, \forall_{k\geq 0}$. From assumption (P3) there cannot exist an initial condition for the plant such that (50) has a zero output $y_r[k] = 0$ and $y_m[k]$ can be generated by the exosystem. Since by assumption (D3), the system C_D does not cancel signals other than those generated by the exosystem, we obtain that this yields a non-zero output for C_D and thus these eigenvalues of A are also observable for the system $\bar{\mathsf{C}}_A$.

■

4.5 Output regulation

The following result and Lemma 8 allows to conclude the main result of the paper, i.e., Theorem 4.

Theorem 12. Suppose that (P1)-(P3) hold for the plant P. Then one can find C_I such that (I1)-(I4) holds, C_D such that (D1)-(D4) hold, and C_K such that the closed-loop in Figure 1 is stable. Moreover, output regulation is achieved even in the presence of plant uncertainties that do not destroy closed-loop stability.

□

Proof The first part of the Theorem has been established in Proposition 6 and 7 and Lemma 8. Thus, it suffices to prove that output regulation is achieved. We start by writing the equations for the series connection of C_I, P, and C_D, which are given by

$$\begin{aligned} x_A[k+1] &= A_{Ak}x_A[k] + B_{Ak}u_A[k] + B_{Awk}w[k] \\ y_A[k] &= C_{Ak}x_A[k] + D_{Awk}w[k] \end{aligned}$$

where

$$A_{Ak} = \begin{bmatrix} A & BC_I & 0 \\ 0 & A_I & 0 \\ B_{Dk}\Gamma_{mk}C_m & 0 & A_{Dk} \end{bmatrix}$$

$$B_{Ak} = \begin{bmatrix} 0 \\ B_I \\ 0 \end{bmatrix}$$

$$C_{Ak} = \begin{bmatrix} D_{Dk}\Gamma_{mk}C_m & 0 & C_{Dk} \\ \Gamma_{rk}C_r & 0 & 0 \end{bmatrix}$$

$$B_{Awk} = \begin{bmatrix} B_V C_V \\ 0 \\ 0 \end{bmatrix}$$

$$D_{Awk} = \begin{bmatrix} 0 \\ -\Gamma_{rk}C_R \end{bmatrix}$$

Let C_K be a stabilizing controller described by (43), whose existence is established in Lemma 8. Then the closed-loop is described by

$$\begin{bmatrix} x_A[k+1] \\ x_K[k+1] \end{bmatrix} = \begin{bmatrix} A_{Ak} & B_{Ak}C_{Dk} \\ B_{Kk}C_{Ak} & A_{Kk} \end{bmatrix} \begin{bmatrix} x_A[k] \\ x_K[k] \end{bmatrix} + \begin{bmatrix} B_{Awk} \\ B_{Kk}D_{Awk} \end{bmatrix} w[k]. \tag{51}$$

where $w[k]$ is described by (9). Using the fact that the periodic system (51) is stable, since C_K is a stabilizing controller, from Proposition 18 in the Appendix, we conclude that there exists unique $\Pi_k, k \in \{1, \dots h\}$, such that

$$\begin{bmatrix} A_{Ak} & B_{Ak}C_{Kk} \\ B_{Kk}C_{Ak} & A_{Kk} \end{bmatrix} \Pi_k + \begin{bmatrix} B_{Awk} \\ B_{Kk}D_{Awk} \end{bmatrix} = \Pi_{\lfloor k+1 \rfloor} S \tag{52}$$

and for any initial condition $(x_A[0], x_K[0])$ the state of the system tends asymptotically to

$$\begin{bmatrix} x_A[k] \\ x_K[k] \end{bmatrix} = \Pi_k w[k] \tag{53}$$

We provide a solution to (52), which is unique as mentioned above, and see that corresponding to such solution, we have that (53) is such that output regulation is achieved, i.e.,

$$y_{rk} = r[k]$$

or equivalently, using (53),

$$[C_r \;\; 0 \;\; 0]\Pi_k w[k] = C_R w[k]. \tag{54}$$

Such solution Π_k is obtained as follows. Make

$$\Pi_k = \begin{bmatrix} \Pi_{Ak} \\ 0 \end{bmatrix} \tag{55}$$

where

$$\Pi_{Ak} = \begin{bmatrix} \Pi_P \\ \Pi_I \\ \Pi_{Dk} \end{bmatrix}. \tag{56}$$

