Extending LaSalle's Invariance Principle to Switched Linear Systems*

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Abstract

In this paper we extend LaSalle's Invariance Principle to certain classes of switched linear systems. In particular we illustrated how to prove asymptotic stability using multiple Lyapunov functions whose Lie derivatives are only negative semi-definite.

1 Introduction

Several tools exist to prove asymptotic stability[1] of a system described by a linear differential equation of the form

$$\dot{x} = f(x), \qquad x \in \mathbb{R}^n, \tag{1}$$

with f locally Lipschitz and $f(0) = 0$. Lyapunov's Theorem [1] states that global asymptotic stability of (1) can be deduced from the existence of a continuously differentiable, positive definite, and radially unbounded[2] function $V : \mathbb{R}^n \to \mathbb{R}$ for which

$$L_f V(z) < 0, \qquad \forall z \in \mathbb{R}^n \setminus \{0\}, \tag{2}$$

where $L_f V$ denotes the *Lie derivative* of V along the vector field f. LaSalle's Invariance Principle [2] addresses cases when (2) is too restrictive and it states that, when we only have

$$L_f V(z) \leq 0, \qquad \forall z \in \mathbb{R}^n, \tag{3}$$

every solution $x(t)$ to (1) converges to the largest invariant set $\mathcal{M}$ contained in

$$\{z \in \mathbb{R}^n : L_f V(z) = 0\},$$

as time goes to infinity. When $\mathcal{M}$ is the singleton $\{0\}$, we can still conclude that (1) is globally asymptotically stable.

For a linear system

$$\dot{x} = Ax, \qquad x \in \mathbb{R}^n, \tag{4}$$

with a quadratic Lyapunov function $V(x) := x'Px$, $x \in \mathbb{R}^n$, where $P \in \mathbb{R}^{n\times n}$ is a symmetric positive definite matrix, the condition (3) is equivalent to requiring the matrix

$$Q := A'P + PA$$

to be negative semi-definite. When this condition is satisfied, we conclude from LaSalle's Invariance Principle that every solution to (4) converges to the largest invariant set $\mathcal{U}$ in the kernel of Q. From linear geometric theory [3] it is well known that $\mathcal{U}$ is the largest A-invariant subspace[3] in the kernel of Q, which is precisely the unobservable subspace of the pair (Q, A). Note that when Q is negative semi-definite, we can write Q as $-C'C$, with $C \in \mathbb{R}^{m\times n}$ full rank, and conclude that the solution to (4) converges to the unobservable subspace of the pair (C, A). In case (Q, A) or (C, A) is observable then $\mathcal{U}$ is the singleton $\{0\}$ and (4) is globally asymptotically stable.

This paper provides results towards the extension of LaSalle's invariance principle to *switched systems* of the form

$$\dot{x} = f_\sigma(x), \qquad x \in \mathbb{R}^n, \tag{5}$$

where σ is a piecewise constant *switching signal* taking values on an index set $\mathcal{P}$, and $\{f_p : p \in \mathcal{P}\}$ is a parameterized family of locally Lipschitz vector fields in $\mathbb{R}^n$ for which $f_p(0) = 0$, $p \in \mathcal{P}$. The time instants at which σ is discontinuous are called *switching times.* Typically, the switching signal σ is generated by a hybrid system, i.e., a system that combines continuous dynamics with discrete logic [4]. Without loss of generality we assume here that σ is continuous from above

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[1] In this paper we always consider systems for which the origin is an equilibrium point. With some abuse we say that a system is stable, meaning that the origin is a stable equilibrium point of the system.

[2] A function $V : \mathbb{R}^n \to \mathbb{R}$ is called *positive definite* if $V(z) \geq 0$ for all $z \in \mathbb{R}^n$, with equality just for $z = 0$. The function f is also called radially unbounded if $V(z)$ is unbounded as $z \to \infty$.

