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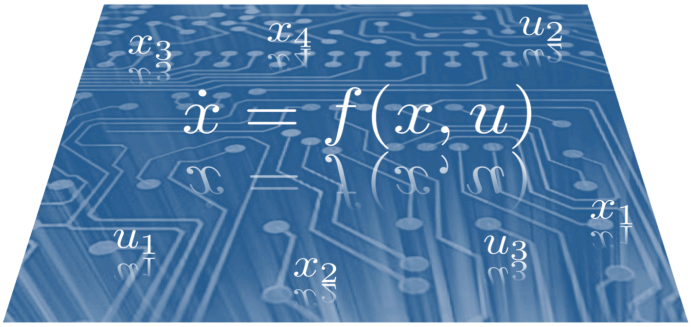

Stochastic, Dynamic, and Periodic Networked Control Systems

Duarte José Guerreiro Tomé Antunes

Supervisor: Doctor Carlos Jorge Ferreira Silvestre
Co-Supervisor: Doctor João Pedro Hespanha

Thesis approved in public session to obtain the PhD Degree in Electrical and Computer Engineering

Juri final classification: Pass with Distinction

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Chairperson: Chairman of the IST Scientific Board

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Abstract

Networked Control Systems are controlled physical systems where sensing and control devices are connected via communication networks. Motivated by the proliferation of embedded sensors, microprocessors, and wireless networks, extensive research has been conducted in recent years on networked control systems, bringing forth important theoretical advances in the automatic control field.

The results of the present thesis spring from networked control applications, considering three different network scenarios. First, considering a network model inspired on the Ethernet and the Wireless 802.11 protocols, where delays and intervals between transmissions are stochastic, we provide a stability result that can be tested using the Nyquist criterion and show how to compute moment Lyapunov exponents to investigate performance. In the same setup, but considering several asynchronous networks, we establish that stability can be asserted by computing the spectral radius of an integral operator. Second, assuming that all the sensing and control devices in the network can run an arbitration algorithm, we propose dynamic protocols, assigning priorities based on data sent by the devices, which can outperform any static protocol where nodes transmit in a prescribed order. Third, considering networks that guarantee fixed transmission rates, although in general different among users-such as circuit-switching networks-, we propose output regulation and gain-scheduling solutions for multi-rate systems.

We use the framework of hybrid systems to model several networked control scenarios, and the machinery of Volterra equations, piecewise deterministic processes, and dynamic programming to establish the main results of the thesis. These results are often stated with enough generality that their interest is shown to exceed the networked control systems scope.

Keywords: Networked Control Systems; Impulsive Systems; Stochastic Hybrid Systems; Asynchronous Systems; Volterra Equations; Dynamic Protocols; Piecewise Deterministic Processes; Gain-Scheduling; Output Regulation; Multi-Rate Control.

Resumo

Sistemas de controlo sobre redes são sistemas de controlo automático em que sensores, atuadores e controladores comunicam entre si através de redes de comunicação. A proliferação de sensores embebidos, micro-processadores e redes de comunicação sem fios despertou o interesse por estes sistemas, o que se traduziu em progressos significativos na área do controlo automático.

Esta tese aborda problemas na área do controlo automático que provêm de três cenários para sistemas de controlo sobre redes. Em primeiro lugar, considera-se modelos para a rede inspirados na rede *Ethernet* e na rede *Wireless* 802.11, obtendo-se resultados de estabilidade que podem ser testados através do critério de Nyquist e mostrando-se também como se pode avaliar o desempenho do sistema. Ainda para este cenário, mas considerando várias redes operando assíncronamente, prova-se que a estabilidade do sistema de controlo sobre redes pode ser testada através do cálculo do raio espetral de um operador integral. Em segundo lugar, assumindo que os vários intervenientes na cadeia de controlo têm capacidade de processamento, propõem-se protocolos dinâmicos, que atribuem prioridades de acesso à rede com base na informação enviada pelos intervenientes na cadeia de controlo, conseguindo superar a performance dos protocolos estáticos, em que uma dada ordem para as transmissões é repetida periodicamente. Em terceiro lugar, considera-se redes que garantem uma determinada taxa de transmissão, possivelmente diferente para cada utilizador-tais como as redes de circuitos comutados-, e propõem-se soluções para os problemas de regulação da saída e de síntese de sistemas de controlo de ganhos comutados para sistemas multi-ritmo.

A capacidade de modelação dos sistemas híbridos e as ferramentas matemáticas para equações de Volterra e de Fredholm, de processos determinísticos por troços, e de programação dinâmica estão na base dos principais resultados da tese. Os resultados são enunciados com generalidade, o que permite mostrar que o seu interesse excede a área dos sistemas de controlo sobre redes.

Palavras Chave: Sistemas de controlo sobre redes; Sistemas Impulsivos; Sistemas Híbridos Estocásticos; Sistemas Assíncronos; Multi-Ritmo; Equações de Volterra; Protocolos Dinâmicos; Processos Determinísticos por Troços; Sistemas de controlo de ganhos comutados; Regulação da Saída.

Ad majorem Dei gloriam

“Àquele que puder ser sábio, não lhe perdoamos que o não seja”
(Josemaria Escrivá).

Acknowledgements

First and foremost, I acknowledge the support and friendship of my two supervisors. I am grateful to Carlos Silvestre for several useful conversations. His trust, especially in difficult moments, has been invaluable. On the other hand, it has been an enormous pleasure to work with such an honest and dedicated researcher as João Hespanha. I am very grateful for his thorough work in our articles, his critical attitude towards the results, his encouragement, and several precious pieces of advice that I shall take with me in my professional future.

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A subset of the work presented in Chapter 7 was done in collaboration with Rita Cunha. Some of the work presented in Chapter 5 was reviewed after a follow-up work with Maurice Heemels. I gratefully acknowledge their help.

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Introduction

In recent years we have witnessed remarkable technological advances in computing, sensing and wireless communication technologies. The combination of these advances made possible pocket-size embedded electronic devices, designed to sense, compute, and communicate information of interest. Sensors and micro-processors are now ubiquitously deployed in vehicles, roads, cell-phones, buildings, and environment, and have the ability to process measurement data in real time and transmit this data to perform adequate control actions. A control system where sensors, micro-controllers, and actuators are connected through communication networks is termed a *Networked Control System* (NCS).

There are abundant new and envisioned applications of networked control systems [43]. As a first example, consider the problem of preventing highways' traffic congestions. A requirement for providing traffic control is to predict traffic conditions. This can be achieved by using a limited number of vehicle detector stations (sensors) deployed along the highway. A networked control estimation problem is to infer on the highway traffic density from these distributed measurements [37], [70]. Another research direction is the control of vehicle platoons [30] to mitigate the so-called butterfly effect, i.e., large traffic jams triggered by minor events, such as an abrupt steering maneuver by a single motorist. Here, the goal is to control the velocities and relative positions of the vehicles based on communication from immediate predecessors, so that a desired behavior for the platoon is achieved, e.g., constant speed cruising or leader following. As a second class of applications, consider the emerging field of smart grids. It is predicted that in a near future renewable energy sources, such as wind and solar power, will play an even more important role in the overall energy production. However, the uncertainty on the availability of such energy sources, along with the increasingly variability of loads, such as electric vehicles, poses challenging problems concerning the transient stability of the network, i.e., achieving synchronism when subjected to these large disturbances on load and generation. In [29], the relation between the transient analysis in a power network and a distributed networked control problem known as the consensus problem is sharply recognized. Another example in the area of energy systems is the use of sensor networks in energy building efficiency, which enables improved control of indoor environment [92]. As a third class of applications, consider automation in manufacturing systems. Here, communication networks are being used more and more for diagnostic and control operations [71]. The use of wireless communications in industry environments can drastically reduce cabling and maintenance costs. However, the measurement and control delays resulting from introducing a shared communication medium have to be taken into account when designing such systems. There are several other applications in many distinct fields, such as remote surgery [67], or thermal control of livestock stables [96]. Also in a luxury car there may exist over 50 embedded computers, running several control algorithms, which include safety critical operations, as well as leisure applications [83]. Using a network to close the control loop has several advantages, including flexibility, high reliability, simple installation and maintenance, and low cost. The demand for small and intelligent sensing devices is steadily growing in our society, which allows one to predict that networked control systems will continue to be a prominent object of interest in coming years.

On the other hand, the interest on networked control system is far from confining itself to the realm of applications. Networked control systems have given rise to a multi-disciplinary research field lying in the intersection of three distinct research areas: control systems, telecommunications, and computer science, and this has promoted the interchange of problem solving techniques and insights from different areas. Moreover, some of the problems that arise in networked control lead to important research advances in the control research field. For example, the celebrated Shannon's results on the maximum bit-rate at which a communication channel can carry information reliably, has inspired several researchers [31, 42, 72, 94], to tackle the problem of determining the minimum bit-rate needed to stabilize a linear system through feedback over a finite capacity channel. Stochastic and deterministic hybrid systems [45], [77], have played an important role in modeling several networked control scenarios, and several theoretical results prompted from this relation [62], [90], [73]. The inherent distributed structure of many problems in networked control system has also inspired many problems in the area of distributed control of agents deployed in a given environment [12]. There are several problems where agents are connected by a communication network and wish to optimize a given performance cost in a distributed way [49] or achieve a common objective [10]. Hence, there is a significant overlap between the research on networked control systems and the research on distributed optimization and decentralized control. Another example are switched systems, where recognizing the connection with networked control system has favored research on both areas [28], [36].

In this thesis, we formulate and address several networked control problems, considering three models for the communication network: (i) networks with stochastic characteristics, where the arbitration process for network access involves stochastic events, such as random delays, packet drops, or random back-off times. Examples of networks where such events may occur are the Ethernet and networks utilizing the Wireless 802.11 protocol; (ii) networks in which access to the network is determined by a protocol based on state information sent by the nodes. This may be achieved in networks where nodes have enough computational resources to run an arbitration algorithm in a distributed way, or in cases where the network itself may provide an arbitration mechanism based on data sent by the nodes, e.g., CAN-BUS networks. Since decisions are dynamically taken at each transmission time based on state information, we denote these protocols by dynamic protocols; (iii) networks with periodic multi-rate data transmissions, where each communication link has a fixed rate, which is in general different from the rates of the other links. Examples of these networks are the circuit switching networks, which guarantee a fixed bandwidth for each link.

The solutions we propose often involve developing novel theoretical and system analytical results. In fact, it is a common trend in the thesis that the underlying networked control problem serves as a motivation or as a starting point to consider mathematical and system analytical problems obtained from raising the level of abstraction of the models that capture the networked control problem. As we shall see, this approach has the advantage that our results often have a broader range of application that exceeds the networked control systems scope. We use the framework of hybrid systems to model the networked control systems that we consider and the machinery of Volterra equations, piecewise deterministic processes, and dynamic programming to establish the main results of the thesis.

In this introductory chapter, we give a general description of the networked control problems that we tackle in Section 1.1 and provide an overview of our main results in Section 1.2. We explain the organization of the thesis in Section 1.3 and enumerate the thesis

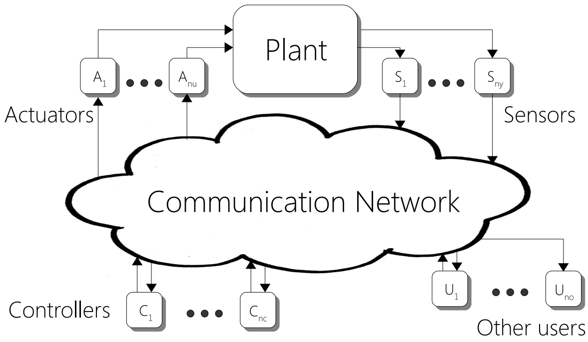


Figure 1.1: General Diagram of a Networked Control System

contributions, as well as the list of publications that substantiate the thesis in Section 1.4. Basic notation is established in Section 1.5.

1.1 Networked Control Systems - General Description

A nuts and bolts description of a networked control system as perceived in the present thesis is depicted in Figure 1.1. We use the nomenclature *nodes*, to denote sensors, actuators, and controllers that are connected by the communication network in a networked control system. Nodes pertaining to the same closed-loop are connected by a shared network. The network is abstractly represented by a cloud due to the uncertainty that may exist on the service it provides, or on the type of communication that agents perform. This cloud can also encapsulate several other subnetworks or dedicated communications links between nodes. The sensors acquire measurement data from the plant, and transmit this data through a communication network to be processed by one or several remote controllers. In the present thesis we shall consider mainly the case where there is only one controller. The controller transmits control values to the actuators through the same shared network. Based on this control values the actuators perform a digital to analog conversion and apply a continuous-time actuations to the plant. The network may be shared with other users, which in general are not related to the closed-loop. The interference that the data transmitted by these other users has on the closed-loop, e.g., whether other users transmit frequently or sporadically, is assumed to be encapsulated in the model for the communication network.

In this thesis we address several problems, considering different models for the agents in the networked control system (plant, controller, sensors, actuators), and for the network itself. These different models are described next.

Plant

The model for the plant may be *linear* or *nonlinear*, certain or uncertain. Uncertainty may be modeled by external *stochastic disturbances*, or by external *deterministic* (but unknown) *disturbances*. It may also be the case that there exists *parameter uncertainty* on the plant.

Controller Synthesis

Tracing a parallel with traditional digital control [15], we consider two methods for obtaining the controller: *emulation* and *direct design*. In emulation, a continuous-time stabilizing controller designed without regard to the network characteristics is assumed to be available. The controller for the networked control system is obtained by running a numerical approximation method of the continuous-time controller based on the measurement values received from the sensors. This numerical approximation is typically obtained by emulating the evolution of the continuous-time controller using as its inputs hold values of the measurement data most recently received from the sensors. Both sensor measurement arrivals and transmissions of actuation updates from the controller to the actuators are dictated by the communication networks availability. From a networked control point of view, for an emulation design, we are simply concerned with *analyzing* the effects of the network in the closed-loop. On the other hand, in direct design the controller for the networked control system is *synthesized* by directly taking into account the plant and network characteristics. The specifications to obtain this controller can be closed-loop *stability*, *optimality* according to some performance criterion, and/or *output regulation* of some outputs of interest.

Sensors and actuators

Sensors perform an analog to digital conversion, which is assumed to be an ideal sampling, while actuators perform a digital to analog conversion which is typically assumed to be a standard hold operation, although other possibilities exist (cf. [69], [63]). Sensors and actuators are assumed to have network adapters to transmit data through the network. Each sensor may be associated with more than one output of the plant's state, and each actuator may be associated with more than one input of the plant. In cases where sensors and actuators have dedicated links to transmit, we may only need to assume that they can be sampled and updated (respectively) at a *fixed rate*, while in general we shall require that they can be sampled and updated on demand, i.e., at any desired transmission time. We denote sensors and actuators by *simple sensors* and *simple actuators* if these sampling and transmitting operations are the only operations they can perform. However, we denote sensors by *smart sensors* if they have enough computational resources to run an arbitration algorithm for network access, and denote controllers by *smart controllers* if they are collocated with the actuators, in which case we also assume that they have enough computational resources to run an arbitration algorithm.

