Optimal Communication Logics in Networked Control Systems

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Typo in proceedings paper { eq. (22) and eq. above (11) }
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Distributed Control

Goal:
• minimize controller communication (stealth, bandwidth)
• study the effect of non-ideal communication (delays, drops, blackouts)
**Communication minimization**

controller node  controller node  controller node

controller couplings supported by a communication network

*The “every bit-counts” paradigm...*

**Goal:** Design each controller to minimize the number of bits/second that need to be exchanged between nodes (quantization, compression, …)

**Domain:** Media with little capacity and low-overhead protocols (bit at-a-time)

E.g., underwater acoustic comm. between a small number of nodes.

*The “cost-per-message” paradigm...*

**Goal:** Design each controller to minimize the number of message exchanges between nodes (scheduling, estimation, …)

**Domain:** Media shared by a large number of nodes with nontrivial media access control (MAC) protocol (packet at-a-time)

E.g., 802.11 wireless comm. between a large number of nodes.

**Scenarios**

Rendezvous in minimum-time or using minimum-energy (in spite of disturbances)

Group of autonomous agents cooperate in searching for a target (perhaps mobile—search & pursuit)
In this talk:

- **decoupled linear processes** (with stochastic disturbance $d$)

$$x^+ = \begin{bmatrix} * & 0 & 0 \\ 0 & * & 0 \\ 0 & 0 & * \end{bmatrix} x + \begin{bmatrix} * & 0 \\ 0 & * \\ 0 & 0 \end{bmatrix} u + d$$

- **coupled control objective**

$$\sum_{k=0}^{\infty} x' = \begin{bmatrix} * & * & 0 \\ * & * & * \\ 0 & * & * \end{bmatrix} x + \|u\|^2$$

E.g., rendez-vous of two vehicles

Notation: $x^+ \equiv x(k+1)$
Prototype problem

In this talk:

- **decoupled linear processes** (with stochastic disturbance $d$)

\[
x^+ = \begin{bmatrix} * & 0 & 0 \\ 0 & * & 0 \\ 0 & 0 & * \end{bmatrix} x + \begin{bmatrix} * & 0 & 0 \\ 0 & * & 0 \\ 0 & 0 & * \end{bmatrix} u + d
\]

Minimum-cost solution (centralized)

\[
u = \begin{bmatrix} * & * & * \\ * & * & * \\ * & * & * \end{bmatrix} x
\]

Completely decentralized solution

\[
u = \begin{bmatrix} * & 0 & 0 \\ 0 & * & 0 \\ 0 & 0 & * \end{bmatrix} x
\]

Communication performance trade-off

- Optimal communication
- Achievable cost/comm. pairs
- Non-achievable cost/comm. pairs
- Minimum-cost (centralized)
- Minimum comm. needed for solvability (zero, when decentralized solution yields finite cost)
Centralized architecture

\[ x_i^\dagger = A_i x_i + B_i u_i + d_i \]

\[ u_i = K_{ii} x_i + K_{ij} x_j \]

constant communication between local and external process(es)

\[ x_j^ \dagger = A_j x_j + B_j u_j + d_j \]

Closed-loop system

\[ x_i^\dagger = (A_i + B_i K_{ii}) x_i + B_i K_{ij} x_j + d_i \]
\[ x_j^\dagger = (A_j + B_j K_{jj}) x_j + B_j K_{ji} x_i + d_j \]

for simplicity here we assume only two processes

Estimator-based distributed architecture

\[ x_i^\dagger = A_i x_i + B_i u_i + d_i \]

\[ u_i = K_{ii} x_i + K_{ij} \hat{x}_j \]

\[ \hat{x}_j^\dagger = A_j \hat{x}_j + B_j \hat{u}_j \]
\[ u_j = K_{jj} \hat{x}_j + K_{ji} \hat{x}_i \]

continuous open-loop estimator with discrete "updates" from the network

\[ x_j^\dagger = A_j x_j + B_j u_j + d_j \]

Closed-loop system

\[ x_i^\dagger = (A_i + B_i K_{ii}) x_i + B_i K_{ij} x_j + d_i - B_i K_{ij} e_j \]
\[ e_i := x_i - \hat{x}_i \]
\[ x_j^\dagger = (A_j + B_j K_{jj}) x_j + B_j K_{ji} x_i + d_j - B_j K_{ji} e_i \]
\[ e_j := x_j - \hat{x}_j \]

additive perturbation w.r.t centralized equations

[Yook & Tilbury, Montestruque & Antsaklis, Xu & JH]
Communication logic

\[ \dot{x}_i = A_i \dot{x}_i + B_i \dot{u}_j \]

Fusion logic

Scheduling logic

\[ x_j^+ = A_j x_j + B_j u_j + d_j \]

How to fuse data?

With no noise & delay (for now...)

\[ x_j(k) \text{ received from network at time } k \]

\[ \dot{x}_j(k+1) = A_j x_j(k) + B_j u_j(k) \]

"best-estimate" based on data received

\[ \epsilon_j(k+1) = -d_j(k) \]

When to send data?

Scheduling logic action

\[ a_j(k) = \begin{cases} 1 & x_j(k) \text{ sent at time } k \\ 0 & \text{no transmission at } k \end{cases} \]

Options...

- periodically
  \[ a_j(k) = \{ k \text{ divisible by } T \}, \quad T \in \mathbb{N} \]
- feedback policy
  \[ a_j(k) = F(x_j(k), \ldots) \]
- "optimal" ...