Then, from (52) we obtain that (56) must be such that

$$\begin{bmatrix} \Pi_P \\ \Pi_I \\ \Pi_{D\lfloor k+1 \rfloor} \\ 0 \\ 0 \end{bmatrix} S = \begin{bmatrix} A & BC_I & 0 \\ 0 & A_I & 0 \\ B_{Dk}\Gamma_{mk}C_m & 0 & A_{Dk} \\ B_{K1k}D_{Dk}\Gamma_{mk}C_m & 0 & B_{K1k}C_{Dk} \\ B_{K2k}\Gamma_{rk}C_r & 0 & 0 \end{bmatrix} \begin{bmatrix} \Pi_P \\ \Pi_I \\ \Pi_{Dk} \end{bmatrix} + \begin{bmatrix} B_V C_V \\ 0 \\ 0 \\ 0 \\ -B_{K2k}\Gamma_{rk}C_R \end{bmatrix} \tag{57}$$

where B_{K1k} and B_{K2k} are appropriate partitions of $B_{Kk} = [B_{K1k} \; B_{K2k}]$.

The matrices Π_P, Π_I, Π_{Dk} are obtained as follows.

(i) Take Π_P to be the solution, along with $E \in \mathbb{R}^{m \times n_w}$, to

$$\begin{aligned} \Pi_P S &= A\Pi_P + BE + B_V C_V \\ C_r \Pi_P &= C_R \end{aligned} \tag{58}$$

which as explained in Byrnes and Isidori (2000) exists if and only if (P2) holds.

(ii) Take Π_I to be the unique solution to

$$\begin{aligned} A_I \Pi_I &= \Pi_I S \\ C_I \Pi_I &= E \end{aligned}$$

where E is, along with Π_P, the solution to (58). Note that such solution Π_I exists due to the Assumption (I3).

(iii) Take Π_{Dk} to be the solution to

$$\begin{aligned} A_{Dk}\Pi_{Dk} + B_{Dk}\Gamma_{mk}C_m\Pi_P &= \Pi_{D\lfloor k+1 \rfloor} S \\ C_{Dk}\Pi_{Dk} + D_{Dk}\Gamma_{mk}C_m\Pi_P &= 0 \end{aligned} \tag{59}$$

which exists due to the Assumption (D3) and is unique due to Proposition 18 and the fact that C_D is a stable system.

By construction, we conclude that Π_k given by (55), where Π_{Ak} is given by (56), and Π_P, Π_I and Π_{Dk} are described by (i), (ii) and (iii), respectively, satisfies (52) and (54) and therefore output regulation is achieved.

Note that in the proof we only required the controller C_K to stabilize the closed-loop. If the plant describing matrices are not known but C_K still stabilizes the closed-loop the proof remains unchanged.

■

Proof (of Theorem 4) Follows as a Corollary of Lemma 8 and Theorem 12, provided that we notice that the matrices B_J and C_J can be chosen such that C_I, given by (13), satisfies (I1)-(I4) (cf.Proposition 6) and that we notice that C_D, given by (16), (17), (18) satisfies (D1)-(D3) (cf.Proposition (7)).

■

5. EXAMPLE

Example 13. The following continuous-time linear system is considered in Antunes et al. (2008).

$$\mathsf{P}_C = \begin{cases} \dot{x}_1(t) = -x_1(t) - x_2(t) \\ \dot{x}_2(t) = -x_2(t) + u(t) \\ \dot{x}_3(t) = -x_2(t) + 0.5x_3(t) + u(t) \end{cases} \tag{60}$$

A sensor measuring $x_1(t)$ works at a fixed sampling period of $t_{s1} = 0.25$, while the actuator update mechanism can be done at a sampling period of $t_u = 0.05$. We wish that $x_1(t)$ tracks a prescribed reference signal. However, one can verify that P_C is not detectable from $x_1(t)$. Therefore we cannot use the solution provided in Scattolini and Schiavoni (1993) for output regulation of square multi-rate systems, since this solution would not guarantee closed-loop stability. We consider that $x_3(t)$ is also available for feedback at a sampling period of $t_2 = 0.1$. According to the framework of Section 2 we have that $t_s = 0.05$, $h = 10$. The system is now detectable from $(x_1(t), x_3(t))$, and since the sampling period $t_s = 0.05$ is not pathological (see, e.g., Chen and Francis (1995), the discretization of (60) is also detectable with respect to the multi-rate outputs $y_r[k] := \Gamma_{rk} x_1(kt_s)$, and $y_m[k] := \Gamma_{mk} x_3(kt_s)$ where the h-periodic matrices $\Gamma_{mk},\ \Gamma_{rk},\ \Omega_k$ are determined by