Network

The network access in a shared network typically introduces delays, since nodes may have to wait until the network becomes available. Depending on the implementation or context, these delays may either affect the times between consecutive sampling, in cases where sensors and controllers respond on demand to network availability, or may introduce significant delays on the data received from plant and controller, in case the sensors hold past information until the network becomes available for transmission. Further delays may be taken into account such as transmission and processing delays. In this thesis, we shall mainly consider the following model for a network with stochastic characteristics, which we denote by *renewal network*, a nomenclature justified in the sequel.

(i) The time intervals between transmissions from nodes pertaining to the networked control system are independent and identically distributed;

(ii) The transmission delays are small when compared to the time intervals between transmissions;

(iii) Packet may me dropped with a given probability.

Assumption (i) holds for scenarios in which nodes attempt to do periodic transmissions of data, but these regular transmissions may be perturbed by the medium access protocol. It is typically the case in CSMA protocols that nodes may be forced to back-off for a typically random amount of time until the network becomes available. The probability distribution of the time interval between transmissions, which can be estimated experimentally or by running Monte Carlo simulations of the protocol, is determined by two factors: the congestion of the network and the delay introduced by the medium access protocol. Assumption (ii) also holds in general in local area networks. In fact, as explained in [90], in local area networks, the transmission delays are typically small when compared to the times that nodes take to gain access to the communication medium. We shall consider general stochastic models for the delays (see Chapter 3, Sec. 3.1, and Chapter 5, Sec 5.1.1).

Assumption (iii) is typical and our modeling framework allows to consider general models where packet drop are correlated and modeled by Markov Chains (see Chapter 3) such as the well-known Gilbert-Elliot model. We use the nomenclature renewal network, since the state variables associated with the sources of randomness (intervals between transmissions, delays, packet drops) restart or are renewed when a transmission occurs.

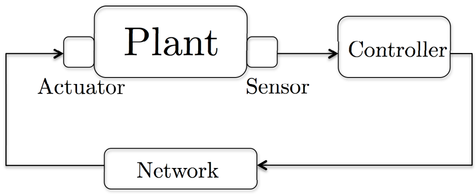
Another model that will be used in the present thesis is to assume that each sensor and each actuator use a different communication link to exchange data with the controller. Each communication link may represent a shared network that guarantees a fixed transmission rate, such as in circuit switching networks (cf. [58]). It can also simply represent a sensor, which outputs measurements at a fixed sampling rate, or an actuator which allows actuation updates at a fixed rate. We denote these networks by *multi-rate networks*.

Access Protocol

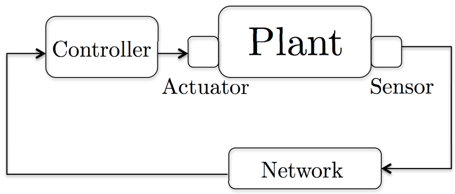
Typical communication networks, such as the wireless 802.11, the Ethernet, and the CAN-BUS, provide a medium access protocol for transmissions. However, what we mean here by access protocol is the high-level protocol that nodes pertaining to the same loop may possibly implement on top of the protocol provided by the network. A *static protocol* is defined as a protocol in which the nodes agree to transmit in a prescribed order, which is repeated periodically. Note that in a static protocol a node is allowed to transmit more than once in a period. On the other hand, one can define *dynamic protocols*, running on-line, i.e., simultaneously with the process, in which nodes are allowed to arbitrate who transmits based on state information about the plant and/or based on the data received from previous transmissions. Dynamic protocols may utilize some mechanism for arbitration provided by some networks, e.g., dynamically changing the arbitration field of messages in CAN-BUS networks. However, in general, nodes run an arbitration algorithm in a distributed way and on top of an underlying communication protocol which offers no direct service for arbitration, in which case we assume that the sensors and actuator nodes are smart.

1.2 Overview of the Main Results

We divide the presentation of our main results according to six works that substantiate the thesis. The first three works focus on analytical properties of networked control systems in which the network has stochastic characteristics. The fourth work proposes a class of dynamic protocols for networked control systems, and the fifth and sixth works study networked control systems with periodic multi-rate transmissions. In each of these works we



(a) Controller collocated with sensor



(b) Controller collocated with actuator

Figure 1.2: Two possible scenarios where a single node transmits through a shared network

consider different models for the plant, controller synthesis, actuators and sensors. These models are chosen typically in a way that we can focus on a particular feature of the networked control system as opposed to considering the most general models for every case. We hope that the reader interested in a problem that combines several of the features considered here can combine and adapt the ideas we propose.

1.2.1 Control Systems over a Renewal Network

We start by considering a simple networked control system where only one node pertaining to the closed-loop transmits through a shared renewal network, and therefore no network access protocol is required. Figure 1.2 depicts two possible scenarios that fit our framework: a) the controller may be collocated with a single sensor of the plant and transmit control data to a single actuator through a shared network; b) The controller may be collocated with the actuator and receive measurement data from the sensor. Delays and packet drops are neglected for now, the plant and the controller are assumed to be linear, and we assume that no disturbances are acting on the plant. The controller synthesis is assumed to be emulation-based. Our assumptions are summarized next.

Plant	Controller Syn.	Sensor & Actuators	Network	Access Protocol
Linear, no disturbances	Emulation	Simple	Renewal, no drops, no delays	Not required

The focus here is to capture the fact that the times between transmissions are independent and identically distributed, asserting stability and performance of networked control systems with this feature. Note that stability should be considered in a stochastic sense. For now it suffices to consider that we declare the networked control system to be stable in a mean square sense, if the expected value of the squared norm of the state variables of the networked control system goes to zero. We shall define performance as the rate at which the expected value of a quadratic positive definite function of the state goes to zero.

We show that the networked control system just described can be modeled by an impulsive renewal system. Impulsive renewal systems are described by a vector field that determines the evolution of the state between transition times at which the state undergoes a jump determined by a reset map. The intervals between transition times are assumed to be independent and identically distributed (i.i.d.) random variables. The nomenclature impulsive renewal system is motivated by the fact that the process that counts the number of transitions up to the current time is a renewal process (cf. [79]). We provide analytical results for impulsive renewal systems, which are deeply rooted in a set of novel results for *Volterra integral equations with positive kernel*, and have applications to the stability analysis of the networked control systems depicted in Figure 1.2. Volterra integral equations with positive kernel shall be defined in Chapter 2, where we also provide general results for this class of equations and show other applications of these results. The main implications of these results for networked control systems can be summarized as follows.

We provide stability conditions for impulsive renewal systems that can be cast in terms of a matrix eigenvalue computation, the feasibility of a set of LMIs, and also tested using the Nyquist criterion. Moreover, we provide a method to compute a second moment Lyapunov exponent, which provides the asymptotic rate of decrease / growth for the expected value of a quadratic function of the systems' state. These results have a direct application to NCSs with a single renewal network.

We also discuss how one can assert the performance of the networked control system with single renewal network by computing the second moment Lyapunov exponent of the impulsive renewal system.

Our framework shares common features with randomly sampled systems [40], [55]. As we shall discuss in Chapter 2, our results, to the best of our knowledge, have not been obtained before, and shed further insight also to randomly sampled systems.

We consider the problem of directly designing a controller in this framework in [AHS09a]. However, since the results one obtains using a dynamic programming optimal control framework are similar to existing results for randomly sampled systems [40], [55], we decide not to include them in the present thesis. The interested reader can consult [AHS09a], [40], [55].

1.2.2 Network Features Modeled by Finite State Machines

Consider now that several nodes pertaining to the same closed-loop are connected to the network, implementing a static protocol, and also the general case where the network may have delays and packet drops. The remaining assumptions on the networked control system are similar to the ones considered in Section 1.2.1 and are summarized next.

Plant	Controller Syn.	Sensor & Actuators	Network	Access Protocol
Linear, no disturbances	Emulation	Simple	Renewal, with drops & delays	Static

We will show that static protocols, delays, packet drops, and other network features which can be captured by finite state machines can be modeled by *stochastic hybrid systems*.

Stochastic hybrid systems (SHSs) are systems with both continuous dynamics and discrete logic. The execution of an SHS is specified by the dynamic equations of the continuous state, a set of rules governing the transitions between discrete modes, and reset maps determining jumps of the state at transition times. We consider SHSs with linear dynamics, linear reset maps, and for which the lengths of times that the system stays in each mode are independent arbitrarily distributed random variables, whose distributions may depend on the discrete mode. The process that combines the transition times and the discrete mode is called a Markov renewal process [48], which motivated us to refer to these systems as *stochastic hybrid systems with renewal transitions*. The class of impulsive renewal systems is a special case of a SHS with renewal transitions, where there is only one discrete mode and only one reset map. The key to model networked control system with SHSs is to capture the sequence of events as a finite state machine. As we will show in Chapter 3, besides delays, packet drops, and static protocols we can capture, e.g., scenarios where nodes try to access the network independently, or scenarios where nodes transmit through more than one renewal network, operating synchronously.

Inspired by the work on impulsive renewal systems, the approach followed to analyze stochastic hybrid systems with renewal transitions is based on a set of Volterra renewal-type equations. As for impulsive renewal systems, we can characterize the asymptotic behavior of the system by providing necessary and sufficient conditions for various stability notions in terms of LMIs, algebraic expressions and Nyquist criterion conditions, and determining the decay or increase rate at which the expected value of a quadratic function of the systems' state converges exponentially fast to zero or to infinity, depending on whether or not the system is mean exponentially stable. We derived these results in [AHS10b]. In the

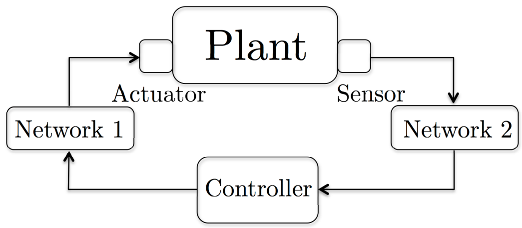


Figure 1.3: Networked Control System with nodes transmitting through different networks operating asynchronously

present thesis, we present other results that differ significantly from the ones described in Subsection 1.2.1. These new results can be summarized as follows:

We give computationally efficient expressions to compute any moment of the state of the SHS, and characterize the asymptotic behavior of special high-order moments. In particular, we provide necessary and sufficient conditions for mean square stability.

Note that these additional results concerning higher order moments also apply to impulsive renewal systems, which are special cases of SHSs. We also discuss how by computing the moments one can provide information about the probability density function of the state of the SHS. Regarding this latter problem, we highlight the advantages of our methods when compared to other general methods in the literature [25], [45], [41].

1.2.3 Asynchronous Renewal Networks and Non-Linear Models

Consider now a networked control system in which a remote controller communicates with a plant through two independent communication links; e.g., the actuation data is sent from the controller to the plant through a shared wired network and the sensor data is sent from the plant to the controller through a shared wireless network. This example is depicted in Figure 1.3. More generally, we can consider networked control systems where several sensors, actuators, and controller are linked through different networks operating asynchronously, i.e., there is no interference between networks. We consider that each of these networks is a renewal network, but for simplicity we ignore delays and packet drops. We consider an emulation based design for controller design and we assume that no delays are present. The fundamental differences between this networked control scenario and the one considered in the previous subsections are two fold: (i) the networks operate asynchronously, (ii) the model for the plant and for the controller are allowed to be non-linear. We still consider no disturbances in the model for the plant, and we assume for simplicity that no two nodes transmit through the same shared network. Our assumptions are summarized next.

Plant	Controller Syn.	S & A	Network	Access Protocol
Linear & non-linear no disturbances	Emulation	Simple	Asynchronous Renewal links, no drops, no delays	Not required

We could model several renewal networks in the networked control system with the framework of stochastic hybrid systems, as long as the protocols could be modeled as finite state machines. However, since each renewal network operates asynchronously, it does not appear to be possible to capture this scenario with finite state machines. This significantly changes the type of mathematical tools used to address these systems and the corresponding results. In fact, the approach we follow for the networked control scenarios considered in Sections 1.2.1 and 1.2.2, based on Volterra integral equations, does not appear to render an easy extension to the present case of asynchronous networks. Piecewise deterministic processes [25] are shown to be an adequate mathematical framework to tackle these systems. Interestingly, they also allow to handle the case of non-linear models for the plant and the controller.

We show that the networked control system at hand can be modeled by an impulsive system triggered by a superposed renewal process, which is a model similar to impulsive renewal systems but allowing several reset maps triggered by independent renewal processes, i.e., the intervals between jumps associated with a given reset map are identically distributed and independent of the other jump intervals. The connection between this class of systems and piecewise deterministic processes is also addressed. We provide stability results for this new class of impulsive systems, which directly entail stability properties for the considered asynchronous networked control systems. Our main results can be summarized as follows.

We provide stability conditions for general non-linear NCSs with asynchronous links. By specializing these stability conditions to linear NCSs, we show that stability for linear NCSs can be asserted by testing if the spectral radius of an integral operator is less than one.

We also show that the origin of the non-linear NCS is stable with probability one if its linearization about zero equilibrium is mean exponentially stable, which justifies the importance of studying the linear case. Stability with probability one is a standard generalization of the notion of local stability to stochastic systems, and will be formally defined in Chapter 4.

1.2.4 Dynamic Protocols

We have already discussed *time-driven* static protocols in which nodes choose a given order to transmit, which is repeated periodically. However, it is reasonable to suspect that in some state configurations of the networked control system it may be more favorable for one of the nodes to transmit, whereas for other states, transmissions from other nodes may lead to more favorable outcomes. This leads us to study dynamic protocols, in which nodes are allowed to arbitrate who transmits based on state information about the plant and/or based on the data received from previous transmissions. We tackle this problem from both the frameworks of emulation and direct design for controller synthesis. The plant is considered to be linear, and in the controller design setup we shall consider a model where the plant is disturbed by Gaussian noise. The sensors and actuators are assumed to be smart, i.e., to have enough computation resources to run an arbitration algorithm, and we also consider delays and packet drops. Our assumptions are summarized next.

Plant	Controller Syn.	S & A	Network	Access Protocol
Linear, with and without disturbances	Emulation & direct design	Smart	Renewal, with drops and delays	Dynamic

Considering an emulation framework, we provide conditions for mean exponential stability of the networked closed-loop in terms of matrix inequalities, both for investigating the stability of given protocols, such as static round-robin protocols and dynamic maximum error first-try once discard protocols [91], and to design new dynamic protocols. The main result entailed by these conditions is that if the networked closed-loop is stable for a static protocol, then we can provide a dynamic protocol for which the networked closed-loop is also stable. The dynamic protocol in this case can be obtained by construction. However, the obtained protocol assumes in general that the full-state is available, which restricts the range of applicability of the results in this emulation framework.

