Control under Clock Offsets and Actuator Saturation

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Abstract— **This paper studies the stability analysis and the stabilization problem for systems with asynchronous sensors and controllers, and actuators subject to saturation. We consider a system with parameter uncertainty caused by clock offsets. By using a polytopic overapproximation, we investigate how large clock offsets affect stability. In addition, we employ a sector characterization approach to address actuator saturation. We see from a numerical example that the range of allowable clock offset bounds drops if the saturation limit becomes smaller than a certain value.**

I. Introduction

The effects of networked-induced delays and variable sampling intervals have been actively studied, as surveyed in, e.g., [1]–[3]. A solution to compensate such uncertainties in the time domain is that the sensor sends the measurement together with its time-stamp [4], [5]. Time-stamping allows the controller to estimate the plant's state under the assumption that the clock of the controller is synchronized that of the sensor. However, perfect clock synchronization is fundamentally impossible [4], [6]. We therefore need to address the existence of a clock offset between the sensor and the controller, as it may degrade the performance of the closed-loop system and can even destabilize the system. Imperfect timing on control has been studied for contentious-time systems in [7], [8] and for the discrete-time systems in [9]. In this work, we consider sampled-data systems, where the plant's state is estimated at each control-updating instant.

The main objective of this paper is to determine how large a clock offset can be allowed until it compromises stability. In our previous work [10], the stabilization problem of systems with *constant* clock offsets has been studied. This paper considers more practical situations in the following two senses: We study systems with *time-varying* clock offsets that take values in a bounded interval. Clock offsets are time varying in practical applications because the oscillator in a clock may be affected by its environment such as temperature and humidity. Our recent work [11] has also investigated systems with time-varying clock offsets and proposed a stabilization method with *causal controllers*, based on the analysis of data rate limitations in *quantized control*. Here we use *controllers consisting of a linear estimator and a static gain*, and study stability analysis and the design of stabilizing gains from the perspective of *robust control*.

The second contribution is to study the stability of systems with actuator saturation in addition to time-varying clock offsets. Actuator saturation is an ubiquitous nonlinearity in engineering applications and could make feedback systems unstable. In [12], [13], control problems with saturation have been studied in the context of networked control systems, in particular, with quantization and variable delay. However, relatively little work has been done towards investigating how actuator saturation affects the range of clock offsets that would be allowed for stability.

First we analyze the effect of clock offsets on the stability of a closed-loop system with ideal actuators. The networked control system we consider is modeled as a discrete-time linear parameter-varying system. In the resulting discretized system, the variable clock offset appears in an exponential form. We therefore use the overapproximation technique in [2], [14], [15] to embed the original model into a larger class of polytopic models with a norm-bounded additive uncertainty. Using this overapproximated system, we provide a sufficient condition for stability in terms of linear matrix inequalities (LMIs). From this condition, two design methods of static stabilizing feedback gains are also presented.

Next we extend the first result to the analysis of local stability for systems with actuator saturation as well as clock offsets. In order to address a saturation nonlinearity, we use the sector characterization proposed in [16], [17], [18, Chapter 3], which can be regarded as an overapproximation technique for actuator saturation. We observe from a numerical study that the allowable offset range decreases sharply when the saturation limit is smaller than a certain value.

This paper is organized as follows. The next section gives the closed-loop system we consider and presents the problem formulation. In Section III, we study the stability analysis and the design of feedback gains for systems using time-varying clock offsets with results in [2], [14]. Section IV is devoted to extend the results in Section III to the regional stability analysis of systems with clock offsets and actuator saturation. Finally, concluding remarks are given in Section V.

Notation: Let $\mathbb{Z}_+$ be the set of non-negative integers. For a vector v, we denote by $\|v\|$ the Euclidean norm of v. For a matrix M, we denote by $\|M\|$ and $\mathrm{trace}(M)$ the Euclidean-induced norm and the trace of M, respectively. Also for square matrices $M_1, \ldots, M_n$, $\mathrm{diag}(M_1, \ldots, M_n)$ means the block diagonal matrix such that the main diagonal blocks starting in the upper left corner are $M_1, \ldots, M_n$ and the off-diagonal blocks are zero matrices.

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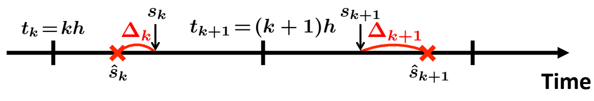

Fig. 1: Sampling instants s_k, reported time-stamps $\hat{s}_k$, and updating instants t_k of the zero-order hold

II. Problem Statement

Consider the following plant:

$$\dot{x}(t) = Ax(t) + Bu(t), \tag{II.1}$$

where $x(t) \in \mathbb{R}^n$ and $u(t) \in \mathbb{R}^m$ are the state and the input of the plant, respectively. To simplify the system representation, we introduce the following assumption:

Assumption 2.1: The matrix A is invertible.

Let $s_0, s_1, \ldots$ be sampling instants. The sensor observes the state $x(s_k)$ and sends it to the controller together with its time-stamp. However, since the sensor and the controller share no global clock, the time-stamp typically includes an unknown offset with respect to the controller clock. In this paper, we assume that the offset is time-varying and bounded, that is, for each $k \in \mathbb{Z}_+$ the sampling instant s_k and the time-stamp $\hat{s}_k$ reported by the sensor have the following relationship: $s_k = \hat{s}_k - \Delta_k$ for some unknown clock offset $\Delta_k \in [\underline{\Delta}, \overline{\Delta}]$.

Let $h > 0$ be the update period of the control signal $u(t)$. The control signal $u(t)$ is piecewise constant and updated at times $t_k = kh$ ($k \in \mathbb{Z}_+$) with values u_k: $u(t) = u_k$ for $t \in [t_k, t_{k+1})$. While the control input is updated periodically, the true sampling times s_k and the reported sampling times $\hat{s}_k$ may not be periodic. We assume that both s_k and $\hat{s}_k$ do not fall behind t_{k+1} by more than h, and that the controller side receives the state measurement $x(s_k)$ and the time-stamp $\hat{s}_k$ transmitted from the sensor by the next control-updating instant t_{k+1}. This assumption is formally stated as follows.

Assumption 2.2: *For $k \in \mathbb{Z}_+$, $s_k, \hat{s}_k \in [t_k, t_{k+1})$. Furthermore, $x(s_k)$ and $\hat{s}_k$ are available to the controller by time $t = t_{k+1}$.*

Fig. 1 shows the timing diagram of the sampling instants s_k, the reported time-stamps $\hat{s}_k$, and updating instants t_k of the control inputs.