$$\Gamma_{mk} = \begin{cases} 1 & k \text{ odd}, \\ 0 & \text{otherwise} \end{cases}, \qquad \Gamma_{rk} = \begin{cases} 1 & k = 1, 6 \\ 0 & \text{otherwise} \end{cases}.$$

and we also used the Proposition 11 to conclude the detectability of the multi-rate discretization. Consider the problem of designing a linear controller for P_C that achieves closed loop stability and such that the output $y_r[k]$ tracks the reference $\Gamma_r r[k]$ with zero steady-state error, where $r[k]$ is described by

$$\begin{aligned} w[k+1] &= Sw[k],\ \ w[0] = w_0, \\ r[k] &= C_R w[k] \end{aligned}$$

where

$$S = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix},\ \ C_R = [1\ 0]$$

Thus, the reference $r[k]$ takes the form

$$r[k] = c_1 + c_2 k \tag{61}$$

where c_1, c_2 can be made arbitrary. Such non-constant references, prevent the use of the results from Antunes et al. (2008). Contrarily to Scattolini and Schiavoni (1993), this problem can be solved with the solution we provide in the present paper. In fact, the discretization of P can be verified to satisfy (P1)-(P3) and therefore a linear controller with the structure depicted in Figure 1 can be synthesized for this system. The stabilizing controller C_K in the Figure 1 can be obtained, e.g., from the solution provided in Colaneri et al. (1990) and the gains c_k^{1k} determining the system C_D are in this special case time-invariant and given by

$$c^{1k} = [-2\ \ 1]^\intercal, \forall_{k \text{ odd}}$$

and do not need to be specified for k even since $y_m[k]$ is only sampled for odd k. In Figure 2, we show the response of the output $x_1[k]$ when a reference signal $r[k]$ consisting of a concatenation of signals taking the general form (61) is applied to the closed-loop system. We see that zero steady-state error is achieved after a transitory period, as desired. In Figure 3 we show several signals of the closed-loop for a short period of time where a transition of the reference signal occurs. Note that before the transition, in steady state, the output $y_D[k]$ of the blocking system C_D is zero as desired, and that at steady state $u[k]$ has the desired value to be applied to the plant so that output regulation can be achieved.

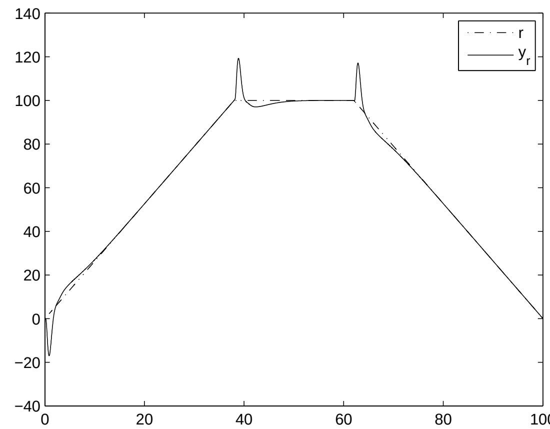

Fig. 2. $y_r[k]$ and $r[k]$.

6. CONCLUSIONS AND FUTURE WORK

We propose a solution to the output regulation problem for non-square multi-rate systems. This solution makes possible to control a plant with several sensors sampled at different frequency taking advantage of all the available measurements for feedback, while driving a subset of the outputs to a prescribed reference.