Considering a direct design framework, we tackle the problem of simultaneously designing the scheduling sequence of transmissions and the control law, so as to optimize a quadratic objective. The plant is assumed to be disturbed by noise. Using the framework of dynamic programming, we propose a rollout strategy [5] by which the scheduling and control decisions are determined at each transmission time as the ones that lead to optimal performance over a given horizon, assuming that from then on controller and sensors transmit in a periodic order and the control law is a standard optimal law for periodic systems. We show that this rollout strategy results in a protocol where scheduling decisions are based on the state estimate and error covariance matrix of a Kalman estimator, and must be determined on-line (closed-loop policy) in the case where both control and measurement values are scheduled. This is in striking contrast with the optimal and analog rollout strategies for the sensor scheduling problem [68], in which only the sensors are to be scheduling while the control variables can be updated at every time step (e.g., controller is collocated with the plant), for which the scheduling sequence can be determined off-line (open loop policy). The resulting protocol obtained from the rollout algorithm can be implemented in a distributed way both in wireless and wired networks, based on previous data sent from sensors and actuators, as opposed to the requirements of the protocol obtained with the emulation framework, where full-state is in general required. It follows by construction of rollout algorithms that our proposed scheduling method can outperform any periodic scheduling of transmissions. Our main results both for emulation and direct design can be summarized as follows.

We propose a class of dynamic protocols for both the cases of emulation and direct design that result directly from stability and optimal solutions to the problem at hand. The protocol for an emulation design depends on state information of the networked control system, whereas the protocol obtained in the direct design framework depends on state estimates of the plant and can be run in a distributed way. In both frameworks, dynamic protocols are shown to outperform periodic protocols.

1.2.5 Output Regulation for Multi-Rate Systems

The output regulation problem consists on controlling the output of a linear time-invariant plant so as to achieve asymptotic tracking of an exogenous signal generated by the free motion of a linear time-invariant system, so-called exosystem, while guaranteeing closed-loop stability. Here we tackle the output regulation problem when sensors transmit over different dedicated links imposing different transmission rates. We consider linear plants

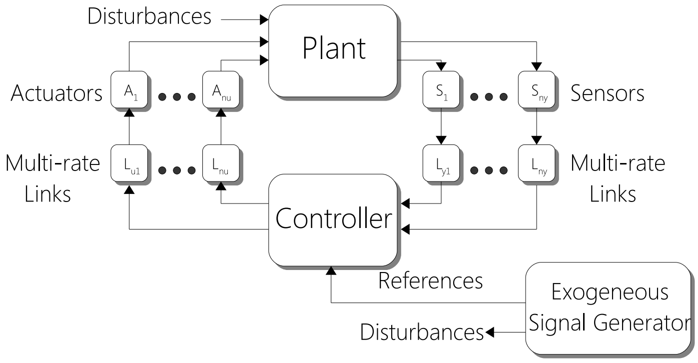


Figure 1.4: Multi-Rate networked control systems setup

which may me subject to unknown but deterministic disturbances and parameter uncertainty. For now, we consider that the actuation is available at a fixed rate, i.e., the links that connect the controller to the actuators have the same rate. The setup is depicted in Figure 1.4, and our assumptions are summarized next

Plant	Controller Syn.	S & A	Network	Access Protocol
Linear, with external and parameter disturbances	Direct Design	Simple	Multi-Rate	Not required

The difficulty here is that these measurement and update rates may be different for the various communication links. A standing assumption is that the rates, although in general different, are rationally related. In principle, one can take advantage of using the maximum allowed sampling rates for sensors and actuators, instead of synchronizing these rates to match the slowest, in which case traditional digital control techniques [15] would be applicable. The set-up considered in Figure. 1.4 matches the one considered for traditional multi-rate systems, which have been the object of research since the early ninetees (cf., e.g., [7], [16]), i.e., before the dawn of the networked control system research area. In the case where the plant has the same number of inputs as outputs, the output regulation problem was solved in [82]. However, as mentioned in [82], the output regulation problem was left unsolved for the general case where more outputs may be available than inputs. Note that providing a solution to this problem entails that in many control problems where the outputs have different measurement rates, one can take advantage of all the measurements available for feedback, and therefore improve the performance of the closed-loop. Moreover, a solution to the output regulation problem for non-square systems that achieves closed-loop stability is required when the non-regulated outputs are needed to guarantee the detectability of the plant. In this thesis we present the following result.

We design a controller that achieves stability for the closed-loop and output regulation for a number of regulated outputs equal to the number of inputs, while taking advantage of the remaining outputs for feedback.

1.2.6 Gain-scheduling Control for Multi-Rate Systems

Gain-scheduling is a useful tool for designing control laws for non-linear plants, using linear control tools [53]. The standard procedure for designing a gain-scheduling controller involves the following steps (cf. [81]): (i) the selection of scheduling variables or parameters; (ii) linearization of the non-linear plant about the equilibrium manifold; (iii) synthesis of controllers for the family of plant linearizations, which typically involves linear controller design for a given set of equilibrium points; and (iv) implementation of the controller. The implementation must be such that the controller verifies the linearization property: at each equilibrium point, the nonlinear gain-scheduled controller must linearize to the linear controller designed for that equilibrium. In fact, it is often the case that for gain-scheduling implementations which would perhaps appear more natural the linearization property does not hold and this mismatch is commonly known as the hidden coupling [81]. A solution to part (iv) is the velocity implementation proposed in [52] and further discussed in [53, Ch. 12], [81]. In the present thesis, we pose the problem of obtaining a solution to the gain-scheduling problem in the case where actuators and sensors have different rates. The plant is assumed to be non-linear and subject to external deterministic but unknown disturbances and we assume that the model for the plant may have parameter uncertainty. Our assumptions are summarized next

Plant	Controller Syn.	S & A	Network	Access Protocol
Non-Linear, with external and parameter disturbances	Direct Design	Simple	Multi-Rate	Not required

In this thesis we have the following results.

We provide a method for the synthesis and implementation of gain-scheduled controllers for multi-rate systems that satisfies the linearization property, and show that the proposed implementation satisfies similar stability properties to the ones considered for linear time *invariant systems [53].*

Gain-scheduling control is a tool per excellence in guidance, navigation and control problems. As an application, we cast the integrated guidance and control problem for an autonomous vehicles as a regulation problem, and using our results we are able to solve in a systematic manner the guidance and control problem for autonomous vehicles equipped with multi-rate sensor suite. The methodology is applied to the trajectory tracking problem of steering an autonomous rotorcraft along a pre-defined trajectory.

1.3 Organization of the Thesis

The organization of the thesis is dictated by the mentioned six networked control problems that we tackle. We devote a chapter to each. Networked control systems with a single node transmitting though a renewal network are addressed in Chapter 2, networked control systems modeled by finite state machines are addressed in Chapter 3, and networked control systems with asynchronous networks are addressed in Chapter 4. Chapter 5 is devoted to dynamic protocols, while Chapters 6 and 7 address the output regulation and gain-scheduling problems for multi-rate systems, respectively. Final conclusions and remarks, as well as future work are provided in Chapter 8.

1.4 Contributions and List of Publications

The publications that substantiate the present thesis are listed after the bibliography. We summarize next the already described contributions for stochastic, dynamic, and periodic networked control systems, and the corresponding publications where these contributions can be found.

- Stability conditions and determination of Lyapunov exponents for NCSs with a single renewal network [AHS11d], [AHS09b], [AHS09a].

- Transitory and asymptotic moment analysis for NCSs with network features modeled by finite state machines [AHS11b], [AHS10b].

- Stability conditions for NCSs with asynchronous renewal networks [AHS10a],[AHS11a].

- Propose a class of dynamic protocols that always outperform static ones [AHS11], [AHS11c], [AHHS11a], [AHHS11b].

- A solution to the output regulation problem for multi-rate systems [ASH11].
- A solution to the gain-scheduling problem for multi-rate systems [ASC07], [ASC08a], [ASC08b], [ASC10].

In the process of solving these networked control problems we found other results which are important per se. These results are listed next, where we also provide the articles and the section of the thesis where they can be found. More details on these results are given therein.

- Definition and stability analysis of Volterra Equations with Positive Kernel [AHS11d], Section 2.7.
- Stability results for piecewise deterministic processes [AHS10a], Section 4.3.1.
- Definition of blocking zero with respect to a matrix both for linear time-invariant systems and for periodically time-varying systems [ASH11], Section 6.4.

1.5 Basic Notation and Nomenclature

For a given matrix A, its transpose is denoted by $A^\intercal$, its hermitian by A^*, its trace by $\mathrm{tr}(A)$, its spectral radius by $r_\sigma(A)$, and an eigenvalue by $\lambda_i(A)$. We use $A > 0$ ($A \geq 0$) to denote that a real or complex symmetric matrix is positive definite (semi-definite). The $n \times n$ identity and zero matrices are denoted by I_n and 0_n, respectively, and the notation 1_n indicates a vector of n ones. The dimensional information is dropped whenever no confusion arises. For dimensionally compatible matrices A and B, we define $(A, B) := [A^\intercal\, B^\intercal]^\intercal$. We denote by $\mathrm{diag}([A_1 \dots A_n])$ a block diagonal matrix with blocks A_i. The Kronecker product is denoted by $\otimes$. The expected value is denoted by $\mathbb{E}(.)$. For a complex number z, $\Re[z]$ and $\Im[z]$ denote the real and complex parts of z, respectively. The notation $x(t_k^-)$ indicates the limit from the left of a function $x(t)$ at the point t_k. When we work in finite dimensional spaces, these are identified with $\mathbb{C}^n$, or $\mathbb{R}^n$, subsumed to be Hilbert spaces with the usual inner product $\langle x, y\rangle = y^* x$, Banach spaces with the usual norm $\|x\|^2 = \langle x, x\rangle$ and endowed with the usual topology inherited by the norm. We consider the usual vector identifications $\mathbb{C}^{n\times n} \cong \mathbb{C}^{n^2}$, $\mathbb{R}^{n\times n} \cong \mathbb{R}^{n^2}$ for matrices and this results in the following inner product $\langle A, B\rangle = \mathrm{tr}(B^* A)$. We denote the value at time $t \in \mathbb{R}_{\geq 0}$ of the continuous-time signals $x : \mathbb{R}_{\geq 0} \mapsto \mathbb{R}^n$ by $x(t)$, and the value at time $k \in \mathbb{N}$ of the discrete time signals $x : \mathbb{N} \mapsto \mathbb{R}^n$ by $x[k]$. We consider scalar *real* measures μ over $\mathbb{R}_{\geq 0}$, and we omit the ’over $\mathbb{R}_{\geq 0}$’, since these are the only scalar measures that we consider. We consider also matrix real measures Θ (each entry θ_{ij} is a scalar real measure), with the usual total variation norm $|\Theta|(E) := \sup \sum_{j=1}^{\infty} \|\Theta(E_j)\|$, where the supremum is taken over all countable partitions $\{E_j\}$ of a set E (cf. [39, Ch.3, Def.5.2], [80, p. 116]), and $\|\Theta(E_j)\|$ denotes the induced matrix norm by the usual vector norm. One can prove that $|\Theta|(E)$ is a positive measure (cf. [39, Ch.3,Th.5.3]). For a measurable vector function $b(s) \in \mathbb{R}^n$ and an interval $I \subseteq [0, \infty]$, the integral $\int_I \Theta(ds)b(s)$ is a vector function with components $\sum_{j=1}^{n} \int_I b_j(s)\theta_{ij}(ds)$. We say that $\int_I \Theta(ds)b(s)$ converges absolutely if $\int_I \|b(s)\||\Theta|(ds) < \infty$, and use the same nomenclature when we replace the real measure $\Theta(ds)$ by the positive Lebesgue measure ds. Further notation will be introduced when necessary.

2

Control Systems over a Renewal Network

In this chapter, we consider networked control systems with a single renewal network and for which only one sensor or actuator transmits through the shared network introducing independent and identically distributed intervals between transmissions. Network induced delays and packet drops are for now neglected and will be addressed in the next chapters.

We show that these networked control scenarios can be modeled by impulsive renewal systems, which are impulsive systems with independent and identically distributed intervals between transmissions. The nomenclature used to address these systems is motivated by the fact that the process that counts the number of transitions up to the current time is a renewal process [79]. We characterize the stability of impulsive renewal systems by providing necessary and sufficient conditions for mean square stability, stochastic stability and mean exponential stability. This result provides a unified treatment for these three stability notions and reveals that these are not equivalent in general. We prove that the stability conditions can be cast in terms of a matrix eigenvalue computation, the feasibility of a set of LMIs, and also tested using the Nyquist criterion. Furthermore we discuss how one can assert the performance of the system by computing a second moment Lyapunov exponent, which provides the asymptotic rate of decrease/growth for the expected value of a quadratic function of the systems' state. We provide a method to compute this Lyapunov exponent. The applicability of the results to networked control is illustrated by an example of a linearized model of an inverted pendulum.

Our results follow from a new approach to analyze impulsive renewal system based on a Volterra integral equation, describing the expected value of a quadratic function of the systems' state. For this specific class of Volterra equations we show that stability can be determined through a matrix eigenvalue computation or through a cone programming problem, and we provide a method to obtain the Lyapunov exponent of the Volterra equation. These two results can be used in problems unrelated to impulsive renewal systems. As an example, we show how they can be used to construct a simple stability condition for a class of LTI closed-loop systems with non-rational transfer functions.

The remainder of the chapter is organized as follows. Section 2.1 establishes the connection between networked control systems and impulsive renewal systems. Section 2.2 defines impulsive renewal systems and introduces appropriate stability notions. The main stability theorems are stated without a proof and discussed in Section 2.3. An example illustrating the applicability of these results is given in Section 2.4. Section 2.5 derives the Volterra integral equation describing a second moment of the impulsive systems' state, and establishes general results for Volterra integral equations with positive kernel, leading to the proof of the results of Section 2.6. Section 2.7 discusses LTI closed-loop systems with non-rational transfer functions. Some technical results are proved in the Section 2.8. Section 2.9 provides further comments and references.