[3] A subspace $\mathcal{S}$ of $\mathbb{R}^n$ is called *A-invariant* if $A\mathcal{S} \subset \mathcal{S}$.

at each time instant, i.e., for every time t the limit from above of $\sigma(s)$ as $s \downarrow t$ is equal to $\sigma(t)$. Throughout this paper we assume that σ is *non-chattering*, i.e., σ has a finite number of discontinuities on any finite interval. At times we shall restrict our attention to "uniformly non-chattering" switching signals for which the number of switchings on any given time interval does not grow faster than linearly with the length of the interval. To be precise, we say that σ is *uniformly non-chattering* if there exist positive constants τ_D, N_0 for which

$$N_\sigma(t,\tau) \leq N_0 + \frac{t-\tau}{\tau_D}, \qquad t > \tau \geq 0, \tag{6}$$

where $N_\sigma(t,\tau)$ denotes the number of discontinuities of σ in the open interval (τ, t). A switching signal that satisfies (6) is said to have *average dwell-time* τ_D and *chatter bound* N_0 [5]. The set of all such switching signals is denoted by $\mathcal{S}_{\text{ave}}[\tau_D, N_0]$. Any *dwell-time* switching signal—i.e., a switching signal with interval between discontinuities uniformly bounded below by a positive constant—is uniformly non-chattering. The converse, of course, may not be true.

The most powerful tools used to prove stability of switched systems employ multiple Lyapunov functions [6, 7]. The following result is of this type [8]: Suppose that there exists a family $\{V_p : p \in \mathcal{P}\}$ of continuously differentiable, radially unbounded, positive definite functions from $\mathbb{R}^n$ to $\mathbb{R}$ such that

$$L_{f_p} V_p(z) \leq W(z), \qquad p \in \mathcal{P},\ z \in \mathbb{R}^n, \tag{7}$$

for some negative definite function $W : \mathbb{R}^n \to \mathbb{R}$ and

$$V_{p_2}\big(x(t)\big) \leq V_{p_1}\big(x(t)\big), \tag{8}$$

at every switching time t at which σ switches from p_1 to p_2. Then (5) is globally asymptotically stable. Note that, when σ is generated by a hybrid system, it is often possible to verify that (8) holds without actually computing the solution x to the switched system. This is because switching typically results from discrete transitions that are triggered by algebraic conditions on x.

As an extension of LaSalle's Invariance Principle, we will show that, for certain classes of switched linear systems, the function W in (7) need only be negative semi-definite to conclude asymptotic stability of the switched system (5). The results presented here go further than existing versions of LaSalle-like theorems for time-varying system, such as the ones in [1, 9], because we explicitly explore the switching structure of (5). Related to this research is also the work reported in [10], where the authors present an invariance principle for discrete-time systems that can be used to design switching controllers. In discrete-time, the closed loop switched system can be viewed as a time-invariant nonlinear system and the authors of [10] are able to use an argument similar to the one in [2] to prove asymptotic stability.

The remainder of this paper is organized as follows. In Section 2 we present the main result, namely that when a certain observability condition holds, we can conclude asymptotic stability of the switched system even when W in (7) is only negative semi-definite. In Section 3 we relax the observability condition. We show in Section 3.1 that when extra structure is available the observability condition can be relaxed to a detectability condition. In Section 3.2, we show that when no extra structure is available then—similarly to what happens in LaSalle's Invariance Principle—we can only conclude that x converges to a certain invariant set. Section 4 contains some concluding remarks and directions for future research.

2 Main Result

Consider the switched linear system

$$\dot{x} = A_\sigma x, \qquad x \in \mathbb{R}^n, \tag{9}$$

where the switching signal σ takes values on the index set $\mathcal{P}$ and $\{A_p : p \in \mathcal{P}\}$ is a parameterized compact family of $n \times n$ matrices, and suppose that there exists a compact family $\{P_p : p \in \mathcal{P}\}$ of symmetric positive definite $n \times n$ matrices such that

