Robust Stability under Asynchronous Sensing and Control

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***Abstract*— We address the stability of networked control systems in which a sensor and an output-feedback controller operate asynchronously, which leads to uncertainty in the sampling instants. In addition, we also consider polytopic uncertainty in the plant model. The analysis is based on transforming the closed-loop system into an impulsive system, by considering an extended state variable that includes the states of the continuous-time plant and the discrete-time controller. We provide a sufficient condition for the robust stability of the closed-loop system in terms of linear matrix inequalities. This condition is based on the construction of a continuous-time Lyapunov functional that also incorporates the discrete-time state of the digital controller. We illustrate the obtained result with numerical simulations.**

I. Introduction

Parameter perturbation and disturbances/noises has been extensively studied in the robust control literature; see, e.g., [1], [2] and many references therein. While control systems also have uncertainty in the time domain [3], [4], relatively little work has been done on time-domain uncertainty. Our goal is to analyze how large uncertainty in both the parameter and time domains can be without compromising the closed-loop stability.

In networked control systems, one of the major sources of time-domain uncertainties is a synchronization error between local subsystems. As surveyed in [5], [6], many synchronization algorithms have been developed, and easy access to global clocks such as GPS and radio clocks leads to high-precision synchronization in practical situations. However, there are fundamental limitations on clock synchronization due to variable delays [7]. Furthermore, the signals of GPS and radio clocks are not ubiquitously available, and it is reported in [8] that the GPS-based synchronization is vulnerable against attacks.

Asynchronous dynamical systems have been investigated in various fields including engineering and biology. An observer-based control has been proposed for networked controlled systems under synchronization errors and parametric uncertainty in [9]. For systems with asynchronous sensing and control, stability analysis [10], L^2-gain analysis [11], and limitations on the clock offset tolerable for stabilization [12], [13] have been studied. The time measure of the optimal controller in [14] is a stochastic process subject to noise. The authors in [15] have compensated clock offsets and skews for the timestamp-based synchronization of multiple plants over networks. The experimental results in [16], [17] indicate that human subjects potentially learn temporal uncertainty.

In this paper, we study the robust stability of systems that have variable offsets between the clocks of the sensor and the digital controller. We assume that the system has polytopic uncertainty as, e.g., in [18], [19]. We provide a sufficient condition for the closed-loop stability via linear matrix inequalities (LMIs). The proposed method is illustrated with a numerical simulation by showing how large space and time-domain uncertainty would be allowed by a given controller.

A significant challenge to the analysis of networked closed-loop systems stems from the fact that such systems have both continuous-time and discrete-time state variables. In the stability analysis of [9], a continuous-time controller is used with the input-delay approach [20]–[22], and hence the closed-loop system has only continuous-time states. The authors in [10] focus on discrete-time states by discretizing the closed-loop system. However, this discretization approach leads to a nonlinear term including both parametric and time-domain uncertainty, which brings conservativeness for robust stability analysis.

In contrast to the references mentioned above, we represent the closed-loop system as an impulsive system as done in [23]–[28] for systems with variable delays and aperiodic sampling. In this representation, space-domain uncertainty appears in an affine form, and therefore allows us to more efficiently address polytopic uncertainty in the original system. We describe the state of the digital controller by a piecewise constant function and construct a Lyapunov functional that incorporates both continuous and discrete-time states.

This paper is organized as follows. In Section II, we introduce the closed-loop system and basic assumptions, and then formulate our problem. Section III is devoted to the main result, and we provide its proof in Section IV. In Section V, we discuss the advantages and the disadvantages of a discretization of the closed-loop system, with respect to modeling the closed loop as an impulse system (as we have done here). We illustrate the proposed method with a numerical simulation in Section VI and give concluding remarks in Section VII.

Notation and definitions: For a real matrix M, let $M^\top$ denote its transpose. For a real square matrix Q, define $\mathrm{He}(Q) := Q + Q^\top$. We denote the Euclidean norm of a

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real vector v by $\|v\| := (v^\top v)^{1/2}$.