The controller estimates the state of the plant at time $t = t_{k+1}$ using the following dynamics after receiving the data $x(s_k)$ and $\hat{s}_k$:

$$\begin{aligned} \dot{\hat{x}}(t) &= A\hat{x}(t) + Bu(t), && (\hat{s}_k \le t < t_{k+1}) \\ \hat{x}(\hat{s}_k) &= x(s_k) && (k \in \mathbb{Z}_+), \end{aligned}$$

where $\hat{x} \in \mathbb{R}^n$ is the estimated state. This estimate leads to the discretized extended system in Fig. 2, with state ξ_k and input u_k given by

$$\xi_k = \begin{bmatrix} x(t_k) - \hat{x}(t_k) \\ \hat{x}(t_k) \end{bmatrix}, \qquad u_k = u(t_k),$$

respectively, which can be shown to evolve according to

$$\xi_{k+1} = F_{\Delta_k}\xi_k + G_{\Delta_k}u_k, \qquad y_k = H\xi_k, \tag{II.2}$$

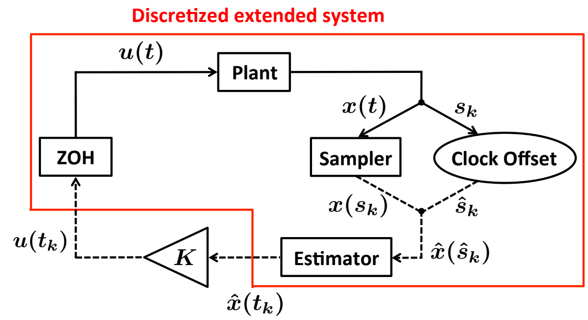

Fig. 2: Closed-loop system with clock offsets

where $p := e^{Ah}$, $\theta(\Delta_k) := e^{-A\Delta_k} - I$, and

$$\begin{aligned} F_{\Delta_k} &:= \begin{bmatrix} -p\theta(\Delta_k) & -p\theta(\Delta_k) \\ p(I + \theta(\Delta_k)) & p(I + \theta(\Delta_k)) \end{bmatrix} \\ G_{\Delta_k} &:= \begin{bmatrix} -p\theta(\Delta_k)A^{-1}B \\ (p(I + \theta(\Delta_k)) - I)A^{-1}B \end{bmatrix} \\ H &:= \begin{bmatrix} 0 & I \end{bmatrix}. \end{aligned} \tag{II.3}$$

For the stabilization of the system (II.2), we use a static output feedback $u(t) = u_k = Ky_k = K\hat{x}(t_k)$ for $t \in [t_k, t_{k+1}), k \ge 1$ and $u(t) = u_0 = 0$ for $t \in [0, h)$. The measurement $x(s_0)$ cannot be used for control in $[0, h)$ from Assumption 2.2. We therefore put no control signal in $[0, h)$, which makes the initial condition of the plant's state simple when we study the stability of systems with actuator saturation in Section IV.

The main objective of the present paper is to determine whether, given a feedback gain K and a clock offset range $[\underline{\Delta}, \overline{\Delta}]$, the closed-loop system is stable for every time-varying clock offset sequence Δ_k in $[\underline{\Delta}, \overline{\Delta}]$. We also propose methods to design a feedback gain K that stabilizes the system for every clock offset in a given range.

III. Stability Analysis and Stabilization Based on Polytopic Overapproximation

In this section, we study the stability of the system (II.2) with static feedback control and the design of stabilizing feedback gains, by using the *polytopic overapproximation* in [2], [14].

A. Stability Analysis

Define $\boldsymbol{\Theta}$ by $\boldsymbol{\Theta} := \{\theta(\Delta) : \Delta \in [\underline{\Delta}, \overline{\Delta}]\}$. Construct matrices $T_i \in \mathbb{R}^{n \times n}$, $U_i \in \mathbb{R}^{n \times \phi}$, and $V_i \in \mathbb{R}^{\phi \times n}$ overapproximating $\boldsymbol{\Theta}$ as follows:

$$\boldsymbol{\Theta} \subset \left\{ \sum_{i=1}^{N} \alpha_i (T_i + U_i \Phi V_i) : \ \alpha := \{\alpha_i\}_{i=1}^{N} \in \mathcal{A}, \ \Phi \in \boldsymbol{\Phi} \right\}, \tag{III.1}$$

where $\mathcal{A}$ and $\boldsymbol{\Phi}$ are given by

$$\begin{aligned} \mathcal{A} &:= \left\{ \{\alpha_i\}_{i=1}^{N} : \ \sum_{i=1}^{N} \alpha_i = 1, \ \alpha_i \ge 0 \ \text{ for } \ i = 1, \ldots, N \right\} \\ \boldsymbol{\Phi} &:= \Big\{ \mathrm{diag}(\Phi^1, \ldots, \Phi^L) : \\ &\qquad \Phi^j \in \mathbb{R}^{\phi_j \times \phi_j}, \ \|\Phi^j\| \le 1 \ \text{ for } j = 1, \ldots, L \Big\} \end{aligned}$$

for some fixed $\{\phi_i\}_{i=1}^L$ satisfying $\phi = \sum_{j=1}^L \phi_j$. See [2], [14] for a construction method for T_i, U_i, V_i, and $\{\phi_i\}_{i=1}^L$ satisfying (III.1).

Since $\theta(\Delta_k) \in \mathbf{\Theta}$ for every $k \in \mathbb{Z}_+$, we have

$$\theta(\Delta_k) = \sum_{i=1}^{N} \alpha_{k,i}(T_i + U_i \Phi_k V_i) \tag{III.2}$$

for appropriately selected $\Phi_k \in \mathbf{\Phi}$ and $\{\alpha_{k,i}\}_{i=1}^N \in \mathcal{A}$. Using the representation (III.2) for $\theta(\Delta_k)$, we can embed the system (II.2) into the new polytopic system with structured uncertainty given by

$$\begin{aligned}\xi_{k+1} &= \\ &\sum_{i=1}^{N} \alpha_{k,i}\left(\begin{bmatrix} -p(T_i+U_i\Phi_kV_i) & -p(T_i+U_i\Phi_kV_i) \\ p(I+T_i+U_i\Phi_kV_i) & p(I+T_i+U_i\Phi_kV_i)\end{bmatrix}\xi_k \right.\\ &\qquad \left. + \begin{bmatrix} -p(I+T_i+U_i\Phi_kV_i)A^{-1}B \\ (p(I+T_i+U_i\Phi_kV_i)-I)A^{-1}B\end{bmatrix} u_k\right) \\ y_k &= \begin{bmatrix} 0 & I \end{bmatrix}\xi_k.\end{aligned} \tag{III.3}$$

The system (III.3) with the static feedback controller $u_k = Ky_k$ can be expressed as

$$\xi_{k+1} = \sum_{i=1}^{N} \alpha_{k,i}(A_{cl,i} + B_{cl,i}\Phi_k C_{cl,i})\xi_k \tag{III.4}$$

with $\Phi_k \in \mathbf{\Phi}$ and $\{\alpha_{k,i}\}_{i=1}^N \in \mathcal{A}$ for all $k \in \mathbb{Z}_+$, where

$$\begin{aligned} A_{cl,i} &:= \begin{bmatrix} -pT_i & -pT_i(I+A^{-1}BK) \\ p(I+T_i) & p(I+T_i)(I+A^{-1}BK) - A^{-1}BK \end{bmatrix} \\ B_{cl,i} &:= \begin{bmatrix} -p \\ p \end{bmatrix} U_i, \qquad C_{cl,i} := V_i \begin{bmatrix} I & I + A^{-1}BK \end{bmatrix}. \end{aligned}$$

Using a parameter-dependent Lyapunov function

$$V(k,\xi) = \xi^\top \left(\sum_{i=1}^{N} \alpha_{k,i} P_i\right)\xi, \tag{III.5}$$

we can analyze the stability of the derived system (III.4) in terms of LMIs.