$$A_p' P_p + P_p A_p \leq 0, \qquad p \in \mathcal{P},$$

and

$$z' P_{p_2} z \leq z' P_{p_1} z, \tag{10}$$

at every point z in the state space, at which σ can switch from p_1 to p_2. Since all $A_p' P_p + P_p A_p$ are negative semi-definite we can write these matrices as

$$A_p' P_p + P_p A_p = -C_p' C_p, \qquad p \in \mathcal{P}. \tag{11}$$

for appropriate matrices $C_p \in \mathbb{R}^{m \times n}$. Defining

$$v(t) := x'(t) P_{\sigma(t)} x(t), \qquad t \geq 0,$$

because of (11) we conclude that between switching times

$$\dot{v}(t) = -\|y(t)\|^2 \leq 0, \qquad t \geq 0, \tag{12}$$

where $y(t) := C_{\sigma(t)} x(t)$. Moreover, at switching times $v(t)$ may be discontinuous but it is non-increasing because of (10). This means that v is uniformly bounded by $v(0)$ and

$$\|x(t)\| \leq \mu \|x(0)\|,\ \forall t \geq 0, \quad \mu := \frac{\max\limits_{p \in \mathcal{P}} \sigma_{\max}[P_p]}{\min\limits_{p \in \mathcal{P}} \sigma_{\min}[P_p]}, \tag{13}$$

where $\sigma_{\max}[P_p]$ and $\sigma_{\min}[P_p]$ denote the largest and smallest singular values of P_p, respectively. Uniform stability of the switched linear system (9) follows directly from (13). From (12) we can also conclude that $y(t)$ is an $\mathcal{L}_2$ signal[4]. Indeed, because of (12), for every $t \geq 0$,

$$v(t) \leq v(0) + \int_0^t \dot{v}(\tau)d\tau = v(0) - \int_0^t \|y(\tau)\|^2 d\tau,$$

and therefore

$$\int_0^t \|y(\tau)\|^2 d\tau \leq v(0) - v(t) \leq v(0),$$

which shows that $y \in \mathcal{L}_2$. Although $y \in \mathcal{L}_2$ and its derivative is bounded wherever it exists, we cannot use the so called Barbalat's Lemma [11, 12] to conclude that $y(t) \to 0$ as $t \to \infty$. Barbalat's Lemma cannot be applied because y is not continuous. However, this lemma can easily be extended to discontinuous signals, provided that the interval between consecutive discontinuities is uniformly bounded below by a positive constant. This happens, e.g, when σ is uniformly non-chattering. It turns out that when σ is uniformly non-chattering and, in addition, every pair (C_p, A_p), $p \in \mathcal{P}$ is observable, we can actually conclude that the whole state $x(t)$ converges to zero as $t \to \infty$ and not just $y(t) \to 0$. This is stated in the following theorem:

Theorem 1 *Suppose that there exists a compact family $\{P_p : p \in \mathcal{P}\}$ of symmetric positive definite $n \times n$ matrices such that*[5]

$$A_p' P_p + P_p A_p \leq -C_p' C_p \quad (\leq 0) \qquad p \in \mathcal{P}, \tag{14}$$

for appropriately defined matrices $C_p \in \mathbb{R}^{m \times n}$, and

$$z' P_{p_2} z \leq z P_{p_1} z, \tag{15}$$

at every point z in the state space, at which σ can switch from p_1 to p_2. Then (9) is uniformly stable. Moreover, if every pair (C_p, A_p), $p \in \mathcal{P}$ is observable and σ is uniformly non-chattering then (9) is uniformly asymptotically stable.

Remark 1 In hybrid control systems, σ is often generated by a supervisory logic that guarantees, by construction, that (14)–(15) hold. Typically, (15) only needs to hold over subsets of the state-space that are defined by simple algebraic conditions (often linear or affine subspaces of $\mathbb{R}^n$, cf. [8] and references therein).

To prove Theorem 1 we need the following two results. The first is taken from [5][6] and the second is the Squashing Lemma taken from [13].

[4]A signal y is said to be $\mathcal{L}_2$ if $\int_0^\infty \|y(t)\|^2 dt < \infty$.