For a piecewise continuous function ϕ, we denote its left-sided limit at time t by

$$\phi(t^-) := \lim_{\epsilon \searrow 0} \phi(t-\epsilon)$$

The upper right-hand derivative of ϕ with respect to time t is denoted by $\dot{\phi}$, that is,

$$\dot{\phi}(t) := \lim_{\epsilon \searrow 0} \frac{\phi(t+\epsilon)-\phi(t)}{\epsilon}.$$

Let $\mathcal{W}_h$ denote the space of functions $\phi : [-h,0] \to \mathbb{R}^n$ that are absolutely continuous in $[-h,0)$ and have the square integrable first-order derivatives in $[-h,0]$. The norm of $\mathcal{W}_h$ is defined by

$$\|\phi\|_{\mathcal{W}_h} := \max_{\theta\in[-h,0]} \|\phi(\theta)\| + \left(\int_{-h}^{0} \left\| \frac{d\phi}{ds}(s) \right\|^2 ds \right)^{1/2}.$$

We denote by $\mathcal{U}_h$ the space of functions $\psi : [-h,0] \to \mathbb{R}^n$ that are piecewise continuous in $[-h,0)$. The norm of $\mathcal{U}_h$ is defined by

$$\|\psi\|_{\mathcal{U}_h} := \max_{\theta\in[-h,0]} \|\psi(\theta)\|.$$

For two normed linear spaces W and U, we define the direct sum $W \oplus U$ by

$$W \oplus U := \left\{ \begin{bmatrix} w \\ u \end{bmatrix} : w \in W,\ u \in U \right\},$$

which becomes a normed linear space with the norm

$$\left\| \begin{bmatrix} w \\ u \end{bmatrix} \right\| = \sqrt{\|w\|_W^2 + \|u\|_U^2},$$

where $\|\cdot\|_W$ and $\|\cdot\|_U$ are the norms of W and U.

II. Problem Formulation

A. Plant and controller

Consider a linear time-invariant system:

$$\Sigma_P : \begin{cases} \dot{x}(t) = Ax(t) + Bu(t) \\ y(t) = Cx(t), \end{cases}$$

where $x(t) \in \mathbb{R}^{n_s}$, $u(t) \in \mathbb{R}^{n_u}$, and $y(t) \in \mathbb{R}^{n_y}$ are the state, input, and output of the plant, respectively. This plant Σ_P is connected with a digital controller Σ_C through a zero-order hold and a sampler:

$$\Sigma_C : \begin{cases} \xi[k+1] = A_c\xi[k] + B_c y[k] \\ \quad u[k] = C_c\xi[k], \end{cases} \tag{1}$$

where $\xi[k] \in \mathbb{R}^{n_c}$ is the controller state.

We place the following assumptions on the zero-order hold and the sampler:

Assumption 2.1 (Periodic update of zero-order hold): The control input $u(t)$ is periodically generated through the zero-order hold with period h:

$$u(t) = u[k] \qquad (kh \le t < (k+1)h).$$

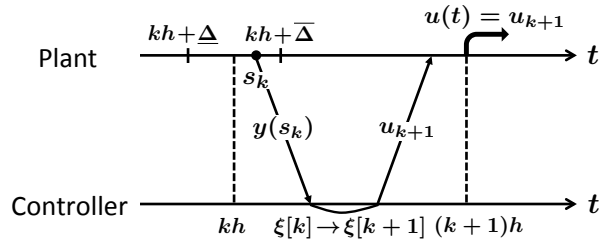

Fig. 1: Timing diagram.

Assumption 2.2 (Perturbed sampling times): The sensor samples the plant output $y(t)$ at a perturbed sampling time $s_k = kh + \Delta_k$, where $\Delta_k \in [\underline{\Delta}, \overline{\Delta}] \subset (-h,h)$ represents the sensor clock offset at time $t = kh$. The output y_k received by the controller is therefore defined by

$$y[k] := y(s_k) = Cx(kh + \Delta_k).$$

Fig. 1 illustrates the timing diagram.

B. Delayed impulsive system

Extending the discrete-time state $\xi[k]$ to a piecewise constant function $\xi(t)$

$$\xi(t) := \xi[k] \qquad (kh \le t < (k+1)h),$$

we can represent the closed-loop system Σ_{cl} as the following delayed impulsive system:

$$\Sigma_{cl} : \begin{cases} \dot{z}(t) = \begin{bmatrix} A & BC_c \\ 0 & 0 \end{bmatrix} z(t), \quad (kh \le t < (k+1)h) \\ z(kh) = \begin{bmatrix} I & 0 \\ 0 & A_c \end{bmatrix} z(kh^-) + \begin{bmatrix} 0 & 0 \\ B_cC & 0 \end{bmatrix} z(s_{k-1}), \end{cases} \tag{2}$$

where the closed-loop state z is defined by

$$z(t) := \begin{bmatrix} x(t) \\ \xi(t) \end{bmatrix}.$$

For the delayed impulsive system Σ_{cl}, the initial state z_0 can be defined by

$$z_0 := \begin{bmatrix} x_0 \\ \xi_0 \end{bmatrix} \in \mathcal{W}_{2h+\underline{\Delta}} \oplus \mathcal{U}_h,$$