Theorem 3.1 ([2], [14]): *Let Assumptions 2.1 and 2.2 hold. Define the set of matrices*

$$\begin{aligned}\mathcal{R} := \{&\mathrm{diag}(r_1 I_1, \ldots, r_L I_L) \in \mathbb{R}^{\phi\times\phi} : \\ & r_j > 0 \ \text{for } j = 1, \ldots, L\},\end{aligned} \tag{III.6}$$

where I_j is the identity matrix of size ϕ_j. Fix $\gamma \in (0, 1]$. If there exist matrices $P_i > 0$ and $R_i \in \mathcal{R}$ for $i = 1, \ldots, N$, such that

$$\begin{bmatrix} \gamma P_i & 0 & A_{cl,i}^\top P_j & C_{cl,i}^\top R_i \\ * & R_i & B_{cl,i}^\top P_j & 0 \\ * & * & P_j & 0 \\ * & * & * & R_i \end{bmatrix} > 0 \tag{III.7}$$

for all $i, j = 1, \ldots, N$, then the closed-loop system (III.4) *is exponentially stable with a decay rate less than γ.*

Proof: Only asymptotic stability is discussed in [2], [14], but the extension to exponential stability is easy. We therefore omit the proof. ■

Remark 3.2: Theorem 3.1 provides a sufficient condition for stability. However, the polytopic overapproximation does not introduce conservatism as follows: If the original system (II.2) is quadratically stable in the sense that there exists a continuous parameter-dependent quadratic Lyapunov function, then this gridding approach guarantees exponential stability with a sufficiently fine grid size with a large N; see [2], [14] for details.

In the example below, by using Theorem 3.1, we obtain a range $[\underline{\Delta}, \overline{\Delta}]$ of offsets for which a given feedback system is stable.

Example 3.3 (*Vehicle suspension [19]*): An active suspension system is given in [19] as follows:

$$\frac{d}{dt}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1 \\ -\frac{k_s}{m_s} & 0 & -\frac{c_s}{m_s} & \frac{c_s}{m_s} \\ \frac{k_s}{m_u} & -\frac{k_t}{m_u} & \frac{c_s}{m_u} & \frac{c_s+c_t}{m_u} \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \frac{1}{m_s} \\ -\frac{1}{m_u} \end{bmatrix} u,$$

where x_1, x_2, x_3, and x_4 correspond to the suspension deflection, the tire deflection, the sprung mass speed, and the unsprung mass speed, respectively. Also m_s is the sprung mass, which represents the car chassis; m_u is the unsprung mass, which represents the wheel assembly; c_s and k_s are damping and stiffness of the passive suspension system, respectively; c_t and k_t stand for the damping and compressibility of the pneumatic tire, respectively. Here we omit the input disturbance, which represents the vertical ground velocity of the road profile. These parameters have the following values: $m_s = 972.2$ kg, $m_u = 113.6$ kg, $c_s = 1096$ Ns/m, $c_t = 14.6$ Ns/m, $k_s = 42719.6$ N/m, and $k_t = 101115$ N/m. As in [19], we took the feedback control $u = Kx$ with

$$K = -10^4 \times \begin{bmatrix} 0.3292 & 0.6361 & 1.0125 & 0.0020 \end{bmatrix}. \tag{III.8}$$

We set the sampling period $h = 0.1$ sec, the decay rate $\gamma = 1$, and the number of vertices in the overapproximation $N = 26$ in (III.1) with equal partitioning. We can conclude from Theorem 3.1 that the closed-loop system is stable whenever the time-varying offset Δ_k belongs to $[-0.0760, 0.0818]$. On the other hand, if the clock offset is constant, then the maximum allowable offset interval is $[-0.0792, 0.0955]$ for the given gain (III.8), which was obtained by checking the eigenvalue of the closed-loop system for each constant clock offset Δ and can be regarded as a necessary condition for stability with time-varying offsets.

Fig. 3 shows the relationship between the number of segments N with equal partitioning and the allowable offset interval $[\underline{\Delta}, \overline{\Delta}]$. We observe that as N increases, the numerical result becomes less conservative, but that there is almost no difference on $[\underline{\Delta}, \overline{\Delta}]$ between $N = 21$ and $N = 26$. As stated in Remark 3.2, if N is sufficiently large, then Theorem 3.1 gives the exact offset bound for quadratic stability. Fig. 3 illustrates this fact.

Remark 3.4: We used a static stabilizer here, but we can extend Theorems 3.1 to the case of a general-order stabilizer as follows.

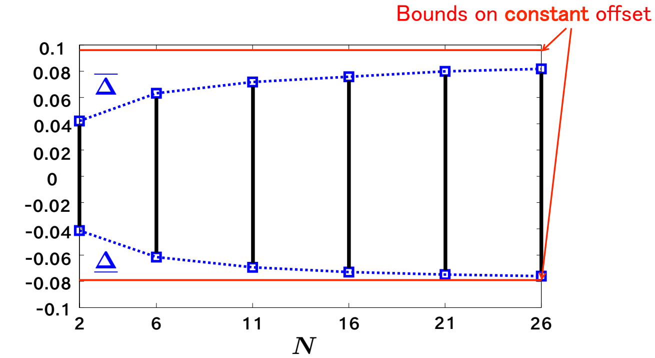

Fig. 3: Number of segments N versus allowable offset interval $[\underline{\Delta}, \overline{\Delta}]$

Let a stabilizable and detectable realization of the dynamic stabilizer be $z_{k+1} = A_c z_k + B_c y_k$, $u_k = C_c z_k + D_c y_k$. Then the state equation of the closed-loop system is given by

$$\begin{bmatrix}\xi_{k+1}\\ z_{k+1}\end{bmatrix} = \begin{bmatrix} F_{\Delta_k}+G_{\Delta_k}D_cH_{\Delta_k} & G_{\Delta_k}C_c \\ B_cH_{\Delta_k} & A_c\end{bmatrix}\begin{bmatrix}\xi_k\\ z_k\end{bmatrix}.$$

We can rewrite this state equation in the form (III.4) with

$$A_{cl,i} = \left[\begin{array}{c|c} A_{cl,i}^{[1]} & A_{cl,i}^{[2]} \\ \hline A_{cl,i}^{[3]} & A_{cl,i}^{[4]}\end{array}\right], \quad B_{cl,i} = \left[\begin{array}{c} -pU_i \\ pU_i \\ \hline 0\end{array}\right]$$
$$C_{cl,i} = \left[\begin{array}{cc|c} V_i & V_i(I + A^{-1}BD_c) & V_iA^{-1}BC_c\end{array}\right],$$

where

$$A_{cl,i}^{[1]} := \begin{bmatrix} -pT_i & -pT_i(I+A^{-1}BD_c) \\ p(I+T_i) & p(I+T_i)(I+A^{-1}BD_c)-A^{-1}BD_c\end{bmatrix},$$
$$A_{cl,i}^{[2]} := \begin{bmatrix} -pT_iA^{-1}BC_c \\ (p(I+T_i)-I)A^{-1}BC_c\end{bmatrix}$$
$$A_{cl,i}^{[3]} := \begin{bmatrix}0 & B_c\end{bmatrix}, \quad A_{cl,i}^{[4]} := A_c.$$

B. Stabilization

We propose two approaches for the design of a stabilizing feedback gain K. The main difficulty is the LMI in (III.7) that has multiple product terms $A_{cl,i}^{\top}P_j$ and $C_{cl,i}^{\top}R_i$, where the unknown K and P_j, R_i appear multiplied. To circumvent this difficulty, the first approach fixes the positive definite matrices of the quadratic Lyapunov function V in (III.5) as in [20], [21]. The second approach uses the cone complementarity linearization (CCL) algorithm [22].

