Stochastic difference inclusions: results on recurrence and asymptotic stability in probability

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***Abstract*— Stability theory for stochastic difference inclusions is discussed. We summarize a framework for stochastic difference inclusions that has been introduced recently and for which a variety of new stability theory results have been obtained. In particular, we study recurrence and global asymptotic stability in probability. For these properties, we review new results on robustness, converse Lyapunov theorems, and Matrosov-function-based sufficient conditions. We also discuss input-to-state stability in probability. Examples are used to illustrate the framework and results.**

I. Introduction

For discrete-time stochastic systems with unique solutions, stability theory has been developed extensively. For example, [1] contains a comprehensive treatment of a stability property called "positive recurrence", including a variety of necessary and sufficient conditions, including converse Lyapunov theorems. Similar results have been pursued for hybrid switching diffusions in [2]. An excellent source for stability theory for stochastic differential equations is [3].

There has also been a long history of interest in closed stochastic systems with nonunique solutions or open stochastic systems with external disturbances, including the study of Markov decision processes [4], probabilistic automata [5], set-valued random processes [6], the recent Labelled Markov processes [7], and adversary-induced Markov decision processes [8] as considered in the context of machine learning. Nevertheless, stability theory has not been pursued as extensively in this setting.

For non-stochastic systems, non-uniqueness of solutions manifests itself in several settings. One situation comes in defining generalized solutions to systems with discontinuities [9], [10], [11], which are appropriate for assessing robustness of asymptotic stability [12], [10], [13]. Another setting includes modeling the behavior of a switched system where switching satisfies an average dwell-time condition [14] using a hybrid modeling framework [15]. Closed systems with nonunique solutions also arise in equivalent characterizations of input-to-state stability [16]. Stability theory for non-stochastic systems with non-unique solutions has been developed thoroughly, including results on robustness, converse Lyapunov theorems, and a wide variety of sufficient conditions for stability. For example, many such results are contained in [17].

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The purpose of this paper is to review a framework for stochastic difference inclusions introduced recently [18], to describe some basic mathematical constructs that are used to characterize recurrence and global asymptotic stability in probability, and to recall some results pertaining to robustness and converse Lyapunov theorems [19], [20], as well as Matrosov-function-based sufficient conditions [21], [22].

The paper is organized as follows. The next section contains a summary of notation and basic definitions. Section III presents the class of systems that we consider. In Section IV, V, and VI, we define quantities that are used to characterize recurrence, stability, and asymptotic stability in probability. In these sections, we also review robustness results and converse Lyapunov theorems from [19], [20], and Matrosov-based sufficient conditions from [21], [22]. Examples are presented in Section VIII.

II. Notation and basic definitions

$\mathbb{R}_{\geq 0}$ denotes the nonnegative real numbers; $\mathbb{Z}_{\geq 0}$ denotes the nonnegative integers. For a closed set $S \subset \mathbb{R}^n$ and $x \in \mathbb{R}^n$, $|x|_S := \inf_{y \in S} |x - y|$ is the Euclidean distance to S. $\mathbb{B}$ (resp., $\mathbb{B}^\circ$) denotes the closed (resp., open) unit ball in $\mathbb{R}^n$. For a closed set $S \subset \mathbb{R}^n$ and $\varepsilon > 0$, $S + \varepsilon\mathbb{B}$ (resp., $S + \varepsilon\mathbb{B}^\circ$) denotes the set $\{x \in \mathbb{R}^n : |x|_S \leq \varepsilon\}$ (resp., $\{x \in \mathbb{R}^n : |x|_S < \varepsilon\}$). The function $\mathbb{I}_S : \mathbb{R}^n \to \{0,1\}$ satisfies $\mathbb{I}_S(x) = 1$ for $x \in S$ and $\mathbb{I}_S(x) = 0$ otherwise. A function $\phi : \mathbb{R}^n \to \mathbb{R}$ is *upper semicontinuous* if $\limsup_{i \to \infty} \phi(x_i) \leq \phi(x)$ whenever $\lim_{i \to \infty} x_i = x$. The function $\mathbb{I}_S$ is upper semicontinuous for closed S. A set-valued mapping $M : \mathbb{R}^p \rightrightarrows \mathbb{R}^n$ is *outer semicontinuous* if, for each $(x_i, y_i) \to (x, y) \in \mathbb{R}^p \times \mathbb{R}^n$ satisfying $y_i \in M(x_i)$ for all $i \in \mathbb{Z}_{\geq 0}$, $y \in M(x)$. A mapping M is *locally bounded* if, for each bounded set $K \subset \mathbb{R}^p$, $M(K) := \bigcup_{x \in K} M(x)$ is bounded. $\mathbf{B}(\mathbb{R}^m)$ denotes the Borel field, the subsets of $\mathbb{R}^m$ generated from open subsets of $\mathbb{R}^m$ through complements and finite and countable unions. A set $F \subset \mathbb{R}^m$ is *measurable* if $F \in \mathbf{B}(\mathbb{R}^m)$. A mapping $M : \mathbb{R}^p \rightrightarrows \mathbb{R}^n$ is *measurable* [23, Def. 14.1] if for each open set $\mathcal{O} \subset \mathbb{R}^n$ the set $M^{-1}(\mathcal{O}) := \{v \in \mathbb{R}^p : M(v) \cap \mathcal{O} \neq \varnothing\}$ is measurable. When the values of M are closed, measurability is equivalent to $M^{-1}(\mathcal{C})$ being measurable for each closed set $\mathcal{C} \subset \mathbb{R}^n$ [23, Thm. 14.3]. A function $\alpha : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is *of class* $\mathcal{K}$ if it is continuous, strictly increasing and $\alpha(0) = 0$. It is *of class* $\mathcal{K}_\infty$ if it is of class $\mathcal{K}$ and unbounded. A function $\psi : \mathbb{Z}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is *of class* $\mathcal{L}$ if it is nonincreasing and $\lim_{\ell \to \infty} \psi(\ell) = 0$. A function $\gamma : \mathbb{R}_{\geq 0} \times \mathbb{Z}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is *of class* $\mathcal{KL}$ if $\gamma(\cdot, k) \in \mathcal{K}$ for each $k \in \mathbb{Z}_{\geq 0}$ and $\gamma(s, \cdot) \in \mathcal{L}$ for each $s \in \mathbb{R}_{\geq 0}$.