[5]Although we assumed an equality in (11), the argument above also holds for the inequality in (14).

[6]In [5], (16) is formulated slightly differently using Lyapunov functions. For completeness, we prove in the Appendix this reformulation of what is essentially the same result.

Theorem 2 *Suppose that there exist positive constants γ and λ_0 such that*

$$\|e^{A_p t}\| \leq e^{\gamma - \lambda_0 t}, \qquad t \geq 0,\ p \in \mathcal{P}. \tag{16}$$

For every $\lambda \in (0, \lambda_0)$, and $N_0 \geq 0$ there exists a constant c such that the state transition matrix $\Phi_\sigma(t, \tau)$ of $\dot{z} = A_\sigma z$ can bounded by

$$\|\Phi_\sigma(t, \tau)\| \leq c e^{-\lambda(t-\tau)}, \qquad t \geq \tau \geq 0, \tag{17}$$

for every switching signal $\sigma \in \mathcal{S}_{\text{ave}}[\tau_D, N_0]$, where $\tau_D \geq \frac{\gamma}{\lambda_0 - \lambda}$.

Lemma 1 (Squashing Lemma) *Given any observable matrix pair (C, A) and any positive constants τ_0 and δ, there exists a positive constant λ^* such that for any $\lambda \geq \lambda^*$ it is possible to find an output-injection matrix K for which*

$$\|e^{(A+KC)t}\| \leq \delta e^{-\lambda(t-\tau_0)}, \qquad t \geq 0. \tag{18}$$

Remark 2 The statement of the Squashing Lemma in [13] simply asserts the existence of a single λ and K for which (18) holds. However, in [13] it is actually proved that for a sufficiently large λ a gain K can always be found so that (18) holds. In the statement of the Squashing Lemma above we denote by λ^* the smallest value of λ for which the desired output-injection matrix K can be found.

Proof of Theorem 1. We have already shown that (9) is uniformly stable. It remains to show that when every pair (C_p, A_p), $p \in \mathcal{P}$ is observable and σ is uniformly non-chattering then $x(t) \to 0$ as $t \to \infty$.

Since σ is uniformly non-chattering, there exist constants τ_D and N_0 for which (6) holds. Because all the pairs (C_p, A_p), $p \in \mathcal{P}$ are observable, we conclude from Lemma 1 (with $\delta = 1$) that there exists a (sufficiently large) positive constant λ_0 and output-injection matrices K_p, $p \in \mathcal{P}$ for which

$$\|e^{(A_p + K_p C_p)t}\| \leq e^{-\lambda_0 (t - \frac{\tau_D}{2})} = e^{\gamma - \lambda_0 t}, \quad t \geq 0,\ p \in \mathcal{P},$$

where $\gamma := \frac{\tau_D \lambda_0}{2}$. Selecting $\lambda := \frac{\lambda_0}{2}$, we obtain $\frac{\gamma}{\lambda_0 - \lambda} = \tau_D$ and we can therefore use Theorem 2 to conclude that

$$\dot{z} = (A_\sigma + K_\sigma C_\sigma) z$$

is an exponentially stable system. To finish the proof, it remains to note that (9) can also be written as

$$\dot{x} = (A_\sigma + K_\sigma C_\sigma) x - K_\sigma y. \tag{19}$$

Since we already established that $y \in \mathcal{L}_2$ and that the homogeneous part of (19) is exponentially stable, it follows that x must converge to zero as $t \to \infty$. ∎

3 Relaxing the Observability Assumption

In this section we present two alternative methods to relax the observability assumption in Theorem 1. The first explores the case where the matrices A_p, $p \in \mathcal{P}$ have additional structure. The second does not require any extra structure but we are no longer able to conclude that the state x of the switched system converges to the origin. Instead, we conclude that x converges to a particular "invariant" set.

