Root-Mean-Square Gains of Switched Linear Systems: A Variational Approach

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Abstract— **We consider the problem of computing the root-mean-square (RMS) gain of switched linear systems. We develop a new approach which is based on an attempt to characterize the "worst-case" switching law (WCSL), that is, the switching law that yields the maximal possible gain. Our main result provides a sufficient condition guaranteeing that the WCSL can be characterized explicitly using the differential Riccati equations corresponding to the linear subsystems. This condition automatically holds for first-order systems, so we obtain a complete solution to the RMS gain problem in this case. In particular, we show that in the first-order case there always exists a WCSL with no more than two switches.**

I. Introduction

Consider the switched linear system

$$\begin{aligned}\dot{\boldsymbol{x}} &= A_{\sigma(t)}\boldsymbol{x} + B_{\sigma(t)}\boldsymbol{u} \\ \boldsymbol{y} &= C_{\sigma(t)}\boldsymbol{x},\end{aligned} \tag{1}$$

where $\boldsymbol{x} \in \mathbb{R}^n$, $\boldsymbol{u} \in \mathbb{R}^m$, and $\boldsymbol{y} \in \mathbb{R}^k$. The *switching signal* $\sigma : \mathbb{R}_+ \to \{1, 2\}$ is a piecewise constant function specifying at each time instant t, the index of the currently active system. Roughly speaking, (1) models a system that can switch between the two linear sub-systems

$$\dot{\boldsymbol{x}} = A_1\boldsymbol{x} + B_1\boldsymbol{u}, \qquad \boldsymbol{y} = C_1\boldsymbol{x}, \tag{2}$$

and

$$\dot{\boldsymbol{x}} = A_2\boldsymbol{x} + B_2\boldsymbol{u}, \qquad \boldsymbol{y} = C_2\boldsymbol{x}. \tag{3}$$

Note that we consider subsystems with no direct input-to-output term. To avoid some technical difficulties, we assume from here on that both linear subsystems are minimal realizations.

Let $\mathcal{S}$ denote the set of all piecewise constant switching laws. Many important problems in the analysis and design of switched systems can be phrased as follows.

Problem 1: Given $\mathcal{S}' \subseteq \mathcal{S}$ and a property P of dynamic systems, determine whether the switched system (1) satisfies property P for **every** $\sigma \in \mathcal{S}'$.

For example, when $\boldsymbol{u} \equiv 0$, P is the property of asymptotic stability of the equilibrium $\boldsymbol{x} = \boldsymbol{0}$, and $\mathcal{S}' = \mathcal{S}$, Problem 1 specializes into the following problem.

Problem 2: [1], [2] Is the switched system (1) asymptotically stable under arbitrary switching laws?

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Solving Problem 1 is difficult for two reasons. First, the set $\mathcal{S}'$ is usually huge, so exhaustively checking the system's behavior for each $\sigma \in \mathcal{S}'$ is impossible. Second, it is entirely possible that each of the subsystems satisfies property P, yet the switched system admits a solution that does not satisfy property P. This implies that it is not enough to merely check the behaviors of the subsystems.

A useful approach for addressing Problem 1 is based on studying the "worst-case" scenario. We say that $\tilde{\sigma} \in \mathcal{S}'$ is a "worst-case" switching law in $\mathcal{S}'$, with respect to property P, if the following condition holds: if the switched system satisfies property P for $\tilde{\sigma}$, then it satisfies property P for *any* $\sigma \in \mathcal{S}'$. Thus, the analysis of property P under *arbitrary* switching signals from $\mathcal{S}'$ is reduced to analyzing the behavior of the switched system for the *specific* switching signal $\tilde{\sigma}$.

The worst-case switching law for Problem 2 (that is, the "most destabilizing" switching law) can be characterized using variational principles (see the survey paper [3]). This approach originated in the pioneering work of E. S. Pyatnitskii on the celebrated *absolute stability problem* [4], [5] (see also [6], [7], [8]). The basic idea is to embed the switched system in a more general bilinear control system.[1] Then, the "most destabilizing" switching law can be characterized as the solution to a suitable optimal control problem. For second-order systems, this problem can be explicitly solved using the generalized first integrals of the subsystems [10], [11].

In this paper, we use a similar approach for studying the RMS gain problem. Our main result (see Theorem 5 below) shows that if a certain condition holds, then the worst case switching law can be explicitly characterized. Roughly speaking, the WCSL is obtained by following the solution to a switched differential Riccati equation (DRE), where the switching is between the two DREs corresponding to the two linear subsystems.

This condition automatically holds for first-order systems (that is, when $n = m = k = 1$), so we obtain a complete solution to the RMS gain problem for this case.

The remainder of this paper is organized as follows. In the next section, we briefly review some known results on the RMS gain of (non-switched) linear systems that will be used later on. Section III defines the RMS gain of switched systems. Section IV describes the variational approach and uses it to derive our main result. As a corollary, in Section V we obtain a complete solution to the RMS gain problem for

[1] For a recent and comprehensive presentation of bilinear systems, see [9].

first-order switched systems. The final section concludes.

II. Preliminaries

In this section, we describe some known results on the RMS gain of (non-switched) linear systems and the associated Riccati equations. For more details and the proofs, see [12] and the references therein.