2.1 Modeling Networked Control Systems with Impulsive Renewal Systems

Suppose that a linear plant and a state-feedback controller are connected by a communication network. The plant and the controller are described by:

$$\text{Plant:} \quad \dot{x}_P(t) = A_P x_P(t) + B_P \hat{u}(t), \tag{2.1}$$

$$\text{Controller:} \quad u(t) = K x_P(t) \tag{2.2}$$

where $\hat{u}$ is the input to the plant and u is the output of the controller. The controller has direct access to the state of the plant. However, a network connects the controller to a standard sample and hold actuator, which holds the actuation value between transmission times denoted by t_k, i.e.,

$$\hat{u}(t) = \hat{u}(t_k), t \in [t_k, t_{k+1}), \quad k \in \mathbb{Z}_{\geq 0},$$

where

$$\hat{u}(t_k) = u(t_k^-) = K x_P(t_k^-)$$

is the value sent from the controller to the plant at time t_k. Let

$$e(t) := (\hat{u}(t) - u(t)) \tag{2.3}$$

and note that $e(t)$ is reset to zero each time a transmission occurs, i.e.,

$$e(t_k) = 0. \tag{2.4}$$

From (2.1), (2.2), (2.3), and (2.4) we obtain

$$\begin{aligned} \begin{bmatrix} \dot{x}_P(t) \\ \dot{e}(t) \end{bmatrix} &= \begin{bmatrix} I \\ -K \end{bmatrix} \begin{bmatrix} A_P + B_P K & B_P K \end{bmatrix} \begin{bmatrix} x_P(t) \\ e(t) \end{bmatrix} \\ \begin{bmatrix} x_P(t_k) \\ e(t_k) \end{bmatrix} &= \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_P(t_k^-) \\ e(t_k^-) \end{bmatrix}, \end{aligned} \tag{2.5}$$

The time intervals between transmissions are assumed to be independent and identically distributed and described by a given probability measure characterizing the network. The model (2.5) is an impulsive renewal system, as defined in the next section. Typically one assumes that the controller has been designed to stabilize the closed-loop, when the process and the controller are directly connected, i.e., when $\hat{u}(t) = u(t)$, or in other words, one assumes that $A_P + B_P K$ is Hurwitz, and the objective is to analyze for which probability measures of the times between transmissions does the closed-loop remains stable. In [AHS09a], we proved that if this probability measure assigns high probability to fast sampling then the closed-loop remains stable, as intuition suggests.

Besides this scenario, other simple scenarios could be modeled by impulsive renewal systems. Namely, we could consider an output feedback controller and that only the sensors send their information through a shared network while continuous-time actuation is provided to the plant. The modeling framework will however be significantly augmented in the next chapters, and the scenarios just described are special cases of the modeling frameworks provided in Sections 3.1, and 4.1.

2.2 Definition of Impulsive Renewal Systems

An impulsive renewal system is defined by the following equations

$$\begin{aligned} \dot{x}(t) &= Ax(t), & t \neq t_k,\ \ t \geq 0,\ \ k \in \mathbb{Z}_{>0}, \\ x(t_k) &= Jx(t_k^-), & t_0 = 0,\ \ x(t_0) = x_0, \end{aligned} \tag{2.6}$$

where the state x evolves in $\mathbb{R}^n$ and the notation $x(t_k^-)$ indicates the limit from the left of a function $x(t)$ at the transition time t_k. The intervals between consecutive transition times $\{h_k := t_{k+1} - t_k, k \geq \ 0\}$ are assumed to be i.i.d.. The matrices A and J are real. The value at time t of a sample path of (2.6) is given by $x(t) = T(t)x_0$, where

$$T(t) = e^{A(t-t_r)} J e^{Ah_{r-1}} \dots J e^{Ah_0},\ r = \max\{k \in \mathbb{Z}_{\geq 0} : t_k \leq t\},$$

is the transition matrix.

The probability measure of the random variables h_k is denoted by μ. The support of μ may be unbounded but we assume that $\mu((0,\infty)) = 1$, $\mu(\{\infty\}) = 0$. We also assume that $\mu(\{0\}) = 0$. The measure μ can be decomposed into a continuous and a discrete component as in $\mu = \mu_c + \mu_d$, with $\mu_c([0,s)) = \int_0^s f(r)dr$, for some density function $f(r) \geq 0$, and μ_d is a discrete measure that captures possible point masses $\{b_i > 0, i \geq 1\}$ such that $\mu(\{b_i\}) = w_i$. The integral with respect to the measure μ is defined as

$$\int_0^t W(s)\mu(ds) = \int_0^t W(s)f(s)ds + \sum_{i:b_i \in [0,t]} w_i W(b_i). \tag{2.7}$$

In the present chapter we address two problems for the system (2.6). The first, is to obtain stability conditions for the following three stability notions, which are consistent with those appearing in the literature (e.g. [32, Def. 2.1]).

Definition 1. The system (2.6) is said to be

(i) *Mean Square Stable (MSS)* if for every x_0,

$$\lim_{t\to+\infty} \mathbb{E}[x(t)^\intercal x(t)] = 0,$$

(ii) *Stochastic Stable (SS)* if for every x_0,

$$\int_0^{+\infty} \mathbb{E}[x(t)^\intercal x(t)]dt < \infty,$$

(iii) *Mean Exponentially Stable (MES)* if there exists constants $c > 0$ and $\alpha > 0$ such that for every x_0,

$$\mathbb{E}[x(t)^\intercal x(t)] \leq ce^{-\alpha t} x_0^\intercal x_0, \forall_{t\geq 0}.$$

□

The second problem is to compute a second order Lyapunov exponent for (2.6). The following definition of second order Lyapunov exponent is adapted from [66, Ch.2].

Definition 2. Suppose that $\mathbb{E}[x(t)^\intercal x(t)] \neq 0, \forall t_{\geq 0}$ and that for every $x_0 \neq 0$ the following limit exists

$$\lambda_L(x_0) := \lim_{t\to\infty} \frac{1}{t} \log \mathbb{E}[x(t)^\intercal x(t)].$$

Then the second order Lyapunov exponent λ_L for the system (2.6) is defined as

$$\lambda_L := \sup_{x_0 \in \mathbb{R}^n} \lambda_L(x_0).$$

Moreover, if $\exists_{b>0} : \mathbb{E}[x(t)^\intercal x(t)] = 0, \forall t > b$ then $\lambda_L := -\infty$. □

The Lyapunov exponent provides a measure of how quickly the probability of $\|x(t)\|$ being large decays with time. One can see this, e.g., through the Chebyshev's inequality

$$\text{Prob}[\|x(t)\| > \epsilon] \leq \frac{\mathbb{E}[x(t)^\intercal x(t)]}{\epsilon^2}.$$

A small second moment Lyapunov exponent corresponds to a fast exponential decrease of $\mathbb{E}[x(t)^\intercal x(t)]$ and consequently to a fast decrease of the probability that $\|x(t)\|$ is larger than some positive constant ϵ.

2.3 Stability and Performance Analysis of Impulsive Renewal Systems

In this section we state the two main theorems of the chapter. The proofs are deferred to Section 2.6, and build upon the general results provided in Section 2.5 for Volterra Equations. The first main theorem characterizes the stability of (2.6) and to state it we need to introduce the complex function

$$\hat{\Theta}(z) := \int_0^\infty (Je^{As})^\intercal \otimes (Je^{As})^\intercal e^{-zs}\mu(ds),$$

which can be partitioned as in (2.7), $\hat{\Theta}(z) = \hat{\Theta}_c(z) + \hat{\Theta}_d(z)$, where

$$\hat{\Theta}_c(z) := \int_0^\infty (Je^{As})^\intercal \otimes (Je^{As})^\intercal e^{-zs} f(s)ds,$$
$$\hat{\Theta}_d(z) := \sum_{i=1}^\infty w_i (Je^{Ab_i})^\intercal \otimes (Je^{Ab_i})^\intercal e^{-zb_i}.$$

The following two technical conditions on the function $\hat{\Theta}$ will be needed:

(T1) $\hat{\Theta}(-\epsilon)$ converges absolutely for some $\epsilon > 0$.

(T2) $\inf_{z \in \mathcal{C}(R,\epsilon)}\{|\det(I - \hat{\Theta}_d(z))|\} > 0$ for some $\epsilon > 0$, and $R > 0$, where $\mathcal{C}(R,\epsilon) := \{z : |z| > R, \Re[z] > -\epsilon\}$.

These conditions hold trivially when μ has bounded support and no discrete component ($\mu_d([0,t]) = 0, \forall_{t \geq 0}$), but they also hold for much more general classes of probability measures. Let $r_\sigma(M)$ be the spectral radius of a $m \times m$ matrix M, i.e.,

$$r_\sigma(M) := \max\{|\lambda| : Mz = \lambda z, \text{ for some } z \in \mathbb{C}^m\}.$$

The following is our first main theorem. Each of the conditions appearing in its statement will be commented in the sequel.

Theorem 3. Suppose that (T1) and (T2) hold. Then, the following conditions are equivalent

$$
\begin{aligned}
&\text{(A)}\quad \det(I-\hat{\Theta}(z)) \neq 0,\ \ \Re[z] \geq 0,\\
&\text{(B)}\quad r_\sigma(M) < 1,\ \text{ where}\\
&\qquad\qquad M := \int_0^\infty (Je^{As})^\intercal \otimes (Je^{As})^\intercal \mu(ds),\\
&\text{(C)}\quad \exists_{P>0} : L(P) - P < 0,\ \text{ where}\\
&\qquad\qquad L(P) := \int_0^\infty (Je^{As})^\intercal P J e^{As} \mu(ds).
\end{aligned}
$$

Moreover, (2.6) is

(i) MSS if and only if (A), (B) and (C) hold and

$$e^{2\lambda_\Re(A)t} t^{2(m_\Re(A)-1)} r(t) \to 0 \text{ as } t \to \infty; \tag{2.8}$$

(ii) SS if and only if (A), (B) and (C) hold and

$$\int_0^\infty e^{2\lambda_\Re(A)t} t^{2(m_\Re(A)-1)} r(t) dt < \infty; \tag{2.9}$$

(iii) MES if and only if (A), (B) and (C) hold and

$$e^{2\lambda_\Re(A)t} t^{2(m_\Re(A)-1)} r(t) \leq c e^{-\alpha_1 t} \text{ for some } c>0,\ \alpha_1>0, \tag{2.10}$$

where $r(t) := \mu((t,\infty])$ denotes the survivor function, $\lambda_\Re(A)$ denotes the real part of the eigenvalue of A with largest real part and $m_\Re(A)$ the dimension of the largest Jordan block associated with this eigenvalue. □

Proof. See Section 2.6. □

The second main result provides a method to compute the Lyapunov exponent λ_L for (2.6). For simplicity, we restrict ourselves to the case where μ has bounded support. We recall that if $\hat{\Theta}(b)$ converges absolutely for a real b then $\hat{\Theta}(a)$ converges absolutely for every $a > b$ (cf. [39, Ch. 3, Th.8.2]).

Theorem 4. Suppose that μ has bounded support, and let

$$b := \inf\{a : \hat{\Theta}(a) \text{ converges absolutely}\}.$$

Then, the spectral radius $r_\sigma(\hat{\Theta}(a))$ of $\hat{\Theta}(a)$ is a non-increasing function of a for $a > b$ and the second-order Lyapunov exponent for (2.6) is given by

$$\lambda_L = \begin{cases} a \in \mathbb{R} : r_\sigma(\hat{\Theta}(a)) = 1, \text{ if such } a \text{ exists} \\ -\infty \text{ otherwise} \end{cases} \tag{2.11}$$

□

Proof. See Section 2.6. □

Note that, since $r_\sigma(\hat{\Theta}(a))$ is non-increasing, one can compute λ_L by performing a simple binary search on $\mathbb{R}$.

We comment next on each condition of Theorem 3.

Condition (A)

When $\det(I - \hat{\Theta}(i\omega)) \neq 0$, $\forall\, \omega \in \mathbb{R}$, the Nyquist criterion can be used to check if (A) holds, i.e., the number of zeros of $\det(I - \hat{\Theta}(z))$ in the closed-right half complex plane counted according to their multiplicities equals the number of times that the curve $\det(I - \hat{\Theta}(i\omega))$ circles anticlockwise around the origin as ω goes from ∞ to $-\infty$.

Eigenvalue Condition (B)

Using the properties

$$(AB) \otimes (CD) = (A \otimes B)(C \otimes D), \tag{2.12}$$

and $(A \otimes B)^\intercal = A^\intercal \otimes B^\intercal$ (cf. [46]), and considering the Jordan normal form decomposition of $A = VDV^{-1}$ we obtain

$$M^\intercal = (JV) \otimes (JV) \int_0^\infty e^{Ds} \otimes e^{Ds} \mu(ds)(V^{-1} \otimes V^{-1}).$$

Thus M can typically be obtained by integrating exponentials with respect to the measure μ.

LMI Condition (C)

By choosing a basis C_i for the linear space of symmetric matrices, we can write $P = \sum_{i=1}^m c_i C_i$, $m \coloneqq \frac{n \times (n+1)}{2}$, and express (C) it terms of the LMIs:

$$\exists_{\{c_i, i=1,\dots,m\}} : \sum_{i=1}^m c_i C_i > 0, \quad \sum_{i=1}^m c_i (L(C_i) - C_i) < 0. \tag{2.13}$$

Let ν denote the operator that transforms a matrix into a column vector, i.e.,

$$\nu(A) = \nu([a_1 \, \dots \, a_n]) = [a_1^\intercal \, \dots \, a_n^\intercal]^\intercal.$$

Recalling that

$$\nu(ABC) = (C^\intercal \otimes A)\nu(B) \tag{2.14}$$

(cf. [46]), we conclude that

$$\nu(L(C_i)) = M\nu(C_i). \tag{2.15}$$

This shows that the integrals that appear in (2.13) can be obtained by computing the matrix M, which can be done efficiently, as noted in the discussion regarding the condition (B).

A generalized version of [21, Th.3.9], can be used to prove that (C) is the stability condition for the discrete-time stochastic process $v_k := x(t_k)$ obtained by sampling (2.6) at jump times, which is described by $v_{k+1} = Je^{Ah_k}v_k$. Related stability condition are also available for randomly sampled systems [40], [55]. However, Theorem 3 shows that, although being closely related, the stability of (2.6) is not equivalent to the stability of the process v_k, when μ does not have a bounded support. Moreover, even when μ has bounded support, Theorem 3 provides novel alternative stability conditions.

Conditions (i), (ii) and (iii)

These conditions pertain to the inter-jump behavior of (2.6). To gain intuition on why (A) could not suffice for stability, and on why there may exist systems for which the stability notions differ, suppose that $A = a \in \mathbb{R}_{>0}$, $J = 0$ in (2.6) (in which case (A), (T1), and (T2) hold trivially). For this system, $x(t) = 0$ if one or more transitions have occurred up to time t, and $x(t) = e^{at}x_0$ if no jump has occurred up to time t. This latter event is equivalent to the first interval between transitions being greater than t, and has probability $\text{Prob}[h_0 > t] = r(t)$. Thus, $\mathbb{E}[x(t)^\intercal x(t)] = e^{2at}x_0^2 r(t)$ and (2.6) is:

(i) MSS but not SS nor MES if $r(t) = e^{-2at}\frac{1}{1+t}$,

(ii) MSS and SS but not MES if $r(t) = e^{-2at}\frac{1}{1+t^2}$.