In the first approach, we use the following partitioned positive definite matrix: Let a positive definite matrix Q be partitioned into

$$Q = \begin{bmatrix} Q_{11} & Q_{12} \\ Q_{12}^{\top} & Q_{22}\end{bmatrix} > 0.$$

If we define $\Psi := (Q_{12}Q_{22}^{-1})^{\top}$, $Q_1 := Q_{11} - Q_{12}Q_{22}^{-1}Q_{12}^{\top}$, and $Q_2 := Q_{22}$, then we can rewrite Q as

$$Q = \begin{bmatrix} Q_1 + \Psi^{\top}Q_2\Psi & \Psi^{\top}Q_2 \\ * & Q_2\end{bmatrix} = T_{\Psi}Q_dT_{\Psi}^{\top}, \qquad \text{(III.9)}$$

where $Q_d = \operatorname{diag}(Q_1, Q_2)$ and

$$T_{\Psi} := \begin{bmatrix} I & \Psi \\ 0 & I\end{bmatrix}. \qquad \text{(III.10)}$$

We see from (III.9) that if $Q > 0$, then Q_d defined above satisfies $Q_d > 0$.

Conversely, for every $Q_d := \operatorname{diag}(Q_1, Q_2) > 0$ and T_{Ψ} defined by (III.10) with an arbitrary matrix Ψ, $Q := T_{\Psi}Q_dT_{\Psi}^{\top} > 0$.

The following theorem gives a sufficient condition for the gain synthesis, using the structured positive definite matrix Q in (III.9) with a fixed Ψ:

Theorem 3.5: *Let Assumptions 2.1 and 2.2 hold. Define the set $\mathcal{R}$ of matrices as in* (III.6). *Let γ be in $(0,1]$ and fix a matrix $\Psi \in \mathbb{R}^{n\times n}$. If there exist matrices $Q_i^{(1)} > 0$ and $S_i \in \mathcal{R}$ for $i = 1,\dots,N$, $Q^{(2)} > 0$, and $X \in \mathbb{R}^{m\times n}$ such that for all $i,j = 1,\dots,N$,*

$$\begin{bmatrix} \gamma Q_i & 0 & \Gamma_{A,i}^{\top} & \Gamma_{C,i}^{\top} \\ * & S_i & S_iB_{cl,i}^{\top} & 0 \\ * & * & Q_j & 0 \\ * & * & * & S_i\end{bmatrix} > 0, \qquad \text{(III.11)}$$

where

$$Q_i := \begin{bmatrix} I & \Psi \\ 0 & I\end{bmatrix}\begin{bmatrix} Q_i^{(1)} & 0 \\ 0 & Q^{(2)}\end{bmatrix}\begin{bmatrix} I & \Psi \\ 0 & I\end{bmatrix}^{\top} \qquad \text{(III.12)}$$
$$\Gamma_{A,i} := \begin{bmatrix} -pT_i & -pT_i \\ p(I+T_i) & p(I+T_i)\end{bmatrix}Q_i + \begin{bmatrix} -pT_iA^{-1}B \\ (p(I+T_i)-I)A^{-1}B\end{bmatrix}\begin{bmatrix} X\Psi^{\top} & X\end{bmatrix}$$
$$\Gamma_{C,i} := \begin{bmatrix} V_i & V_i\end{bmatrix}Q_i + V_iA_i^{-1}B\begin{bmatrix} X\Psi^{\top} & X\end{bmatrix},$$

then $K := X(Q^{(2)})^{-1}$ stabilizes the closed-loop system (III.4).

Proof: We aim to transform (III.7) into (III.11) with the structured Lyapunov matrix defined by Q_i in (III.12). We see that the LMI in (III.7) is equivalent to

$$\begin{bmatrix} \gamma P_i^{-1} & 0 & P_i^{-1}A_{cl,i}^{\top} & P_i^{-1}C_{cl,i}^{\top} \\ * & R_i^{-1} & R_i^{-1}B_{cl,i}^{\top} & 0 \\ * & * & P_j^{-1} & 0 \\ * & * & * & R_i^{-1}\end{bmatrix} > 0$$

by using the congruence transformation $T = \operatorname{diag}(P_i^{-1}, R_i^{-1}, P_j^{-1}, R_i^{-1})$. Fix the structure of $P_i^{-1} =: Q_i$ by (III.12) and define $S_i := R_i^{-1}$. A routine calculation shows that $A_{cl,i}Q_i = \Gamma_{A,i}$ and $C_{cl,i}Q_i = \Gamma_{C,i}$ with $X = KQ^{(2)}$, and hence we obtain (III.11). ■

The second approach is based on the CCL algorithm [22]. The following theorem provides a stability condition in terms of the feedback gain K, which can be found using the CCL algorithm:

Theorem 3.6: *Let Assumptions 2.1 and 2.2 hold. Define the set $\mathcal{R}$ of matrices as in* (III.6). *Fix $\gamma \in (0,1]$. If there exist matrices $P_i, Q_i > 0$ and $S_i \in \mathcal{R}$ for $i = 1,\dots,N$, and*

$K \in \mathbb{R}^{m\times n}$ *such that for all* $i,j = 1,\dots,N$,

$$\begin{bmatrix} \gamma P_i & 0 & A_{cl,i}^\top & C_{cl,i}^\top \\ * & S_i & S_i B_{cl,i}^\top & 0 \\ * & * & Q_j & 0 \\ * & * & * & S_i \end{bmatrix} > 0, \qquad \begin{bmatrix} P_i & I \\ * & Q_i \end{bmatrix} \geq 0, \tag{III.13}$$

and $\mathrm{trace}(P_iQ_i) = 2n$, *then* K *stabilizes the closed-loop system* (III.4).

Proof: For all $P_i, Q_i > 0$ satisfying the second inequality of (III.13), it turns out that $\mathrm{trace}(P_iQ_i) \geq 2n$. Furthermore $\mathrm{trace}(P_iQ_i) = 2n$ if and only if $P_iQ_i = I$.

Define $S_i := R_i^{-1}$. Since $P_i = Q_i^{-1}$, if we apply the congruence transformation $T = \mathrm{diag}(I, S_i, Q_j, S_i)$ to (III.7), we see that (III.7) is equivalent to the first condition of (III.13). ∎

Since $\min(\mathrm{trace}(P_iQ_i)) = 2n$ under (III.13) as shown in the proof above, the conditions in Theorem 3.6 are feasible if the problem of minimizing trace $\left(\sum_{i=1}^N P_iQ_i\right)$ under (III.13) has the solution $2nN$. The CCL algorithm solves this constrained minimization problem. Although the CCL algorithm does not always find the global optimal solution, the non-linear minimization problem is easier to solve than the original non-convex feasibility problem [23].

