Control of Impulsive Renewal Systems: Application to Direct Design in Networked Control

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***Abstract*— We consider the control of impulsive systems with independent and identically distributed intervals between jumps. The control action and output measurement are assumed to take place only at jump times. We give necessary and sufficient conditions, in the form of LMIs, for mean square stabilizability and detectability and solve an infinite horizon quadratic optimal control problem, under appropriate stabilizability and detectability properties of the system. The class of systems considered is especially suited to model networked control systems utilizing CSMA-type protocols, with stochastic intervals between transmissions and packet drops. In this setting, the analysis and synthesis tools mentioned above are used to (i) prove that for an emulation-based design, stability of the closed-loop is preserved if the distribution of the intervals between transmissions assigns high probability to fast sampling (ii) illustrate through a benchmark example the potential advantages of controller direct-design over an emulation-based design.**

I. Introduction

Networked control systems are spatially distributed systems for which the communication between sensors, actuators, and controllers is supported by a shared communication network. In [1], a class of systems is proposed which allows for the modeling of networked control systems with stochastic intervals between transmission and packet drops. We consider a controlled version of this class taking the form

$$\begin{aligned} \dot{x}(t) &= a(x(t)), \quad t \neq t_k \\ x(t_k) &= j(k, x(t_k^-), z_k) + b(k, u_k, z_k), \\ y_k &= c(k, x(t_k^-), z_k) \\ & \quad x(t_0^-) = x_0, \;\; t_0 = 0, t \in \mathbb{R}_{\geq 0}, \, k \in \mathbb{Z}_{\geq 0}, \end{aligned} \tag{1}$$

where the state $x(t)$ evolves in $\mathbb{R}^n$, the control u_k takes values in $\mathbb{R}^m$ and is applied at jump times, and the measurement signal y_k takes values in $\mathbb{R}^p$ and is only available at jump times. The notation $x(t_k^-)$ indicates the limit from the left of a function $x(t)$ at the point t_k, except at t_0 where, by convention, $x(t_0^-) = x_0$. The times between consecutive jumps $\{h_k := t_{k+1} - t_k, k \geq 0\}$ are assumed independent and identically distributed (i.i.d.) with a common cumulative distribution F. The variables $\{z_k, k \geq 0\}$, called *jump points*, are i.i.d. random variables taking values on a discrete finite set $\mathcal{Z}$, and are assumed to be independent of the $\{h_k, k \geq 0\}$. We restrict our analysis to linear maps $a,\ j,\ b$ and c.

The class of systems (1) was considered in the homogenous case ($b = 0$) in [1], [2] and [3]. In [1], the nomenclature impulsive system driven by renewal processes was used for this class of systems, motivated by the fact that the process that counts the number of jumps up to time t is a renewal process. Here, we abbreviate the nomenclature to impulsive renewal systems. In [3], necessary and sufficient conditions are given for mean square stability, stochastic stability, and mean exponential stability for the homogenous version of (1). The present paper builds upon these conditions. For related work on the analysis and control of systems with i.i.d. parameters see, e.g., [4], [5], [6], and the references therein.

In the first part of this paper, we give synthesis results pertaining to a controlled impulsive renewal system taking the form (1). We start by introducing notions of mean square stabilizability and mean square detectability for this class of systems. Our definitions parallel those for LTI systems (e.g. [7]) and Markov Jump Linear Systems (MJLS, e.g. [8]). We give necessary and sufficient conditions for verifying these two properties in terms of LMIs and show that if (1) exhibits these two properties, then there exists an output stabilizing controller. Finally, we consider an infinite horizon quadratic cost optimal control problem that parallels the well-known LQR problem. We show that if a Riccati-like equation has an appropriately defined stabilizing solution, then the optimal control law is given by a simple state-feedback law. Stabilizability and detectability conditions guarantee the existence of a stabilizing solution to the Riccati equation.

In the second part of the paper, we focus on the application of our results to networked control systems. We address scenarios in networked control systems for which the impulsive renewal system is an appropriate model for analysis and controller synthesis, in a similar fashion to the recent work [9]. These scenarios are based on Carrier Sense Multiple Access (CSMA) type protocols, which are prevalent in network links. By considering i.i.d. inter-transmission intervals, we can broaden the class of scenarios considered in [9], in which the analysis was restricted to exponentially distributed inter-transmission intervals. We then show how the tools provided in the first part of the paper can be used to compare an emulation-based approach with a controller direct-design approach. Regarding emulation-based design, we assume that a continuous-time stabilizing controller has been designed, without regard to the network characteristics.

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We show that stability of the closed-loop is preserved if the distribution of the inter-sampling times assigns high probability to fast sampling. As with periodic constant sampling, this result shows that an emulation-based design is an appropriate choice when the sampling is sufficiently fast. However, as illustrated in a benchmark example, for less frequent sampling it may fail to guarantee closed-loop stability in a mean square sense. For the same example, we show that if one directly designs a controller that takes into account the network constraints (direct design), it is possible to guarantee mean square closed-loop stability even when sampling does not occur frequently and emulation would fail.