where $x_0(\theta) := x(\theta)$ for all $\theta \in [-(2h+\underline{\Delta}), 0]$ and $\xi_0(\theta) := \xi(\theta) = \xi[-1]$ for all $\theta \in [-h, 0]$. In fact, for all $t \in [kh, (k+1)h)$, the plant dynamics is given by

$$\begin{aligned} \dot{x}(t) &= Ax(t) + BC_c\xi[k] \\ &= Ax(t) + BC_c(A_c\xi(t-h) + B_c y[k]) \\ &= Ax(t) + BC_cB_cCx(t-(t-s_{k-1})) + BC_cA_c\xi(t-h), \end{aligned} \tag{3}$$

and the delay $t - s_{k-1}$ satisfies $t - s_{k-1} \le 2h + \underline{\Delta}$ for all $t \in [kh, (k+1)h)$. In what follows, we omit the subscripts $2h + \underline{\Delta}$ of $\mathcal{W}_{2h+\underline{\Delta}}$ and h of $\mathcal{U}_h$ for simplicity of notation.

Before stating our control problem, we define exponential stability for the closed-loop system.

Definition 2.3: The delayed impulsive system Σ_{cl} in (2) is exponentially stable with decay rate $\gamma > 0$ if there exists $\Omega \geq 1$ such that $\|z(t)\| \leq \Omega e^{-\gamma t}\|z_0\|$ for every $t \geq 0$ and for every $z_0 \in \mathcal{W} \oplus \mathcal{U}$.

In this paper, we study the following problem:

Problem 2.4: Let $\mathcal{P}$ be the set to which the triple of matrices (A, B, C) belong. Given a controller (A_c, B_c, C_c) and a clock-offset bound $[\underline{\Delta}, \overline{\Delta}]$, determine whether the closed-loop system Σ_{cl} in (2) is exponentially stable for all system matrices $(A, B, C) \in \mathcal{P}$ and all time-varying clock offsets $\Delta_k \in [\underline{\Delta}, \overline{\Delta}]$.

III. Main Results

First, we consider the case where there is no uncertainty in the input matrix B.

Assumption 3.1 (Polytopic uncertainty in A and C): The input matrix B is fixed and the matrices A and C satisfy

$$\begin{bmatrix} A \\ C \end{bmatrix} \in \mathcal{P} := \left\{ \sum_{i=1}^{n} \alpha_i \begin{bmatrix} A_i \\ C_i \end{bmatrix} : \ \alpha_i \geq 0, \ \sum_{i=1}^{n} \alpha_i = 1 \right\}.$$

Theorem 3.2: Let Assumptions 2.1, 2.2, and 3.1 hold. Fix $\gamma > 0$. The closed-loop system is exponentially stable with decay rate $\gamma/2$ if there exist positive definite matrices P_i, $R_{k,i}$ $(k = 1, \ldots, 4, i = 1, \ldots, n)$, Z_i, X_i, and Q_i and (not necessarily symmetric) matrices $N_{k,i}$ $(k = 1, \ldots, 4, i = 1, \ldots, n)$, V_l, U_l, and W_l $(l = 1, 2)$ such that

$$\begin{bmatrix} \Gamma_{1,i} & \Lambda_{1,i} \\ \star & \Xi_{1,i} \end{bmatrix} \geq 0, \quad \begin{bmatrix} \Gamma_{2,i} & \Lambda_{2,i} \\ \star & \Xi_{2,i} \end{bmatrix} \geq 0, \tag{4}$$

where the matrices Γ, Λ, and Ξ are defined by (\$) with $E_1 := [I \ 0 \ 0 \ 0 \ 0] \in \mathbb{R}^{n_s \times (4n_s + n_c)}, \ldots, E_4 := [0 \ 0 \ 0 \ I \ 0] \in \mathbb{R}^{n_s \times (4n_s + n_c)}$, $E_5 := [0 \ 0 \ 0 \ 0 \ I] \in \mathbb{R}^{n_c \times (4n_s + n_c)}$, $\tau_1 := h + \underline{\Delta}$, $\tau_2 := h$, $\tau_3 := h - \overline{\Delta}$, $\tau_4 := h + (\underline{\Delta} + \overline{\Delta})$, $\bar{\tau}_1 := 2h + \underline{\Delta}$, $\bar{\tau}_4 := \underline{\Delta} + \overline{\Delta}$,