III. Stochastic difference inclusions

A stochastic difference inclusion is written formally as

$$x^+ \in G(x, v) \tag{1}$$

where $x \in \mathbb{R}^n$ is the state and $v \in \mathbb{R}^m$ is the random input, eventually specified as a random variable, that is, a measurable function from a probability space to $\mathbb{R}^m$. The set-valued mapping $G : \mathbb{R}^n \times \mathbb{R}^m \rightrightarrows \mathbb{R}^n$ specifies the admissible next values of the state as determined by the current state value and the random input. We focus on mappings G that are guaranteed to generate random processes and for which a variety of probabilistic stability notions are robust. The former property calls for certain measurability conditions, motivated by results on measurable set-valued mappings as described in [23]. Informed by non-stochastic results, the desire for robustness leads to stronger regularity with respect to the state x in the form of outer semicontinuity. Specifically, we impose the following regularity conditions.

Standing Assumption 1: The following conditions hold:

1) $x \mapsto G(x, v)$ is outer semicontinuous for each $v \in \mathbb{R}^m$.
2) $v \mapsto \{(g, x) \in \mathbb{R}^n \times \mathbb{R}^n : g \in G(x, v)\}$ is measurable.
3) G is locally bounded.

This assumption guarantees that the set-valued mapping $(v_0, \ldots, v_{r-1}) \mapsto G(\ldots, G(G(x, v_0), v_1), \ldots, v_{r-1}) =: G^r(x, (v_0, \ldots, v_{r-1}))$ is closed-valued and measurable for each $x \in \mathbb{R}^n$. See [23, Theorem 14.13]. This property is appealing since a closed-valued measurable set-valued mapping admits a measurable selection [23, Corollary 14.6], that is, a measurable function ϕ defined on the domain of $G^r(x, \ldots)$ and satisfying $\phi(v_0, \ldots, v_{r-1}) \in G^r(x, (v_0, \ldots, v_{r-1}))$ for all $(v_0, \ldots, v_{r-1})$ in the domain of $G^r(x, \ldots)$. These measurable selections, when composed with random variables, produce random processes. However, as we will demonstrate later through an example, we are interested only in random variables that exhibit a causal dependence on v. In particular, we are interested in the existence of measurable functions $\phi_i : \mathrm{dom}\ \phi_i \subset (\mathbb{R}^m)^i \to \mathbb{R}^n$ that satisfy $\phi_0 = x$ and

$$\phi_{i+1}(v_0, \ldots, v_i) \in G(\phi_i(v_0, \ldots, v_{i-1}), v_i) \qquad \forall i \in \mathbb{Z}_{\geq 0}\ .$$

According to [23, Corollary 14.14], such functions exist and can be chosen so that $\mathrm{dom}\ \phi_{i+1} = (\phi_i \times \mathbb{R}^m) \cap \{(v_0, \ldots, v_i) : (\phi_i(v_0, \ldots, v_{i-1}), v_i) \in \mathrm{dom}\ G\}$. They are called *maximal pre-random solutions to (1) from x*.

We emphasize that we allow $G(x, v)$ to be multi-valued for certain values of (x, v) and empty for other values of (x, v). We may even be interested in difference inclusions where $G(x, v)$ is empty at points x that are reachable. For example, we may be interested in studying the behavior of random processes that remain in some closed neighborhood of an equilibrium point, acknowledging that not every solution that starts in that set remains there for all time.

We now introduce a probability structure that is used to generate the inputs in (1). Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space. For $i \in \mathbb{Z}_{\geq 0}$, let $\mathbf{v}_i : \Omega \to \mathbb{R}^m$ be a sequence of independent, identically distributed (i.i.d.) random variables. Thus, $\mathbf{v}_i^{-1}(F) := \{\omega \in \Omega : \mathbf{v}_i(\omega) \in F\} \in \mathcal{F}$ for each $F \in \mathbf{B}(\mathbb{R}^m)$. Let $(\mathcal{F}_0, \mathcal{F}_1, \ldots)$ denote the natural filtration of the sequence $(\mathbf{v}_0, \mathbf{v}_1, \ldots)$. That is, $\mathcal{F}_i \subset \mathcal{F}$ is all sets of the form $\{\omega \in \Omega : (\mathbf{v}_0(\omega), \ldots, \mathbf{v}_i(\omega)) \in F\}$, $F \in \mathbf{B}((\mathbb{R}^m)^{i+1})$. Due to the i.i.d. property, each random variable has the same probability measure $\mu : \mathbf{B}(\mathbb{R}^m) \to [0, 1]$ defined as $\mu(F) := \mathbb{P}\{\omega \in \Omega : \mathbf{v}_i(\omega) \in F\}$ and, for almost all $\omega \in \Omega$,

$$\begin{aligned}\mathbb{E}\left[f(\mathbf{v}_0, \ldots, \mathbf{v}_i, \mathbf{v}_{i+1}) | \mathcal{F}_i\right](\omega) \\ = \int_{\mathbb{R}^m} f(\mathbf{v}_0(\omega), \ldots, \mathbf{v}_i(\omega), v)\mu(dv)\end{aligned}$$

for each $i \in \mathbb{Z}_{\geq 0}$ and each measurable $f : (\mathbb{R}^m)^{i+2} \to \mathbb{R}$.