3.1 Switched State Feedback

We consider here the case when all the matrices A_p, $p \in \mathcal{P}$ only differ by a state feedback matrix gain, i.e., all the A_p, $p \in \mathcal{P}$ are of the form

$$A_p = A + BF_p, \tag{20}$$

where A and B are given matrices and $\{F_p : p \in \mathcal{P}\}$ is a compact set of state-feedback matrix gains. This type of structure arises, e.g., when a fixed time-invariant process is controlled using a switched state feedback gain.

The following result explores the added structure provided by (20) to relax the observability assumption and simply demand detectability. However, now the left-hand-side of (14) is also required to be constant. This result is inspired by the Switching Theorem in [14].

Theorem 3 *Suppose that* (20) *holds, that there exists a compact family* $\{P_p : p \in \mathcal{P}\}$ *of symmetric positive definite* $n \times n$ *matrices such that*

$$A_p' P_p + P_p A_p \le -C'C \quad (\le 0) \qquad p \in \mathcal{P},$$

for an appropriately defined matrix C, *and that*

$$z' P_{p_2} z \le z P_{p_1} z,$$

at every point z *in the state space, at which* σ *can switch from* p_1 *to* p_2. *Then* (9) *is uniformly stable. Moreover, if* (A, B, C) *is a left-invertible system, every pair* (C, A_p), $p \in \mathcal{P}$ *is detectable, and* σ *is uniformly non-chattering then* (9) *is uniformly asymptotically stable.*

The following result from [14, Lemma 5] will be used in the proof of Theorem 3:

Lemma 2 *Let* (A, B, C) *be a left-invertible system. There exist matrices* $A_-, \bar{A}, \bar{B}, \bar{C}, \bar{K}, \bar{F}, T$, *with* T *nonsingular such that*

$$TAT^{-1} = \begin{bmatrix} \bar{A} & \bar{B}\bar{F} \\ \bar{K}\bar{C} & A_- \end{bmatrix}, \tag{21}$$

$$TB = \begin{bmatrix} \bar{B} \\ 0 \end{bmatrix}, \qquad CT^{-1} = \begin{bmatrix} \bar{C} & 0 \end{bmatrix}, \tag{22}$$

where A_-*'s spectrum is the set of transmission zeros of* (A, B, C) *with negative real part, and* $(\bar{A}, \bar{B}, \bar{C})$ *does not have any transmission zero with negative real part.*

Proof of Theorem 3. We have already shown for the general case that (9) is uniformly stable and that $y := Cx \in \mathcal{L}_2$. It remains to show that, under the assumptions of the theorem, $x(t) \to 0$ as $t \to \infty$ for any σ that is uniformly non-chattering.

Let $A_-, \bar{A}, \bar{B}, \bar{C}, \bar{K}, \bar{F}, T$ be the matrices whose existence is guaranteed by Lemma 2. Since detectability is invariant under state-coordinate transformations and $(C, A + BF_p)$, $p \in \mathcal{P}$ is detectable then so is the pair

$$\big(CT^{-1}, T(A + BF_p)T^{-1}\big) = \Bigg(\begin{bmatrix} \bar{C} & 0 \end{bmatrix}, \begin{bmatrix} \bar{A} + \bar{B}\bar{F}_p & \bar{B}(\bar{F} + \tilde{F}_p) \\ 0 & A_- \end{bmatrix} + \begin{bmatrix} 0 \\ \bar{K} \end{bmatrix} \begin{bmatrix} \bar{C} & 0 \end{bmatrix} \Bigg),$$

where $\begin{bmatrix} \bar{F}_p & \tilde{F}_p \end{bmatrix} := F_p T^{-1}$, $p \in \mathcal{P}$. Moreover, since detectability is also invariant under output-injection transformations and A_- is a stability matrix, we conclude that each pair $(\bar{C}, \bar{A} + \bar{B}\bar{F}_p)$, $p \in \mathcal{P}$ must be detectable. But $(\bar{A}, \bar{B}, \bar{C})$ does not have any transmission zero with negative real part, therefore $(\bar{C}, \bar{A}+\bar{B}\bar{F}_p)$ must actually be observable. On the other hand, defining $\begin{bmatrix} z_1' & z_2' \end{bmatrix}' := Tx$, from (9), (20), and (21)–(22) we conclude that

$$\begin{bmatrix} \dot{z}_1 \\ \dot{z}_2 \end{bmatrix} = \begin{bmatrix} \bar{A} + \bar{B}\bar{F}_\sigma & \bar{B}(\bar{F} + \tilde{F}_\sigma) \\ \bar{K}\bar{C} & A_- \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix}. \tag{23}$$