$$\begin{aligned}
F_i &:= \begin{bmatrix} A_i & 0 & 0 & BC_cB_cC_i & BC_cA_c \end{bmatrix} \\
G_i &:= \begin{bmatrix} 0 & 0 & 0 & B_cC_i & A_c \end{bmatrix}, \quad \Pi_i := \begin{bmatrix} F_i \\ G_i \end{bmatrix} \\
M_{0,i} &:= \mathrm{He}(N_{1,i}^\top (E_2 - E_4)) + \mathrm{He}(N_{2,i}^\top (E_1 - E_2)) \\
&\quad + \mathrm{He}(N_{3,i}^\top (E_1 - E_3)) + \mathrm{He}(N_{4,i}^\top (E_3 - E_4)) \\
&\quad - E_1^\top Z_i E_1 + E_3^\top (e^{-\gamma\tau_3} Z_i) E_3 \\
&\quad + (E_1 - E_2)^\top X_i (E_1 - E_2) + E_5^\top (e^{-\gamma\tau_2} Q_i) E_5 \\
M_{1,i} &:= M_{0,i} + \mathrm{He}\left(\begin{bmatrix} E_1 \\ 0 \end{bmatrix}^\top P_i \begin{bmatrix} 0 \\ E_5 \end{bmatrix} \right) - \gamma \begin{bmatrix} E_1 \\ 0 \end{bmatrix}^\top P_i \begin{bmatrix} E_1 \\ 0 \end{bmatrix} \\
M_{2,i} &:= M_{0,i} + \mathrm{He}\left(\begin{bmatrix} E_1 \\ hE_5 \end{bmatrix}^\top P_i \begin{bmatrix} 0 \\ E_5 \end{bmatrix} \right) \\
&\quad - \gamma \begin{bmatrix} E_1 \\ hE_5 \end{bmatrix}^\top P_i \begin{bmatrix} E_1 \\ hE_5 \end{bmatrix} - \gamma h (E_1 - E_2)^\top X_i (E_1 - E_2) \\
\Omega_{11,i} &:= \begin{bmatrix} I \\ 0 \end{bmatrix} (\bar{\tau}_1 R_{1,i} + \tau_2 R_{2,i} + \tau_3 R_{3,i} + \tau_4 R_{4,i}) \begin{bmatrix} I & 0 \end{bmatrix} \\
&\quad + \begin{bmatrix} 0 \\ I \end{bmatrix} Q_i \begin{bmatrix} 0 & I \end{bmatrix} \\
\Omega_{12,i} &:= \mathrm{He}\left(P_i \begin{bmatrix} 0 & 0 \\ 0 & hI \end{bmatrix} \right) + \gamma \begin{bmatrix} 0 & 0 \\ 0 & hI \end{bmatrix} P_i \begin{bmatrix} 0 & 0 \\ 0 & hI \end{bmatrix} \\
\Omega_{1,i} &:= \Omega_{11,i} + \Omega_{12,i}, \quad \Omega_{2,i} := \Omega_{11,i} \\
\Upsilon_{1,i} &:= P_i \begin{bmatrix} E_1 \\ 0 \end{bmatrix} - \begin{bmatrix} 0 & 0 \\ 0 & hI \end{bmatrix} P_i \begin{bmatrix} 0 \\ E_5 \end{bmatrix} + \gamma \begin{bmatrix} 0 & 0 \\ 0 & hI \end{bmatrix} P_i \begin{bmatrix} E_1 \\ 0 \end{bmatrix} \\
\Upsilon_{2,i} &:= P_i \begin{bmatrix} E_1 \\ hE_5 \end{bmatrix} + h \begin{bmatrix} I \\ 0 \end{bmatrix} X_i (E_1 - E_2).
\end{aligned}$$

We will sketch the proof of Theorem 3.2 in Section IV, which will be based on the constructions of an appropriate Lyapunov functional.

We next study the case where there may be uncertainty in the input matrix B, but not in the output matrix C.

Assumption 3.3 (Polytopic uncertainty in A and B): The output matrix C is fixed and the matrices A and B satisfy

$$\begin{bmatrix} A & B \end{bmatrix} \in \mathcal{P} := \left\{ \sum_{i=1}^{n} \alpha_i \begin{bmatrix} A_i & B_i \end{bmatrix} : \ \alpha_i \geq 0, \ \sum_{i=1}^{n} \alpha_i = 1 \right\}.$$

Similarly to Theorem 3.2, we can obtain the following result under Assumption 3.3.