Optimal Scheduling Logic

\[ d \text{ is Gaussian i.i.d. with zero mean and covariance } \Sigma \]

Goals:

- minimize the estimation error \( \Rightarrow \) minimize cost-penalty w.r.t. centralized
- minimize the number of transitions \( \Rightarrow \) minimize communication bandwidth

\[ \limsup_{T \to \infty} E \left[ \frac{1}{T} \sum_{k=0}^{T-1} \epsilon_j(k)'^Q \epsilon_j(k) \right] \]

Average L-2 norm

\[ \limsup_{T \to \infty} E \left[ \frac{1}{T} \sum_{k=0}^{T-1} a_j(k) \right] \]

Average transmission rate

\[ \min_{a(k)} \limsup_{T \to \infty} E \left[ \frac{1}{T} \sum_{k=0}^{T-1} \epsilon_j(k)'^Q \epsilon_j(k) + \lambda a_j(k) \right] \]

Relative weight of two criteria (will lead to Pareto-optimal solution)
Dynamic Programming solution

\[ \min_{a(k)} \lim_{T \to \infty} \sup_{a(k)} \mathbb{E} \left[ \frac{1}{T} \sum_{k=0}^{T-1} e(k)^T Q e(k) + \lambda a(k) \right] \]

for simplicity of notation
1. dropped j/i subscripts
2. \( A \) denotes \( A_j + B_j K_j \)
3. \( d \) is Gaussian with zero mean and covariance \( \Sigma \)

Theorem

\[
(TV)(e) := \min_{a} \mathbb{E} \left[ e^{\pi^*(e) + \lambda a + V(e^*)} \mid e \right] \quad \text{for dynamic programming (DP) operator}
\]

1. There exists \( J^* \in \mathbb{R} \) and bounded \( h^* : \mathbb{R}^n \to \mathbb{R} \) such that
   \[ h^*(0) = 0, \quad h^* + J = Th^* \]
2. \( J^* \) is the optimal cost and is achieved by the \((deterministic)\) static policy
   \[
a(k) = \pi^*(e(k)), \quad \pi^*(e) := \begin{cases} 1 & \mathbb{E}[h^*(Ae + d)] + e^T Q Ae \geq \mathbb{E}[h^*(d)] + \lambda \\ 0 & \text{otherwise} \end{cases}
   \]
3. \( h \) can be found by value iteration
   \[
h_{i+1} = Th_i - (Th_i)(0) \xrightarrow{i \to \infty} h^*
   \]

Proof outline:

1. \( e(k) \) is Markov and its transition distribution satisfies an Ergodic property
   (requires a mild restriction on the set of admissible policies omitted here)
2. \( T \) is a span-contraction [Hernandez-Lerma 96]
3. Result follows using standard arguments based on Banach’s Fixed-Point Theorem for semi-norms.

Theorem

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   \]
Example (2-dim)

\[
\limsup_{T \to \infty} \mathbb{E} \left[ \frac{1}{T} \sum_{k=0}^{T-1} e(k) e(k) + \lambda a(k) \right] = 0
\]

local process

\[
A := A_j + B_j K_j = \begin{bmatrix} 1 & 1 \\ 1 & .9 \end{bmatrix}
\]

\[
\mathbb{E} \left[ d(k) d(k) \right] = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}
\]

not ellipses!

Example (2-dim)

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\limsup_{T \to \infty} \mathbb{E} \left[ \frac{1}{T} \sum_{k=0}^{T-1} e(k) e(k) + \lambda a(k) \right] = 0
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\]

large weight in comm. cost

large error threshold

only communicate when error is very large
Communication with latency

\[
\dot{x}_i^+ = A_j \dot{x}_j + B_j \dot{u}_j
\]

jth external process

\[
x_j^+ = A_j x_j + B_j u_j + d_j
\]

\(\tau\) is incorporated into estimate

\(\tau = 0\) in previous case

Dynamic Programming solution

\[
\min_{a(k)} \limsup_{T \to \infty} \mathbb{E} \left[ \frac{1}{T} \sum_{k=0}^{T-1} e(k)' Q e(k) + \lambda a(k) \right]
\]

undiscounted average-cost problem

Theorem

\[(TV)(e) := \min_{a} \mathbb{E} \left[ (1 - a)e' A^{r+1} Q A^{-1} e + \lambda a + \rho^2 + V((1 - a)A e + d) \mid e \right]
\]

1. There exists \(J^* \in \mathbb{R}\) and bounded \(h^* : \mathbb{R}^n \to \mathbb{R}\) such that

\[
h^*(0) = 0, \quad h^* + J = Th^*
\]

2. \(J^*\) is the optimal cost and is achieved by the deterministic static policy

\[
a(k) = \pi^*(e(k)), \quad \pi^*(e) := \begin{cases} 1 & \mathbb{E}[h^*(A e + d)] + e' A' Q A e \geq \mathbb{E}[h^*(d)] + \lambda \\ 0 & \text{otherwise} \end{cases}
\]

3. \(h^*\) can be found by value iteration

\[
h_{i+1} = Th_i - (Th_i)/(0) \quad \text{exp.} \quad h^*
\]
Example (1-dim)

\[
\limsup_{T \to \infty} E \left[ \frac{1}{T} \sum_{k=0}^{T-1} e(k)^2 + \lambda a(k) \right]
\]

\[
A := A_j + B_j K_j = 2
\]

\[
E[d(k)^2] = .01, \ \forall k
\]

optimal scheduling

\[
a(k) = \begin{cases} 
1 & |\hat{e}(k)| \geq \epsilon_{\text{threshold}} \\
0 & \text{otherwise}
\end{cases}
\]

estimation error, given all information enroute

with network latency
same error variance
requires more bandwidth

Conclusions

We constructed communications logics that minimize communication (measured in messages sending rate)

We considered networks with (fixed) latency

Study the effect of packet losses
(especially important in wireless networks)

Coupled control/communication-logic design

Nonlinear processes

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