A *random process* $\mathbf{x}$ is a sequence of random variables $\mathbf{x}_i : \mathrm{dom}\ \mathbf{x}_i \subset \Omega \to \mathbb{R}^n$ with $\mathrm{dom}\ \mathbf{x}_{i+1} \subset \mathrm{dom}\ \mathbf{x}_i$ for all $i \in \mathbb{Z}_{\geq 0}$. A random process $\mathbf{x}$ is *adapted to the natural filtration of* $\mathbf{v}$ if $\mathbf{x}_{i+1}$ is $\mathcal{F}_i$-measurable for each $i \in \mathbb{Z}_{\geq 0}$. That is, $\mathbf{x}_{i+1}^{-1}(F) \subset \mathcal{F}_i$ for each $F \in \mathbf{B}(\mathbb{R}^n)$. It is a *maximal random solution of (1)* if, for each $i \in \mathbb{Z}_{\geq 0}$,

$$\mathbf{x}_{i+1}(\omega) \in G(\mathbf{x}_i(\omega), \mathbf{v}_i(\omega)) \qquad \forall \omega \in \mathrm{dom}\ \mathbf{x}_{i+1}$$

and $\mathrm{dom}\ \mathbf{x}_{i+1} = \{\omega \in \mathrm{dom}\ \mathbf{x}_i : G(\mathbf{x}_i(\omega), \mathbf{v}_i(\omega)) \neq \varnothing\}$. We use $\mathcal{S}(x)$ to denote the set of maximal random processes adapted to the natural filtration of $\mathbf{v}$ that satisfy (1) from x, and we call these random processes "random solutions". As noted in [19], $\mathbf{x} \in \mathcal{S}(x)$ if and only if there exists a maximal pre-random solution ϕ of (1) from x such that

$$\begin{aligned}&\mathbf{x}_i(\omega) = \phi_i(\mathbf{v}_0(\omega), \ldots, \mathbf{v}_{i-1}(\omega)) \\ &\forall \omega \in \mathrm{dom}\ \mathbf{x}_i = \{\omega : (\mathbf{v}_0(\omega), \ldots, \mathbf{v}_{i-1}(\omega)) \in \mathrm{dom}\ \phi_i\}\ .\end{aligned}$$

For $\mathbf{x} \in \mathcal{S}(x)$, we use the convention that $\mathbb{I}_S(\mathbf{x}_i(\omega)) = 0$ for $\omega \notin \mathrm{dom}\ \mathbf{x}_i$ and we define

$$\mathrm{graph}(\mathbf{x}(\omega)) := \cup_{i \in \mathbb{Z}_{\geq 0}} (\{i\} \times \mathbf{x}_i(\omega))\ .$$

IV. Weak viability and strong recurrence

A. Weak viability

Upcoming definitions of recurrence and of asymptotic stability in probability entail that certain closed sets have zero probability of being forever viable, that is, having the property that solutions remain in the set for all time. For a closed set $S \subset \mathbb{R}^n$, $x \in \mathbb{R}^n$, and $k \in \mathbb{Z}_{\geq 1}$, we define the *weak viability probabilities*

$$m_{\subset S}(k, x) := \sup_{\mathbf{x} \in \mathcal{S}(x)} \mathbb{E}\left[\prod_{i=1}^{k} \mathbb{I}_S(\mathbf{x}_i)\right]. \tag{2}$$

These functions quantify the largest probability, over all random solutions, of remaining in the closed set S for all time up to time k. We use the term "*weak* viability" since *only one* random solution from x needs to have high probability of remaining in S for k steps in order for $m_{\subset S}(k, x)$ to be close to one. The following facts have been established in [18], [19]: 1) $m_{\subset S}(k, \cdot)$ is upper semicontinuous for each $k \in \mathbb{Z}_{\geq 1}$, 2) with $m_{\subset S}(0, x) = 1$ for all $x \in \mathbb{R}^n$, we have

$$m_{\subset S}(k+1, x) = \int_{\mathbb{R}^m} \max_{g \in G(x,v)} \mathbb{I}_S(g) m_{\subset S}(k, g)\mu(dv) \tag{3}$$

for all $(k, x) \in \mathbb{Z}_{\geq 0} \times \mathbb{R}^n$, and 3) there exists $\mathbf{x} \in \mathcal{S}(x)$ such that $m_{\subset S}(k, x) = \mathbb{E}\left[\prod_{i=1}^{k} \mathbb{I}_S(\mathbf{x}_i)\right]$. It is also clear that

$0 \le m_{\subset S}(k+1,x) \le m_{\subset S}(k,x)$ for all $(k,x) \in \mathbb{Z}_{\ge 0} \times \mathbb{R}^n$. Thus, we can define $\widehat{m}_{\subset S}(x) := \lim_{k\to\infty} m_{\subset S}(k,x)$.

B. Strong recurrence

1) Robustness and a converse theorem: An open, bounded set $\mathcal{O} \subset \mathbb{R}^n$ is said to be *globally recurrent* if, for each $x \in \mathbb{R}^n \backslash \mathcal{O}$ and $\mathbf{x} \in \mathcal{S}(x)$, $\mathbb{E}\left[\prod_{i\in\mathbb{Z}_{\ge 1}} \mathbb{I}_{\mathbb{R}^n\backslash\mathcal{O}}(\mathbf{x}_i)\right] = 0$. Equivalently, $\widehat{m}_{\subset \mathbb{R}^n\backslash\mathcal{O}}(x) = 0$ for all $x \in \mathbb{R}^n\backslash\mathcal{O}$. Yet another equivalent characterization of global recurrence is the condition that, for each $x \in \mathbb{R}^n$ and each $\mathbf{x} \in \mathcal{S}(x)$,

$$\lim_{k\to\infty} \mathbb{P}\Big((\text{graph}(\mathbf{x}) \subset (\mathbb{Z}_{\le k} \times \mathbb{R}^n)) \vee (\text{graph}(\mathbf{x}) \cap (\mathbb{Z}_{\le k} \times \mathcal{O})) \neq \varnothing \Big) = 1$$

where $\vee$ denotes the logical "or" operation. Since *every* random solution must have high probability of reaching $\mathcal{O}$ in finite time (or stopping), we have labeled this section "*strong* recurrence". As shown in [20], the upper semicontinuity of $m_{\subset\mathbb{R}^n\backslash\mathcal{O}}(k,\cdot)$ enables establishing that $\mathcal{O}$ is globally recurrent if and only if for each compact set $K \subset \mathbb{R}^n$ and each $\varrho > 0$ there exists k such that $m_{\subset\mathbb{R}^n\backslash\mathcal{O}}(k,x) \le \varrho$ for all $x \in K$. Recurrence also has an equivalent Lyapunov function characterization. The main result of [20] establishes that the open, bounded set $\mathcal{O} \subset \mathbb{R}^n$ is globally recurrent if and only if there exists a smooth, radially unbounded function $V : \mathbb{R}^n \to \mathbb{R}_{\ge 0}$ and a continuous function $\rho : \mathbb{R}^n \to \mathbb{R}_{>0}$ such that

$$\int_{\mathbb{R}^m} \max_{g\in G(x,v)} V(g)\mu(dv) \le V(x) - \rho(x) + \mathbb{I}_{\mathcal{O}}(x) \quad \forall x \in \mathbb{R}^n. \tag{4}$$

This result is similar to results reported in [1] that pertain to a stronger form of recurrence, called positive recurrence, for discrete-time systems with unique solutions.