The remainder of the paper is organized as follows. Section II introduces some preliminaries and reviews the results of [2]. Section III presents a set of basic general results pertaining to the stability and stabilization of (1). Section IV illustrates the application of these results to networked control systems. Section V contains final conclusions.

Notation: For two column vectors x, y, $(x, y) := [x'\, y']'$. The notation $\mathrm{diag}([A_1 \dots A_n])$ indicates a block diagonal matrix with blocks A_i. The $n \times n$ identity and zero matrices are denoted by I_n and 0_n, respectively. The probability of an event A is denoted by $\mathbf{P}[A]$ and the expected value by $\mathbb{E}[.]$.

II. PRELIMINARIES AND BACKGROUND

The i.i.d. random variables h_k follow a common distribution F with support on a given interval $[0, T], T \in \mathbb{R}_{>0} \cup \{+\infty\}$. We assume some regularity on F, namely that $F(0) = 0$, $F(\infty) = 1$, and that this function can be written as $F = F_1 + F_2$, where F_1 is an absolutely continuous function $F_1(t) = \int_0^t f(s)ds$, for some density function $f(x) \geq 0$, and F_2 is a piecewise constant increasing function that captures possible atom points $\{a_i\}$ where the distribution places mass $\{w_i\}$.

The random variables z_k belong to a finite discrete set taking the form $\mathcal{Z} := \{1, \dots, n_z\}$ and are assumed i.i.d. with $\mathbf{P}[z_k = i] = p_i$ for any k, where $\sum_{i=1}^{n_z} p_i = 1$.

In [3], the following simplified version (1) is considered

$$\begin{aligned} \dot{x}(t) &= Ax(t), \quad t \neq t_k \\ x(t_k) &= Jx(t_k^-), \quad x(t_0^-) = x_0, \;\; t_0 = 0. \end{aligned} \tag{2}$$

Since the system (2) is fully specified by the pair (A, J) and the distribution F, for short we use the notation $(A, J)_F$ to refer to this system.

We say that (2) is *Mean Square Stable* (MSS), which is abbreviated as $(A, J)_F$ is MSS, if for any x_0, $\lim_{t\to+\infty} \mathbb{E}[x(t)'x(t)] = 0$.

We assume that there exists $\lambda > \bar{\lambda}(A)$, where $\bar{\lambda}(A)$ is the real part of the eigenvalues of A with largest real part, s.t.,

$$e^{\lambda t}(1 - F(t)) \leq ce^{-\alpha_1 t} \text{ for some } c,\ \alpha_1 > 0. \tag{3}$$

Notice that this condition is met if T is finite or if A is Hurwitz. As explained in [3], this condition guarantees that mean square stability is equivalent to two other stability notions stochastic stability and mean exponentially stability.

In the following theorem we summarize the results of [3], relevant to the present paper. Let

$$\mathcal{E}_A(P) := \int_0^T e^{A's} P e^{As} F(ds).$$

Theorem 1: Assuming that (3) holds, (2) is MSS if and only if any one of the following conditions hold

$$\exists_{P>0} : \; J'\mathcal{E}_A(P)J - P < 0; \tag{4}$$

$$\exists_{P>0} : \; J\mathcal{E}_{A'}(P)J' - P < 0. \tag{5}$$

Note that the conditions (4) and (5) can be viewed as LMIs on the unknown matrix P since the left-hand side of (4) and (5) is an affine function of P.

III. DISCRETE CONTROL OF IMPULSIVE RENEWAL SYSTEMS

We start by considering the following version of (1)

$$\begin{aligned} \dot{x}(t) &= Ax(t), \quad t \neq t_k \\ x(t_k) &= Jx(t_k^-) + Bu_k, \\ y_k &= Cx(t_k^-), \quad x(t_0^-) = x_0, \;\; t_0 = 0. \end{aligned} \tag{6}$$

A. Stabilizability and Detectability

Intuitively, the system (6) is stabilizable if one can find a feedback control law of the form $u_k = Kx(t_k^-)$ that stabilizes the resulting closed-loop. Formally, the system (6) is said to be *mean square stabilizable*, which is abbreviated as $((A, J), B)_F$ is stabilizable, if there exists a matrix $K \in \mathbb{R}^{m\times n}$ such that $(A, J + BK)_F$ is MSS.

The system (6) is said to be *mean square detectable*, which is abbreviated as $(C, (A, J))_F$ is detectable, if there exists a matrix $L \in \mathbb{R}^{n\times p}$ such that $(A, J + LC)_F$ is MSS. If the system (6) is detectable it is possible to estimate in a mean square sense the state of the system (6) given the output. In fact, defining the estimator

$$\begin{aligned} &\dot{\hat{x}}(t) = A\hat{x}(t), \quad t \neq t_k \\ &\hat{x}(t_k) = J\hat{x}(t_k^-) + Bu_k + L(C\hat{x}(t_k^-) - y_k), \;\; \hat{x}(t_0^-) = \hat{x}_0, \end{aligned}$$

and the error variable $e := \hat{x} - x$, we have that

$$\begin{aligned} &\dot{e}(t) = Ae(t), \quad t \neq t_k \\ &e(t_k) = (J + LC)e(t_k^-), \quad e(t_0^-) = e_0, t_0 = 0, \end{aligned}$$

and therefore the estimation error goes to zero in a mean square sense.