A topic for future work is to consider the case where some or all the components of the actuation update mechanism may not be available at the sampling period at which the controller operates, i.e., the actuator may also be multi-rate. Note that due to the linearity of the plant, if output regulation of some variables of the output is to be achieved then in general the actuation signal should be composed of the same frequency content signals as the desired output values. Thus, if one considers standard sample and hold device, as in the present paper, but working at different rates, it does not appear to be simple in general to guarantee that such signal is generated. Note that we were able to consider a multi-rate actuation updating mechanism in Antunes et al. (2010), where we

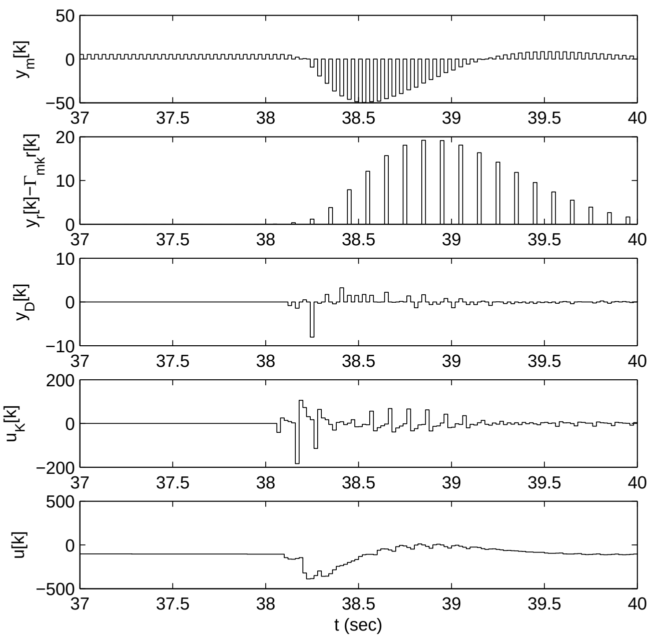

Fig. 3. Various closed-loop signals

restrict the reference to constant signal, since the standard sample and hold mechanism is capable of generating constant references signals. One solution that appears to be promising is to consider a generalized sample-and-hold device, as in Lawrence and Medina (2001), i.e., a hold device that provides signals to the plant between sampling instants generated by a linear time invariant system, instead of simply holding its input as a standard zero order hold. It is expectable that if one combines the solution we provide here with the work of Lawrence and Medina (2001), that as in Lawrence and Medina (2001) asymptotic tracking can be achieved with zero steady-state error also in continuous-time.

Appendix A. BLOCKING ZEROS WITH RESPECT TO A MATRIX

Consider an LTI system

$$\begin{aligned} x[k+1] &= Ax[k] + Bu[k] \\ y[k] &= Cx[k] + Du[k], \;\; k \geq 0, \end{aligned} \tag{A.1}$$

and a periodic linear system

$$\begin{aligned} x[k+1] &= A_k x[k] + B_k u[k] \\ y[k] &= C_k x[k] + D_k u[k], \;\; k \geq 0, \end{aligned} \tag{A.2}$$

where both (A.1) and (A.2) have the same number of inputs and outputs, and $x[k] \in \mathbb{R}^{n_A}$, $u[k] \in \mathbb{R}^{n_B}$, $y[k] \in \mathbb{R}^{n_B}$.

We generalize next the definition of blocking zero for LTI systems (see, e.g. Zhou et al. (1995)) and subsequently introduce the notion for periodic systems. This generalization lies at the heart of the solution we propose in Figure 1 since it is the property that the key system C_D is required to satisfy (cf. Subsection 4.3).

Definition 14. We say that (A.1) has a *blocking zero with respect to a matrix* $R \in \mathbb{R}^{n_R}$, if there exists $\Pi \in \mathbb{R}^{n_A \times n_R}$ such that for every $E \in \mathbb{R}^{n_B \times n_R}$ we have that

$$\Pi R = A\Pi + BE \tag{A.3}$$

$$0 = C\Pi + DE \tag{A.4}$$

□

To interpret the nomenclature blocking zero used in the previous definition, we need the following proposition. Consider the system

$$\begin{aligned} w_R[k+1] &= R w_R[k], \;\; k \geq 0, \\ u[k] &= C_U w_R[k]. \end{aligned} \tag{A.5}$$

Proposition 15. Suppose that the input of the system (A.1), is generated by (A.5). Then there exists a solution $\Pi \in \mathbb{R}^{n_A \times n_R}$ to

$$\Pi R = A\Pi + BC_U \tag{A.6}$$

if and only if the solution to (A.2) satisfies $x[k] = \Pi w[k]$ when $x[0] = \Pi w[0]$ for an arbitrary $w[0] \in \mathbb{R}^{n_R}$. Moreover, if A has all its eigenvalues inside the open unit dist and R has all its eigenvalues outside the open unit disk then the solution to (A.6) is unique, and $x[k] \to \Pi w[k]$ as $k \to \infty$ for every initial condition $x[0]$, $w[0]$.