One of the intermediate results of [20] is that global recurrence of an open, bounded set $\mathcal{O}$ is robust. By robust, we mean that there exists a continuous function $\delta : \mathbb{R}^n \to \mathbb{R}_{>0}$, $\varepsilon > 0$, and an open, bounded set $\mathcal{O}' \subset \mathbb{R}^n$ such that $\mathcal{O}' + \varepsilon\mathbb{B} \subset \mathcal{O}$ and, with the definition $H_\delta(x) := \{x\} + \delta(x)\mathbb{B}$, the set $\mathcal{O}'$ is globally recurrent for the system

$$x^+ \in G_\delta(x,v) := H_\delta(G(H_\delta(x),v)) \ . \tag{5}$$

Noteworthy is the fact, established in [19], that if δ is continuous then G_δ inherits the properties in Standing Assumption 1 from G. Other results in the literature that are related to robustness, in this case to perturbations on the probability measure associated with v, can be found in [24] and [25].

2) Matrosov-based sufficient conditions: While Lyapunov functions are necessary for recurrence, there exist weakened sufficient conditions for recurrence. We describe conditions that have been derived recently in [22] based on Matrosov functions, as developed for nonstochastic systems in [26], [27], [28]. We specialize the discussion here to the case of time-invariant systems and recurrence of open, bounded sets, which covers time-varying, periodic systems. In particular, the open, bounded set $\mathcal{O}$ is globally recurrent if 1) there exists an upper semicontinuous, radially unbounded function $V : \mathbb{R}^n \to \mathbb{R}_{\ge 0}$ such that

$$\int_{\mathbb{R}^m} \max_{g\in G(x,v)\cap(\mathbb{R}^n\backslash\mathcal{O})} V(g)\mu(dv) \le V(x) \quad x \in \mathbb{R}^n\backslash\mathcal{O}$$

and 2) for each $R > 0$ there exist $N \in \mathbb{Z}_{\ge 1}$ upper semicontinuous functions $W_i : \mathbb{R}^n \to \mathbb{R}_{\ge 0}$ and continuous functions $Y_i : \mathbb{R}^n \to \mathbb{R}$, $i \in \{1,\ldots,N\}$ such that, for all $x \in \mathbb{R}^n\backslash\mathcal{O}$,

$$\int_{\mathbb{R}^m} \max_{g\in G(x,v)\cap(\mathbb{R}^n\backslash\mathcal{O})\cap R\mathbb{B}} W_i(g)\mu(dv) - W_i(x) \le Y_i(x)$$

and, with the definitions, $Y_0(x) := 0$ for all $x \in \mathbb{R}^n$ and $Y_{N+1}(x) := 1$ for all $x \in \mathbb{R}^n$, we have the following property for each $j \in \{0,\ldots,N\}$: if $x \in (\mathbb{R}^n\backslash\mathcal{O}) \cap R\mathbb{B}$ and $Y_i(x) = 0$ for all $i \in \{0,\ldots,j\}$ then $Y_{j+1}(x) \le 0$.

The condition (4) is a special case of these conditions with $N = 1$ and $Y_1(x) := -\rho(x)$ since ρ is continuous and never negative. We illustrate Matrosov conditions, in the context of global asymptotic stability in probability in Section VIII-A.

Related sufficient conditions for recurrence can be found in [29] and [30].

V. Weak reachability and strong stability

A. Weak reachability

Proofs of recurrence from Lyapunov or Matrosov conditions, as well as characterizations of stability in probability, use the probabilities of reaching certain closed sets S. For a closed set S, $k \in \mathbb{Z}_{\ge 1}$ and $x \in \mathbb{R}^n$, we define the *weak reachability probabilities*

$$m_{\cap S}(k,x) := \sup_{\mathbf{x}\in\mathcal{S}(x)} \mathbb{E}\left[\max_{i\in\{1,\ldots,k\}} \mathbb{I}_S(\mathbf{x}_i)\right]. \tag{6}$$

These functions quantify the largest probability, over all random solutions, of reaching (intersecting) the closed set S within k time steps. We use the term "*weak* reachability" since *only one* random solution from x needs to have high probability of reaching S in k steps in order for $m_{\cap S}(k,x)$ to be close to one. The following facts have been established in [18], [19]: 1) $m_{\cap S}(k,\cdot)$ is upper semicontinuous for each $k \in \mathbb{Z}_{\ge 1}$, 2) with $m_{\cap S}(0,x) = 0$ for all $x \in \mathbb{R}^n$,

$$m_{\cap S}(k+1,x) = \int_{\mathbb{R}^m} \max_{g\in G(x,v)} \max\left\{\mathbb{I}_S(g), m_{\cap S}(k,g)\right\}\mu(dv) \tag{7}$$

for all $(k,x) \in \mathbb{Z}_{\ge 0} \times \mathbb{R}^m$, and 3) there exists $\mathbf{x} \in \mathcal{S}(x)$ such that $m_{\cap S}(k,x) = \mathbb{E}\left[\max_{i\in\{1,\ldots,k\}} \mathbb{I}_S(\mathbf{x}_i)\right]$. It is also clear that $m_{\cap S}(k,x) \le m_{\cap S}(k+1,x) \le 1$. Thus, the limit $\lim_{k\to\infty} m_{\cap S}(k,x)$ is well defined.