Taking into account the two characterizations of MSS (4) and (5), one concludes that detectability of $(C, (A, J))_F$ is equivalent to stabilizability of $((A', J'), C')_F$, and therefore these two concepts are dual.

The detectability and stabilizability properties of (6) can be used to analyze the relation between MSS and the input-output properties of (6). Let $\mathcal{C}^q$ be the set of sequences of r.v.s $v = (v_0, v_1, \dots)$, such that (i) each $v_i \in \mathbb{R}^q$ is causally obtained from $\{h_k,\, k \leq i\}$, i.e., v_i is adapted to the natural filtration associated with the $\{h_k,\, k \leq i\}$; (ii) v has bounded norm, defined as $\|v\|^2 = \sum_{k\geq 0} \mathbb{E}(\|v_k\|^2)$. We say that the system (6) is input-output stable if $y \in \mathcal{C}^p$ when $u \in \mathcal{C}^m$.

Lemma 2: Suppose that the system (6) is stabilizable and detectable. Then the system (6) is input-output stable if and only it is MSS.

As with LTI systems, a separation principle also holds for the impulsive renewal system (6). In particular, given L such that $(A, J+LC)_F$ is MSS, and K such that $(A, J+BK)_F$ is MSS, the following output feedback controller results in a MSS closed-loop system.

$$\begin{aligned} \dot{\hat{x}}(t) &= A\hat{x}(t), \quad t \neq t_k \\ \hat{x}(t_k) &= (J+BK+LC)\hat{x}(t_k^-) - Ly_k, \\ u_k &= K\hat{x}(t_k^-), \qquad \hat{x}(t_0^-) = \hat{x}_0, \;\; t_0 = 0. \end{aligned} \tag{7}$$

Theorem 3: If the system (6) is detectable and stabilizable there exists an output feedback controller that makes the closed-loop MSS. One such controller is given by (7).

1) Conditions for testing stabilizability and detectability: The next theorem provides necessary and sufficient conditions for testing stabilizability in terms of LMIs

Theorem 4: The system (6) is stabilizable if and only if

$$\exists_{P>0,W} : \begin{bmatrix} \mathcal{E}_{A'}(P) & \mathcal{E}_{A'}(P)J' + W'B' \\ J\mathcal{E}_{A'}(P) + BW & P \end{bmatrix} > 0 \tag{8}$$

In this case, $(A, J+BK)_F$ is MSS for $K = W\mathcal{E}_{A'}(P)^{-1}$.

The dual result for detectability is stated as follows.

Theorem 5: The system (6) is detectable if and only if

$$\exists_{P>0,N} : \begin{bmatrix} \mathcal{E}_{A}(P) & \mathcal{E}_{A}(P)J + NC \\ J'\mathcal{E}_{A}(P) + C'N' & P \end{bmatrix} > 0. \tag{9}$$

In this case, $(A, J+LC)_F$ is MSS for $L = \mathcal{E}_A(P)^{-1}N$.

B. Infinite horizon quadratic optimal control

In this section, we consider the following infinite horizon quadratic optimal control problem for the system (6)

Problem 6: $\min \int_0^{+\infty} \mathbb{E}[x(t)'Q_c x(t)]dt \; + \; \sum_{k=0}^{+\infty} \mathbb{E}[u_k' u_k]$ subject to (6)

where the minimization is taken over control sequences $\{u_k, k \geq 0\}$ that (i) are causally obtained from the $x(t_k^-)$, i.e., u_k is adapted to the natural filtration associated with the sequence $x(t_k^-)$; (ii) stabilize the system (6) in a mean square sense, i.e., $\mathbb{E}[x(t)'x(t)] \to 0$.

To solve Problem 6 we introduce the following Ricatti-type equation

$$\begin{aligned} &J'\mathcal{E}_A(P)J - P + Q - \\ &\quad J'\mathcal{E}_A(P)B(I + B'\mathcal{E}_A(P)B)^{-1}B'\mathcal{E}_A(P)J = 0, \end{aligned} \tag{10}$$

where $Q := \int_0^T \int_0^y J' \exp(A't)\; Q_c \; \exp(At)\, J dt\, F(dy)$. We say that a symmetric solution P of (10) is *stabilizing* if $(A, J+B\mathcal{K}(P))$ is MSS, where

$$\mathcal{K}(P) := -(I + B'\mathcal{E}_A(P)B)^{-1}B'\mathcal{E}_A(P)J. \tag{11}$$

The next theorem, gives the optimal control law for the Problem 6.