Conditions (T1) and (T2)

Note that (T1) holds if μ has bounded support or if A is Hurwitz, or, more generally, if for some $\lambda > \lambda_{\Re}(A)$, we have that $\int_0^\infty e^{2\lambda s}\mu(ds) < \infty$. Moreover, whenever (T1) holds, the following proposition provides a simple condition to verify if (T2) holds. The proof can be found in Section 2.8.

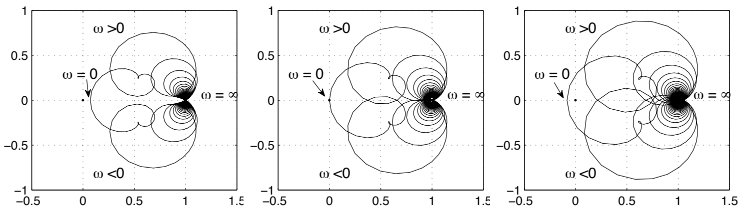
Proposition 5. Assuming that (T1) holds, the condition (T2) also holds provided that

$$r_\sigma(\hat{\Theta}_d(0)) = r_\sigma\Big(\sum_{i=1}^{\infty} w_i (Je^{Ab_i})^\intercal \otimes (Je^{Ab_i})^\intercal\Big) < 1.$$

2.4 Example-Inverted Pendulum

In this section we illustrate the applicability of the main results in the previous section through a simple example. Suppose that the plant (2.1) is described by

$$A_P = \begin{bmatrix} 0 & 1 \\ \frac{1}{2} & 0 \end{bmatrix}, \quad B_P = \begin{bmatrix} 0 \\ 1 \end{bmatrix},$$



(a) $\tau = 1.35(MSS)$ (b) $\tau = 1.391$ (c) $\tau = 1.43$ (NOT MSS)

Figure 2.1: Nyquist plot illustration of stability for the impulsive renewal pendulum system.

which by properly scaling the state and input can be viewed as a linearized model of a damp-free inverted pendulum. The controller is synthesized assuming $\hat{u}(t) = u(t)$, $\hat{y}(t) = y(t)$ and it is chosen to be an LQR controller given by $u(t) = K_{\text{opt}}\hat{y}(t)$, $K_{\text{opt}} = \begin{bmatrix} -1.618 - 2.058 \end{bmatrix}$ which is the solution to the minimization of $\int_0^\infty [x_P(t)^\intercal x_P(t) + u(t)^2]dt$.

Suppose that the intervals between transmission $t_{k+1} - t_k$ are uniformly distributed on $[0, \tau]$. Then, we wish to: (i) determine for which values of the maximum sampling time τ the system (2.5) is stable (MSS sense); (ii) assess the performance of the networked control system by computing the second-moment Lyapunov exponent of (2.5). The equations for $(x_P(t), e_u)$ take the form (2.5). We can test for which values of τ the system is MSS (or equivalently SS and MES since τ is finite, cf. Theorem 3), using the algebraic condition (B) or the LMI condition (C) of the Theorem 3. Performing a binary search we conclude that the maximum value of τ for which the system is MSS is $\tau_c = 1.391$. Using the condition (A) of the Theorem 3 and the Nyquist Theorem, we can illustrate this graphically in Fig. 2.1 by drawing the Nyquist plot of $\det(I - \hat{\Theta}(z))$. In Figure 2.1 we see that for a value $\tau = 1.35$ less than τ_c the number of anti-clockwise encirclements of the curve $\det(I - \hat{\Theta}(j\omega))$, when ω varies from ∞ to $-\infty$ is zero, while for a value $\tau = 1.43$ greater than τ_c, the number of encirclements is one, which by the Nyquist Theorem implies that the stability condition (A) does not hold.

To assess the closed-loop performance we consider

$$\mathbb{E}[x_P(t)^\intercal x_P(t) + \hat{u}(t)^2 dt], \tag{2.16}$$

which is the function whose integral is minimized when designing the controller in the absence of a network. Notice that, since $\hat{u}(t) = e(t) + K_{\text{opt}}x_P(t)$, we can re-write (2.16) as $\mathbb{E}[[x_P(t)^\intercal \ \ e(t)]U^\intercal U[x_P(t)^\intercal \ \ e(t)]^\intercal dt]]$, with $U = \begin{bmatrix} I & 0 \\ K_{\text{opt}} & 1 \end{bmatrix}$. Using the Theorem 4 we can compute the Lyapunov exponent of this function for different values of τ. The results are summarized in Table 2.1, quantifying the performance degradation as the distribution of times between consecutive transmissions assigns high probability to slow sampling.

Table 2.1: Variation of the Lyapunov exponent of (2.16) with the support τ of for the inter-sampling distribution

τ	0.1	0.3	0.6	0.9	1.2
Lyap. exponent	-2.152	-2.111	-1.904	-1.836	-0.709

2.5 Volterra Integral Equations with Positive Kernel

In this section we establish general results for Volterra Equations, which lie at the heart of the proofs of the main results in Section 2.3, which are established in Section 2.6. In Section 2.7, we discuss additional applications of these general results.

We start by showing that for the system (2.6),

$$\mathbb{E}[x(t)^\intercal x(t)], \tag{2.17}$$

can be obtained as the solution to a Volterra equation of the form (2.21). which motivates the study of Volterra Equations. Using (2.12) and (2.14), we can write (2.17) as

$$\mathbb{E}[x(t)^\intercal I x(t)] = \mathbb{E}[(x(t)^\intercal \otimes x(t)^\intercal)\nu(I)] = (x_0^\intercal \otimes x_0^\intercal)w(t)$$

where

$$w(t) := \mathbb{E}[(T(t)^\intercal \otimes T(t)^\intercal)\nu(I)], \tag{2.18}$$

and I is the identity matrix. The next result shows that $w(t)$ satisfies a Volterra equation.

Proposition 6. The function $w(t)$ satisfies

$$w(t) = \int_0^t \Theta(ds)w(t-s) + h(t),\, t \geq 0 \tag{2.19}$$

where $h(t) := e^{A^\intercal t} \otimes e^{A^\intercal t}\nu(I)r(t)$ and

$$\Theta([0,t]) := \int_0^t (Je^{As})^\intercal \otimes (Je^{As})^\intercal \mu(ds). \tag{2.20}$$

□

Proof. See Section 2.8. □

The kernel Θ has a special property which we term of being positive with respect to a cone, as defined below. Since our results hold for general Volterra equations with positive kernel we consider throughout this chapter a general Volterra equation taking the general form

$$y(t) = \int_0^t \Psi(ds)y(t-s) + g(t), \quad t \geq 0 \tag{2.21}$$

where $g : \mathbb{R}_{\geq 0} \mapsto \mathbb{R}^m$ and the *kernel* Ψ is a finite real measure on $\mathbb{R}_{\geq 0}$ (i.e., $\int_0^\infty \Psi(ds)$ converges absolutely). We assume that Ψ has no singular part with respect to the Lebesgue measure other than a possible set of point masses $\{c_i, i \geq 1\}$, which means that we can decompose the integral of a measurable function $a(t)$ with respect to Ψ into an absolutely continuous part and a discrete part, i.e.,

$$\int_0^t \Psi(ds)a(s) = \int_0^t K_c(s)a(s)ds + \sum_{i:c_i \in [0,t]} K_{di}a(c_i), \tag{2.22}$$

where $K_c(s)$ is a real matrix valued measurable function that only needs to be defined a.e., i.e., up to a set of zero Lebesgue measure, and $\{K_{di}, i \geq 1\}$ are real matrices (cf. [39, p.79]). We assume there are no point masses at zero, i.e., $c_i > 0$.

2.5.1 Cones and Positive Linear Maps

Following [56, p. 6], a *cone* in $\mathcal{K} \subset \mathbb{R}^m$ is a closed convex set such that if $x, y \in \mathcal{K}$ then $\alpha_1 x + \alpha_2 y \in \mathcal{K}$, for $\alpha_1 \geq 0$, $\alpha_2 \geq 0$ and such that the set $-\mathcal{K} := \{-x : x \in \mathcal{K}\}$ intersects $\mathcal{K}$ only at the zero vector. A cone is said to be *solid* if the cone contains a ball of $\mathbb{R}^m$, which in finite-dimensional spaces is equivalent to being *reproducing*, i.e., any element $v \in \mathbb{R}^m$ can be written as $v = v_1 - v_2$ where $v_1, v_2 \in \mathcal{K}$ (cf. [56, p. 10]). A linear operator $A : \mathbb{R}^m \mapsto \mathbb{R}^m$ is said to be *positive* with respect to (w.r.t.) a cone $\mathcal{K}$ if it maps elements in $\mathcal{K}$ into elements in $\mathcal{K}$. For a complex vector $u = u_R + iu_I \in \mathbb{C}^m$, $u_R, u_I \in \mathbb{R}^m$, define $Au = Au_R + iAu_I$. Let $r_\sigma(A)$ be the spectral radius of A, i.e.,

$$r_\sigma(A) := \max\{|\lambda| : Au = \lambda u, \text{ for some } u \in \mathbb{C}^m\}. \tag{2.23}$$

If $A : \mathbb{R}^m \mapsto \mathbb{R}^m$ is positive w.r.t. the solid cone $\mathcal{K}$ then

$$\exists_{x \in \mathcal{K}} : Ax = r_\sigma(A)x \tag{2.24}$$

(cf. [56, Ch.8,9]). The following are examples of cones which will be useful in the sequel.

Examples 7.

1. The positive orthant $\mathbb{R}^m_{\geq 0}$ consisting of all vectors with non-negative components is a solid cone in $\mathbb{R}^m$;
2. The set $\mathcal{S}^k_{\geq 0}(\mathbb{R})$ of $k \times k$ real positive semi-definite matrices is a solid (and therefore reproducing) cone when viewed as a subset of the linear space of symmetric $k \times k$ matrices with dimension $m := k(k+1)/2$, denoted by $\mathcal{S}^k(\mathbb{R})$. Note however, that it is not a solid cone if we view it as a subset of the linear space of $k \times k$ matrices with dimension $m := k^2$. To verify that this is the case, note that any symmetric matrix can be written as the difference between two positive semidefinite matrices (which confirms the reproducing property), but a non-symmetric matrix cannot be decomposed in this way.

□

We can extend the above definition of positivity to the kernel of Volterra equations by defining Ψ in (2.22) to be *positive w.r.t. the cone* $\mathcal{K}$ if $K_c(s)$ is positive w.r.t. to $\mathcal{K}$ for s a.e. in $[0,\infty)$ and K_{di} is positive for every $i \geq 1$. Since the kernel Ψ is not affected by the values of $K_c(s)$ on sets of measure zero, we can arbitrate hereafter that $K_c(s)$ is positive w.r.t. K for every value of $s \in [0,\infty)$.

We introduce the Laplace transform of Ψ,

$$\hat{\Psi}(z) := \int_0^\infty e^{-zs}\Psi(ds), \tag{2.25}$$

which can be partitioned into $\hat{\Psi}(z) = \hat{\Psi}_d(z) + \hat{\Psi}_c(z)$ where

$$\hat{\Psi}_c(z) := \int_0^\infty K_c(s)e^{-sz}ds, \;\; \hat{\Psi}_d(z) := \sum_{i\geq 1} K_{di}e^{-c_i z}. \tag{2.26}$$

We state next that for a real a, $\hat{\Psi}(a)$ is a positive operator. The proof can be found in Section 2.8.

Proposition 8. If Ψ is positive w.r.t. a cone $\mathcal{K}$ then, for a given real a such that $\hat{\Psi}(a)$ converges absolutely, $\hat{\Psi}(a)$ is a positive operator w.r.t. $\mathcal{K}$. □

2.5.2 Main results for Volterra Integral Equations

We establish in this subsection two theorems that characterize the stability of the Volterra equation (2.21) when Ψ is positive w.r.t. a solid cone.

We recall that the stability condition for a Volterra equation (see [39, p.195] for the definition) taking the form (2.21) is $\det(I - \hat{\Psi}(z)) \neq 0, \;\; \Re[z] \geq 0$. The first of the two theorems in the present section provides computationally efficient alternatives to this stability condition when the kernel of (2.21) is positive. We denote by $\text{int}(B)$ the interior of a set $B \in \mathbb{R}^m$.

Theorem 9. Suppose that Ψ is positive w.r.t. a solid cone $\mathcal{K}$. Then, the following conditions are equivalent

$$\begin{aligned}
&\text{(A)} \;\; \det(I - \hat{\Psi}(z)) \neq 0, \;\; \Re[z] \geq 0,\\
&\text{(B)} \;\; r_\sigma(N) < 1, \;\text{where } N := \int_0^\infty \Psi(ds),\\
&\text{(C)} \;\; \exists_{x\in \text{int}(\mathcal{K})} : x - Nx \in \text{int}(\mathcal{K}) \quad \square
\end{aligned}$$

Note that (A) can be tested by using the Nyquist criterion, and (B) by computing the eigenvalues of N. The condition (C) is a cone optimization problem for which numerical efficient algorithms can be found in [74].

As a preliminary to prove Theorem 9 we state two lemmas both proved in Section 2.8. The first lemma establishes a monotone property for the spectral radius of the Laplace transform of Ψ. We recall that if $\hat{\Psi}(a)$ converges absolutely for some real a then $\hat{\Psi}(z)$ converges absolutely in the set $\{z \in \mathbb{C} : \Re[z] \geq a\}$(cf. [39, Ch.3,Th.8.2]).

Lemma 10. Suppose that Ψ is positive w.r.t. a solid cone $\mathcal{K}$, and that $\hat{\Psi}(a)$ converges absolutely for some $a \in \mathbb{R}$. Then for any $z : \Re[z] \geq a$, the following holds $r_\sigma(\hat{\Psi}(z)) \leq r_\sigma(\hat{\Psi}(a))$. □

The second Lemma characterizes the geometric placement of the zeros of $\det(I - \hat{\Psi}(z))$.