Theorem 3.4: Let Assumptions 2.1, 2.2, and 3.3 hold. The closed-loop system is exponentially stable with decay rate $\gamma/2$ if there exist positive definite matrices P_i, $R_{k,i}$ $(k = 1, \ldots, 4, i = 1, \ldots, n)$, Z_i, X_i, and Q_i and (not necessarily symmetric) matrices $N_{k,i}$ $(k = 1, \ldots, 4, i = 1, \ldots, n)$, V_l, U_l, and W_l $(l = 1, 2)$ such that LMIs (4) are feasible with the matrices Γ, Λ, and Ξ are defined by (\$), where E, τ, $\bar{\tau}$, and $M_{0,i}$ are defined as in Theorem 3.2 and

$$\begin{aligned}
F_i &:= \begin{bmatrix} A_i & 0 & 0 & B_iC_cB_cC & B_iC_cA_c \end{bmatrix} \\
G &:= \begin{bmatrix} 0 & 0 & 0 & B_cC & A_c \end{bmatrix}, \quad \Pi_i := F_i \\
M_{1,i} &:= M_{0,i} - G^\top Q_i G - \mathrm{He}\left(\begin{bmatrix} 0 \\ G - E_5 \end{bmatrix}^\top P_i \begin{bmatrix} E_1 \\ hG \end{bmatrix} \right) \\
&\quad - \gamma \begin{bmatrix} E_1 \\ hG \end{bmatrix}^\top P_i \begin{bmatrix} E_1 \\ hG \end{bmatrix} \\
M_{2,i} &:= M_{0,i} - G^\top Q_i G - \mathrm{He}\left(\begin{bmatrix} E_1 \\ hE_5 \end{bmatrix}^\top P_i \begin{bmatrix} 0 \\ G - E_5 \end{bmatrix} \right) \\
&\quad - \gamma \begin{bmatrix} E_1 \\ hE_5 \end{bmatrix}^\top P_i \begin{bmatrix} E_1 \\ hE_5 \end{bmatrix} \\
\Omega_{1,i} &:= \bar{\tau}_1 R_{1,i} + \tau_2 R_{2,i} + \tau_3 R_{3,i} + \tau_4 R_{4,i}, \ \Omega_{2,i} := \Omega_{1,i} \\
\Upsilon_{1,i} &:= \begin{bmatrix} I \\ 0 \end{bmatrix}^\top P_i \begin{bmatrix} E_1 \\ hG \end{bmatrix} \\
\Upsilon_{2,i} &:= \begin{bmatrix} I \\ 0 \end{bmatrix}^\top P_i \begin{bmatrix} E_1 \\ hE_5 \end{bmatrix} + hX_i(E_1 - E_2).
\end{aligned}$$

Remark 3.5: If we consider the state feedback case, then the state estimator is given by

$$\xi[k+1] = A_d x(s_k) + B_d u[k],$$

where A_d and B_d are defined by

$$A_d := e^{Ah}, \quad B_d := \int_0^h e^{A(h-\tau)} B d\tau.$$

$$
\begin{aligned}
\Gamma_{l,i} &:= \begin{bmatrix} M_{l,i} - \mathrm{He}(\Pi_i^\top V_l) & -\Pi_i^\top U_1 & V_l^\top - \Upsilon_{l,i}^\top - \Pi_i^\top W_l \\ \star & \mathrm{He}(U_l) - \Omega_{l,i} & 0 \\ \star & \star & \mathrm{He}(W_l) \end{bmatrix} \qquad (l = 1, 2) \\
\Lambda_{1,i} &:= \begin{bmatrix} \tau_1 N_{1,i}^\top & \tau_2 N_{2,i}^\top & \tau_3 N_{3,i}^\top & \tau_4 N_{4,i}^\top \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \qquad \Lambda_{2,i} := \begin{bmatrix} \tau_1 N_{1,i}^\top & \tau_3 N_{3,i}^\top & \bar{\tau}_4 N_{4,i}^\top \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \\
\Xi_{1,i} &:= \mathrm{diag}(\tau_1 e^{-\gamma\bar{\tau}_1} R_{1,i}, \tau_2 e^{-\gamma\tau_2}(R_{1,i} + R_{2,i}), \tau_3 e^{-\gamma\tau_3} R_{3,i}, \tau_4 e^{-\gamma\bar{\tau}_1} R_{4,i}) \\
\Xi_{2,i} &:= \mathrm{diag}(\tau_1 e^{-\gamma\bar{\tau}_1} R_{1,i}, \tau_2 e^{-\gamma\tau_3} R_{3,i}, \bar{\tau}_4 e^{-\gamma\bar{\tau}_1} R_{4,i})
\end{aligned} \tag{\$}
$$

We set the control input $u[k] = K\xi[k]$. Note that if the actual sampling time s_k satisfies $s_k = kh$, then this estimator achieves perfect state reconstruction $\xi[k+1] = x((k+1)h)$. Under Assumption 3.3, if we use F_i and G_i defined by

$$
\begin{aligned}
F_i &:= \begin{bmatrix} A_i & 0 & 0 & B_i K A_d & B_i K B_d K \end{bmatrix} \\
G_i &:= \begin{bmatrix} 0 & 0 & 0 & A_d & B_d K \end{bmatrix},
\end{aligned}
$$

then the counterpart of Theorem 3.4 can be obtained in the state feedback case.