Theorem 7: Suppose that there exists a stabilizing solution to (10). Then the optimal control law to the Problem 6 is given by

$$u_k^{opt} = \mathcal{K}(P)x_k. \tag{12}$$

1) Conditions for the existence of a stabilizing solution to (10) : The presentation in this subsection follows closely the presentation in [8] for a related MJLS problem. Defining

$$\begin{aligned} \mathcal{P}(P) := \; & J'\mathcal{E}_A(P)J + Q - \\ & J'\mathcal{E}_A(P)B(I + B'\mathcal{E}_A(P)B)^{-1}B'\mathcal{E}_A(P)J \end{aligned} \tag{13}$$

we can rewrite (10) as $\mathcal{P}(P) - P = 0$. We say that P is the *maximal* solution to (10) if for any other $\bar{P}$ verifying $\mathcal{P}(\bar{P}) - \bar{P} \geq 0$, we have $P \geq \bar{P}$. The next theorem, gives conditions, in the form of LMIs, for the existence of a maximal solution.

Theorem 8: Suppose that $((A,J),B)_F$ is stabilizable. Then the maximal solution to (10) coincides with the solution to the following problem

$$\begin{aligned} &\max \quad \mathrm{tr}(P) \\ &\text{s.t.} \begin{bmatrix} J'\mathcal{E}_A(P)J - P + Q & J'\mathcal{E}_A(P)B \\ B'\mathcal{E}_A(P)J & B'\mathcal{E}_A(P)B + I \end{bmatrix} \geq 0 \end{aligned} \tag{14}$$

The following lemma establishes the relation between the maximal and the stabilizing solution.

Lemma 9: There exists at most one stabilizing solution to (10). When such stabilizing solution exists, it coincides with the maximal solution.

This lemma gives a simple way of testing if a stabilizing solution exists. Namely, we can solve the LMI optimization problem (14) and test if the maximal solution obtained is stabilizing. If so, it coincides with the stabilizing solution, if not there is no stabilizing solution. Finally, we give conditions for a stabilizing solution to exist.

Theorem 10: Suppose that $((A,J),B)_F$ is stabilizable and $(Q^{\frac{1}{2}},(A,J))_F$ is detectable. Then there exists a unique stabilizing solution to (10).

C. Extensions to the index and jump points dependent case

The next definition and theorem provide extensions of the previous results stated for the system (6), to the more general system (1). We take the linear maps in (1) to be of the form

$$\begin{aligned} b(k,u_k,z_k) = B_k^{z_k}u_k, \;\; c(k,u_k,z_k) = C_k^{z_k}u_k, \\ \text{and } j(k,x(t_k^-),z_k) = J_k^{z_k}x(t_k^-). \end{aligned} \tag{15}$$

Since the homogeneous version of system (1) is fully specified by the matrix A, the set of matrices $J_k^{z_k}$, the distribution F, and the probability vector p, for short we use the notation $(A, J_k^{z_k})_{(F,p)}$ to refer to this system. The dependency of the matrices $B_k^{z_k}$, $C_k^{z_k}$ and $J_k^{z_k}$ on the jump index k is assumed to be periodic with period K, e.g., $B_k^i = B_{k+K}^i$ for any $i \in \mathcal{Z}$. We define $\mathcal{S} := \{1,\dots,K\}$ and $[\kappa] := \kappa,$ if $\kappa \in \mathcal{S}, [\kappa] := 1,$ if $\kappa = K+1$.

Analogously to the definitions for (2), we say that $(A, J_k^{z_k})_{(F,p)}$ is MSS if for every (x_0, k_0), $\lim_{t\to+\infty} \mathbb{E}[x(t)'x(t)] = 0$. Moreover, we say the system (1) with maps (15) is stabilizable if there exists matrices $\{K_\kappa, \kappa \in \mathcal{S}\}$ such that $(A, J_k^{z_k} + B_k^{z_k}K_k)_{(F,p)}$ is MSS, and that it is detectable if there exists matrices $\{L_\kappa, \kappa \in \mathcal{S}\}$ such that $(A, J_k^{z_k} + L_k C_k^{z_k})_{(F,p)}$ is MSS.

Theorem 11: Assuming that (3) holds, the system (1) with maps (15) is

(i) Mean Square Stable if and only if there exists $\{P_\kappa > 0, \kappa \in \mathcal{S}\}$ such that one of the following holds,

$$\sum_{i=1}^{n_z} p_i (J_\kappa^i)' \mathcal{E}_A(P_{[\kappa+1]}) J_\kappa^i - P_\kappa < 0,\, \kappa \in \mathcal{S};$$

$$\sum_{i=1}^{n_z} p_i (J_\kappa^i) \mathcal{E}_A(P_\kappa) (J_\kappa^i)' - P_{[\kappa+1]} < 0,\, \kappa \in \mathcal{S};$$

(ii) Stabilizable if and only if there exists $\{P_\kappa > 0, Y_\kappa, Z_\kappa, \kappa \in \mathcal{S}\}$ such that, for $\kappa \in \mathcal{S}$,

$$\sum_{i=1}^{n_z} p_i [J_\kappa^i \mathcal{E}_{A'}(P_\kappa)(J_\kappa^i)' + B_\kappa^i Y_\kappa (J_\kappa^i)' + J_\kappa^i Y_\kappa' (B_\kappa^i)' + (B_\kappa^i) Z_\kappa (B_\kappa^i)'] - P_{[\kappa+1]} < 0,$$

$$\begin{bmatrix} \mathcal{E}_{A'}(P_\kappa) & Y_\kappa' \\ Y_\kappa & Z_\kappa \end{bmatrix} > 0$$