Lemma 11. Suppose that Ψ is positive w.r.t. a solid cone $\mathcal{K}$, and that $\hat{\Psi}(a)$ converges absolutely for some $a \in \mathbb{R}$. Then, if $r_\sigma(\hat{\Psi}(a)) \geq 1$, there exists $a_1 \geq a$ with the following properties:

(i) a_1 is the unique real number such that $r_\sigma(\hat{\Psi}(a_1)) = 1$,

(ii) $\det(I - \hat{\Psi}(a_1)) = 0$,

(iii) if $z : \det(I - \hat{\Psi}(z)) = 0$ then $\Re[z] \leq a_1$. □

Proof. (of Theorem 9)
(B)⇒(A) Note that $N = \hat{\Psi}(0)$. From Lemma 10, $r_\sigma(\hat{\Psi}(z)) \leq r_\sigma(\hat{\Psi}(0)) = r_\sigma(N)$ for every $z : \Re[z] \geq 0$. This implies that if (B) holds, all the eigenvalues of $\hat{\Psi}(z)$ have a modulus strictly less than one, and therefore $\det(I - \hat{\Psi}(z))$ cannot be zero in $\Re[z] \geq 0$, which implies (A).
(A)⇒(B) If $r_\sigma(N) = r_\sigma(\hat{\Psi}(0)) \geq 1$, from Lemma 11 we have that there exists some $a_1 \geq 0$ such that $\det(I - \hat{\Psi}(a_1)) = 0$, and therefore condition (A) does not hold at $z = a_1$.
(B)⇒(C) If $r_\sigma(N) < 1$ the system of equations

$$Nx - x = -v$$

has a unique solution given by

$$x = \sum_{k=0}^{+\infty} N^k(v),$$

which satisfies $x \in \mathcal{K}$ if $v \in \mathcal{K}$ due to the fact that N is a positive operator (since $N = \hat{\Psi}(0)$ and due to Proposition 8). If $v \in \text{int}(\mathcal{K})$ then since $x = v + w$, where

$$w = \sum_{k=1}^{+\infty} N^k(y) \in \mathcal{K}$$

and $v \in \text{int}(\mathcal{K})$, we have that $x \in \text{int}(\mathcal{K})$. In fact, since $v \in \text{int}(\mathcal{K})$ there exists an ϵ such that

$$B_\epsilon := \{z : \|z - v\| < \epsilon\} \subseteq \mathcal{K}.$$

Thus the set $\{w+z, z \in B_\epsilon\}$ is a ball of radius ϵ centered at x which belongs to $\mathcal{K}$ since w and z belong to $\mathcal{K}$, and thus $x \in \text{int}(\mathcal{K})$.
(C)⇒(B) If (C) holds, then $x - Nx = v$ for some $v \in \text{int}(\mathcal{K})$, from which we conclude that, for each $k \geq 1$,

$$x - \sum_{j=0}^{k-1} N^j v = N^k x \in \mathcal{K}.$$

This implies that $\| \sum_{j=0}^{k} N^j v\| \leq c\|x\|$ (cf [56, p.37]). Letting $k \to \infty$ we conclude that $\sum_{j=0}^{\infty} N^j v$ converges which implies that it converges absolutely, a general fact in Banach spaces (as it is the case since $\mathcal{K} \in \mathbb{R}^m$), and this implies that $N^k v \to 0$ as $k \to \infty$. Given any $w \in \mathcal{K}$, there exists $\alpha > 0$ such that $v - \alpha w \in \mathcal{K}$ because $v \in \text{int}(\mathcal{K})$. Then

$$N^k v - \alpha N^k w = N^k(v - \alpha w) \in \mathcal{K},$$

and hence $N^k w \to 0$ as $k \to \infty$ because $N^k v \to 0$ as $k \to \infty$ and $\|\alpha N^k w\| \leq c\|N^k v\|$. Finally, since $\mathcal{K}$ is reproducing, for every $u \in \mathbb{R}^m$, there exists $u_1, u_2 \in \mathcal{K}$ such that $u = u_1 - u_2$, and therefore $N^k u = N^k u_1 - N^k u_2 \to 0$ as $k \to \infty$, which implies that $r_\sigma(N) < 1$. □

The *Lyapunov exponent* of the Volterra equation (2.21) is defined as follows.

Definition 12. Suppose that the solution to (2.21) satisfies $\|y(t)\| \neq 0, \forall t_{\geq 0}$. Then the second order Lyapunov exponent for (2.21) is defined as

$$\lambda_V := \lim_{t\to\infty} \frac{1}{t} \log \|y(t)\| \tag{2.27}$$

when such limit exists. Moreover, if $\exists_{b>0} \colon \|y(t)\| = 0, \forall t > b$ then $\lambda_V := -\infty$. □

The next Theorem provides a method to determine the Lyapunov exponent λ_V for (2.21). Recall that a singularity z_i of a complex function is said to be removable if the function is bounded in a neighborhood of z_i. Consider the following assumptions.

(V1) $\hat{\Psi}(-\epsilon)$ converges absolutely for some $\epsilon > 0$.

(V2) $\inf_{z\in\mathcal{C}(R,\epsilon)}\{|\det(I - \hat{\Psi}_d(z))|\} > 0$ for some $\epsilon > 0$, and $R > 0$, where $\mathcal{C}(R,\epsilon) := \{z : |z| > R, \Re[z] > -\epsilon\}$.

(V3) g is bounded and $g(t) = 0, t \in [b, \infty)$ for some $b > 0$.

The same arguments utilized in the proof of Proposition 5 can be used to prove that $r_\sigma(\hat{\Psi}_d(0)) < 1$ implies (V2), provided that (V1) holds.

Theorem 13. Suppose that Ψ is positive w.r.t. a solid cone $\mathcal{K}$ and suppose that (V1), (V2) and (V3) hold, $\hat{\Psi}(a)$ converges absolutely for some real a and $r_\sigma(\hat{\Psi}(a)) \geq 1$. Then there is a unique

$$\lambda > a : r_\sigma(\hat{\Psi}(\lambda)) = 1 \tag{2.28}$$

which is a singularity of $[I - \hat{\Psi}(z)]^{-1}\hat{g}(z)$, and, if it is not removable, the Lyapunov exponent λ_V is given by $\lambda_V = \lambda$. Moreover, if there does not exists a real a such that $r_\sigma(\hat{\Psi}(a)) \geq 1$, then $\lambda_V = -\infty$. □

Note that $r_\sigma(\hat{\Psi}(a))$ is a non-increasing function of a (cf. Lemma 10), and therefore (2.28) can be easily computed by performing a binary search.

To prove the Theorem 13 we need to derive two results. The first states how a special perturbation of $g(t)$ and Ψ affects the solution of (2.21). We introduce the following Laplace transforms

$$\hat{y}(z) := \int_0^\infty y(s)e^{-zs}ds, \qquad \hat{g}(z) := \int_0^\infty g(s)e^{-zs}ds. \tag{2.29}$$

The following proposition can be obtained by direct substitution.

Proposition 14. Let $g_\delta(t) := g(t)e^{\delta t}$ and Ψ_δ be a measure such that,

$$\int_0^t \Psi_\delta(ds)a(s) = \int_0^t K_c(s)e^{\delta s}a(s)ds + \sum_{i:c_i\in[0,t]} K_{di}e^{\delta c_i}a(c_i), \tag{2.30}$$

for a measurable function $a(t)$, where K_c and K_{di} are specified by (2.22). Then the solution to

$$y_\delta(t) = \int_0^t \Psi_\delta(ds)y_\delta(t-s) + g_\delta(t), t \geq 0 \tag{2.31}$$

satisfies $y_\delta(t) = y(t)e^{\delta t}$. Moreover

$$\hat{y}_\delta(z) = \hat{y}(z-\delta),\ \hat{g}_\delta(z) = \hat{g}(z-\delta),\ \hat{\Psi}_\delta(z) = \hat{\Psi}(z-\delta), \tag{2.32}$$

where $\hat{y}_\delta(z)$, $\hat{g}_\delta(z)$, and $\hat{\Psi}_\delta(z)$ are the Laplace transforms of w_δ and g_δ and Ψ_δ defined as in (2.25) and (2.29). □

The second is an instability result that takes into account the critical case, i.e, it allows the characteristic equation $(\det(I - \hat{\Psi}(z)) = 0)$ to have zeros on the imaginary axis. It is important to emphasize that this result avoids the hard-to-test assumptions typically used to handle the critical case (e.g. [39, Ch.7,Th.3.7]).

Lemma 15. Consider a general Volterra equation taking the form (2.21) and suppose that (V1), (V2), (V3) hold. Then

(i) there exists at most a finite number of $z_i \in \mathbb{C}$ such that $\Re[z_i] \geq 0$ and $\det(I - \hat{\Psi}(z_i)) = 0$.

(ii) The solution to (2.21) is given by

$$y(t) = w_s(t) + \sum_{i=1}^{n_z} \sum_{j=0}^{m_i-1} s_{i,j} t^j e^{z_i t}, \quad t \geq 0, \tag{2.33}$$

and its Laplace transform is given by $\hat{y}(z) = [I - \hat{\Psi}(z)]^{-1}\hat{g}(z)$, where $w_s(t)$ is a bounded function that converges to zero and $s_{i,j}$ are such that in a neighborhood of z_i,

$$\hat{y}(z) = u_i(z) + \sum_{j=0}^{m_i-1} s_{i,j} \frac{j!}{(z - z_i)^{j+1}}, \tag{2.34}$$

where $u_i(z)$ are analytic vector functions. □

The proof of Lemma 15 is given in Section 2.8. Note that the second term on the right hand side of (2.34) is the principal part of $\hat{y}(z)$ at z_i and $s_{i,j}$ can be uniquely determined by the characterization (2.34) (see [80, Ch.10, Th.10.21]).

Proof. (of Theorem 13)

If there exists a real number a such that $r_\sigma(\hat{\Psi}(a)) \geq 1$, then there exists a unique $\lambda \in \mathbb{R}$ such that $r_\sigma(\hat{\Psi}(\lambda)) = 1$, which satisfies $\det(I - \hat{\Psi}(\lambda)) = 0$ (cf. Lemma 11), and therefore λ is a singularity of $[I - \hat{\Psi}(z)]^{-1}\hat{g}(z)$. Consider a perturbation to the Volterra equation (2.21) as in Proposition (14) with $\delta = -\lambda$. Since λ is the zero of $\det(I - \hat{\Psi}(z))$ with largest real part (cf. Lemma 10), we have that $z_{1\delta} = 0$ is the zero of $\det(I - \hat{\Psi}_\delta(z)) = \det(I - \hat{\Psi}(z - \delta))$ with largest real part, where we used (2.32). Also, due to (2.32), the principal parts of $\hat{y}(z)$ about z_i and of $\hat{y}_\delta(z)$ about $z_i + \delta$ have the same coefficients $s_{i,j}$. Thus, from Lemma 15, the solution to the Volterra equation (2.31) becomes $y_\delta(t) = v_\delta(t) + u_\delta(t) + w_\delta(t)$ where $v_\delta(t) = \sum_{j=0}^{m_i-1} s_{1,j} t^j$ corresponds to the zero of $\det(I - \hat{\Phi}(z))$ labeled $z_{1\delta} = 0$, u_δ is a sum of exponent and polynomial functions such that $u_\delta(t)e^{-ct} \to 0$, as $t \to \infty$ for every $c > 0$ and $w_\delta(t)$ is a bounded function that converges to zero. Note that u_δ may not tend to zero since there may exist zeros of $\det(I - \hat{\Phi}_\delta(z))$ on the imaginary axis other than $z_{1\delta}$. Since the singularity $z_1 = \lambda$ is not removable, $\exists_j : s_{1,j} \neq 0$, we have that $y(t) = y_\delta(t)e^{-\delta t}$ is such that

$$\lambda_V = \lim_{t\to\infty} \frac{\log(\|y(t)\|)}{t} = \lim_{t\to\infty} \frac{\log(\|y_\delta(t)\|)}{t} - \delta = 0 - \delta = \lambda.$$

If there does not exists a real number a such that $r_\sigma(\hat{\Psi}(a)) \geq 1$, then using Lemma 10, we have that $r_\sigma(\hat{\Psi}(z)) < 1$ for every complex z and therefore $\det(I - \hat{\Phi}(z)) \neq 0$ for all z. Using the arguments above we can choose δ arbitrarily large, and prove that $y(t) = y_\delta(t)e^{-\delta t}$ where $y_\delta(t)$ is a bounded function that converges to zero. This implies that $\Lambda_V = -\infty$. Note that, this latter case encompasses the case where, e.g., $\Psi([0, t]) = 0, \forall_{t \geq 0}$ and $y(t) = g(t)$ is such that $y(t) = 0, t > b$ for some $b > 0$. □

2.6 Proofs of the Main Results

As a preliminary, we rewrite (2.19) in two forms. Applying ν^{-1} to both sides of (2.19), and using (2.14), we obtain

$$W(t) = K(W)(t) + H(t), \quad t \geq 0 \tag{2.35}$$

where $W(t) := \nu^{-1}(w(t))$, $H(t){:=}e^{A^\intercal t}Xe^{At}r(t)$ and

$$K(W)(t) := \int_0^t (Je^{As})^\intercal W(t-s)Je^{As}\mu(ds).$$

Let $K^j(W)(t)$ denote the composition operator obtained by applying j times K. The unique solution to (2.35) is given by

$$W(t) = \sum_{j=1}^{\infty} K^j(H)(t) + H(t), \quad t \geq 0, \tag{2.36}$$

(cf. [39, Ch.4, Th.1.7]). Note that $H(t)$ and $K(H)(t)$ are symmetric and therefore $W(t)$ is symmetric. Let $\{C_i,\, 1 \leq i \leq m\}$, with $m := \frac{n(n+1)}{2}$, be an orthogonal basis for the space of symmetric matrices $\mathcal{S}^n(\mathbb{R})$ and let $\Pi(x) : \mathbb{R}^m \mapsto \mathcal{S}^n(\mathbb{R})$ be the invertible map defined by

$$\Pi(x) = \sum_{i=1}^{m} x_i C_i.$$

Then (2.35) (or equivalently (2.19)) can be written in terms of $u(t) := \Pi^{-1}(W(t))$ as

$$u(t) = \int_0^t \Phi(ds)u(t-s) + e(t) \tag{2.37}$$

where $e(t) = \Pi^{-1}(H(t))$ and $\Phi(ds)$ is such that

$$\int_0^t \Phi(ds)a(s) = \int_0^t F_c(s)a(s)ds + \sum_{i:c_i\in[0,t]} F_{di}a(c_i)$$

where $a(s)$ is a measurable function and

$$\begin{aligned} F_c(s) &= \Pi^{-1}\nu^{-1}(Je^{As})^\intercal \otimes (Je^{As})^\intercal f(s)\nu\Pi, \\ F_{di} &= \Pi^{-1}\nu^{-1}w_i(Je^{Ab_i})^\intercal \otimes (Je^{Ab_i})^\intercal \nu\Pi. \end{aligned} \tag{2.38}$$

The kernel Φ is positive w.r.t. the cone in $\mathbb{R}^m$ corresponding to $\mathcal{S}^k_{\geq 0}(\mathbb{R})$, i.e., $\mathcal{K}^S := \{v \in \mathbb{R}^m : \sum_{i=1}^m v_i C_i > 0\}$. In fact, if $x \in \mathcal{K}^S$, i.e., $X = \Pi(x) \geq 0$ then using (2.38) and (2.15)

$$F_c(s)x = \Pi^{-1}(Y) \in \mathcal{K}^S, \text{ where } Y = (Je^{As})^\intercal XJe^{As}f(s).$$

Similarly one can prove that all the F_{di} are positive operators.