B. Strong stability

A compact set $\mathcal{A} \subset \mathbb{R}^n$ is said to be *stable in probability* for (1) if for each $\varepsilon > 0$ and $\varrho > 0$ there exists $\delta > 0$ such that, for each $x \in \mathcal{A} + \delta\mathbb{B}$ and $\mathbf{x} \in \mathcal{S}(x)$, we have $\mathbb{E}\left[\max_{i\in\mathbb{Z}_{\ge 1}} \mathbb{I}_S(\mathbf{x}_i)\right] \le \varrho$. This property is equivalent to the condition that for each each $\varepsilon > 0$ and $\varrho > 0$ there exists $\delta > 0$ such that $\lim_{k\to\infty} m_{\cap(\mathbb{R}^n\backslash(\mathcal{A}+\varepsilon\mathbb{B}^\circ))}(k,x) \le \varrho$ for $x \in \mathcal{A} + \delta\mathbb{B}$. It can also be stated as follows: for each each $\varepsilon > 0$

and $\varrho > 0$ there exists $\delta > 0$ such that, for each $x \in \mathcal{A} + \delta\mathbb{B}$ and each $\mathbf{x} \in \mathcal{S}(x)$,

$$\mathbb{P}\left(\text{graph}(\mathbf{x}) \subset \left(\mathbb{Z}_{\geq 0} \times (\mathcal{A} + \varepsilon\mathbb{B}^{\circ})\right)\right) \geq 1 - \varrho. \quad (8)$$

Since *every* random solution must exhibit stable behavior, we have labeled this section "*strong* stability". The compact set $\mathcal{A} \subset \mathbb{R}^n$ is said to be *globally stable in probability* for (1) if it is stable in probability and for each $\delta > 0$ and $\varrho > 0$ there exists $\varepsilon > 0$ such that (8) holds for each $x \in \mathcal{A} + \delta\mathbb{B}$ and each $\mathbf{x} \in \mathcal{S}(x)$.

VI. Asymptotic stability and post-transient weak reachability

A. Asymptotic stability and robustness

A compact set $\mathcal{A} \subset \mathbb{R}^n$ is said to be *uniformly globally attractive in probability for (1)* if for each $\varepsilon > 0$, $\Delta > 0$, and $\varrho > 0$ there exists $k \in \mathbb{Z}_{\geq 0}$ such that, for each $x \in \mathcal{A} + \Delta\mathbb{B}$ and $\mathbf{x} \in \mathcal{S}(x)$,

$$\mathbb{P}\left((\text{graph}(\mathbf{x}) \cap (\mathbb{Z}_{\geq k} \times \mathbb{R}^n)) \subset (\mathbb{Z}_{\geq 0} \times (\mathcal{A} + \varepsilon\mathbb{B}^{\circ}))\right) \geq 1 - \varrho$$

with the convention that the empty set is a subset of any set. A compact set $\mathcal{A} \subset \mathbb{R}^n$ is *uniformly globally asymptotically stable in probability for (1)* if it is stable in probability and uniformly globally attractive in probability for (1). It has been established in [21] that a compact set is uniformly globally attractive in probability if it is stable in probability and each open neighborhood of the set is globally recurrent.

It has been established in [19] that uniform global asymptotic stability in probability is robust. Namely, there exists a continuous function $\delta : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ that is positive definite with respect to $\mathcal{A}$ such that $\mathcal{A}$ is globally asymptotically stable in probability for the system (5).

B. Post-transient weak reachability

According to [21, Lemma 5], when the compact set $\mathcal{A}$ is uniformly globally asymptotically stable in probability for (1) and $S = \mathbb{R}^n \backslash (\mathcal{A} + \varepsilon\mathbb{B}^{\circ})$ with $\varepsilon > 0$, the function $\widehat{m}_{\cap S}(x) := \max\left\{\mathbb{I}_S(x), \lim_{k\to\infty} m_{\cap S}(k, x)\right\}$ is upper semicontinuous and satisfies $\int_{\mathbb{R}^m} \max_{g\in G(x,v)} \widehat{m}_{\cap S}(g)\mu(dv) \leq \widehat{m}_{\cap S}(x)$ for all $x \in \mathbb{R}^n$. Then we can define post-transient weak reachability probabilities for such sets S as follows. For each $\xi \in \mathbb{R}^n$, define $\widetilde{m}_{\cap S}(0, \xi) := \widehat{m}_{\cap S}(\xi)$ and, for $j \in \mathbb{Z}_{\geq 0}$, define $\widetilde{m}_{\cap S}(j+1, \xi) := \int_{\mathbb{R}^m} \max_{g\in G(\xi,v)} \widetilde{m}_{\cap S}(j, g)\mu(dv)$. According to results in [19], the compact set $\mathcal{A}$ is uniformly globally asymptotically stable in probability for (1) if and only if for each $\varepsilon > 0$ there exists $\gamma_\varepsilon \in \mathcal{KL}$ such that

$$\widetilde{m}_{\cap(\mathbb{R}^n \backslash (\mathcal{A}+\varepsilon\mathbb{B}^{\circ}))}(k, \xi) \leq \gamma_\varepsilon(|\xi|_{\mathcal{A}}, k) \quad \forall (k, \xi) \in \mathbb{Z}_{\geq 0} \times \mathbb{R}^n.$$

C. A converse theorem

These post-transient weak reachability probabilities can then be used to construct Lyapunov functions for systems with uniformly globally asymptotically stable sets. In particular, a main result of [19] is that if the compact set $\mathcal{A}$ is globally asymptotically stable in probability for (1) then there exists a continuous function $V : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ that is smooth on $\mathbb{R}^n \backslash \mathcal{A}$, positive definite with respect to $\mathcal{A}$, and radially unbounded, and a continuous function $\rho : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ that is positive definite with respect to $\mathcal{A}$ such that

$$\int_{\mathbb{R}^m} \max_{g\in G(x,v)} V(g)\mu(dv) \leq V(x) - \rho(x) \quad \forall x \in \mathbb{R}^n. \quad (9)$$