In this case, $(A, J_k^{z_k} + B_k^{z_k} K_k)_{(F,p)}$ is MSS for $K_\kappa = Y_\kappa \mathcal{E}_{A'}(P_\kappa)^{-1}$;

(iii) Detectable if and only if there exists $\{P_\kappa > 0, V_\kappa, W_\kappa, \kappa \in \mathcal{S}\}$ such that, for $\kappa \in \mathcal{S}$,

$$\sum_{i=1}^{n_z} p_i [(J_\kappa^i)' \mathcal{E}_A(P_{[\kappa+1]}) J_\kappa^i + (C_\kappa^i)' V_\kappa' J_\kappa^i (J_\kappa^i)' V_\kappa C_\kappa^i + (C_\kappa^i)' W_\kappa C_\kappa^i] - P_\kappa < 0,$$

$$\begin{bmatrix} \mathcal{E}_A(P_{[\kappa+1]}) & V_\kappa \\ V_\kappa' & W_\kappa \end{bmatrix} > 0$$

In this case, $(A, J_k^{z_k} + L_k C_k^{z_k})_{(F,p)}$ is MSS for $L_\kappa = \mathcal{E}_A(P_{[\kappa+1]})^{-1} V_\kappa$.

Moreover, if the system (1) with maps (15) is stabilizable and detectable, there exists an output feedback controller that makes the closed-loop MSS.

IV. Emulation and Direct Design in Networked Control

Consider a networked control system in which a remote controller receives and processes sensor information and sends actuation signals through a communication network, possibly shared with other users. Suppose that there are n_y sensors, indexed by i running from 1 through n_y, and n_u actuators, indexed by j running from 1 through n_u. Assume that the plant is linear and described by

$$\dot{x}_P(t) = A_P x_P(t) + B_P \hat{u}(t) \qquad y_P(t) = C_P x_P(t) \tag{16}$$

where $\hat{u}(t)$ and $y_P(t)$, are partitioned into components associated which each of the sensor and actuation nodes as follows

$$\hat{u}(t) = (\hat{u}^1(t), \dots, \hat{u}^{n_u}(t)),\; y_P(t) = (y_P^1(t), \dots, y_P^{n_y}(t)). \tag{17}$$

We assume that the actuation mechanism is a simple hold operation of the form

$$\hat{u}^j(t) = \hat{u}^j(\tau_l^j), \qquad t \in [\tau_l^j, \tau_{l+1}^j), \tag{18}$$

where τ_l^j are the times at which an actuation signal corresponding to actuator j is sent by the controller. This signal is received by the actuator j also at τ_l^j, assuming that the transmission delays are negligible. Denoting the transmitted actuation signal by v_l^j, we have

$$\hat{u}^j(\tau_l^j) = \begin{cases} v_l^j, & \text{packet not dropped} \\ \hat{u}^j(\tau_l^{j-}), & \text{packet dropped} \end{cases}, \tag{19}$$

where by "packet dropped" we mean that the data transmitted might arrive corrupted at the receiver. The sensor i sends the measurement signal pertaining to the plant output $y_P^i(t)$ at times σ_r^i. If we denote this signal by w_r^i, this means

$$w_r^i = y_P^i(\sigma_r^i). \tag{20}$$

This data is received, corrupted or not, at the controller also at time σ_r^i, assuming again that the transmission delays are negligible. We define $\{t_k\} = \{\sigma_r^i\} \cup \{\tau_l^j\}$, to be the set of times at which a packet from any of the nodes pertaining the feedback loop is transmitted. We are interested in scenarios for which the following assumptions hold:

(i) The time intervals $\{t_{k+1} - t_k\}$ are i.i.d. random variables;
(ii) The transmission delays are negligible when compared to the time constants of the system dynamics, which is already implicit in expressions (19) and (20);
(iii) Corresponding to each transmission there is a probability p_{drop} of a packet being dropped, which might be zero. The events of packet drop at different transmission times are independent.

Assumptions (ii) and (iii) are common. Assumption (iii) holds, at least approximately, for scenarios in which nodes attempt to do periodic transmissions of data, but these regular transmissions may be perturbed by the medium access protocol. For example, nodes using CSMA for medium access, may be forced to back-off for a random amount of time until the network becomes available. However, note that for the model used here to be valid, if a sensor node is forced to back off, after the waiting time it should transmit the most recent measured data, instead of the one collected at the time it initially tried to transmit data. This approach is the most reasonable when transmitting dynamic data.

The distribution of the time interval between transmission is determined by two factors: the congestion of the network and the delay introduced by the medium access protocol. In [9], it is argued that for variants of CSMA protocols such as slotted p-persistent and pure ALOHA, the intervals between consecutive node transmissions are i.i.d. random variables. This argument is especially compelling if one does not need to restrict the distribution to exponential, which was necessary in [9]. In the example of the next two subsection, we take the distribution to be uniform, for illustration purposes. In general, the distribution can be determined experimentally or estimated by running Monte Carlo simulations of the protocol.