IV. Systems with Clock Offsets and Actuator Saturation

In this section, the results in Theorem 3.1 are extended to the case when the system has both clock offsets and actuator saturation. To this end, we use the regional sector characterization of the saturation nonlinearity in [16], [17], [18, Chapter 3].

Let us consider the system (II.2) with input saturation:

$$\begin{aligned} \xi_{k+1} &= F_{\Delta_k}\xi_k + G_{\Delta_k}\sigma_k \\ u_k &= K_H\xi_k + L\sigma_k \\ \sigma_k &= \mathrm{sat}(u_k), \end{aligned} \tag{IV.1}$$

with F_{Δ_k}, G_{Δ_k}, and H defined by (II.3) and $K_H := (I-L)KH$, where K and L are a control feeback gain and an anti-windup gain, respectively. We obtained $u_k = K_H\xi_k + L\sigma_k$ from $u_k = Ky_k + L(\sigma_k - Ky_k)$.

The function $\mathrm{sat} : \mathbb{R}^m \to \mathbb{R}^m$ is defined by $\mathrm{sat}(u) := [\sigma_1(u) \ \cdots \ \sigma_m(u)]$, where σ_i depends only on the i-th input component u_i as follows:

$$\sigma_i(u) := \begin{cases} -\bar{u}_i & u_i < -\bar{u}_i \\ u_i & -\bar{u}_i \leq u_i \leq \bar{u}_i \\ \bar{u}_i & u_i > \bar{u}_i \end{cases} \tag{IV.2}$$

with $\bar{u}_i > 0$.

In this section, we make the following assumption:

Assumption 4.1: The matrix $I - L$ is invertible.

The invertibility of $I-L$ is necessary for the wellposedness of the closed-loop system, because $u_k - L\,\mathrm{sat}(u_k) = K_H\xi_k$.

We are now ready to state the main result in this section: A sufficient condition for the regional stability of the closed-loop system with clock offset and actuator saturation.

Theorem 4.2: *Consider the system* (IV.1). *Let Assumptions 2.1, 2.2, and 4.1 hold. Define the set* $\mathcal{R}$ *of matrices as in* (III.6) *and let* $e_l \in \mathbb{R}^m$ *be a column vector such that the* l*-th element is one and the other elements are zero. Fix* $\gamma \in (0,1]$ *and a positive semidefinite matrix* $\Omega \in \mathbb{R}^{n\times n}$. *If there exist matrices* $Q_i > 0$, $S_i, \hat{S}_i \in \mathcal{R}$, *and* $Y_i \in \mathbb{R}^{m\times 2n}$ *for* $i = 1,\dots,N$, *and a diagonal matrix* $\Lambda_i > 0$ *in* $\mathbb{R}^{m\times m}$ *such that the following three LMIs hold for all* $i,j = 1,\dots,N$:

$$\begin{bmatrix} \tilde{Q}_i & 0 & \begin{bmatrix} Q_i & 0 \\ 0 & \Lambda \end{bmatrix}\tilde{A}_{cl,i}^\top & \begin{bmatrix} Q_i & 0 \\ 0 & \Lambda \end{bmatrix}\tilde{C}_{cl,i}^\top \\ * & S_i & S_i\tilde{B}_{cl,i}^\top & 0 \\ * & * & Q_j & 0 \\ * & * & * & S_i \end{bmatrix} > 0 \tag{IV.3}$$

$$\begin{bmatrix} Q_i & Y_i^\top e_l \\ * & \bar{u}_i^2 \end{bmatrix} \geq 0 \tag{IV.4}$$

$$\begin{bmatrix} \Omega & 0 & \hat{A}_{cl,i}^\top & \hat{C}_{cl,i}^\top \\ * & \hat{S}_i & \hat{S}_i\hat{B}_{cl,i}^\top & 0 \\ * & * & Q_j & 0 \\ * & * & * & \hat{S}_i \end{bmatrix} \geq 0, \tag{IV.5}$$

where

$$\tilde{K}_H := K\begin{bmatrix} 0 & I \end{bmatrix}, \quad \tilde{L} := -(I-L)^{-1}L$$

$$J := \tilde{L} - I, \qquad \tilde{Q}_i := \begin{bmatrix} \gamma Q_i & -Y_i^\top - Q_i\tilde{K}_H^\top \\ * & 2U - \tilde{L}\Lambda - \Lambda\tilde{L}^\top \end{bmatrix}$$

$$\tilde{A}_{cl,i} := \left[\begin{array}{cc|c} 0 & 0 & 0 \\ p & p(I + A^{-1}BK) - A^{-1}BK & (p-I)A^{-1}BJ \end{array}\right] + \begin{bmatrix} -p \\ p \end{bmatrix} T_i \left[\begin{array}{cc|c} I & I + A^{-1}BK & A^{-1}BJ \end{array}\right]$$

$$\tilde{B}_{cl,i} := \begin{bmatrix} -p \\ p \end{bmatrix} U_i, \ \tilde{C}_{cl,i} := V_i \left[\begin{array}{cc|c} I & I + A^{-1}BK & A^{-1}BJ \end{array}\right]$$

$$\hat{A}_{cl,i} := \begin{bmatrix} -p \\ p \end{bmatrix} T_i + \begin{bmatrix} 0 \\ p \end{bmatrix}, \quad \hat{B}_{cl,i} = \begin{bmatrix} -p \\ p \end{bmatrix} U_i, \quad \hat{C}_{cl,i} = V_i,$$

then the system (IV.1) *is well posed, and for every initial state* $x(0) \in \{x \in \mathbb{R}^n : x^\top\Omega x \leq 1\}$, *the system* (IV.1) *with* $\hat{x}(0) = 0$ *is exponentially stable with a decay rate less than* γ.

Proof: In order to prove that the Lyapunov function V in (III.5) with $P_i > 0$ ($i = 1,\dots,N$) decreases along the trajectory (IV.1), we use the sector characterization of the saturation nonlinearity for stability analysis over a bounded region in [16], [17], [18, Chapter 3]: If the pair (u,σ) satisfies $\sigma = \mathrm{sat}(u)$, then for every matrix $M \in \mathbb{R}^{m\times 2n}$ and every diagonal positive definite matrix $W \in \mathbb{R}^{m\times m}$, we have

$$\begin{aligned} (u,\sigma) \in \big\{(u,\sigma) :\ & (\sigma + M\xi)^\top W(u-\sigma) \geq 0 \\ & \text{for all } \xi \text{ satisfying } \mathrm{sat}(M\xi) = M\xi\big\}. \end{aligned} \tag{IV.6}$$