Consider the linear system

$$\dot{\boldsymbol{x}} = A\boldsymbol{x} + B\boldsymbol{u}, \qquad \boldsymbol{y} = C\boldsymbol{x}, \tag{4}$$

with (A, B) controllable and (A, C) observable.

Fix some $T \in (0, +\infty]$, and let $\mathcal{L}_{2,T}$ denote the set of functions $\boldsymbol{f}(\cdot)$ such that $||\boldsymbol{f}||_{2,T} := \left(\int_0^T \boldsymbol{f}^T(t)\boldsymbol{f}(t)dt\right)^{1/2} < \infty$. The RMS gain over $[0, T]$ of the linear system (4) is defined by

$$g(T) := \inf\{\gamma \geq 0 : ||\boldsymbol{y}||_{2,T} \leq \gamma||\boldsymbol{u}||_{2,T}, \quad \forall \boldsymbol{u} \in \mathcal{L}_{2,T}\},$$

where $\boldsymbol{y}$ is the output of (4) corresponding to $\boldsymbol{u}$ with $\boldsymbol{x}(0) = \boldsymbol{0}$.

It is well-known that $g(\infty) = ||C(sI - A)^{-1}B||_\infty$, where $||Q(s)||_\infty := \sup_{Re(s)\geq 0} ||Q(s)||$, that is, the $\mathbb{H}_\infty$ norm of the transfer matrix $Q(s)$.

The RMS gain can also be computed by solving a suitable Riccati equation [13]. This equation arises because the Hamilton-Jacobi-Bellman equation, characterizing a suitable dissipation function [14], admits a solution with a quadratic form for linear systems. Fix $T \in (0, \infty)$, a symmetric matrix $P_T \in \mathbb{R}^{n\times n}$, and consider the *differential Riccati equation* (DRE):

$$\dot{P}(t) = -S(P(t); \gamma), \qquad P(T) = P_T, \tag{5}$$

where $S(P; \gamma) := PA + A^T P + C^T C + \gamma^{-2} PBB^T P$.

Theorem 1: $g(T) = \inf\{\gamma \geq 0 :$ solution to (5) with $P_T = 0$ exists on $[0, T]\}$.

The *algebraic Riccati equation* (ARE) associated with (5) is:

$$S(P; \gamma) = 0. \tag{6}$$

Theorem 2: Consider the linear system (4), where A is Hurwitz. Fix $\gamma > 0$ and denote $R := \gamma^{-2}BB^T$. If $\gamma > g(\infty)$ then the ARE (6) admits solutions $P^-, P^+ \in \mathbb{R}^{n\times n}$, referred to as the *stabilizing* and *antistabilizing* solutions, respectively, with the following properties: P^- is symmetric, positive definite, and $A + RP^-$ is Hurwitz. P^+ is symmetric, positive definite, and $-(A + RP^+)$ is Hurwitz. Moreover, $P^+ - P^- > 0$.

It is possible to express the solution $P(t)$ to the DRE (5) using P^- and P^+.

Theorem 3: Suppose that the conditions of Theorem 2 hold, and that $P_T - P^+$ is nonsingular. Define $Q := (P_T - P^+)^{-1} + (P^+ - P^-)^{-1}$, and $\Lambda(t) := e^{(A+RP^+)(t-T)}Qe^{(A+RP^+)^T(t-T)} - (P^+ - P^-)^{-1}$, for $t \leq T$.

1) The solution $P(t)$ to (5) is given by
$$P(t) = P^+ + (\Lambda(t))^{-1}, \qquad t \in I,$$
where $I \subset (-\infty, T]$ is an interval on which Λ is nonsingular.
2) If $P_T - P^+ < 0$, the solution $P(t)$ exists for all $t \leq T$ and $P(t) \to P^-$ as $t \downarrow -\infty$.
3) If $P_T - P^+ \not\leq 0$, the solution $P(t)$ has a finite escape time, that is, there exist $\tau \in (-\infty, T)$ and $\boldsymbol{z} \in \mathbb{R}^n$ such that $\lim_{t\downarrow\tau} \boldsymbol{z}^T P(t)\boldsymbol{z} = +\infty$.

III. RMS gain of switched systems

The RMS gain of (1), over some set of switching signals $\mathcal{S}' \subseteq \mathcal{S}$, is defined by

$$g_{\mathcal{S}'}(T) := \inf\{\gamma \geq 0 : ||\boldsymbol{y}||_{2,T} \leq \gamma||\boldsymbol{u}||_{2,T},\ \forall \boldsymbol{u} \in \mathcal{L}_{2,T},\ \forall \sigma \in \mathcal{S}'\},$$

where $\boldsymbol{y}$ is the solution to (1) corresponding to $\boldsymbol{u}, \sigma$, with $\boldsymbol{x}(0) = \boldsymbol{0}$. An important open problem in the design and analysis of switched systems is the computation of $g_{\mathcal{S}}(T)$ [15]. Calculating induced gains is also the first step toward the application of robust control techniques to switched systems [16], [17], [18].