□

Proof If there exists Π such that (A.6) has a solution, then we can argue by induction that $x[k] = \Pi w[k]$. In fact this is valid by hypothesis for $k = 0$, and if it is valid for $k = r$, then

$$\begin{aligned} x[r+1] – \Pi w[r+1] & \qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{(A.7)} \\ &= Ax[r] + BC_U w[r] - \Pi R w[r] \\ &= (A\Pi + BC_U - \Pi R) w[r] + A(x[r] - \Pi w[r]) \\ &= A(x[r] - \Pi w[r]) \qquad\qquad\qquad\qquad\qquad\qquad \text{(A.8)} \\ &= 0. \end{aligned}$$

Conversely if $x[k] = \Pi w[k]$ when $x[0] = \Pi w[0]$ for an arbitrary $w[0] \in \mathbb{R}^{n_R}$, then $0 = x[1] - \Pi w[1] = (A\Pi + BC_U - \Pi R)w[0]$, which implies (A.6) since $w[0]$ is arbitrary.

Now suppose that A has all its eigenvalues inside the open unit disk and R has all its eigenvalues outside the open unit disk, and that there exists two solutions Π and $\tilde{\Pi}$, which satisfy (A.6). Let $E := \Pi - \tilde{\Pi}$. Then $ER = AE$, which implies that $ER^k = A^k E$ for every $k \geq 0$ and this implies that $E = 0$. Moreover from (A.7), (A.8) and the stability of A we conclude that for every $x[0], w[0]$, we have that $x[k] \to \Pi w[k]$ when $k \to \infty$.

■

From Proposition 15, we can conclude that according to the Definition 14, the system (A.1) has a blocking zero with respect to R if the following hold. If the input of (A.1) is generated by (A.5) for an arbitrary matrix $C_R = E$ and the initial condition of (A.1) satisfies $x[0] = \Pi w[0]$, then the output of (A.1) is identically zero. This suggests the nomenclature blocking zero with respect to the matrix R.

According to (Zhou et al., 1995, Def. 3.14) the system (A.1) has a blocking zero at a complex number $z_0 \in \mathbb{C}$ that does not belong to the spectrum of A, if

$$C(z_0 I - A)^{-1} B + D = 0. \tag{A.9}$$

As stated in the next proposition, in the case where R is scalar our definition coincides with the one from (Zhou et al., 1995, Def. 3.14),

Proposition 16. Suppose that $R = z_0 \in \mathbb{C}$ is a complex number that does not belong to the spectrum of A. Then, the system (A.1) has a blocking zero according to the Definition 14 if and only if (A.9) holds.

□

Proof To prove sufficiency it suffices to multiply (A.3) by $C(z_0 I - A)^{-1}$ and sum the result to (A.4). One obtains $(C(z_0 I - A)^{-1} B + D)E = 0$ for every E, which implies (A.9). To prove necessity take $\Pi = (z_0 I - A)^{-1} BE$, and see that (A.3)-(A.4) holds if (A.9) holds.

■

Note that if the matrix R has only simple eigenvalues, then having a blocking zero with respect to R, is equivalent to having n_R zeros with respect to all the eigenvalues of R. Our definition of a blocking zero is broader since it allows to consider a matrix $R \in \mathbb{R}^{n_R}$ with a Jordan block structure

$$R = \begin{bmatrix} \lambda & 1 & 0 & \dots & 0 \\ 0 & \lambda & 1 & 0 & \dots \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \dots & 0 & \lambda \end{bmatrix} \tag{A.10}$$

and conclude if (A.1) has a blocking zero with respect to R, then every signal taking the form

$$u[k] = \sum_{l=0}^{n_R - 1} c_l \binom{k}{l} \lambda^{k-l}, \tag{A.11}$$

is blocked (yields a zero output) by (A.1).

We extend the definition of blocking zero to periodic systems as follows. Recall that $\lfloor r \rfloor$ denotes the remainder of the division by h.