A. Analysis Tools and Emulation

Consider the following continuous-time controller for (16)

$$\dot{x}_P(t) = A_C x_C(t) + B_C \hat{y}(t), \quad y_C(t) = C_C x_C(t) + D_C \hat{y}(t), \tag{21}$$

and suppose it has been designed to stabilize the closed-loop, when

$$\hat{u}(t) = y_C(t), \quad \hat{y}(t) = y_P(t), \tag{22}$$

i.e., when the process and the controller are directly connected. We partition $\hat{y}$ and $y_C(t)$ accordingly to (17), i.e.,

$$\hat{y}(t)=(\hat{y}^1(t),\ldots,\hat{y}^{n_y}(t)),\ y_C(t)=(y_C^1(t),\ldots,y_C^{n_u}(t)), \tag{23}$$

and consider that

$$\begin{aligned}
\hat{y}^i(t) &= \hat{y}^i(\sigma_r^i), && t \in [\sigma_r^i, \sigma_{r+1}^i) && (24)\\
\hat{y}^i(\sigma_r^i) &= \begin{cases} w_r^i, & \text{packet not dropped} \\ \hat{y}^i(\sigma_r^{i-}), & \text{packet dropped} \end{cases}, && && (25)\\
v_l^j &= y_C^j(\tau_l^j). && && (26)
\end{aligned}$$

This means that the sensor data received by the controller is also held constant, between updating times σ_r^i. The controller integrates its differential equation with this frozen values at the input, and sends the control laws, at times τ_l^j. Defining

$$e(t) = \begin{bmatrix} e_u(t) \\ e_y(t) \end{bmatrix} = \begin{bmatrix} \hat{u}(t) - y_C(t) \\ \hat{y}(t) - y_P(t) \end{bmatrix} \tag{27}$$

and partitioning this vector accordingly to (17) and (23), we can conclude from (19), (20), (25), and (26) that

$$e_u^j(t_k) = \pi_{jk}^{z_k} e_u^j(t_k^-) \tag{28}$$
$$e_y^i(t_k) = \nu_{ik}^{z_k} e_y^i(t_k^-) \tag{29}$$

where

$$\pi_{jk}^{z_k} = \begin{cases} 0 \text{ if } t_k = \tau_j^l \text{ for some } l \text{ and packet not dropped} \\ 1 \text{ otherwise} \end{cases} \tag{30}$$

$$\nu_{ik}^{z_k} = \begin{cases} 0 \text{ if } t_k = \sigma_r^i \text{ for some } r \text{ and packet not dropped} \\ 1 \text{ otherwise} \end{cases} \tag{31}$$

and $\{z_k \in \mathcal{Z}, k \geq 0\}$ is a set of i.i.d. random variables which allow for introducing random effects in the transmissions.

We can encode all this information into a single matrix

$$\Lambda_k^{z_k} = \text{diag}(\begin{bmatrix} \Omega_k^{z_k} & \Gamma_k^{z_k} \end{bmatrix})$$

where, considering that the actuation signal j has dimension q_j and the sensor signal i has dimension s_i,

$$\begin{aligned}
\Omega_k^{z_k} &= \text{diag}([\pi_{1k}^{z_k} I_{q_1} \ldots \pi_{n_u k}^{z_k} I_{q_{n_u}}]), \\
\Gamma_k^{z_k} &= \text{diag}([\nu_{1k}^{z_k} I_{s_1} \ldots \nu_{n_y k}^{z_k} I_{s_{n_y}}]).
\end{aligned} \tag{32}$$

From (28) and (29) we obtain

$$e(t_k) = \Lambda_k^{z_k} e(t_k^-). \tag{33}$$

The matrices $\Lambda_k^{z_k}$ specify the protocol used by the nodes, which might be subject to random effects modeled by the z_k. Motivated by [1], we say the protocol is *mean square stable* if (33) is MSS when $\dot{e}(t) = 0, t \in [t_k, t_{k+1})$. Note that Theorem 11 can be used to test if (33) is MSS, although this is typically simpler. Two examples of stable protocols are:

Protocol (i)- The nodes transmit in a round-robin fashion. We define that $z_k \in \mathcal{Z} := \{1, 2\}$ equals 2 if a packet drop occurred at t_k and 1 otherwise, with $\mathbf{P}[z_k = 2] = p_{\text{drop}} < 1$. Let

$$M_\kappa := \text{diag}([0_{d_1} \ldots 0_{d_{\kappa-1}} 1_{d_\kappa} 0_{d_{\kappa+1}} \ldots 0_{d_{n_y+nu}}])$$

be a set of $n_y + n_u$ block diagonal matrices, with blocks partitioned according to the partition of sensors and actuators nodes explicit in (32), i.e., $d_j = q_j$ and $d_{i+n_u} = s_i$. The matrices $\Lambda_k^{z_k}$ are given by

$$\Lambda_k^{z_k} = I_{m+p} - M_{\rho(k)}, \text{ if } z_k = 1, \quad \Lambda_k^{z_k} = I_{m+p}, \text{ if } z_k = 2, \tag{34}$$

where the map $\rho : \mathbb{N} \mapsto \{1, \ldots, nu + ny\}$ determines the order by which nodes transmit. We assume that ρ is $K-$periodic and onto, implying that all the nodes transmit in a period.