Let $\hat{\Phi}(z) := \int_0^\infty e^{-zs}\Phi(ds)$. Since Φ is a positive kernel w.r.t. the solid cone $\mathcal{K}^S$ (cf. Example 7.2), from Theorem 9 we obtain that the following are equivalent:

$$
\begin{aligned}
&\text{(A')}\ \det(I - \hat{\Phi}(z)) \neq 0,\ \ \Re[z] \geq 0,\\
&\text{(B')}\ r_\sigma(\hat{\Phi}(0)) < 1,\\
&\text{(C')}\ \exists_{x\in \mathrm{int}(\mathcal{K}^S)} : x - \hat{\Phi}(0)x \in \mathrm{int}(\mathcal{K}^S).
\end{aligned}
$$

We shall also need the following facts: Using (2.38), we have

$$\hat{\Phi}(z) = \Pi^{-1}\nu^{-1}\hat{\Theta}(z)\nu\Pi. \tag{2.39}$$

Since Φ is positive w.r.t. a solid cone we have that $\hat{\Phi}(a)$ for real a is a positive operator (cf. Proposition 8), and therefore from (2.24),

$$\exists_{x\in\mathcal{K}^S} : \hat{\Phi}(a)x = r_\sigma(\hat{\Phi}(a))x \tag{2.40}$$

which, using (2.39), can be written as

$$\exists_{X\geq 0}\hat{\Theta}(a)\nu(X) = r_\sigma(\hat{\Phi}(a))\nu(X). \tag{2.41}$$

Proof. (of Theorem 3)

Since, as discussed above, (A')-(C') are equivalent, to prove that (A)-(C) are equivalent we can show that

$$\text{(C)} \Leftrightarrow \text{(C')}\ ,\ \text{(C)} \Rightarrow \text{(B)},\ \text{(B)} \Rightarrow \text{(A)},\ \text{(A)} \Rightarrow \text{(A')}.$$

(C)$\Leftrightarrow$(C') Using (2.39) with $z = 0$ we write (C') as

$$\exists_{P>0} : \nu(P) - M\nu(P) \in \{\nu(Q) : Q > 0\}. \tag{2.42}$$

Applying ν^{-1} to (2.42), and using (2.15), we conclude that (2.42) is equivalent to (C).

(C) $\Rightarrow$(B) Let $\mathcal{S}^n_{\geq 0}(\mathbb{C})$ be the set of $n\times n$ complex positive semi-definite matrices and consider the system

$$u_{k+1} = M^\intercal u_k,\ \ U_{k+1} = L'(U_k),\ \ \nu(U_k) = u_k,\ \ U_0 \in \mathcal{S}^n_{\geq 0}(\mathbb{C}), \tag{2.43}$$

where $L'(U_k) := \nu^{-1}\circ M^\intercal \circ \nu(U_k)$ is given by

$$L'(U_k) = \int_0^T Je^{As}U_k(Je^{As})^\intercal F(ds),$$

and for any $Y, Z \in S^n_+(\mathbb{C})$ satisfies

$$\mathrm{tr}(L(Z)^*Y) = \mathrm{tr}(Z^*L'(Y)). \tag{2.44}$$

Note that $U_k \geq 0$ for a given k implies that $U_{k+1} = L'(U_k) \geq 0$, and therefore, by induction, we conclude that $U_k \geq 0$ for all k. We show that (C) implies that this system is stable by considering a Lyapunov function

$$V(u_k) := \operatorname{tr}(P\nu^{-1}(u_k))$$

for (2.43), where P satisfies (C), that is $P > 0$ and $L(P) - P < 0$. In fact, this function V is radially unbounded and positive definite for $U_k \geq 0$, and satisfies $V(0) = 0$. Using (C) and (2.44) we have that for every $U_k \in \mathcal{S}^n_{\geq 0}(\mathbb{C}) - \{0\}$,

$$\begin{aligned}
V(u_{k+1}) - V(u_k) &= \operatorname{tr}(P\nu^{-1}(M^\intercal \nu(U_k))) - \operatorname{tr}(PU_k)\\
&= \operatorname{tr}(PL'(U_k)) - \operatorname{tr}(PU_k)\\
&= \operatorname{tr}((L(P) - P)U_k)\\
&= -\operatorname{tr}(ZU_k) < 0,
\end{aligned}$$

where $Z := -(L(P) - P) > 0$. Therefore (2.6) is stable for any $U_0 \in \mathcal{S}^n_{\geq 0}(\mathbb{C})$, i.e., $(M^\intercal)^k U_0 \to 0$. Since any complex matrix Z can be written as

$$Z = Z_1 - Z_2 + i(Z_3 - Z_4),$$

where $Z_i \in \mathcal{S}^n_{\geq 0}(\mathbb{C}), 1 \leq i \leq 4$ (cf. [57, Rem. 2]) this implies that $(M^\intercal)^k Z \to 0$, which implies that

$$r_\sigma(M) = r_\sigma(M^\intercal) < 1.$$

(B)$\Rightarrow$(A) For $z : \Re[z] \geq 0$ and $X, Y \in \mathcal{S}^n_{\geq 0}(\mathbb{C})$, we have

$$\begin{aligned}
&|\langle Y, \nu^{-1}(\hat{\Theta}^k(z)\nu(X))\rangle|\\
&= |\int_0^T \dots \int_0^T \operatorname{tr}(Y^*(Je^{Ah_k})^\intercal \dots (Je^{Ah_1})^\intercal X J e^{Ah_1} \dots Je^{Ah_k} e^{-zh_1} \dots e^{-zh_k} \mu(dh_1) \dots \mu(dh_k)|\\
&\leq \int_0^T \dots \int_0^T |\operatorname{tr}(Y^*(Je^{Ah_k})^\intercal \dots (Je^{Ah_1})^\intercal X (Je^{Ah_1}) \dots Je^{Ah_k}| \mu(dh_1) \dots \mu(dh_k)\\
&= \langle Y, \nu^{-1}(M^k \nu(X))\rangle
\end{aligned} \tag{2.45}$$

where the last equality holds because $\operatorname{tr}(Y^*(Je^{Ah_k})^\intercal \dots (Je^{Ah_1})^\intercal X J e^{Ah_1} \dots Je^{Ah_k})$ is a non-negative function since it can be written as the inner product of two (complex) positive semi-definite matrices. If (B) holds then $r_\sigma(M) < 1$, $M^k\nu(X)$ converges to zero as k tends to infinity and by the inequality (2.45) this implies that $|\langle Y, \nu^{-1}(\hat{\Theta}^k(z)\nu(X))\rangle|$ converges to zero for any matrices $Y, X \in \mathcal{S}^n_{\geq 0}(\mathbb{C})$. Given $Z^1, Z^2 \in \mathbb{C}^{n\times n}$ we can write $Z^j = Z^j_1 - Z^j_2 + i(Z^j_3 - Z^j_4)$, where for $1 \leq l \leq 4$ and $1 \leq j \leq 2$, $Z^j_l \in \mathcal{S}^n_{\geq 0}(\mathbb{C})$ (cf. [57, Rem. 2]). Thus,

$$|\langle Z^1, \nu^{-1}(\hat{\Theta}^k(z)\nu(Z^2))\rangle| \leq \sum_{1\leq l\leq 4, 1\leq j\leq 2} c_{lj} |\langle Z^j_l, \nu^{-1}(\hat{\Theta}^k(z)\nu(Z^j_l))\rangle|$$

also converges to zero as k tends to infinity, where c_{lj} are constants. This implies $r_\sigma(\hat{\Theta}(z)) < 1$ in $\Re[z] \geq 0$ and thus $\det(I - \hat{\Theta}(z))$ cannot be zero in $\Re[z] \geq 0$, which is (A).
(A) $\Rightarrow$ (A') If (A') does not hold then there must exist a $z_1 : \Re[z_1] \geq 0$ such that $\det(I - \hat{\Phi}(z_1)) = 0$ and this implies that

$$\exists_{x \in \mathbb{C}^m} : \hat{\Phi}(z)x = x. \tag{2.46}$$

Using (2.39) we conclude that (2.46) can be rewritten as

$$\exists_{X=Y+iZ:Y,Z \in \mathcal{S}^n(\mathbb{R})} : \hat{\Theta}(z_1)\nu(X) = \nu(X)$$

and this implies that

$$\det(I - \hat{\Theta}(z_1)) = 0,$$

which means that (A) does not hold.

Sufficiency of (A) and (2.8)*,* (2.9)*,* (2.10) *for MSS, SS and MES*

We start by noticing that (2.8) is equivalent to $h(t)$ (or $H(t)$) converges to zero, and (2.9) is equivalent to $\int_0^T h(t)dt$ (or $\int_0^T H(t)dt$) converging absolutely. Moreover, since

$$\mathbb{E}[x(t)^\intercal x(t)] = x_0^\intercal \otimes x_0^\intercal w(t) = x_0^\intercal W(t) x_0,$$

where $W(t)$, given by (2.36), is non-zero and positive semi-definite, MSS is equivalent to $w(t) \to 0$ (or $W(t) \to 0$), SS is equivalent to $\int_0^\infty w(t)dt$ (or $\int_0^\infty W(t)dt$) converges absolutely.

Sufficiency of conditions (A) and (2.8) for MSS and of conditions (A) and (2.9) for SS follow then directly from [39, Ch.4,Th.4.9], which states that for a Volterra equation taking the form (2.19) if

$$\inf_{\Re[z] \geq 0} |\det(I - \hat{\Theta}(z))| > 0$$

then, $w(t)$ is bounded and converges to zero if $h(t)$ is bounded and converges to zero, $\int_0^\infty w(t)dt$ converges absolutely if $\int_0^\infty h(t)dt$ converges absolutely. It remains to prove that $\inf |\det(I - \hat{\Theta}(z))| > 0$ for z in $\Re[z] \geq 0$ is equivalent to $\det(I - \hat{\Theta}(z)) \neq 0$ in the same complex region. Necessity is trivial. To prove sufficiency note that, $\det(I - \hat{\Theta}(z)) \neq 0$ in $\Re[z] \geq 0$ is equivalent to $r_\sigma(\hat{\Theta}(0)) < 1$, due to the equivalence between (A) and (B) of the present theorem. If $r_\sigma(\hat{\Theta}(0)) < 1$, then by Lemma 10, $r_\sigma(\hat{\Theta}(z)) < 1$ in the closed right half plane and therefore $\inf(|\det(I - \hat{\Theta}(z))|) > 0$ in $\Re[z] \geq 0$. To prove sufficiency of the conditions (A) and (2.10) for MES, we use a perturbation as in Proposition 14. For $\delta < \min(\epsilon, \alpha_1)$ where ϵ is such that (T1) holds and α_1 is such that (2.10) holds, we obtain that $h_\delta(t) = h(t)e^{\delta t}$, and Θ_δ, satisfy the conditions (A) and (2.8) of the present theorem for MSS and therefore $w_\delta(t)$ converges to zero, whereas $w(t) = w_\delta(t)e^{-\delta t}$ converges to zero exponentially fast.

Necessity of (A) and (2.8), (2.9), (2.10) *for MSS, SS and MES*

From (2.36), $W(t) \geq H(t)$. Hence, $W(t) \to 0$ (MSS), $\int_0^\infty W(t)dt$ converges absolutely (SS), and $W(t)$ converges to zero exponentially fast (MES), imply that the respective condition must hold for $H(t)$ (or equivalently for $h(t)$) and therefore that (2.8), (2.9), and (2.10) must hold, respectively. Necessity of (A) for MSS, SS and MES, follows from Lemma 15, which requires (V1) and (V2), implied by (T1) and (T2), and also (V3), which is assumed for now to hold, i.e., $h(t) = 0, \forall_{t>b}$ for some $b > 0$, which holds for example if μ has bounded support. If (A) does not hold, from Lemma 11 there exists a real $a > 0$ such that $\det(I - \hat{\Theta}(a)) = 0$. Thus, from Lemma 15, $w(t)$ converges to infinity exponentially fast, since the zero $z_i = a$ is not a removable singularity ($\exists_j : s_{i,j} \geq 0$ (cf. Proof of Theorem 4). If (V3) does not hold, we can use a similar reasoning to prove that (A) is necessary for MSS, SS, and MES. Let $W_2(t)$ be the solution to (2.35) when $H(t)$ is replaced by $H_2(t) := H(t)$ if $t > b, H(t) := 0$ otherwise for some $b > 0$. Then, $W_2(t) \leq W(t)$ since

$$W(t) - W_2(t) = K(W - W_2)(t) + H(t) - H_2(t),$$

whose solution is positive semi-definite (cf. (2.36)). Thus, since (A) is necessary for $W_2(t)$ to be MSS, SS and MES, it is also for $W(t)$. □

Proof. (of Theorem 4) Note that

$$\mathbb{E}[x(t)^\intercal x(t)] = (x_0^\intercal \otimes x_0^\intercal)w(t) = x_0^\intercal W(t)x_0 = \sum_{i=1}^m u_i(t)x_0^\intercal C_i x_0.$$

Note also that $\|W(t)\| = \|w(t)\| = \|u(t)\|$. It is then clear that the limit

$$\lambda_L(x_0) = \lim_{t\to\infty} \frac{1}{t} \log(\mathbb{E}[x(t)^\intercal x(t)])$$

exists for every x_0 if and only if the limit $\lambda_V = \lim_{t\to\infty} \frac{1}{t} \log(\|u(t)\|)$ exists, and $\lambda_V = \sup_{x_0} \lambda_L(x_0) = \lambda_L$. Note also that $\mathbb{E}[x(t)^\intercal x(t)] = 0, \forall_{x_0}, t > b > 0$ if and only if $u(t) = 0, t > b > 0$ in which case both λ_L and λ_V equal $-\infty$ according to our definitions. We use Theorem 13 applied to the Volterra equation (2.37) to obtain that λ_V is determined by $r_\sigma(\hat{\Psi}(\lambda_V)) = 1$, when such a λ_V exists, and $\lambda_V = -\infty$ otherwise. Note that λ_L in (2.11) equals λ_V since we can obtain that $r_\sigma(\hat{\Phi}(a)) = r_\sigma(\hat{\Theta}(a))$, for real a such that $\hat{\Phi}(a)$ and $\hat{\Theta}(a)$ converge. In fact, similarly to the equivalence between (B') and (B) established in the proof of Theorem 4, we can prove that $r_\sigma(\hat{\Phi}(a)) < 1$ is equivalent to $r_\sigma(\hat{\Theta}(a)) < 1$, for every real a such that $\hat{\Phi}(a)$ and $\hat{\Theta}(a)$ converge, which implies that $r_\sigma(\hat{\Theta}(a)) = r_\sigma(\hat{\Phi}(a))$. To apply

Theorem 13, we note that the the fact that μ has bounded support implies conditions (T1) and (T2), which in turn, imply (V1) and (V2). Moreover, the fact that μ has bounded support implies (V3). It suffices to prove that the real zero a of $\det(I - \hat{\Phi}(a))$ such that $r_\sigma(\hat{\Phi}(a)) = 1$, is not a removable singularity of $R(a)\hat{e}(a)$ where $R(a) := [I - \hat{\Phi}(a)]^{-1}$ and $\hat{e}(z) := \int_0^\infty e(t)e^{-zt}dt$ is the Laplace transform of $e(t)$.