D. Matrosov-based sufficient conditions

While Lyapunov functions are necessary for global asymptotic stability in probability, there exist weakened sufficient conditions for global asymptotic stability in probability. The following discussion reviews results that appear in [21], specialized to the case of time-invariant systems and globally asymptotic stability in probability for a compact set, which covers time-varying periodic systems. According to the main result of [21], the compact set $\mathcal{A}$ is globally stable in probability for (1) and uniformly globally attractive in probability for (1) (thus, uniformly globally asymptotically stable in probability for (1)) if 1) there exists an upper semicontinuous function $V : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ that is radially unbounded and positive definite with respect to $\mathcal{A}$ such that

$$\int_{\mathbb{R}^m} \max_{g\in G(x,v)} V(g)\mu(dv) \leq V(x) \quad \forall x \in \mathbb{R}^n$$

and 2) for each pair (r, R) satisfying $0 < r < R$ there exists $N \in \mathbb{Z}_{\geq 1}$ upper semicontinuous functions $W_i : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ and continuous functions $Y_i : \mathbb{R}^n \to \mathbb{R}$, $i \in \{1, \ldots, N\}$, such that, for all $x \in (\mathcal{A} + R\mathbb{B}) \backslash (\mathcal{A} + r\mathbb{B}^{\circ}) =: S_{r,R}$,

$$\int_{\mathbb{R}^m} \max_{g\in G(x,v)\cap S_{r,R}} W_i(g)\mu(dv) - W_i(x) \leq Y_i(x)$$

and, with the definitions, $Y_0(x) := 0$ for all $x \in \mathbb{R}^n$ and $Y_{N+1}(x) := 1$ for all $x \in \mathbb{R}^n$, we have the following property for each $j \in \{0, \ldots, N\}$: if $x \in S_{r,R}$ and $Y_i(x) = 0$ for all $i \in \{0, \ldots, j\}$ then $Y_{j+1}(x) \leq 0$. The condition (9) is a special case of these conditions with $N = 1$ and $Y_1(x) := -\rho(x)$, since ρ is continuous and positive definite with respect to $\mathcal{A}$.

VII. Input-to-state stability

We consider systems of the form

$$x^+ \in \widehat{G}(x, u, v) \quad (10)$$

where u is a worst-case, exogenous disturbance. The set-valued mapping $\widehat{G} : \mathbb{R}^n \times \mathbb{R}^p \times \mathbb{R}^m \rightrightarrows \mathbb{R}^n$ is assumed to satisfy a generalization of Standing Assumption 1:

Assumption 1: The following conditions hold:

1) $(x, u) \mapsto G(x, u, v)$ is outer semicontinuous for each $v \in \mathbb{R}^m$.
2) $v \mapsto \{(g, x, u) \in \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^p : g \in G(x, u, v)\}$ is measurable.
3) G is locally bounded.

The system (10) is *input-to-state stable relative to the compact set* $\mathcal{A}$ if the following properties hold:

1) the compact set $\mathcal{A}$ is stable in probability for the system $x^+ \in \widehat{G}(x, 0, v)$.

2) there exists $\gamma \in \mathcal{K}_\infty$ such that, for each $c > 0$, the set $\mathcal{A} + \gamma(c)\mathbb{B}^\circ$ is globally recurrent for the system $x^+ \in \widehat{G}(x, c\mathbb{B}, v)$.

The main result of [31] is that the system (10) is input-to-state stable relative to the compact set $\mathcal{A}$ if and only if there exists $\alpha \in \mathcal{K}_\infty$ such that the compact set $\mathcal{A}$ is uniformly globally asymptotically stable in probability for the system $x^+ \in \widehat{G}(x, \alpha(|x|_\mathcal{A})\mathbb{B}, v)$. Then, from the converse Lyapunov theorem discussed above, we conclude that the system (10) is input-to-state stable relative to the compact set $\mathcal{A}$ if and only if there exists a continuous function $V : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ that is smooth on $\mathbb{R}^n \backslash \mathcal{A}$, positive definite with respect to $\mathcal{A}$, and radially unbounded, a function $\alpha \in \mathcal{K}_\infty$, and a continuous function $\rho : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ that is positive definite with respect to $\mathcal{A}$ such that

$$\int_{\mathbb{R}^m} \max_{g \in \widehat{G}(x, \alpha(|x|_\mathcal{A})\mathbb{B}, v)} V(g)\mu(dv) \leq V(x) - \rho(x) \ .$$

VIII. Examples

A. The role of causality and Matrosov conditions for global asymptotic stability in probability

We start with an example, adapted from [21, Example 1] for the case of periodic time variations, that also emphasizes the role causality plays in our results. Consider the system (1) with $(x, v) \in \mathbb{R}^3 \times \mathbb{R}$ where $\mathrm{dom}(G) = (\{-\gamma, \gamma\} \times \mathbb{R} \times \{0, \ldots, N\}) \times \{-\gamma, \gamma\}$, where $N \in \mathbb{Z}_{\geq 0}$ and

$$G(x, v) = \begin{bmatrix} \{-\gamma, \gamma\} \\ (1 - \psi(x_3))x_2 + \psi(x_3)(v + x_1)kx_2 \\ (x_3 + 1) \mod (N + 1) \end{bmatrix}$$

for all $(x, v) \in \mathrm{dom}(G)$. Suppose $\mu(\{-\gamma\}) = \mu(\{\gamma\}) = 0.5$. Also suppose $1 - k^2 2\gamma^2 > 0$. For $\psi : \mathbb{Z}_{\geq 0} \to \{0, 1\}$, we assume $\sum_{j=0}^N \psi(j) \geq 1$. We study stability for the compact set $\mathcal{A} := \{-\gamma, \gamma\} \times \{0\} \times \{0, \ldots, N\}$. Consider the Lyapunov function candidate $V(x) = x_2^2$. For each $x \in \mathbb{R}^3$, we get

$$\begin{aligned} &\int_\mathbb{R} \max_{g \in G(x,v)} V(g)\mu(dv) \\ &= (1 - \psi(x_3))x_2^2 + \psi(x_3)k^2 x_2^2 \sum_{i=1}^2 0.5((-1)^i\gamma + x_1)^2 \\ &= (1 - \psi(x_3))x_2^2 + \psi(x_3)k^2 x_2^2 0.5\left(2\gamma^2 + 2x_1^2\right) \\ &= (1 - \psi(x_3))x_2^2 + \psi(x_3)k^2 x_2^2 2\gamma^2 \\ &= V(x) - \psi(x_3)(1 - 2k^2\gamma^2)x_2^2 \leq V(x) \ . \end{aligned}$$