By definition, $g_{\mathcal{S}}(T) \geq \max(g_1(T), g_2(T))$, where $g_i(T)$ is the RMS gain of the ith linear subsystem. It is well-known and easy to demonstrate that global asymptotic stability of the individual linear subsystems is necessary, but not sufficient for global asymptotic stability of the switched system (1) for every $\sigma \in \mathcal{S}$. This implies that $g_{\mathcal{S}}(T)$ may be arbitrarily large even when $g_1(T)$ and $g_2(T)$ are bounded. We assume from here on that the switched system is *globally uniformly asymptotically stable* (GUAS). For linear switched systems, this implies exponential convergence [19], that is, there exist $\lambda_1, \lambda_2 > 0$ such that

$$||\boldsymbol{x}(t)|| \leq \lambda_1||\boldsymbol{x}(0)||e^{-\lambda_2 t}, \quad \forall t \geq 0, \forall \sigma \in \mathcal{S}, \forall \boldsymbol{x}(0) \in \mathbb{R}^n.$$

In particular, this implies of course that A_1 and A_2 are Hurwitz. The next example, adapted from [20], shows that even in this case, $g_{\mathcal{S}}(T)$ can be very different from $g_1(T)$ and $g_2(T)$.

Example 1: Consider the system (1) with $n = k = m = 1$ and $a_1, a_2 < 0$. Each subsystem is asymptotically stable and since $n = 1$ it is clear that the switched system is GUAS.

Fix $T > 0$ and consider the behavior of the switched system for the switching signal $\sigma(t) = 1$ for $t \in [0, T/2)$, $\sigma(t) = 2$ for $t \in [T/2, T]$, the control $u(t) = 1$ for $t \in [0, T/2)$, $u(t) = 0$ for $t \in [T/2, T]$, and the initial condition $x(0) = 0$.

A calculation shows that the corresponding output is $y(t) = c_1 b_1(\exp(a_1 t) - 1)/a_1$ for $t \in [0, T/2)$, and $y(t) = c_2 b_1 \exp(a_2(t - T/2))(\exp(a_1 T/2) - 1)/a_1$ for $t \in [T/2, T]$. Note that $y(t)$ does not depend on b_2. Thus, for this particular switching signal and control

$$\begin{aligned}
&\frac{||y||_{2,T}^2}{||u||_{2,T}^2} \\
&= (2/T)\int_0^T y^2(t)dt \\
&= c_1^2 b_1^2(3 + Ta_1 + \exp(Ta_1) - 4\exp(a_1 T/2))/(Ta_1^3) \\
&+ c_2^2 b_1^2(\exp(a_1 T/2) - 1)^2(\exp(a_2 T) - 1)/(Ta_2 a_1^2).
\end{aligned}$$

Now suppose that $c_1b_1 = a_1$, $c_2 > 0$, and $b_2 = a_2/c_2$. In this case, $g_1(\infty) = g_2(\infty) = 1$, so $g_1(T), g_2(T) \leq 1$. Yet, for any $v > 0$, we can make $\frac{||y||_{2,T}}{||u||_{2,T}} > v$ by taking c_2 large enough. Thus, $g_{\mathcal{S}}(T)$ can be made arbitrarily large, even though both subsystems have RMS gain ≤ 1. ■

Several authors considered the RMS gain of switched systems over sets of sufficiently slow switching signals, i.e., switching signals with sufficiently large dwell time between consecutive discontinuities. Upper bounds for the RMS gain were derived in [21], [22], [23]. Hespanha [20] provided a complete solution to the problem of computing $g_{\mathcal{S}_1}(\infty)$, where $\mathcal{S}_1$ is the set of switching signals with no more than a single switch.

Theorem 4: [20] Fix $\gamma > \max(g_1(\infty), g_2(\infty))$. If $P_1^+ - P_2^- \not\geq 0$ or $P_2^+ - P_1^- \not\geq 0$ then $g_{\mathcal{S}_1}(\infty) \geq \gamma$. If $P_1^+ - P_2^- > 0$ and $P_2^+ - P_1^- > 0$ then $g_{\mathcal{S}_1}(\infty) \leq \gamma$.

Thus it is possible to determine whether $g_{\mathcal{S}_1}(\infty) \leq \gamma$ or $g_{\mathcal{S}_1}(\infty) \geq \gamma$ by analyzing the relationship between the stabilizing and antistabilizing solutions to the AREs associated with the subsystems.

In this paper, we apply a variational approach to the problem of computing the RMS gain for arbitrary switching laws.

IV. VARIATIONAL APPROACH

Denote $A(v) := A_1 + (A_2 - A_1)v$, $B(v) := B_1 + (B_2 - B_1)v$, and $C(v) := C_1 + (C_2 - C_1)v$. Our starting point is to embed the switched system in the more general control system:

$$\begin{aligned} \dot{\boldsymbol{x}} &= A(v)\boldsymbol{x} + B(v)\boldsymbol{u}, \\ \boldsymbol{y} &= C(v)\boldsymbol{x}, \end{aligned} \tag{7}$$

where $v \in \mathcal{V}$, the set of measurable controls taking values in $[0, 1]$.

The RMS gain of (7) over $[0, T]$ is defined by

$$\begin{aligned} g_b(T) := \inf\{\gamma \geq 0 : \\ ||\boldsymbol{y}||_{2,T} \leq \gamma ||\boldsymbol{u}||_{2,T},\ \forall \boldsymbol{u} \in \mathcal{L}_{2,T},\ \forall v \in \mathcal{V}\}, \end{aligned}$$

where $\boldsymbol{y}$ is the solution to (7) corresponding to $\boldsymbol{u}, v$, with $\boldsymbol{x}(0) = \boldsymbol{0}$.