Therefore,

$$\dot{z}_2 = A_- z_2 + \bar{K} y,$$

because $\bar{C} z_1 = y$. Since A_- is a stability matrix and $y \in \mathcal{L}_2$, we conclude that $z_2 \to 0$ as $t \to \infty$. As for z_1, from (23) we also conclude that

$$\dot{z}_1 = (\bar{A} + \bar{B}\bar{F}_\sigma) z_1 + \bar{B}(\bar{F} + \tilde{F}_\sigma) z_2. \tag{24}$$

Since each pair $(\bar{C}, \bar{A} + \bar{B}\bar{F}_p)$, $p \in \mathcal{P}$ is observable, reasoning as in the proof of Theorem 1 we conclude that there exists a set of output-injection matrices $\{\bar{K}_p : p \in \mathcal{P}\}$ such that

$$\dot{z} = (\bar{A} + \bar{B}\bar{F}_\sigma + \bar{K}_\sigma \bar{C}) z$$

is an exponentially stable system. Since (24) can also be written as

$$\dot{z}_1 = (\bar{A} + \bar{B}\bar{F}_\sigma + \bar{K}_\sigma \bar{C}) z_1 + \bar{B}(\bar{F} + \tilde{F}_\sigma) z_2 - \bar{K}_\sigma y,$$

and we already established that $z_2 \to 0$ as $t \to \infty$ and $y \in \mathcal{L}_2$, it follows that z_1 also converges to zero as $t \to \infty$. This finishes the proof because $x = T^{-1} \begin{bmatrix} z_1' & z_2' \end{bmatrix}'$ and we established that both z_1 and z_2 converge to zero as $t \to \infty$. ∎

3.2 Convergence to an Invariant Set

We proceed now to generalize Theorem 1 to the case when the observability assumption fails and we do not have additional structure. In this case, we will not be able to conclude that the state x of the switched system converges to zero. However, we will be able to show that it converges to a particular "invariant" set.

Take the basic assumptions of Theorem 1—except for the observability of the pairs (C_p, A_p), $p \in \mathcal{P}$—and consider a time interval $[t_k, t_{k+1})$ of length no smaller than τ_D on which $\sigma = p$. It is well known that there exists a coordinate transformation $\begin{bmatrix} x_u & x_o \end{bmatrix}' := Tx$ for which the system (9) can be represented as

$$\begin{bmatrix} \dot{x}_u \\ \dot{x}_o \end{bmatrix} = \begin{bmatrix} A_1 & A_2 \\ 0 & A_o \end{bmatrix} \begin{bmatrix} x_u \\ x_o \end{bmatrix}, \quad y = \begin{bmatrix} 0 & C_o \end{bmatrix} \begin{bmatrix} x_u \\ x_o \end{bmatrix}, \tag{25}$$

$t \in [t_k, t_{k+1})$, with (C_o, A_o) an observable pair. Moreover, $T^{-1} \begin{bmatrix} x_u' & 0 \end{bmatrix}'$, belongs to the unobservable subspace of the pair (C_p, A_p). From Lemma 1 we conclude that there exists a constant λ_δ and an output-injection matrix K_δ for which

$$\|e^{(A_o + K_\delta C_o)t}\| \leq \delta e^{-\lambda_\delta (t - \tau_D)}, \quad t \geq 0, \ p \in \mathcal{P}. \tag{26}$$