To obtain the LMI (IV.3), we need to change the variable $\sigma = \mathrm{sat}(u)$ to $q := u - \sigma$, which indicates the deadzone nonlinearity. Then the system (IV.1) is transformed into

$$\begin{aligned} \xi_{k+1} &= \tilde{F}_{\Delta_k}\xi_k + \tilde{G}_{\Delta_k}q_k \\ u_k &= \tilde{K}_H\xi_k + \tilde{L}q_k \\ q_k &= u_k - \mathrm{sat}(u_k), \end{aligned} \tag{IV.7}$$

where $\tilde{K}_H$ and $\tilde{L}$ are defined as in the statement of Theorem 4.2 and $\tilde{F}_{\Delta_k} := F_{\Delta_k} + G_{\Delta_k} KH$, $\tilde{G}_{\Delta_k} := -G_{\Delta_k}((I-L)^{-1}L + I)$. Here we used the invertibility of $I - L$ in Assumption 4.1. Under the relationship $q = u - \sigma$, (IV.6) is equivalent to

$$(u,q) \in \big\{(u,q):\ (u - q + M\xi)^\top W q \geq 0 \\ \text{for all } \xi \text{ satisfying } \mathrm{sat}(M\xi) = M\xi\big\}. \qquad \text{(IV.8)}$$

Using the sector characterization in (IV.8), we show that the Lyapunov function V along the trajectory (IV.7) decreases if the LMI (IV.3) is feasible and if ξ_k satisfies $\mathrm{sat}(M^{[k]}\xi_k) = M^{[k]}\xi_k$:

Lemma 4.3: *If there exist matrices $Q_i > 0$, $S_i \in \mathcal{R}$, $Y_i \in \mathbb{R}^{m\times 2n}$ for $i = 1, \dots, N$, and a diagonal matrix $\Lambda > 0$ in $\mathbb{R}^{m\times m}$ such that the LMI* (IV.3) *holds for every $i, j = 1, \dots, N$, then the system* (IV.7) *is well posed and the following statement on the Lyapunov function V in* (III.5) *along the trajectory* (IV.7) *is true:*

$$(u_k - q_k + M^{[k]}\xi_k) W q_k \geq 0 \ \ \textit{for all } (\xi_k, q_k) \neq 0 \\ \Rightarrow \quad V(k+1, \xi_{k+1}) - \gamma V(k, \xi_k) < 0, \qquad \text{(IV.9)}$$

where we define $P_i := Q_i^{-1}$, $W := \Lambda^{-1}$, $M_i := Y_i Q_i^{-1}$, and using $\{\alpha_{k,i}\}_{i=1}^N$ in (III.2), *$M^{[k]} := \sum_{i=1}^N \alpha_{k,i} M_i$.*

Proof: See Section IV. A. ■

Furthermore, we see from the next lemma that the remaining condition $\mathrm{sat}(M^{[k]}\xi_k) = M^{[k]}\xi_k$ for the decrease of the Lyapunov function V holds if $V(k,\xi_k) \leq 1$ and the LMI (IV.4) is feasible. The condition $V(1,\xi_1) \leq 1$ is implied by $x_0^\top \Omega x_0 \leq 1$ on the initial state x_0 and the LMI (IV.5).

Lemma 4.4: *Consider the system* (IV.7). *Suppose that $x_0^\top \Omega x_0 \leq 1$ and $\hat{x}(0) = 0$. If there exist $Q_i > 0$ and $\hat{S}_i \in \mathcal{R}$ for $i = 1, \dots, N$, such that the LMI* (IV.5) *holds for every $i, j = 1, \dots, N$, then ξ_1 satisfies $V(1, \xi_1) \leq 1$ in* (III.5) *with $P_i = Q_i^{-1}$. Furthermore, if such Q_i satisfies the LMI* (IV.4) *with some matrix $Y_i \in \mathbb{R}^{m\times 2n}$ for $i = 1, \dots, N$, then $V(k,\xi_k) \leq 1$ implies $\mathrm{sat}(M^{[k]}\xi_k) = M^{[k]}\xi_k$ for every $k \in \mathbb{N}$, where $M^{[k]}$ is defined as in Lemma 4.3.*

Proof: See Section IV. B. ■

Since the well-posedness of the systems (IV.7) and (IV.1) have been proved in Lemma 4.3, we only need to show that exponential stability follows from Lemmas 4.3 and 4.4.

Since $\mathrm{sat}(M^{[1]}\xi_1) = M^{[1]}\xi_1$ by Lemma 4.4, it follows that (u_1, q_1) in (IV.7) satisfies $(u_1 - q_1 + M^{[1]}\xi_1)^\top W q_1 \geq 0$ for every diagonal positive definite matrix $W \in \mathbb{R}^{m\times m}$. If we use $W := \Lambda^{-1}$, then Lemma 4.3 shows that $V(2,\xi_2) \leq \gamma V(1,\xi_1) \leq 1$, and hence the second statement of Lemma 4.4 gives $\mathrm{sat}(M^{[2]}\xi_2) = M^{[2]}\xi_2$. Continuing in this way, we achieve the exponential decrease of the Lyapunov function $V(k,\xi_k)$. Since $P_i > 0$ for every $i = 1, \dots, N$, it follows that $V(k,\xi_k) \geq \epsilon \|\xi_k\|^2$ for some $\epsilon > 0$. Thus ξ_k converges to the origin exponentially. ■

Using Theorem 4.2, the following example shows how the offset interval $[\underline{\Delta}, \overline{\Delta}]$ varies with the saturation limit $\bar{u}$:

Example 4.5 (*Aircraft [24]*): Let us consider the longitudinal dynamics of the TRANS3 aircraft in [24]. The aircraft model is given by $\dot{x} = Ax + Bu$, where

$$A := \begin{bmatrix} 0 & 14.3877 & 0 & -31.5311 \\ -0.0012 & -0.4217 & 1 & -0.0284 \\ 0.0002 & -0.3816 & -0.4658 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix},$$

$B := [4.526 \ \ -0.0337 \ \ -1.4566 \ \ 0]^\top$, and $[x_1\ x_2\ x_3\ x_4] := x^\top$ are the velocity, the angle of attack, the pitch rate, and the Euler angle rotation of the aircraft about the inertial y-axis respectively, and u is the elevator input.

Let the sampling period be $h = 0.02$ sec and the feedback gain K be the linear quadratic regular for (A, B) with state weight I and input weight 600. In addition, let the anti-windup gain L be $L = 0$. Here we set the matrix $\Omega = \mathrm{diag}(0.05, 20, 20, 20)$ for the initial condition and the decay rate $\gamma = 1$. We used $N \leq 5$ in the overapproximation (III.1) with equal partitioning. We note that in this example, an increase in N did not make the analysis by Theorem 4.2 less conservative since h is small. For small values of h, U_i in (III.1) and hence $\tilde{B}_{cl,i}, \hat{B}_{cl,i}$ in Theorem 4.2 become small as well as the case of a large N.

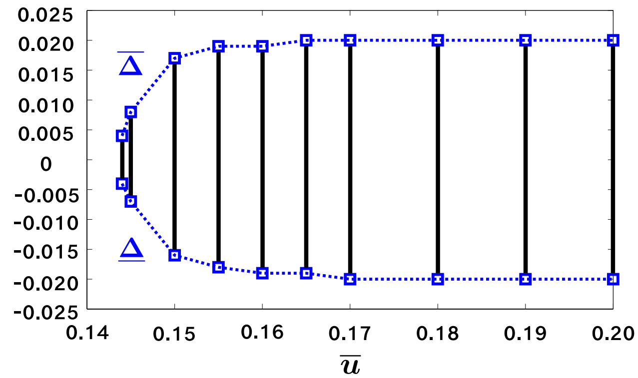

Fig. 4: Allowable offset interval $[\underline{\Delta}, \overline{\Delta}]$ and saturation limit $\bar{u}$ versus

Fig. 4 shows the allowable offset interval $[\underline{\Delta}, \overline{\Delta}]$ obtained by Theorem 4.2 for each saturation limit $\bar{u}$. We see that if $\bar{u} \geq 0.170$, then the saturation does not affect the allowable offset. However, we have almost no margin on clock offsets for $\bar{u} \leq 0.144$. This is because the plant has two unstable poles $0.0212 \pm j0.1670$ and the closed-loop system easily becomes unstable for a small saturation limit $\bar{u}$.