Note that trajectories of the original switched system (1) correspond to piecewise constant controls v taking values in the set $\{0, 1\}$. In particular, for $v(t) = 0$ (7) yields (2), and for $v(t) = 1$ (7) yields (3). Every solution of the switched system is also a solution of (7), so $g_{\mathcal{S}}(T) \leq g_b(T)$ for all T.

On the other hand, it is well-known [24, Theorem 8.7] that we can approximate the effect of any measurable control using a piecewise constant control with "sufficiently fast" switching. This implies that for any $\epsilon > 0$:

$$g_{\mathcal{S}}(T) \leq g_b(T) \leq g_{\mathcal{S}}(T) + \epsilon.$$

Thus, the problem of computing the RMS gain of the switched system for arbitrary switching laws is equivalent to the problem of computing g_b. To do this, fix arbitrary $T \in [0, \infty)$ and $\gamma > 0$, and consider the problem of maximizing

$$J(\boldsymbol{u}, v) := \int_0^T (\boldsymbol{y}^T(\tau)\boldsymbol{y}(\tau) - \gamma^2 \boldsymbol{u}^T(\tau)\boldsymbol{u}(\tau))d\tau$$

along the trajectories of (7), with $\boldsymbol{x}(0) = \boldsymbol{0}$, $\boldsymbol{u} \in \mathcal{L}_{2,T}$ and $v \in \mathcal{V}$.

Applying the *maximum principle*, and guessing that the adjoint vector is linear in the state, yields that the candidates for optimality $(\boldsymbol{u}^*, v^*)$, and the corresponding trajectory $\boldsymbol{x}^*$, satisfy:

$$\begin{aligned} \dot{\boldsymbol{x}}^* &= A(v^*)\boldsymbol{x}^* + B(v^*)\boldsymbol{u}^*, \qquad \boldsymbol{x}(0) = \boldsymbol{0}, \\ \dot{P}^* &= -S(P^*, v^*; \gamma), \qquad P^*(T) = 0, \\ v^* &= \arg\max_{z \in [0,1]} (\boldsymbol{x}^*)^T S(P^*, z; \gamma)\boldsymbol{x}^*, \\ \boldsymbol{u}^* &= \gamma^{-2} B^T(v^*) P^* \boldsymbol{x}^*, \end{aligned} \tag{8}$$

where $S(P, z; \gamma) := PA(z) + A^T(z)P + \gamma^{-2} PB(z)B^T(z)P + C^T(z)C(z)$.

We can now state our main result.

Theorem 5: Fix arbitrary $T \in [0, \infty)$ and $\gamma > \max(g_1(T), g_2(T))$. Define $\boldsymbol{x}^*(t)$, $P^*(t)$, $\boldsymbol{u}^*(t)$ and $v^*(t)$ as the solutions to (8), and suppose that

$$S(P^*(t), v^*(t); \gamma) \geq S(P^*(t), z; \gamma), \quad \forall t \in I,\ \forall z \in [0, 1], \tag{9}$$

where I is the maximal interval of existence of the solution $P^*(t)$. Then the following properties hold.

1) If the solution $P^*(t)$ to (8) exists for all $t \in [0, T]$ then: $P^*(t) > 0$ for all $t \in [0, T)$, the problem of maximizing $J(\boldsymbol{u}, v)$ is well-defined, $(\boldsymbol{u}^*, v^*)$ is a maximizing pair, and $g_b \leq \gamma$.
2) If the solution $P^*(t)$ to (8) does not exist for all $t \in [0, T]$ then J is unbounded and $g_b \in [\gamma, +\infty]$.

Remark 1: It is possible to give an intuitive explanation of the WCSL v^* as follows. Fix an arbitrary $t \in int(I)$ and $\epsilon > 0$ sufficiently small, and denote $\tau := t - \epsilon$. Then:

$$\begin{aligned} P^*(\tau) &\approx P^*(t) - \epsilon \dot{P}(t) \\ &= P^*(t) + \epsilon S(P^*(t), v; \gamma). \end{aligned}$$

Thus, when condition (9) holds, the choice v^* maximizes the value $(\boldsymbol{x}^*(\tau))^T P^*(\tau)\boldsymbol{x}^*(\tau)$.