For the time being we do not pick a particular value for δ but we recall that λ_δ and K_δ in (26) will depend on the particular choice of δ to be specified below. Since we can also write (25) as

$$\begin{bmatrix} \dot{x}_u \\ \dot{x}_o \end{bmatrix} = \begin{bmatrix} A_1 & A_2 \\ 0 & A_o + K_\delta C_o \end{bmatrix} \begin{bmatrix} x_u \\ x_o \end{bmatrix} - \begin{bmatrix} 0 \\ K_\delta \end{bmatrix} y,$$

we conclude that

$$x_o(t) = e^{(A_o + K_\delta C_o)(t - t_k)} x_o(t_k) - \int_{t_k}^{t} e^{(A_o + K_\delta C_o)(t - \tau)} K_\delta y(\tau) d\tau,$$

$t \in [t_k, t_{k+1}]$. Taking norms and using the Cauchy-Schwartz inequality together with (26), we conclude that

$$\begin{aligned} \|x_o(t)\| &\leq \|e^{(A_o + K_\delta C_o)(t - t_k)}\| \, \|x_o(t_k)\| \\ &+ \Big(\int_{t_k}^{t} \|e^{(A_o + K_\delta C_o)(t - \tau)} K_\delta\|^2 d\tau \Big)^{\frac{1}{2}} \Big(\int_{t_k}^{t} \|y(\tau)\|^2 d\tau \Big)^{\frac{1}{2}} \\ &\leq \delta e^{-\lambda_\delta (t - t_k - \tau_D)} \|x_o(t_k)\| + k_\delta \Big(\int_{t_k}^{t} \|y(\tau)\|^2 d\tau \Big)^{\frac{1}{2}}, \end{aligned}$$

where

$$k_\delta := \delta \|K_\delta\| \Big(\int_0^\infty e^{-\lambda_\delta (s - \tau_D)} ds \Big)^{\frac{1}{2}} = \frac{\delta \|K_\delta\| e^{\lambda_\delta \tau_D}}{\lambda_\delta}.$$

In particular, for $t \in [t_k + \tau_D, t_{k+1}]$, we obtain

$$\|x_o(t)\| \leq \delta \|x_o(t_k)\| + k_\delta \Big(\int_{t_k}^{t_{k+1}} \|y(\tau)\|^2 d\tau \Big)^{\frac{1}{2}}.$$

Since we already established that the overall state of the system is bounded, $\|x_o(t_k)\|$ is bounded and therefore we can make the term $\delta \|x_o(t_k)\|$ arbitrarily small by choosing δ sufficiently small. Since it has also been established that $y \in \mathcal{L}_2$, once δ is chosen (and therefore k_δ takes a specific value), the term $\int_{t_k}^{t_{k+1}} \|y(\tau)\|^2 d\tau$ can be made arbitrarily small by considering an interval that starts at a time t_k sufficiently large. We therefore conclude that, given any $\epsilon > 0$, there is a time T_ϵ sufficiently large so that for any interval $[t_k, t_{k+1})$ of length no smaller than τ_D on which $\sigma = p$, with $t_k \geq T_\epsilon$, the distance from $x(t)$, $t \in [t_k + \tau_D, t_{k+1})$ to the unobservable subspace of the pair (C_p, A_p) is smaller than ϵ. Because of the compactness of the $\{A_p : p \in \mathcal{P}\}$, the time T_ϵ can be selected independent of p.

In case $\sigma \in \mathcal{S}_{\text{ave}}[\tau_D, N_0]$, one cannot have more than $N_0 - 1$ consecutive intervals between switchings that are smaller than τ_D. This means that there is a sequence of times $\{\tau_k\}$ spaced by less than $N_0 \tau_D$ with the property that, for every k, the distance from $x(\tau_k)$ to the unobservable subspace of the pair (C_p, A_p), $p := \sigma(\tau_k)$ is smaller than ϵ.