Given $0 < r \leq R < \infty$, we define $W_1(x) := V(x)$, $\phi(x, \sigma) := \psi(\sigma)$, and $Y_1(x) := -(1 - 2k^2\gamma^2)r^2\psi(x_3)$, Next, let $\lambda \in (0, 1)$ and define $W_2(x_3) := \frac{1}{1-\lambda} - \sum_{j=x_3}^\infty \lambda^{j-x_3}\psi((j)\mathrm{mod}(N + 1))$. It can be verified that $0 \leq W_2(x_3) \leq \frac{1}{1-\lambda} - \lambda^N$ for all $x_3 \in \{0, \ldots, N\}$. Moreover, with $g_3(x_3) := (x_3 + 1)\mathrm{mod}(N + 1)$, we get for all $x_3 \in \{0, \ldots, N\}$,

$$\begin{aligned} &W_2(g_3(x_3)) \\ &= \frac{1}{1-\lambda} - \sum_{j=g_3(x_3)}^\infty \lambda^{j-g_3(x_3)}\psi((j)\mathrm{mod}(N + 1)) \\ &= \frac{1}{1-\lambda} - \lambda^{-1}\sum_{j=x_3}^\infty \lambda^{j-x_3}\psi((j)\mathrm{mod}(N + 1)) + \lambda^{-1}\psi(x_3) \\ &\leq W_2(x_3) - (\lambda^{-1} - 1)\lambda^N + \lambda^{-1}\psi(x_3) \ . \end{aligned}$$

Let $Y_2(x) := -(1 - \lambda)\lambda^{N-1} + \lambda^{-1}\psi(x_3)$. It can be verified that the functions Y_1, Y_2 satisfy the Matrosov conditions spelled out earlier. Thus, the set $\mathcal{A}$ is uniformly globally asymptotically stable in probability.

Now suppose that *non-causal* measurable selections were considered and suppose $k^2 \in [1/(4\gamma^2), 1/(2\gamma^2))$. The upper bound of k^2 guarantees global asymptotic stability in probability, as established above. However, there exists a non-causal random solution for which no sample path converges to $\mathcal{A}$. Indeed, consider the random solution that satisfies $\mathbf{x}_{1,k} = \mathbf{v}_k$. In this case,

$$\mathbf{x}_{2,k+1}^2 \geq \min\left\{\mathbf{x}_{2,k}^2, 4\gamma^2 k^2 \mathbf{x}_{2,k}^2\right\} \geq \mathbf{x}_{2,k}^2 \ .$$

B. From non-Markovian to Markovian processes

Stochastic difference inclusions provide a convenient methodology to study processes that are not Markov. To see how this can be done, consider the process $\mathbf{x}_k \in \mathbb{R}$, $k \in \mathbb{Z}_{\geq 0}$ defined recursively by

$$\mathbf{x}_0 = x, \qquad \mathbf{x}_{k+1} = \mathbf{v}_{\lfloor \frac{k}{2} \rfloor}\mathbf{v}_k\mathbf{x}_k, \quad \forall k \in \mathbb{Z}_{\geq 0},$$

with $x \in \mathbb{R}$ an arbitrary initial condition and with the $\mathbf{v}_k$ independent and uniformly distributed in the interval $[0, 1 + \epsilon]$, $\epsilon > 0$. This process is not Markov for the natural filtration $\widetilde{\mathcal{F}} := \{\widetilde{\mathcal{F}}_k : k \in \mathbb{Z}_{\geq 0}\}$ of $\mathbf{x}_k$, $k \in \mathbb{Z}_{\geq 0}$ since

$$\begin{aligned} \mathbb{E}[\mathbf{x}_{k+1} \mid \widetilde{\mathcal{F}}_k] &= \mathbb{E}[\mathbf{x}_{k+1} \mid \mathbf{x}_k, \mathbf{x}_{k-1}, ..., \mathbf{x}_0] \\ &= \mathbf{v}_{\lfloor \frac{k}{2} \rfloor}\mathbf{x}_k\mathbb{E}[\mathbf{v}_k \mid \mathbf{x}_k, \mathbf{x}_{k-1}, ..., \mathbf{x}_0] = \mathbf{v}_{\lfloor \frac{k}{2} \rfloor}\mathbf{x}_k\mathbb{E}[\mathbf{v}_k] \\ &\neq \mathbb{E}[\mathbf{x}_{k+1} \mid \mathbf{x}_k] = \mathbf{x}_k\mathbb{E}[\mathbf{v}_{\lfloor \frac{k}{2} \rfloor}\mathbf{v}_k \mid \mathbf{x}_k] \ . \end{aligned}$$

However, it satisfies the stochastic difference inclusion

$$x^+ \in \left\{\nu\, vx : \nu \in [0, 1 + \epsilon]\right\} =: G(x, v).$$

Defining $V(x) := x^2$, $\forall x \in \mathbb{R}$ we have that

$$\int_\mathbb{R} \max_{g \in G(x,v)} V(g)\mu(dv) = \frac{(1 + \epsilon)^4}{3}V(x) \qquad \forall x \in \mathbb{R}$$

and therefore V is a Lyapunov function that establishes global asymptotic stability in probability for the origin provided that $\frac{(1+\epsilon)^4}{3} < 1 \Leftrightarrow \epsilon < \sqrt[4]{3} - 1 \approx .316$.

C. Input-to-state stability

Consider the system

$$x^+ = vx + u \ . \tag{11}$$

The random variable v is such that, for some $p > 0$, $\int_\mathbb{R} v^p\mu(dv) =: \lambda_1 < 1$. Pick $\varepsilon > 0$ and $\delta > 0$ small enough so that

$$(1 + \varepsilon)^p\lambda_1 + (1 + \varepsilon^{-1})^p\delta^p =: \lambda_2 < 1 \ . \tag{12}$$

Consider $V(x) = |x|^p$ and the system

$$x^+ \in \{vx\} + \delta|x|\mathbb{B} =: G(x,v) \ .$$

Using that

$$(a+b)^p \le (1+\varepsilon)^p a^p + (1+\varepsilon^{-1})^p b^p \ ,$$

we get

$$\begin{aligned}&\int_{\mathbb{R}} \max_{g\in G(x,v)} V(g)\mu(dv) \\ &\quad \le (1+\varepsilon)^p \lambda_1 |x|^p + (1+\varepsilon^{-1})^p \delta^p |x|^p \\ &\quad = V(x) - (1-\lambda_2)|x|^p \ .\end{aligned}$$

We conclude from the results stated earlier that the system (11) is input-to-state stable relative to the origin.