Remark 2: Note that

$$\begin{aligned} S(P^*(T), z; \gamma) &= C^T(z)C(z) \\ &= z^2 C_{21}^T C_{21} + z(C_{21}^T C_1 + C_1^T C_{21}) + C_1^T C_1, \end{aligned}$$

where $C_{21} := C_2 - C_1$. Thus, there exist k_0, k_1, k_2, with $k_2 \geq 0$, such that

$$(\boldsymbol{x}^*(T))^T S(P^*(T), z; \gamma)\boldsymbol{x}^*(T) = k_2 z^2 + k_1 z + k_0. \tag{10}$$

It is easy to verify that the maximum of (10) over $z \in [0, 1]$ is obtained for $z \in \{0, 1\}$. Hence, $v^*(T) \in \{0, 1\}$. More generally, a calculation shows that

$$\frac{d^2}{dz^2} S(P^*, z; \gamma) = 2C_{21}^T C_{21} + 2\gamma^{-2} P^* B_{21} B_{21}^T P^*.$$

Hence, $P^*(t) \geq 0$ implies that

$$\begin{aligned} &\max_{z \in [0,1]} (\boldsymbol{x}^*(t))^T S(P^*(t), z; \gamma)\boldsymbol{x}^*(t) \\ &\quad = \max_{z \in \{0,1\}} (\boldsymbol{x}^*(t))^T S(P^*(t), z; \gamma)\boldsymbol{x}^*(t), \end{aligned}$$

and there exists a WCSL $v^*(t)$ that takes values in $\{0,1\}$ only. Specifically,

$$v^*(t) = \begin{cases} 1, & (\boldsymbol{x}^*(t))^T D(P^*(t);\gamma)\boldsymbol{x}^*(t) > 0, \\ 0, & (\boldsymbol{x}^*(t))^T D(P^*(t);\gamma)\boldsymbol{x}^*(t) < 0, \end{cases}$$

where $D(P;\gamma) := S(P,1;\gamma) - S(P,0;\gamma)$. The solution $P^*(t)$ is then obtained by switching appropriately between the DREs $\dot{P} = -S_1(P)$ and $\dot{P} = -S_2(P)$, corresponding to the two linear subsystems (see [25] for some related considerations).

The next example demonstrates a simple application of Theorem 5.

Example 2: Consider the switched system (1) with $A_2 = A_1 - \alpha I$, $\alpha > 0$, $B_2 = B_1$, and $C_2 = C_1$. Recall that we assume that A_1 is Hurwitz, so $QA_1 + A_1^T Q = -I$ admits a solution $Q > 0$. It is easy to verify that the switched system is GAUS using the common Lyapunov function $V(\boldsymbol{x}) := \boldsymbol{x}^T Q \boldsymbol{x}$.

In this case, $S(P^*, z;\gamma) = P^*A_1 + A_1^T P^* + \gamma^{-2} P^* B_1 B_1^T P^* + C_1^T C_1 - 2z\alpha P^*$, so (9) holds for $v^*(t) \equiv 0$. Thus, Theorem 5 implies that $\sigma(t) \equiv 1$ is a "worst-case" switching law, and that $g_b(T) = g_1(T)$. This is, of course, what we may expect for this particular example. ■

Proof of Theorem 5. We consider two cases.

Case 1: Suppose that $P^*(t)$ exists for all $t \in [0,T]$.

Fix arbitrary $\boldsymbol{u} \in \mathcal{L}_{2,T}$, $v \in \mathcal{V}$, and let $\boldsymbol{x}$ denote the corresponding trajectory of (7), with $\boldsymbol{x}(0) = \boldsymbol{0}$. Define

$$m(t) := V(\boldsymbol{x}(t),t) + \int_0^t (\boldsymbol{y}^T(\tau)\boldsymbol{y}(\tau) - \gamma^2 \boldsymbol{u}^T(\tau)\boldsymbol{u}(\tau))d\tau, \tag{11}$$

where

$$V(\boldsymbol{x},t) := \boldsymbol{x}^T P^*(t)\boldsymbol{x}.$$

Note that $\boldsymbol{x}$ and $\boldsymbol{y}$ are evaluated along the trajectory corresponding to $(\boldsymbol{u},v)$. However, P^* is the matrix defined in (8).

Differentiating m yields

$$\begin{aligned} \dot{m} &= \boldsymbol{x}^T \dot{P}^* \boldsymbol{x} + 2\boldsymbol{x}^T P^*(A(v)\boldsymbol{x} + B(v)\boldsymbol{u}) + \boldsymbol{y}^T\boldsymbol{y} - \gamma^2 \boldsymbol{u}^T\boldsymbol{u} \\ &= -\gamma^2(\boldsymbol{u} - \gamma^{-2}B^T(v)P^*\boldsymbol{x})^T(\boldsymbol{u} - \gamma^{-2}B^T(v)P^*\boldsymbol{x}) \\ &\quad + \boldsymbol{x}^T(\dot{P}^* + S(P^*,v;\gamma))\boldsymbol{x} \\ &\le \boldsymbol{x}^T(-S(P^*,v^*;\gamma) + S(P^*,v;\gamma))\boldsymbol{x}. \end{aligned} \tag{12}$$

Using (9) yields

$$\dot{m}(t) \le 0, \qquad \forall t,$$

Hence, $m(T) \le m(0) = 0$ for any admissible pair $(\boldsymbol{u},v)$, and (11) yields $J(\boldsymbol{u},v) \le -V(\boldsymbol{x}(T),T) = -(\boldsymbol{x}(T))^T P^*(T)\boldsymbol{x}(T) = 0$. Recalling that $\boldsymbol{u}$ and v were arbitrary, this implies that $g_b \le \gamma$. Also, (12) implies that $\dot{m}(t) \equiv 0$ for $(\boldsymbol{u},v) = (\boldsymbol{u}^*,v^*)$, so $J(\boldsymbol{u}^*,v^*) = 0$.