Definition 17. We say that (A.2) has a *blocking zero with respect to* $R \in \mathbb{R}^{n_R}$, if there exists $\Pi_r \in \mathbb{R}^{n_A \times n_R}$ such that for every $E_r \in \mathbb{R}^{n_B \times n_R}$, $1 \leq r \leq h$, we have that

$$\begin{aligned} \Pi_{\lfloor r+1 \rfloor} R &= A_r \Pi_r + B_r E_r \\ 0 &= C_r \Pi_r + D_r E_r, \;\; 1 \leq r \leq h. \end{aligned} \tag{A.12}$$

Moreover, if (A.12) hold only when $E_r = E$ for an arbitrary matrix $E \in \mathbb{R}^{n_B \times n_R}$ we say that (A.2) has a *time-invariant blocking zero with respect to R*

□

To interpret the nomenclature used in the previous definition, we need the following proposition. Consider the system

$$\begin{aligned} w_R[k+1] &= R w_R[k], \;\; k \geq 0, \\ u[k] &= C_{Uk} w_R[k]. \end{aligned} \tag{A.13}$$

where $C_{Uk} = C_{U(k+h)}, \forall_{k \geq 0}$ and R is now assumed to be invertible.

Proposition 18. Suppose that the input of the system (A.2), is generated by (A.13). Then, there exists a solution $\Pi_r \in \mathbb{R}^{n_A \times n_R}$, $1 \leq r \leq h$ to

$$\Pi_{\lfloor r+1 \rfloor} R = A_r \Pi_r + B_r C_{Ur}, \;\; 1 \leq r \leq h \tag{A.14}$$

if and only if the solution to (A.2) satisfies $x_k = \Pi_{\lfloor k \rfloor} w_k$, $k \geq 1$ when $x[0] = \Pi_h w[0]$ for an arbitrary $w[0] \in \mathbb{R}^{n_R}$. Moreover, if (A.2) is stable, and R has all its eigenvalues outside the open unit disk, then the solution to (A.14) is unique, and $x[k] \to \Pi_{\lfloor k \rfloor} w[k]$ as $k \to \infty$ for every initial condition $x[0]$, $w[0]$.

□

Proof If there exists Π_r such that (A.14) has a solution, then we can argue by induction that $x[k] = \Pi_{\lfloor k \rfloor} w[k]$. In fact, this is valid by hypothesis for $k = 0$, and if it is valid for $k = l$, then

$$\begin{aligned} x[l+1] - &\Pi_{\lfloor l+1 \rfloor} w[l+1] && \text{(A.15)} \\ &= A_l x[l] + B_l C_{Ul} w[l] - \Pi_{\lfloor l+1 \rfloor} R w[r] \\ &= (A_l \Pi_{\lfloor l \rfloor} + B_l C_{Ul} - \Pi_{\lfloor l+1 \rfloor} R) w[l] + A(x[l] - \Pi_{\lfloor l \rfloor} w[l]) \\ &= A_l (x[l] - \Pi_{\lfloor l \rfloor} w[l]) && \text{(A.16)} \\ &= 0. \end{aligned}$$

Conversely if $x[k] = \Pi_{\lfloor k \rfloor} w[k]$ when $x[0] = \Pi_T w[0]$ for an arbitrary $w[0] \in \mathbb{R}^{n_R}$, then $0 = x[r] - \Pi_r w[r] = (A\Pi_r + BC_U - \Pi_r) R w[r] = (A\Pi_r + BC_U - \Pi_r R) R^r w[0]$, which implies (A.6) since $w[0]$ is arbitrary and R is invertible.

Now suppose that (A.2) is stable, R has all its eigenvalues outside the open unit disk, and that there exists two solutions Π_r and $\tilde{\Pi}_r$, which satisfy (A.6). Let $E_r := \Pi_r - \tilde{\Pi}_r$ for $1 \leq r \leq h$. Then $E_{\lfloor k \rfloor} R = A_k E_{\lfloor k \rfloor}$, which implies that $E_r = 0$ for every $1 \leq r \leq h$. Moreover from (A.15), (A.16) and the stability of (A.2) we conclude that for every $x[0], w[0]$, we have that $x[k] \to \Pi_{\lfloor k \rfloor} w[k]$ when $k \to \infty$.

■

From Proposition 18, we can conclude that according to Definition 17, the system (A.2) has a blocking zero with respect to R if the following hold. If the input of (A.2) is generated by (A.13) for an arbitrary matrices $C_{Uk} = E_k$, then the output of (A.1) is identically zero. If this hold when (A.13) is time-invariant, then (A.2) with $n_A = n_B$ has a time-invariant blocking zero with respect to R.