Remark 3.6: If the controller is composed of a Luenberger observer and a feedback gain, then the controller parameters A_c, B_c, C_c in (1) are given by $A_c = A_d + B_d K + LC_d$, $B_c = -L$, and $C_c = K$, where K and L are a feedback gain and an observer gain, respectively.

Remark 3.7: In this paper, we do not consider the case when both of B and C have uncertainty, because the state equation of the plant in (3) has a quadratic term BC_cB_cC with respect to B and C; see also the definition of the matrix F_i in Theorems 3.2 and 3.4.

IV. Construction of Lyapunov Functional

A. Preliminaries

Define $\rho_1(t) := t - s_{k-1}$ and $\rho_2(t) := t - kh$ for all $t \in [kh, (k+1)h)$, and define their supremums

$$
\rho_{1,\sup} := \sup_{t \geq 0} \rho_1(t), \quad \rho_{2,\sup} := \sup_{t \geq 0} \rho_2(t) = h.
$$

Define a function v by

$$
v(t) := \int_{t-h}^{t} \xi(s) ds = (h - \rho_2(t))\xi[k-1] + \rho_2(t)\xi[k]
$$

for every $t \in [kh, (k+1)h)$.

For each $t \geq 0$, define $x_t(\theta) := x(t+\theta)$ for all $\theta \in [-(2h + \underline{\Delta}), 0]$ and define $\xi_t(\theta) := \xi(t+\theta)$ for all $\theta \in [-h, 0]$. Since $dx_t/d\theta$ is integrable in $[-h, 0]$ from the linearity of the plant Σ_P, we have $x_t \in \mathcal{W}$. Also, since ξ_t is piecewise constant, it follows that $\xi_t \in \mathcal{U}$.

B. Lyapunov functional

Define a Lyapunov functional V by

$$
\begin{aligned}
&V(\rho_1(t), \rho_2(t), x_t, \xi_t) \\
&\quad := V_{c,d}(x_t, \xi_t) + V_c(\rho_1(t), \rho_2(t), x_t) + V_d(\xi_t)
\end{aligned}
$$

where $V_{c,d}$ is a Lyapunov functional for both of the continuous-time state x and the discrete-time state ξ, V_c is for the continuous-time state x, and V_d is for the discrete-time state ξ. For positive definite matrices P, R_i, Z, X, and Q, these Lyapunov functionals are defined by $V_{c,d} := V_1$, $V_c := \sum_{i=2}^{8} V_i$, and $V_d := V_9$ with

$$
\begin{aligned}
V_1 &:= \begin{bmatrix} x(t) \\ v(t) \end{bmatrix}^\top P \begin{bmatrix} x(t) \\ v(t) \end{bmatrix} \\
V_2 &:= \int_{t-\rho_1(t)}^{t} (\rho_{1,\sup} - t + s) e^{\gamma(s-t)} \dot{x}(s)^\top R_1 \dot{x}(s) ds \\
V_3 &:= \int_{t-\rho_2(t)}^{t} (\rho_{2,\sup} - t + s) e^{\gamma(s-t)} \dot{x}(s)^\top R_2 \dot{x}(s) ds \\
V_4 &:= \int_{t-(h-\overline{\Delta})}^{t} ((h - \overline{\Delta}) - t + s) e^{\gamma(s-t)} \dot{x}(s)^\top R_3 \dot{x}(s) ds \\
V_5 &:= \int_{t-\rho_1(t)}^{t-(h-\overline{\Delta})} (\rho_{1,\sup} - t + s) e^{\gamma(s-t)} \dot{x}(s)^\top R_4 \dot{x}(s) ds \\
V_6 &:= (\rho_{1,\sup} - (h - \overline{\Delta})) \int_{t-(h-\overline{\Delta})}^{t} e^{\gamma(s-t)} \dot{x}(s)^\top R_4 \dot{x}(s) ds \\
V_7 &:= \int_{t-(h-\overline{\Delta})}^{t} e^{\gamma(s-t)} x(s)^\top Z x(s) ds \\
V_8 &:= (\rho_{2,\sup} - \rho_2(t))(x(t) - x(kh))^\top X (x(t) - x(kh)) \\
V_9 &:= \int_{t-h}^{t} e^{\gamma(s-t)} \xi(s)^\top Q \xi(s) ds.
\end{aligned}
$$

We employ a continuous-time Lyapunov functional V for the stability analysis of the impulsive system Σ_{cl} in (2) that has both the continuous-time state x and the discrete-time state ξ. The Lyapunov functional V_c is used for the continuous-time state x and is inspired by [24].