Note that (11) implies that for any $t \in [0,T]$:

$$V(\boldsymbol{z},t) = \int_t^T (\boldsymbol{y}^T(\tau)\boldsymbol{y}(\tau) - \gamma^2\boldsymbol{u}^T(\tau)\boldsymbol{u}(\tau))d\tau + m(t) - m(T), \tag{13}$$

where $\boldsymbol{y} : [t,T] \to \mathbb{R}^k$ is the solution to (7) for the initial condition $\boldsymbol{x}(t) = \boldsymbol{z}$. Since the linear subsystems are controllable, there exists a control taking $\boldsymbol{x}(0) = \boldsymbol{0}$ to $\boldsymbol{x}(t) = \boldsymbol{z}$, for any $\boldsymbol{z} \in \mathbb{R}^n$, so (13) holds for any $\boldsymbol{z} \in \mathbb{R}^n$. Combining (13) with the analysis of $m(t)$ above yields

$$\begin{aligned} &V(\boldsymbol{z},t) \\ &= \sup_{\boldsymbol{u}\in\mathcal{L}_{2,T}, v\in\mathcal{V}, \boldsymbol{x}(t)=\boldsymbol{z}} \int_t^T (\boldsymbol{y}^T(\tau)\boldsymbol{y}(\tau) - \gamma^2\boldsymbol{u}^T(\tau)\boldsymbol{u}(\tau))d\tau, \end{aligned} \tag{14}$$

for all $t \in [0,T]$ and all $\boldsymbol{z} \in \mathbb{R}^n$. In other words, V is the finite-horizon "cost-to-go" function.

Taking the particular case $\boldsymbol{u}(s) = \boldsymbol{0}$ and $v(s) = 0$, $s \in [t,T]$, in (14) yields $\boldsymbol{z}^T P^*(t)\boldsymbol{z} \ge \boldsymbol{z}^T \int_t^T \exp(A_1^T(\tau - t))C_1^T C_1 \exp(A_1(\tau - t))d\tau \boldsymbol{z}$, and using the fact that (A_1, C_1) is observable, we conclude that $P^*(t) > 0$ for all $t \in [0,T)$.

Case 2: We now consider the case where $P^*(t)$ is not defined for all $t \in [0,T]$. Let $(s,T]$, with $s > 0$, denote the maximal interval of existence of the solution $P^*(t)$, so there exists $\boldsymbol{z} \in \mathbb{R}^n$ such that $\lim_{t\downarrow s} \boldsymbol{z}^T P^*(t)\boldsymbol{z} = +\infty$. It follows from (14) that for $\boldsymbol{x}(s) = \boldsymbol{z}$ there exist $(\boldsymbol{u},v)$ defined on $[s,T]$ that make $\int_s^T (\boldsymbol{y}^T(\tau)\boldsymbol{y}(\tau) - \gamma^2\boldsymbol{u}^T(\tau)\boldsymbol{u}(\tau))d\tau$ arbitrarily large. Since each linear subsystem is controllable, there exists a pair $(\boldsymbol{u},v)$ that takes the state from $\boldsymbol{x}(0) = \boldsymbol{0}$ to $\boldsymbol{x}(s) = \boldsymbol{z}$. Summarizing, we can find inputs $(\boldsymbol{u},v)$ defined on $[0,T]$ that make $J(\boldsymbol{u},v)$ arbitrarily large, so $\gamma \le g_b(T)$. This completes the proof of Theorem 5. ■

V. The case $n = 1$

In this section, we show that Theorem 5 provides a complete and computable solution to the RMS gain problem for first-order switched linear systems

By the definition of v^*, (9) automatically holds for the case $n = 1$. We can easily integrate $p^*(t)$ backwards in time from $t = T$ to $t = 0$, and then $g_b \le \gamma$ ($g_b \ge \gamma$) if $p^*(t)$ exists (does not exist) for all $t \in [0,T]$. This yields a simple bisection algorithm for determining the RMS gain of the switched system. It is also possible to derive some interesting theoretical results.

Proposition 1: Suppose that $n = 1$. Fix $T > 0$ and $\gamma > \max(g_1(T), g_2(T))$. Then

$$p^*(t_1) > p^*(t_2) \tag{15}$$

for any $t_1 < t_2$ such that $[t_1,t_2] \subseteq I$

Proof. See the Appendix.

Using this result, we can immediately bound the number of switches in the WCSL.

Corollary 1: Consider the system (1) with $n = 1$. For any $T \in (0,\infty)$, there exists a worst-case switching law $v^* : [0,T] \to \{0,1\}$ that contains no more than two switches.