Remark 4.6: **(1)** If we fix the anti-windup gain L, then the synthesis technique in Theorem 3.5 can be used in the context of Theorem 4.2 to compute a stabilizing feedback gain K.

(2) As in Remark 3.4, we can generalize the result in this section to the case of the following general-order stabilizer:

$$\begin{aligned} z_{k+1} &= A_c z_k + B_c y_k + L_1(\sigma_k - C_c z_k - D_c y_k) \\ u_k &= C_c z_k + D_c y_k + L_2(\sigma_k - C_c z_k - D_c y_k) \\ \sigma_k &= \mathrm{sat}(u_k). \end{aligned}$$

In this case, the close-loop system is given by

$$\begin{aligned}\begin{bmatrix}\xi_{k+1}\\ z_{k+1}\end{bmatrix} &= \begin{bmatrix}F_{\Delta_k} & 0\\ (B_c-L_1D_c)H & A_c-L_1C_c\end{bmatrix}\begin{bmatrix}\xi_k\\ z_k\end{bmatrix}+\begin{bmatrix}G_{\Delta_k}\\ L_1\end{bmatrix}\sigma_k\\ u_k &= (I-L_2)\begin{bmatrix}D_cH & C_c\end{bmatrix}\begin{bmatrix}\xi_k\\ z_k\end{bmatrix}+L_2\sigma_k \qquad \text{(IV.10)}\\ \sigma_k &= \mathrm{sat}(u_k),\end{aligned}$$

which is the same form as in (IV.1). Thus we can apply the approach in this section to (IV.10).

A. *Proof of Lemma 4.3*

Define $P^{[k]} := \sum_{i=1}^N \alpha_{k,i}P_i$. By the S-procedure, in order to guarantee (IV.9), it is enough to check that for all $(\xi, q) \neq 0$,

$$\begin{bmatrix}\xi\\ q\end{bmatrix}^\top \begin{bmatrix}\tilde F_{\Delta_k}^\top\\ \tilde G_{\Delta_k}^\top\end{bmatrix} P^{[k+1]} \begin{bmatrix}\tilde F_{\Delta_k} & \tilde G_{\Delta_k}\end{bmatrix}\begin{bmatrix}\xi\\ q\end{bmatrix} - \xi^\top\gamma P^{[k]}\xi$$
$$+ 2(M^{[k]}\xi_k+\tilde K_H\xi_k+\tilde L q_k-q_k)^\top W q_k < 0,$$

that is,

$$\begin{bmatrix}\tilde F_{\Delta_k}^\top\\ \tilde G_{\Delta_k}^\top\end{bmatrix} P^{[k+1]} \begin{bmatrix}\tilde F_{\Delta_k} & \tilde G_{\Delta_k}\end{bmatrix} - \tilde P^{[k]} < 0, \qquad \text{(IV.11)}$$

where

$$\tilde P^{[k]} := \begin{bmatrix}\gamma P^{[k]} & -(M^{[k]}+\tilde K_H)^\top W\\ * & 2W - W\tilde L - \tilde L^\top W\end{bmatrix}. \qquad \text{(IV.12)}$$

Since $P^{[k+1]} > 0$, by the Schur complement, we have that (IV.9) is equivalent to

$$\begin{bmatrix}\tilde P^{[k]} & \begin{bmatrix}\tilde F_{\Delta_k}^\top\\ \tilde G_{\Delta_k}^\top\end{bmatrix} P^{[k+1]}\\ * & P^{[k+1]}\end{bmatrix} > 0. \qquad \text{(IV.13)}$$

Also (II.3) and (III.1) give

$$\begin{aligned}\begin{bmatrix}\tilde F_{\Delta_k} & \tilde G_{\Delta_k}\end{bmatrix} &= \begin{bmatrix}F_{\Delta_k}+G_{\Delta_k}KH & G_{\Delta_k}J\end{bmatrix}\\ &= \sum_{i=1}^N \alpha_{k,i}\left(\tilde A_{cl,i}+\tilde B_{cl,i}\Phi_k\tilde C_{cl,i}\right),\end{aligned}$$

for some $\Phi_k \in \mathbf{\Phi}$, where $\tilde A_{cl,i}$, $\tilde B_{cl,i}$, and $\tilde C_{cl,i}$ are defined as in the statement of Theorem 4.2. It follows from the property

$$\sum_{i=1}^N \alpha_{k,i} = 1, \quad \alpha_{k,i} \geq 0 \qquad \text{(IV.14)}$$

that (IV.13) holds for all $\{\alpha_{k,i}\}_{i=1}^N \in \mathcal{A}$ if and only if we have

$$\begin{bmatrix}\tilde P_i & (\tilde A_{cl,i}+\tilde B_{cl,i}\Phi_k\tilde C_{cl,i})^\top P^{[k+1]}\\ * & P^{[k+1]}\end{bmatrix} > 0 \qquad \text{(IV.15)}$$

for all $i = 1,\dots,N$, where $\tilde P_i$ is defined by (IV.12) with P_i, M_i in place of $P^{[k]}$, $M^{[k]}$. Similarly, using the property (IV.14) for $\{\alpha_{k+1,i}\}_{i=1}^N$ again, we show that (IV.15) is satisfied for all $\{\alpha_{k+1,i}\}_{i=1}^N \in \mathcal{A}$ if and only if for all $i,j = 1,\dots,N$,

$$\begin{bmatrix}\tilde P_i & (\tilde A_{cl,i}+\tilde B_{cl,i}\Phi_k\tilde C_{cl,i})^\top P_j\\ * & P_j\end{bmatrix} > 0. \qquad \text{(IV.16)}$$

As shown in [15], [25], since $R - \Phi^\top R\Phi \geq 0$ for all $\Phi \in \mathbf{\Phi}$ and $R \in \mathcal{R}$, it follows that (IV.16) holds for all $\Phi_k \in \mathbf{\Phi}$ if

$$\begin{aligned}&\begin{bmatrix}\tilde P_i - \tilde C_i^\top(R_i - \Phi_k^\top R_i\Phi_k)\tilde C_i & (\tilde A_{cl,i}+\tilde B_{cl,i}\Phi_k\tilde C_{cl,i})^\top P_j\\ * & P_j\end{bmatrix}\\ &= \Pi^\top\begin{bmatrix}\tilde P_i & 0 & \tilde A_{cl,i}^\top P_j & \tilde C_{cl,i}^\top R_i\\ * & R_i & \tilde B_{cl,i}^\top P_j & 0\\ * & * & P_j & 0\\ * & * & * & R_i\end{bmatrix}\Pi > 0, \qquad \text{(IV.17)}\end{aligned}$$

where

$$\Pi := \begin{bmatrix}I & 0\\ \Phi_k\tilde C_{cl,i} & 0\\ 0 & I\\ -\tilde C_{cl,i} & 0\end{bmatrix}.$$