On the other hand, for the discrete-time state ξ, we employ the Lyapunov functional V_d. In fact, if $x(s_k) = 0$, then $V_d = V_9$ with $\gamma = 0$ satisfies

$$
\dot{V}_d(\xi_t) = \xi[k]^\top (A_c^\top Q A_c - Q) \xi[k]^\top
$$

for all $t \in [(k+1)h, (k+2)h)$. Therefore, $\dot{V}_d < 0$ if and only if the discrete-time Lyapunov inequality

$$
A_c^\top Q A_c - Q < 0
$$

holds.

Due to space constraints, we only sketch the robust stability proof. Assume that the LMI (4) holds. Using the

uncertainty parametes, $\{\alpha_i\}_{i=1}^n$, in Assumptions 3.1 and 3.3, we define

$$P := \sum_{i=1}^{n} \alpha_i P_i$$

and R_k, Z, X, Q ($k = 1, \dots, 4$) in the same way. Then there exist positive constant c_1, c_2, and γ such that the Lyapunov functional V and its time derivative $\dot{V}$ along the trajectory of Σ_{cl} in (2) satisfy

$$c_1 \left\| \begin{bmatrix} x(t) \\ v(t) \end{bmatrix} \right\|^2 < V(\rho_1(t), \rho_2(t), x_t, \xi_t) < c_2(\|x_t\|_{\mathcal{W}}^2 + \|\xi_t\|_{\mathcal{U}}^2)$$

for all $t \geq 0$, and

$$\dot{V}(\rho_1(t), \rho_2(t), x_t, \xi_t) \leq -\gamma V(\rho_1(t), \rho_2(t), x_t, \xi_t)$$

for all $t \in [kh, (k+1)h)$ ($k = 0, 1, 2, \dots$). Moreover, by construction,

$$V(\rho_1(kh), \rho_2(kh), x_{kh}, \xi_{kh}) \leq \lim_{t \nearrow kh} V(\rho_1(t), \rho_2(t), x_t, \xi_t)$$

at each $t = kh$ ($k = 1, 2, \dots$). Hence the closed-loop system Σ_{cl} is exponential stable with decay rate $\gamma/2$ as shown in [24], [26].

V. Discretization of the Closed-loop System

In this paper, we represent the closed-loop system as the impulsive system Σ_{cl} in (2), whereas the authors of [10], [12] have analyzed stability by discretizing the closed-loop system. In the remainder of this section, we discuss the merits of each approach.

Let δA, δB, and δC be the uncertainties of A, B, and C, respectively. If we discretize the closed-loop system as in [10], [12], then we have

$$\eta[k+1] = \begin{cases} A_1(\Delta_k)\eta[k] & \text{if } \Delta_k > 0 \\ A_2(\Delta_k)\eta[k] & \text{if } \Delta_k \leq 0, \end{cases} \tag{5}$$

where

$$\eta[k] := \begin{bmatrix} x(kh) \\ \xi[k] \\ \xi[k-1] \end{bmatrix}$$

$$A_1(\Delta) := \begin{bmatrix} \tilde{A}_d & \tilde{B}_d C_c & 0 \\ B_c \tilde{C}_d e^{(A+\delta A)\Delta} & A_c + B_c H_+(\Delta) C_c & 0 \\ 0 & I & 0 \end{bmatrix}$$

$$A_2(\Delta) := \begin{bmatrix} \tilde{A}_d & \tilde{B}_d C_c & 0 \\ B_c \tilde{C}_d e^{(A+\delta A)\Delta} & A_c & -B_c H_-(\Delta) C_c \\ 0 & I & 0 \end{bmatrix}$$

and

$$\tilde{A}_d = e^{(A+\delta A)h}$$

$$\tilde{B}_d := \int_0^h e^{(A+\delta A)(h-\tau)} d\tau (B + \delta B),$$

$$\tilde{C}_d := C + \delta C$$

$$H_+(\Delta) := (C + \delta C) \int_0^{\Delta} e^{(A+\delta A)\tau} d\tau (B + \delta B)$$

$$H_-(\Delta) := (C + \delta C) \int_0^{-\Delta} e^{-(A+\delta A)\tau} d\tau (B + \delta B).$$

In the absence of model uncertainty, for this discretized system, we can employ a gridding and norm-bounded approach [29], [30] and a stochastic approach [31]. However, the discretized system (5) involves exponentials and integrals on the uncertainties δA, Δ, which make it difficult to consider uncertainty both in the time domain and in the system matrices. On the other hand, in the impulsive system representation (2), δA, δB, and δC appear in an affine form, and Δ_k is implicitly represented by the sampling time s_{k-1}. This allows us to analyze robust stability in terms of LMIs. Alternative approaches for robust stability include the input-delay approach [20]–[22], [32] and the loop-functional approach [33], [34].