Proof. The polynomial $d(l) := s_2(l;\gamma) - s_1(l;\gamma)$ is a second-order polynomial in l. Since $p^*(t)$ is monotonic, $d(p^*(t))$ can change sign no more than twice on any interval of time, and this completes the proof. ■

Example 3: Consider the switched system (1) with $n = 1$, $a_1 = -0.8642$, $b_1 = \sqrt{40}$, $c_1 = 1.4402$, $a_2 = -0.0622$, $b_2 = \sqrt{0.9}$, and $c_2 = 0.6831$. Fix the final time $T = 15$. Using the bisection algorithm we find that $10.7573 < g_b < 10.7574$. Fig. 1 shows the worst-case switching law $v^*(t)$

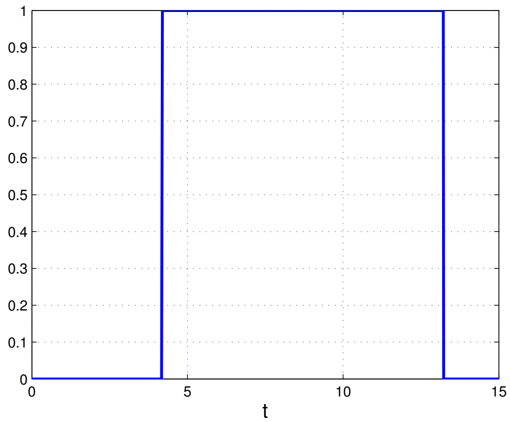

Fig. 1. The worst case switching law $v^*(t)$.

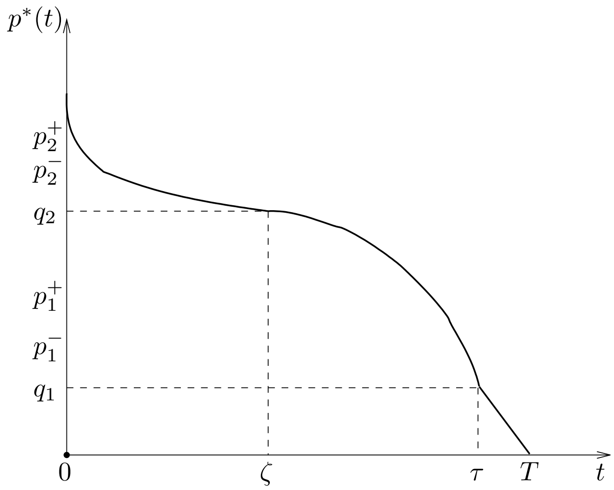

Fig. 2. Schematic behavior of $p^*(t)$.

for $\gamma = 10.7574$. Note that v^* is piecewise constant with two switching points.

We now explain the dynamics of this example in detail (see Fig. 2). A calculation yields: $p_1^- = 2.0000$, $p_1^+ = 3.0003$, $p_2^- = 6.0067$, $p_2^+ = 9.9886$, and that the roots of the polynomial $d(p) := s_1(p;\gamma) - s_2(p;\gamma)$ are $q_1 := 1.4375$ and $q_2 := 3.3098$. Since $d(0) = c_1^2 - c_2^2 > 0$, there exists a minimal time $\tau > 0$ such that $d(p^*(t)) > 0$ for $t \in (\tau, T]$, so $v^*(t) = 0$ for $t \in (\tau, T]$. We know that $p^*(t)$ increases monotonically as $t \downarrow \tau$ and that $p^*(t) \to p_1^- > q_1$ as $t \downarrow -\infty$. At time τ, we have $p^*(\tau) = q_1$, so $d(p^*(t))$ changes sign and the switching law is $v^*(t) = 1$ for some interval $t \in (\zeta, \tau)$. Since $p^*(\tau) < p_2^+$, the solution tends to $p_2^- > q_2$ as $t \downarrow -\infty$. At time ζ, we have $p^*(\zeta) = q_2$, and $d(p^*(t))$ changes sign again. Finally, on the interval $t \in (0, \zeta)$, $p^*(t)$ continues to increase, so there are no more sign changes of d and $v^*(t) = 0$ for $t \in (0, \zeta)$. ■

Example 3 implies that the bound two on the number of switches in Corollary 1 cannot be reduced in general. Surprisingly, perhaps, the situation is simpler for $T = \infty$.

Proposition 2: Consider the system (1) with $n = 1$. Fix $\gamma > \max(g_1(\infty), g_2(\infty))$. If $p_1^+ - p_2^- > 0$ and $p_2^+ - p_1^- > 0$ then $g_b(\infty) \leq \gamma$. If $p_1^+ - p_2^- < 0$ or $p_2^+ - p_1^- < 0$ then $g_b(\infty) \geq \gamma$.

Proof. See the Appendix.

Comparing this with Theorem 4, we conclude that for $T = \infty$, we can always determine the gain using switching signals from $\mathcal{S}_1$ only.

VI. Conclusions

We considered the problem of computing the RMS gain of a switched linear system. Our main result shows that if a certain condition holds, then the switching-law that yields the maximal gain can be described explicitly in terms of the DREs of the linear subsystems. This condition automatically holds when $n = 1$, so our main result provides a complete solution to the problem in this case.

Topics for further research include the following. More work is needed in order to characterize the conditions under which (9) holds. In this case, efficient numerical schemes are needed in order to compute the RMS gain according to the results in Theorem 5.

As noted above, a promising approach for addressing Problem 2 is based on studying the "most unstable" switching-law using variational principles. In particular, it is possible to show that certain geometric properties imply that there exists a worst-case switching-law with a bounded number of switches. The bound on the number of switches is uniform over any time interval $[0, T]$ [26][27]. An interesting problem is to seek geometric conditions guaranteeing that there exists a worst-case switching-law for the RMS gain problem with a uniform bound on the number of switchings. This would imply that, under these conditions, computing the RMS gain could be done efficiently using suitable numerical algorithms (see, e.g., [28], [29] for some related considerations).