Since Π is full column rank, (IV.17) holds if

$$\begin{bmatrix}\tilde P_i & 0 & \tilde A_{cl,i}^\top P_j & \tilde C_{cl,i}^\top R_i\\ * & R_i & \tilde B_{cl,i}^\top P_j & 0\\ * & * & P_j & 0\\ * & * & * & R_i\end{bmatrix} > 0. \qquad \text{(IV.18)}$$

Since there is a term $M_i^\top W$ in $\tilde P_i$, the condition (IV.18) is not an LMI. However, we can transform (IV.18) into an LMI condition. Define $Q_i := P_i^{-1}$, $S_i := R_i$, $\Lambda := W^{-1}$, and $Y_i := M_iQ_i$. Using the congruence transformation $T_1 = \mathrm{diag}(Q_i, \Lambda, S_i, Q_j, S_i)$, we can show that (IV.18) holds if and only if (IV.3) is feasible.

The well-posedness of the closed-loop system, that is, the invertibility of the function $I - (I - \tilde L\,\mathrm{sat}(\bullet))$ is achieved from $2W - W\tilde L - \tilde L^\top W > 0$ in (IV.16) by Proposition 1 of [26]. ■

B. *Proof of Lemma 4.4*

We first prove that if the LMI (IV.5) is feasible, then $V(1, \xi_1) \leq 1$ holds for all the initial state $\xi(0) = [x(0)^\top \ \ \hat x(0)^\top]^\top$ satisfying $x(0)^\top\Omega x(0) \leq 1$ and $\hat x(0) = 0$.

Since we have $u(t) = 0$ for $t \in [0, h)$ from Assumption 4.1, if $\hat x(0) = 0$, then $\xi_1 = \Upsilon(\Delta_0)x(0)$, where

$$\Upsilon(\Delta_0) := \begin{bmatrix}e^{Ah}(I - e^{-A\Delta_0})\\ e^{A(h-\Delta_0)}\end{bmatrix} = \begin{bmatrix}-p\theta(\Delta_0)\\ p(I+\theta(\Delta_0))\end{bmatrix}.$$

In order to prove that $x(0)^\top\Omega x(0) \leq 1$ implies $V(1,\xi_1) \leq 1$, it suffices to show that $x(0)^\top\Omega x(0) \geq V(1,\xi_1)$, namely, $P_j > 0$, $j = 1,\dots,N$, satisfies

$$\Omega - \Upsilon(\Delta_0)^\top P_j\Upsilon(\Delta_0) \geq 0 \qquad \text{(IV.19)}$$

for all $j = 1,\dots,N$ and for all $\Delta_0 \in [\underline{\Delta}, \overline{\Delta}]$. Here we used the property (IV.14) for $\{\alpha_{1,i}\}_{i=1}^N \in \mathcal{A}$.

A sufficient condition for (IV.19) to hold can be obtained in the same way as (IV.11). In fact, $\Upsilon(\Delta_0)$ is given by

$$\Upsilon(\Delta_0) = \sum_{i=1}^N \alpha_{0,i}\left(\hat A_{cl,i}+\hat B_{cl,i}\Phi_0\hat C_{cl,i}\right)$$

for some $\Phi_0 \in \mathbf{\Phi}$, where $\hat A_{cl,i}$, $\hat B_{cl,i}$, and $\hat C_{cl,i}$ are defined as in the statement of Theorem 4.2. It follows from the Schur complement and the property (IV.14) for $\{\alpha_{0,i}\}_{i=1}^N \in \mathcal{A}$ that (IV.19) holds for all $\Delta_0 \in [\underline{\Delta}, \overline{\Delta}]$ if

$$\begin{bmatrix} \Omega & (\hat{A}_{cl,i} + \hat{B}_{cl,i}\Phi_0\hat{C}_{cl,i})^\top P_j \\ * & P_j \end{bmatrix} \geq 0 \tag{IV.20}$$

for all $i = 1, \dots, N$ and $\Phi_0 \in \mathbf{\Phi}$. Moreover, with the transformation similar to that in (IV.17), we have that (IV.20) holds for all $\Phi_0 \in \mathbf{\Phi}$ if there exists $\hat{R}_i \in \mathcal{R}$ such that

$$\begin{bmatrix} \Omega & 0 & \hat{A}_{cl,i}^\top P_j & \hat{C}_{cl,i}^\top \hat{R}_i \\ * & \hat{R}_i & \hat{B}_{cl,i}^\top P_j & 0 \\ * & * & P_j & 0 \\ * & * & * & \hat{R}_i \end{bmatrix} \geq 0. \tag{IV.21}$$

For the consistency of the variables in the LMI (IV.3), we define $Q_i := P_i^{-1}$ and $\hat{S}_i := \hat{R}_i^{-1}$ and apply the congruence transformation $T_2 = \mathrm{diag}(I, \hat{S}_i, Q_j, \hat{S}_i)$ to (IV.21), which gives (IV.5).

Let us next prove that $V(k, \xi_k) \leq 1$ implies $\mathrm{sat}(M^{[k]}\xi_k) = M^{[k]}\xi_k$, To this end, it is enough to show that

$$\begin{aligned} \frac{1}{\bar{u}_l^2}\xi^\top &\left(\sum_{i=1}^N \alpha_{k,i} M_i^{(l)}\right)^\top \left(\sum_{i=1}^N \alpha_{k,i} M_i^{(l)}\right)\xi \\ &\leq V(k,\xi) = \xi^\top \left(\sum_{i=1}^N \alpha_{k,i} P_i\right)\xi \end{aligned} \tag{IV.22}$$

for all $l = 1, \dots, m$, where $M_i^{(l)}$ denotes the l-th row of M_i. Using the Schur complement, we see from the property (IV.14) for $\{\alpha_{k,i}\}_{i=1}^N$ that (IV.22) holds for all $\{\alpha_{k,i}\}_{i=1}^N \in \mathcal{A}$ if and only if

$$\begin{bmatrix} P_i & (M_i^{(l)})^\top \\ * & \bar{u}_l^2 \end{bmatrix} \geq 0 \tag{IV.23}$$

for all $i = 1, \dots, N$ and $l = 1, \dots, m$. For the consistency of the variables in the LMI (IV.3), we use the congruence transformation $T_3 = \mathrm{diag}(Q_i, I)$, which shows that (IV.23) is equivalent to (IV.4) ■

V. Concluding Remarks

We have studied the stability analysis and the stabilization problem for systems with time-varying clock offsets. We have handled the parameter uncertainty caused by clock offsets with a polytopic overapproximation, and have derived a sufficient condition for stability. Two methods for the design of stabilizing feedback gains have been proposed, based on fixing the structure of the matrices in the quadratic Lyapunov function and on the CCL algorithm. Using the sector characterization of the saturation nonlinearity, we have also extended this overapproximation approach to the analysis of regional stability for systems with actuator saturation in addition to clock offsets. The design of dynamic feedback controllers and anti-windup gains is left as a topic for future research.

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