Protocol (ii)- Each node transmits independently. If we assume that at each transmission time each node is equally likely to transmit, we can model this by $z_k \in \mathcal{Z} := \{1, \ldots, n_u + n_y + 1\}$, where the event $z_k = \kappa$ corresponds to node κ transmits at t_k and $z_k = n_u + n_y + 1$ models packet dropped at t_k. The probability of each of these events is

$$\mathbf{P}[z_k = \kappa] = \frac{1 - p_{\text{drop}}}{nu + ny} \quad \text{and} \quad \mathbf{P}[z_k = n_u + n_y + 1] = p_{\text{drop}},$$

and the matrices $\Lambda_k^{z_k} = \Lambda^{z_k}$ are given by

$$\Lambda^{z_k} = I_{m+p} - M_\kappa, \text{if } z_k = \kappa, \quad \Lambda^{z_k} = I_{m+p}, \text{if } z_k = n_u + n_y + 1.$$

The equations for (x, e) where $x = (x_C, x_P)$ can be written as

$$\begin{aligned}
\begin{bmatrix} \dot{x} \\ \dot{e} \end{bmatrix} &= \begin{bmatrix} I \\ A_e \end{bmatrix} \begin{bmatrix} A_{xx} & A_{xe} \end{bmatrix} \begin{bmatrix} x \\ e \end{bmatrix} \\
\begin{bmatrix} x(t_k) \\ e(t_k) \end{bmatrix} &= \begin{bmatrix} I & 0 \\ 0 & \Lambda_k^{n_z} \end{bmatrix} \begin{bmatrix} x(t_k^-) \\ e(t_k^-) \end{bmatrix}
\end{aligned} \tag{35}$$

where A_{xx} is the network-free closed loop matrix, that is

$$A_{xx} = \begin{bmatrix} A_P + B_P D_C C_P & B_P C_C \\ B_C C_P & A_C \end{bmatrix}$$

and the remaining matrices are given by

$$A_e = \begin{bmatrix} 0 & -C_C \\ -C_P & 0 \end{bmatrix} \qquad A_{xe} = \begin{bmatrix} B_P & B_P D_C \\ 0 & B_C \end{bmatrix}$$

Theorem 11(i) can be used to assert mean square stability of the system (35) when the dependence of $\Lambda_k^{z_k}$ on k is periodic, as in the two examples of protocols just described. It is reasonable to expect that if the distribution of the $\{t_{k+1} - t_k\}$ assigns high probability to fast sampling and the network-free closed loop is stable, i.e., A_{xx} is Hurwitz, then the closed loop system (35) taking into account the network characteristics is stable, in a mean square sense. The next result makes this statement precise. We introduce a family of distributions F_α, with support on $[0, \alpha T)$, which are obtained by scaling the original distribution F by a scalar. That is,

$$\int_0^{\alpha T} G(x) F_\alpha(dx) = \int_0^{\alpha T} G(x) \frac{1}{\alpha} f(\frac{x}{\alpha}) dx + \sum_i w_i G(\alpha a_i). \tag{36}$$

Theorem 12: Suppose that A_{xx} is Hurwitz and the protocol (33) is mean square stable. Then there exists a positive constant ϵ, such that for all $\alpha \in [0, \epsilon)$, the system (35) is MSS for $\{t_{k+1} - t_k, k \geq 0\}$ i.i.d. with cumulative distribution F_α.

The following example illustrates an emulation-based controller approach.

Example 13: As in [1], consider the control of a linearized model of an open loop unstable batch reactor. The plant taking the form (16) is described by the matrices

$$A_P = \begin{bmatrix} 1.38 & -0.207 & 6.715 & -5.676 \\ -0.581 & -4.29 & 0 & 0.675 \\ 1.067 & 4.273 & -6.654 & 5.893 \\ 0.048 & 4.273 & 1.343 & -2.104 \end{bmatrix},$$

$$B_P = \begin{bmatrix} 0 & 0 \\ 5.679 & 0 \\ 1.136 & -3.146 \\ 1.136 & 0 \end{bmatrix}, \quad C_P = \begin{bmatrix} 1 & 0 & 1 & -1 \\ 0 & 1 & 0 & 0 \end{bmatrix}.$$

A PI controller yielding the closed-loop stable when (22) holds, takes the form (21), where

$$A_C = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, B_C = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, C_C = \begin{bmatrix} -2 & 0 \\ 0 & 8 \end{bmatrix}, D_C = \begin{bmatrix} 0 & -2 \\ 5 & 0 \end{bmatrix}.$$