Appendix: Proofs

Proof of Proposition 1. For $n = 1$, Theorem 3 yields $\lambda(t) = e^{2(a+rp^+)(t-T)}\alpha(p_T) - 1/(p^+ - p^-)$, $t \leq T$, where $\alpha(p_T) := 1/(p_T - p^+) + 1/(p^+ - p^-)$, $p^+ > p^- > 0$, $a + rp^+ > 0$, and $p(t) = p^+ + 1/\lambda(t)$. Fix arbitrary $t_1 < t_2$ such that $[t_1, t_2] \subset I$. It is easy to verify that

$$p_T \notin [p^-, p^+] \Rightarrow \lambda(t_1) < \lambda(t_2) \Rightarrow p(t_1) > p(t_2), \quad (16)$$

where the last implication is true because, by the definition of I, $\operatorname{sgn}(\lambda(t_1)) = \operatorname{sgn}(\lambda(t_2))$. In other words, $p_T \notin [p^-, p^+]$ implies that $p(t)$ satisfies the desired monotonicity property.

We know that $v^*(t)$ is piecewise constant and takes values in $\{0, 1\}$. Let τ_i denote its switching points such that $\cdots < \tau_3 < \tau_2 < \tau_1 < T$. Assume without loss of generality that $v^*(t) = 0$ for $t \in (\tau_1, T)$, so that $v^*(t) = 1$ for $t \in (\tau_2, \tau_1)$, and so on.

Since $p^*(T) = 0$ and $p_1^- > 0$, (16) implies that $p^*(t_1) > p^*(t_2)$ for all $\tau_1 \leq t_1 < t_2 \leq T$. Also, it follows from Theorem 3 and the monotonicity of $p^*(t)$ that $p^*(\tau_1) < p_1^-$. We now consider two cases.

Case 1: $p^*(\tau_1) \in [p_2^-, p_2^+]$. Using the fact that $s_1(l) = \gamma^{-2}b_1^2(l-p_1^-)(l-p_1^+)$ and $s_2(l) = \gamma^{-2}b_2^2(l-p_2^-)(l-p_2^+)$ yields $s_1(p^*(\tau_1)) > 0$ and $s_2(p^*(\tau_1)) \le 0$. Now (8) implies that τ_1 cannot be a switching point of $v^*(t)$. Thus, this case is impossible.

Case 2: $p^*(\tau_1) \notin [p_2^-, p_2^+]$. In this case, (16) implies that $p^*(t_1) < p^*(t_2)$ also holds for all $\tau_2 \le t_1 < t_2 \le \tau_1$.

Proceeding in this way, we see that $p^*(t)$ satisfies (15) for all $t_1 < t_2$. ∎

Proof of Proposition 2. If $p_1^+ - p_2^- < 0$ or $p_2^+ - p_1^- < 0$ then Theorem 4 implies that $g_{\mathcal{S}_1}(\infty) \ge \gamma$, so $g_{\mathcal{S}}(\infty) \ge \gamma$.

Consider now the case where

$$p_1^+ - p_2^- > 0 \text{ and } p_2^+ - p_1^- > 0. \tag{17}$$

In this case, Theorem 4 implies that $g_{\mathcal{S}_1}(\infty) \le \gamma$, and we need to show that

$$g_{\mathcal{S}}(\infty) \le \gamma. \tag{18}$$

If for every $T > 0$ the WCSL contains up to a single switch, then we are done. Thus, assume that for some T, the WCSL contains two switching points. In other words, the polynomial

$$\begin{aligned} d(l) &:= s_2(l;\gamma) - s_1(l;\gamma) \qquad (19)\\ &= \gamma^{-2}b_2^2(l-p_2^-)(l-p_2^+) - \gamma^{-2}b_1^2(l-p_1^-)(l-p_1^+) \end{aligned}$$

has two roots $0 < q_1 < q_2$. Without loss of generality, we assume that $d(l) < 0$ for $l \in [0, q_1)$, so $d(l) > 0$ for $l \in (q_1, q_2)$, and $d(l) < 0$ for $l > q_2$.

Using (19) yields $d(p_1^-) = \gamma^{-2}b_2^2(p_1^- - p_2^-)(p_1^- - p_2^+)$ and $d(p_2^-) = -\gamma^{-2}b_1^2(p_2^- - p_1^-)(p_2^- - p_1^+)$. Combining this with (17) implies that

$$\operatorname{sgn}(d(p_1^-)) = \operatorname{sgn}(d(p_2^-)). \tag{20}$$

We now consider three cases: If $p_1^- \in (0, q_1]$ then the WCSL will not include any switchings which is a contradiction. If $p_1^- \in (q_1, q_2)$ then $d(p_1^-) > 0$ and (20) implies that $p_2^- \in (q_1, q_2)$. Since $p_2^+ > p_2^-$ the WCSL will include a single switch which is a contradiction. Finally, suppose that $p_1^- \in [q_2, T]$. Then the WCSL may include two switches. However, in this case $p_2^+ > p_1^- \ge q_2$, and it is clear that $p^*(t)$ exists for all $t \in [0, T]$, with $p^*(t) \to p_1^-$ as $t \downarrow -\infty$. Theorem 5 implies that (18) indeed holds. ∎

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