IX. Conclusion

We have summarized a framework for stability theory for stochastic difference inclusions. In particular, we have reviewed recent converse Lyapunov theorems for recurrence and for uniform global asymptotic stability in probability, results on robustness of these properties, and sufficient Matrosov-based conditions for these properties. We have also provided examples to illustrate the main features. The purpose of this paper has been to give a unified presentation of recent results that have been obtained on the subject, with the additional goal of motivating further research in this area.

References

[1] S. P. Meyn and R. L. Tweedie, *Markov Chains and Stochastic Stability*. Springer-Verlag, 1993.

[2] G. G. Yin and C. Zhu, *Hybrid Switching Diffusions*. Springer, 2010.

[3] R. Z. Khasminskii, *Stochastic Stability of Differential Equations*. Sijthoff & Noordhoff, Netherlands, 1980.

[4] R. Bellman, "A Markovian decision process," *Journal of Mathematics and Mechanics*, vol. 6, no. 5, pp. 679–684, 1957.

[5] M. O. Rabin, "Probabilistic automata," *Information and Control*, vol. 6, no. 3, pp. 230 – 245, 1963.

[6] N. Papageorgiou, "Convergence and representation theorems for set-valued random processes," *J. of Mathematical Analysis and Applications*, vol. 150, pp. 129–145, 1990.

[7] P. Panangaden, *Labelled Markov Processes*. Imperial College Press, 2009.

[8] D. Chakraborty and P. Stone, "Online multiagent learning against memory bounded adversaries," in *Machine Learning and Knowledge Discovery in Databases*, ser. Lecture Notes in Computer Science, W. Daelemans, B. Goethals, and K. Morik, Eds. Springer Berlin / Heidelberg, 2008, vol. 5211, pp. 211–226.

[9] N. N. Krasovskii, *Game-Theoretic Problems of Capture*. Nauka, Moscow, 1970, in Russian.

[10] C. M. Kellett and A. R. Teel, "Smooth Lyapunov functions and robustness of stability for differential inclusions," *Sys. Contr. Lett.*, vol. 52, pp. 395–405, 2004.

[11] R. G. Sanfelice, R. Goebel, and A. R. Teel, "Generalized solutions to hybrid dynamical systems," *ESAIM: Control, Optimisation and Calculus of Variations*, vol. 14, no. 4, pp. 699–724, 2008.

[12] F. Clarke, Y. Ledyaev, and R. Stern, "Asymptotic stability and smooth Lyapunov functions." *Journal of Differential Equations*, vol. 149, pp. 69–114, 1998.

[13] R. Goebel and A. R. Teel, "Solutions to hybrid inclusions via set and graphical convergence with stability theory applications," *Automatica*, vol. 42, pp. 573–587, 2006.

[14] J. P. Hespanha and A. S. Morse, "Stability of switched systems with average dwell-time," in *Proc. 38th IEEE Conference on Decision and Control*, Sidney, Australia, 1999, pp. 2655–2660.

[15] R. Goebel, R. G. Sanfelice, and A. R. Teel, "Hybrid dynamical systems," *IEEE Control Systems Magazine*, vol. 29, no. 2, pp. 28–93, April 2009.

[16] E. D. Sontag and Y. Wang, "New characterizations of input-to-state stability," *IEEE Trans. Autom. Control*, vol. 41, no. 9, pp. 1283–1294, 1996.

[17] R. Goebel, R. Sanfelice, and A. Teel, *Hybrid Dynamical Systems*. Princeton University Press, 2012.

[18] A. R. Teel, "Adversarial Markov decision processes: invariance and recurrence principles," 2011, Submitted.

[19] A. R. Teel, J. P. Hespanha, and A. Subbaraman, "A converse Lyapunov theorem for asymptotic stability in probability," 2012, Submitted.

[20] A. Subbaraman and A. R. Teel, "A converse Lyapunov theorem for global recurrence," 2012, Submitted.

[21] A. R. Teel, "A Matrosov theorem for adversarial Markov decision processes," 2012, Submitted.

[22] A. R. Teel and A. Subbaraman, "A Matrosov theorem for global recurrence," 2012, Submitted.

[23] R. T. Rockafellar and R. J.-B. Wets, *Variational Analysis*. Springer, 1998.

[24] R. L. Tweedie, "The robustness of positive recurrence and recurrence of Markov chains under perturbations of the transition probabilities," *J. Appl. Probability*, vol. 12, no. 4, pp. 744–752, 1975.

[25] ——, "Perturbations of countable Markov chains and processes," *Ann. Inst. Statist. Math.*, vol. 32, no. 2, pp. 283–290, 1980.

[26] V. Matrosov, "On the theory of stability of motion," *J. Appl. Math. Mech.*, vol. 26, pp. 1506–1522, 1962.

[27] B. Paden and R. Panja, "Globally asymptotically stable PD+controller for robot manipulators," *International Journal of Control*, vol. 47, no. 6, pp. 1697–1712, 1988.

[28] A. Loria, E. Panteley, D. Popovic, and A. Teel, "A nested Matrosov theorem and persistency of excitation for uniform convergence in stable nonautonomous systems," *IEEE Trans. Autom. Control*, vol. 50, no. 2, pp. 183 – 198, 2005.

[29] W. Wonham, "Liapunov criteria for weak stochastic stability," *Journal of Differential Equations*, pp. 195–207, 1966.

[30] R. Tweedie, "Criteria for classifying general markov chains," *Adv. Appl. Prob.*, pp. 737–771, 1976.

[31] A. R. Teel, J. P. Hespanha, and A. Subbaraman, "Equivalent characterizations of input-to-state stability for stochastic discrete-time systems," 2012, Submitted.