We consider that the round-robin protocol (i) described earlier in this subsection is used, in which the transmission of the bi-dimensional output alternates with the transmission of the bi-dimensional input. The matrices (34) are determined by $\rho(k) = 1, k$ even , $\rho(k) = 2, k$ odd and $M_1 = \text{diag}([1\,1\,0\,0]), M_2 = \text{diag}([0\,0\,1\,1])$. The random variables $\{t_{k+1} - t_k\}$ are assumed uniformly distributed with support T. By Theorem 12 we know that for sufficiently small T the closed-loop (35) taking into account the network characteristics is MSS. Using Theorem 11(i), we can perform a binary search on T to find the maximum value for which the closed-loop remains MSS. In doing so, we conclude that the system (35) is MSS for $T \in [0, T_{\text{MAX-EMULATION}}]$ where for different values of p_{drop}, $T_{\text{MAX-EMULATION}}$ is given by

p_{drop}	0	0.25	0.5
$T_{\text{MAX-EMULATION}}$	0.115	0.0718	0.0408

B. Synthesis Tools and direct design

In this subsection we show how a controller can be designed to stabilize the plant (16) under the network constraints. We do this by re-writing, equations (16), (18), (19), and (20) in the form of an impulsive renewal system (1). To this effect, we define u_k as a discrete-time signal matching the actuation signals sent by the controller to the actuators if they are transmitted at time t_k, and having any other value otherwise. Formally, $u_k = [u_k^1 \dots u_k^{n_u}]$, $u_k^j = v_l^j$ if $t_k = \tau_l^j$ for some l, and any value otherwise. We define y_k as a discrete-time signal that matches the measurement signals sent by the sensors to the controller if they are transmitted at time t_k and the last sent value otherwise. Formally, $y_k = [y_k^1 \dots y_k^{n_y}]$, $y_k^i = w_r^i$ if $t_k = \sigma_r^i$ for some r and $y_k^j = y_{k-1}^j$ otherwise, with the convention $y_0^j = 0$ if $t_0^j \neq \sigma_0^j$. Combining the equations (16), (18), (19), and (20), and using the definitions (30), (31), (32), the augmented system $x = (x_P, \hat{u}, \hat{y})$, takes the form (1) with linear maps (15) given by

$$A = \begin{bmatrix} A_P & B_P & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \; J_k^{z_k} = \begin{bmatrix} I & 0 & 0 \\ 0 & I - \Omega_k^{z_k} & 0 \\ \Gamma_k^{z_k} C_P & 0 & (I - \Gamma_k^{z_k}) \end{bmatrix},$$
$$B_k^{z_k} = \begin{bmatrix} 0 \\ \Omega_k^{z_k} \\ 0 \end{bmatrix} \qquad C_k^{z_k} = \begin{bmatrix} \Gamma_k^{z_k} C_P & 0 & (I - \Gamma_k^{z_k}) \end{bmatrix} \tag{37}$$

The synthesis tools derived in Section III can then be applied to obtain a stabilizing controller.

Example 16 (cont.) Suppose that, in the setting of example (13), there are no packet drops and the support of the uniform distribution that models the inter-transmission times is twice as large as the maximum value obtained with emulation, i.e.,

$$T_{\text{DIRECT-DESIGN}} = 0.23 \text{ and } p_{\text{drop}} = 0.$$

Taking into account the network characteristics, an output mean square stabilizing controller to the system (35) can be obtained by applying Theorem 11(i) to the impulsive renewal system described by matrices (37). One such controller is

$$\begin{aligned} \dot{\hat{x}}(t) &= A\hat{x}(t), \quad t \neq t_k \\ \hat{x}(t_k) &= (J_k^{z_k} + B_k^{z_k} K_k + L_k C_k^{z_k})\hat{x}(t_k^-) - L_k y_k, \\ \hat{u}_k &= K_k \hat{x}(t_k^-), \quad \hat{x}(t_0^-) = \hat{x}_0, \; t_0 = 0, \end{aligned}$$

where

$K_1 = 0_{2\times 8}$, $L_2' = [0_{2\times 6} - I_{2\times 2}]$, $K_2 = [\bar{K}_2 \;\; 0_{2\times 4}]$,

$$\bar{K}_2 = \begin{bmatrix} 0.114 & -0.541 & -0.028 & -0.440 \\ 1.701 & 0.068 & 1.207 & -0.80 \end{bmatrix}, L_1' = [\bar{L}_1' \;\; -I_{2\times 2}],$$

$$L_1' = \begin{bmatrix} -0.667 & -0.005 & -0.243 & 0.101 & -0.003 & 0.0185 \\ -0.011 & -1.005 & -0.576 & -0.583 & -0.199 & 0.0100 \end{bmatrix}.$$

V. Conclusions and Future Work

We tackled the control of impulsive renewal systems deriving a number of basic results which parallel similar problems for LTI systems and MJLS. The results were applied to the analysis and synthesis of feedback loops closed by networks utilizing CSMA-type protocols, both from an emulation-type approach and a direct-design approach. Directions for future work include obtaining realistic distributions for the transmission intervals in CSMA-type protocols.

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