VI. Numerical Example

Consider the following continuous-time system with uncertainty in the matrices A and C:

$$\dot{x}(t) = (A + \delta A)x(t) + Bu(t)$$
$$y(t) = (C + \delta C)x(t),$$

where the nominal system matrices A, B, C are given by

$$A := \begin{bmatrix} -2.6 & 2.9 \\ 3.9 & 4.2 \end{bmatrix}, \quad B := \begin{bmatrix} 0.6 \\ 1 \end{bmatrix}, \quad C := \begin{bmatrix} -3.5 & 4 \end{bmatrix}$$

and the uncertainties $\delta A, \delta C$ are

$$\begin{bmatrix} \delta A \\ \delta C \end{bmatrix} \in \{\alpha(-F_\lambda) + (1-\alpha)F_\lambda : \ 0 \leq \alpha \leq 1\}$$

$$F_\lambda := \begin{bmatrix} 0.05\lambda & \lambda \\ 0 & 0 \\ 0.05 & 0.025 \end{bmatrix}.$$

For these uncertainties, Assumption 3.1 holds. We took the nominal sampling period $h = 0.05$. We use a controller composed of a Luenberger observer and a feedback gain as in Remark 3.6. The feedback gain K and the observer gain L are given by

$$K = -\begin{bmatrix} 3.6549 & 7.6954 \end{bmatrix}, \qquad L = -\begin{bmatrix} 0.0807 \\ 0.2213 \end{bmatrix},$$

These gains correspond to a linear quadratic regulator gain and a Kalman filter gain for the nominal plant (A, B, C), where the state and input weighting matrices and the process and measurement noise covariance are identity matrices with appropriate dimensions. Let the ZOH and the sampler satisfy Assumptions 2.1 and 2.2, respectively. For simplicity, we consider a symmetric offset bound $[-\overline{\Delta}, \overline{\Delta}]$.

Fig. 2 illustrates the uncertainty parameter λ versus the clock-offset bound $[-\overline{\Delta}, \overline{\Delta}]$. The blue line is a lower bound on the allowable *time-varying* clock offsets, which is obtained by Theorem 3.2 with sufficiently small $\gamma > 0$, whereas the red dotted line indicates the exact bound on *constant* clock offsets that would be allowed by K and L without compromising the closed-loop stability. Note that the exact bound on constant offsets can be regarded as an upper bound on time-varying offsets. We can obtain the exact bound on constant offsets from iterative calculations of the eigenvalues of the discretized closed-loop system in (5).

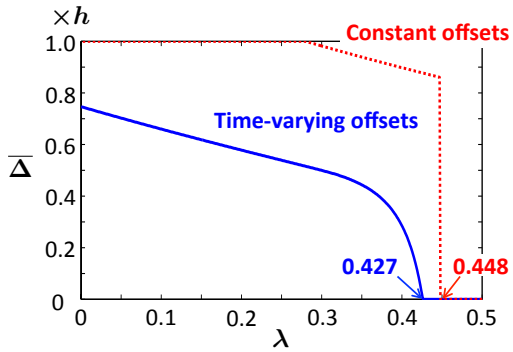

Fig. 2: Uncertainty parameter λ versus offset bound $[-\overline{\Delta}, \overline{\Delta}]$.

The bound on allowable time-varying offsets decreases linearly when $0 \leq \lambda < 0.3$, but it drops rapidly for $\lambda > 0.35$. Similarly, the exact bound on constant offsets decreases linearly from $\lambda = 0.29$, and the closed-loop system suddenly becomes unstable at $\lambda = 0.448$. Theorem 3.2 show that without clock offsets, the closed-loop system may be unstable only for $\lambda > 0.427$. The difference between 0.427 and 0.448 is due to the conservativeness of Theorem 3.2 for the stability analysis of systems with polytopic uncertainties (and no clock offsets).

VII. Conclusion

We studied the robust stability of systems that have polytopic uncertainty and time-varying clock offsets. We represented the closed-loop system as a delayed impulsive system. Through this representation, we constructed a sufficient LMI condition for robust stability by using a Lyapunov functional whose variables are the states of the continuous-time plant and the discrete-time controller. Future work involves constructing less conservative Lyapunov functionals and addressing more general systems by incorporating external disturbances,nonlinear dynamics, and transmission delays larger than one sampling period.

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