Suppose now that we define $\mathcal{M}$ to be the smallest subspace that is A_p-invariant for all $p \in \mathcal{P}$ and contains the unobservable subspaces of all pairs (C_p, A_p), $p \in \mathcal{P}$. Because $\mathcal{M}$ contains all the unobservable subspaces of the pairs (C_p, A_p), $p \in \mathcal{P}$ the distance from the x to $\mathcal{M}$ on the $\{\tau_k\}$ is smaller than ϵ. Since between consecutive τ_k, x evolves according to flows for which $\mathcal{M}$ is invariant, and the separation between the τ_k is bounded by $N_0 \tau_D$, we conclude that between the τ_k the distance from x to $\mathcal{M}$ is bounded by

$$e^{\alpha N_0 \tau_D} \epsilon, \qquad \alpha := \max_{p \in \mathcal{P}} \|A_p\|,$$

(cf. Lemma 3 in the Appendix). Because we can make ϵ arbitrarily small, we conclude that the distance from x to $\mathcal{M}$ actually converges to zero. The following can then be stated:

Theorem 4 *Suppose that there exists a compact family $\{P_p : p \in \mathcal{P}\}$ of symmetric positive definite $n \times n$ matrices such that*

$$A_p' P_p + P_p A_p \leq -C_p' C_p \quad (\leq 0) \qquad p \in \mathcal{P},$$

for appropriately defined matrices $C_p \in \mathbb{R}^{m \times n}$, and

$$z' P_{p_2} z \leq z P_{p_1} z,$$

at every point z in the state space, at which σ can switch from p_1 to p_2. Then (9) is uniformly stable. Moreover, x will converge to the smallest subspace $\mathcal{M}$ that is A_p-invariant for all $p \in \mathcal{P}$ and contains the unobservable subspaces of all pairs (C_p, A_p), $p \in \mathcal{P}$.

Theorem 1 is actually a special case of the more general result that was just proved since when all pairs (C_p, A_p), $p \in \mathcal{P}$ are observable, the set $\mathcal{M}$ only contains the origin.

4 Conclusion

We extended LaSalle's Invariance Principle to certain classes of switched linear systems. In particular, we illustrated how to prove asymptotic stability using multiple Lyapunov functions whose Lie derivatives are only negative semi-definite. We are currently extending these results to nonlinear switched systems.

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Appendix

Proof of Theorem 2. Denoting by $\{t_1, t_2, \dots, t_k\}$ the switching times in the interval (τ, t), because of (16) we conclude that

$$\|\Phi_\sigma(t,\tau)\| = \|e^{A_{\sigma(t_k)}(t-t_k)} e^{A_{\sigma(t_{k-1})}(t_k - t_{k-1})} \cdots \\ e^{A_{\sigma(t_1)}(t_2-t_1)} e^{A_{\sigma(\tau)}(t_1,\tau)}\| \leq e^{(k+1)\gamma - \lambda_0 (t-\tau)}.$$

For (17) to hold, we need to have

$$e^{(k+1)\gamma - \lambda_0(t-\tau)} \leq c e^{-\lambda(t-\tau)}, \tag{27}$$

or, equivalently,

$$k \leq \frac{\log c}{\gamma} - 1 + \frac{\lambda_0 - \lambda}{\gamma}(t-\tau).$$

Since k is precisely the number of discontinuities of σ in (τ, t), we conclude that exponential stability is guaranteed provided that $\sigma \in \mathcal{S}_{\text{ave}}[\tau_D, N_0]$ for any average dwell-time $\tau_D \geq \frac{\gamma}{\lambda_0 - \lambda}$. Any chatter bound N_0 can be accommodated at the expense of having a sufficiently large constant c in (27). ∎

Lemma 3 *[15] Given an $n \times n$ matrix A and a subspace $\mathcal{S}$ of $\mathbb{R}^n$ that is A-invariant, there exists a constant α such that, along trajectories of*

$$\dot{x} = Ax, \tag{28}$$

the distance $d(x, \mathcal{S})$ from x to $\mathcal{S}$ can be bounded by

$$d(x(t), \mathcal{S}) \leq e^{\alpha(t-\tau)} d(x(\tau), \mathcal{S}). \tag{29}$$

The constant α in (29) can be chosen equal to $\|A\|$.

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