The relation between the blocking zeros of a periodic system and the blocking zeros of its lift is provided in the next result.

Proposition 19. The periodic system (A.2) has a blocking zero with respect to R if and only if its LTI lifted system has a blocking zero with respect to R^h.

□

Proof From Proposition 18 we conclude that (A.2) has a blocking zero with respect to R, if and only if for every signal generated by (A.13), the output is zero. This holds if and only if (24) has zero output when the input is generated by

$$\begin{aligned} \hat{w}[l+1] &= R^h \hat{w}[l] \\ \hat{u}[l] &= F w[k], \end{aligned} \tag{A.17}$$

where

$$F = \begin{bmatrix} C_{U0} \\ C_{U1} R \\ \dots \\ C_{U(h-1)} R^{h-1} \end{bmatrix}$$

Since the $C_{U\kappa}, \kappa \in \{1, \ldots, h\}$ are arbitrary and R is invertible, we see that F can be made arbitrary, and using Proposition 15 we conclude that (24) has a blocking zero with respect to the matrix R^h.

REFERENCES

Antunes, D., Silvestre, C., and Cunha, R. (2008). Linear parameter varying control for sampled-data multi-rate systems. In *17th IFAC World Congress*. IFAC, Seoul, Korea.

Antunes, D., Silvestre, C., and Cunha, R. (2010). On the design of multi-rate tracking controllers: Application to rotorcraft guidance and control. *International Journal of Robust and Nonlinear Control*, 20, 1879–1902.

Bittanti, S. and Bolzern, P. (1985). Stabilizability and detectability of linear periodic systems. *System & Control Letters*, 6(2), 141–146.

Bittanti, S. and Colaneri, P. (2000). Invariant representations of discrete-time periodic systems. *Automatica*, 36(12), 1777 – 1793.

Byrnes, C.I. and Isidori, A. (2000). Output regulation for nonlinear systems: an overview. *International Journal of Robust and Nonlinear Control*, 10, 323337.

Chen, T. and Francis, B. (1995). *Optimal Sampled-Data Control Systems*. Springer-Verlag.

Colaneri, P. (1991). Output stabilization via pole placement of discrete-time linear periodic systems. *IEEE Transactions on Automatic Control*, 36(6), 739–742.

Colaneri, P., Scattolini, R., and Schiavoni, N. (1990). Stabilization of multirate sampled-data linear systems. *Automatica*, 26(2), 377 – 380.

Colaneri, P., Scattolini, R., and Schiavoni, N. (1991). Regulation of multirate sampled-data systems. *Control-Theory and Advanced Technology*, 7(3), 429–441.

Colaneri, P., Scattolini, R., and Schiavoni, N. (1992). Lqg optimal control of multirate sampled-data systems. *IEEE Transactions on Automatic Control*, 37(5), 675–682.

Davison, E. (1972). The output control of linear time-invariant multivariable systems with unmeasurable arbitrary disturbances. *Automatic Control, IEEE Transactions on*, 17(5), 621 – 630. doi: 10.1109/TAC.1972.1100084.

Francis, B. and Wonham, W.M. (1976). The internal model principle of control theory. *Automatica*, 12, 457–465.

Francis, B.A. (1977). The linear multivariable regulator problem. *SIAM J. Control Optim.*, 15(3), 486–505.

Lawrence, D. and Medina, E. (2001). Output regulation for linear systems with sampled measurements. volume 3, 2044 –2049 vol.3.

Longhi, S. (1994). Structural properties of multirate sampled-data systems. *Automatic Control, IEEE Transactions on*, 39(3), 692 –696. doi:10.1109/9.280790.

Meyer, R. and Burrus, C. (1975). A unified analysis of multirate and periodically time-varying digital filters. *IEEE Trans. Circuits Systems*, 22, 162–168.

Scattolini, R. and Schiavoni, N. (1993). On the output control of multirate systems subject to arbitrary exogenous signals. *Automatic Control, IEEE Transactions on*, 38(4), 643 –646. doi:10.1109/9.250540.

Wonham, W.M. and Pearson, J.B. (1974). Regulation and internal stabilization in linear multivariable systems. *SIAM Journal on Control*, 12(1), 5–18. doi: 10.1137/0312002.

Zhou, K., Doyle, J.C., and Glover, K. (1995). *Robust and Optimal Control*. Prentice Hall, Inc., Upper Saddle River, NJ.