To this effect, we start by noticing that, using (2.40),

$$\exists_{x \in \mathcal{K}^S} : \hat{\Phi}(a)x = x \tag{2.47}$$

which, by using (2.39), can be written as

$$\exists_{X \geq 0} L_a(X) = X, \tag{2.48}$$

where

$$L_a(X) := \int_0^\infty (Je^{As})^\intercal X (Je^{As}) e^{-as} \mu(ds).$$

Let

$$\hat{H}(b) := \int_0^\infty e^{A^\intercal t} e^{At} r(t) e^{-bt} dt$$

and note that there exists α such that $\alpha X \leq \hat{H}(b)$. For $b > a$, $r_\sigma(\hat{\Phi}(b)) < 1$ due to the characterization of a of Lemma 10 and due to the monotonicity property of Lemma 11. Thus, we can expand $R(b)$ as a von Neumann series and conclude that

$$R(b)\hat{e}(b) = \sum_{k=0}^{\infty} \hat{\Phi}(b)^k \hat{e}(b) = \sum_{k=0}^{\infty} \Pi[L_b^k(\hat{H}(b))] \geq \sum_{k=0}^{\infty} \alpha \Pi(L_b^k(X)), \tag{2.49}$$

where L_b^k denotes the composition of applying L_b, k times. Using (2.48), we have that $\lim_{a \to b} L_b^k(X) = L_a^k(X) = X$. This implies, taking the limit as $b \to a$ in (2.49), that $\lim_{b \to a, b > a} R(b)\hat{e}(z)$ cannot be finite and therefore a is not a removable singularity.

□

2.7 Additional Applications of Theorems 9 and 13

We can apply the results provided in Subsection 2.5.2 to other problems where Volterra equations arise and have the special property of having a positive kernel, e.g., in physics, engineering, biology (cf. [39, Ch.1]). As an example, consider the setup in Fig. 2.2. An LTI system is defined by a convolution product $v(t) = (H * e)(t)$ where $e(t) \in \mathbb{R}^m$ is the input, $v(t) \in \mathbb{R}^m$ the output, $H(t) = [h_{i,j}]_{m \times m}$ is the impulse response, and $*$ denotes convolution. All signals are defined for $t \geq 0$. The Laplace transform of H is given by $\hat{H}(z) := \int_0^\infty H(t)e^{-zt}dt$, and the closed loop is defined by $e(t) = v(t) + r(t)$ for a reference signal $r(t)$, which leads to the following Volterra equation

$$v(t) = \int_0^t H(s)v(t-s)ds + \int_0^t H(s)r(t-s)ds. \tag{2.50}$$

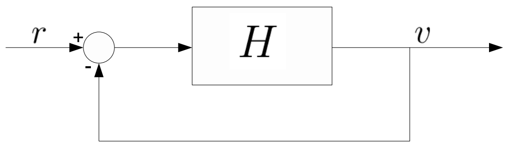


Figure 2.2: MIMO LTI closed loop

The transfer function H is not rational, e.g., when the impulsive response $H(t)$ exhibits delays. Following [39, Ch.1], we say that the closed-loop is stable if $v(t)$ belongs to L^p (i.e., $\int_0^t |v(t)|^p dt < \infty$, if $p < \infty$ and $v(t)$ is bounded a.e. if $p = \infty$) whenever $r(t)$ belongs to L^p. Assume that $\int_0^\infty H(t)dt$ converges absolutely which, in the case where $\hat{H}(z)$ is rational, is equivalent to saying that the open-loop is stable. The stability of the closed-loop can be tested by the condition

$$\det(I - \hat{H}(z)) \neq 0, \forall_{z \in \Re[z] \geq 0}, \tag{2.51}$$

One can conclude this from the stability condition for the Volterra equation (2.50) or from [39, Ch.4, Th.4.9]. However, when $H(t)$, $\forall t$ is a positive operator w.r.t. a given cone $\mathcal{K} \in \mathbb{R}^m$ (e.g., $\mathbb{R}^m_{\geq 0}$), we can use the following very simple algebraic test for stability.

Corollary 16. If $h_{i,j}(t) \geq 0$ for every $t \geq 0$ and $1 \leq i, j \leq m$, then the closed-loop described by (2.50) is stable if and only if

$$r_\sigma(\hat{H}(0)) < 1. \tag{2.52}$$

□

Proof. Since $H(t)$ is positive w.r.t the solid cone $\mathbb{R}^m_{\geq 0}$, Theorem 3 assures that (2.52) and (2.51) are equivalent. □

2.8 Proofs of Auxiliary Results

Proof. (of Proposition 5) The condition (T1) assures that $\hat{\Theta}_d(y)$ converges absolutely for $y > -\epsilon$ and in this region, $r_\sigma(\hat{\Theta}_d(y))$ is continuous (cf. [39, Ch. 3, Th.3.8]). Therefore, $r_\sigma(\hat{\Theta}_d(0)) < 1$ implies that $r_\sigma(\hat{\Theta}_d(-\epsilon_1)) < 1$ for some $\epsilon_1 \in (0, \epsilon)$, and from Lemma 10 we have $r_\sigma(\hat{\Theta}_d(z)) < 1$ in $\Re[z] \geq -\epsilon_1$, which encompasses $\mathcal{C}(\epsilon_1, R)$, $R > 0$. This implies $\inf |\det(I - \hat{\Theta}_d(z))| > 0$ in $\mathcal{C}(\epsilon_1, R)$. □

Proof. (of Proposition 6) Conditioning (2.18) on the time of the first jump t_1, we obtain

$$w(t) = \int_0^\infty \mathbb{E}[T(t)^\intercal \otimes T(t)^\intercal \nu(I) | t_1 = s] \mu(ds). \tag{2.53}$$

The integrand has a different expression for $s > t$ and $s \leq t$,

$$\mathbb{E}[T(t)^\intercal \otimes T(t)^\intercal \nu(I)|t_1 = s] = \begin{cases} (e^{A^\intercal t} \otimes e^{A^\intercal t})\nu(I), \text{ if } s > t \\ \mathbb{E}[(T_1(t-s)Je^{As})^\intercal \otimes (T_1(t-s)Je^{As})^\intercal \nu(I)], \text{ if } s \leq t \end{cases} \tag{2.54}$$

where $T_1(t-s)$ is the transition matrix of (2.6) from $s = t_1$ to t, which depends on $\{h_k : k \geq 1\}$. Due to the i.i.d. assumption on the intervals between transitions $\mathbb{E}[T_1(t)^\intercal \otimes T_1(t)^\intercal \nu(I)] = w(t)$. Thus, partitioning (2.53) using (2.54) we obtain (2.19):

$$\begin{aligned} w(t) =& \int_0^t \mathbb{E}[(T_1(t-s)Je^{As})^\intercal \otimes (T_1(t-s)Je^{As})^\intercal \nu(I)]\mu(ds) + \int_t^\infty \mathbb{E}[(e^{A^\intercal t} \otimes e^{A^\intercal t})\nu(I)]\mu(ds) \\ =& \int_0^t (Je^{As})^\intercal \otimes (Je^{As})^\intercal w(t-s)\mu(ds) + (e^{A^\intercal t} \otimes e^{A^\intercal t})\nu(I)r(t), \end{aligned}$$

where we used the fact that we can write the first term on the right hand side of (2.19) as

$$\int_0^t \Theta(ds)w(t-s) = \int_0^t (Je^{As})^\intercal \otimes (Je^{As})^\intercal w(t-s)\mu(ds),$$

which can be concluded by extending [80, Ch.1, Th.1.29] to real matrix-valued measures. $\square$

Proof. (of Proposition 8) From the fact that

$$\hat{\Psi}(a) = \int_0^\infty K_c(s)e^{-as}ds + \sum_{i \geq 1} K_{di}e^{-ac_i}, \tag{2.55}$$

it suffices to prove that $\int_0^\infty K_c(s)e^{-as}ds$ is a positive operator since this implies that (2.55) is a sum of positive operators and therefore a positive operator. It is clear that the fact that $\hat{\Psi}(a)$ converges absolutely implies that $\int_0^\infty K_c(s)e^{-as}ds$ converges absolutely. Since the space of continuous function with bounded support on $\mathbb{R}_{\geq 0}$, denoted by C_c, is dense in the space of measurable function whose Lebesgue integral on $\mathbb{R}_{\geq 0}$ is absolutely convergent, denoted by L_1, (cf. [80, Th.3.14]), there exists a sequence $K_n(s) \in C_c$ such that

$$\left|\int_0^\infty (K_n(s) - K_c(s))e^{-as}ds\right| < \frac{\epsilon_n}{2},$$

where $\epsilon_n \to 0$. Since $K_n(s)$ is Riemann integrable, we can find a matrix step function $K_n^R(s)$, and h_n-spaced points s_i such that

$$\int_0^\infty K_n^R(s)e^{-as}ds = \sum_{i=1}^{N_n} h_n K_n^R(s_i)e^{-as_i}.$$

The result of this integral is a positive operator since it is the sum of positive operators. Moreover for each n we can choose N_n sufficiently large such that

$$\left|\int_0^\infty (K_n^R(s) - K_n(s))e^{-as}ds\right| < \frac{\epsilon_n}{2}.$$

Thus,

$$|\int_0^\infty K_n^R(s)e^{-as}ds - \int_0^\infty K(s)e^{-as}ds| < \epsilon_n \to 0, \text{ as } n \to \infty$$

and therefore $\int_0^\infty K(s)e^{-as}ds$ is a positive operator since it is the limit of positive operators, and the space of positive operators w.r.t a given cone is closed (cf. [56, p.22]). ☐

Proof. (of the Lemma 10) By Proposition 8, $\hat{\Psi}(a)$ is positive w.r.t. $\mathcal{K}$, and by induction so is $\hat{\Psi}(a)^k$. Let v belong to the dual cone $\mathcal{K}^*$ of $\mathcal{K}$, i.e., the set $v \in \mathbb{R}^m$ such that $\langle x, v\rangle \geq 0, \forall_{x\in\mathcal{K}}$. Such v different from zero always exists if $\mathcal{K}$ does not coincide with the whole space [56, p.22]. For every $v \in \mathcal{K}^*$ and for real a, $\langle\hat{\Psi}(a)^k x, v\rangle$ is a positive number by definition of $\mathcal{K}^*$. For $z : \Re[z] \geq a$ and for every $v \in \mathcal{K}^*$, $w \in \mathcal{K}$ and $k \geq 1$, one can show that

$$|\langle\hat{\Psi}(z)^k w, v\rangle| \leq \langle\hat{\Psi}(a)^k w, v\rangle. \tag{2.56}$$

For example, for $k = 2$, let

$$a(r,s) := \langle K_c(r)K_c(s)w, v\rangle, \;\; b_j(s) := \langle K_c(s)K_{dj}w, v\rangle$$
$$d_i(r) := \langle K_{di}K_c(r)w, v\rangle, \quad h_{ij} := \langle K_{di}K_{dj}w, v\rangle$$

which are positive functions due to the fact that the kernel Ψ is positive and by definition of $\mathcal{K}^*$. Then

$$\begin{aligned}
|\langle\hat{\Psi}(z)^2 w, v\rangle| &= |\int_0^\infty\int_0^\infty a(r,s)e^{-zr}e^{-zs}drds + \int_0^\infty\sum_{j=1}^\infty b_j(s)e^{-zc_j}e^{-zs}ds + \\
&\qquad \int_0^\infty\sum_{i=1}^\infty d_i(r)e^{-zc_i}e^{-zr}dr + \sum_{i=1}^\infty\sum_{j=1}^\infty h_{ij}e^{-zc_j}e^{-zc_i}| \\
&\leq \int_0^\infty\int_0^\infty |a(r,s)|e^{-ar}e^{-as}drds + \int_0^\infty\sum_{j=1}^\infty |b_j(s)|e^{-ac_j}e^{-as}ds + \\
&\qquad \int_0^\infty\sum_{i=1}^\infty |d_i(r)|e^{-ac_i}e^{-ar}dr + \sum_{j=1}^\infty\sum_{i=1}^\infty |h_{ij}|e^{-ac_i}e^{-ac_j} \\
&= \langle\hat{\Psi}(a)^2 w, v\rangle
\end{aligned}$$

where we used the fact that $a(r,s)$, $b_j(s)$, $d_i(r)$, h_{ji} are positive functions.

If $r_\sigma(\hat{\Psi}(a)) < 1$, then $\hat{\Psi}(a)^k$ converges to zero as k tends to infinity, and so does $\langle\hat{\Psi}(a)^k w, v\rangle$ for every $v \in \mathcal{K}^*, w \in \mathcal{K}$. Since $\mathcal{K}$ is solid, and therefore reproducing, so is $\mathcal{K}^*$(cf. [56, Th. 4.5] and use the fact that a cone in $(\mathbb{R}^m, \|.\|)$ is normal [56, p. 37]). Thus, for any $u, y \in \mathbb{R}^m$ we can write $u = u_1 - u_2$ and $y = y_1 - y_2$, for $u_i \in \mathcal{K}$, $y_i \in \mathcal{K}^*$, $i \in \{1,2\}$ and obtain that

$$|\langle\hat{\Psi}(z)^k u, y\rangle| \leq \sum_{1\leq i,j\leq 2} |\langle\hat{\Psi}(z)^k u_i, y_i\rangle|. \tag{2.57}$$

Each of the terms on the right hand side of (2.57) tend to zero as $k \to \infty$ due to (2.56) and the fact that $r_\sigma(\hat{\Psi}(a)) < 1$. This implies that $\langle \hat{\Psi}(z)^k u, y\rangle \to 0$ for every $u, y \in \mathbb{R}^m$ and therefore that $\hat{\Psi}(z)^k \to 0$ as $k \to \infty$, which in turn implies that $r_\sigma(\hat{\Psi}(z)) < 1$.

To conclude the proof we note that for $\alpha > 0$, if $r_\sigma(\hat{\Psi}(a)) < \alpha$ then $r_\sigma(\frac{1}{\alpha}\hat{\Psi}(a)) < 1$. Using the above arguments this implies that $r_\sigma(\frac{1}{\alpha}\hat{\Psi}(z)) < 1$, for $z : \Re[z] \geq a$, or equivalently that $r_\sigma(\hat{\Psi}(z)) < \alpha$. Since α is arbitrary, $r_\sigma(\hat{\Psi}(z)) \leq r_\sigma(\hat{\Psi}(a))$. □

Proof. (of Lemma (11)) To prove (i), note first that, $r_\sigma(\hat{\Psi}(y))$ is continuous along $y \geq a$ and converges to zero as $y \to +\infty$. By the intermediate value theorem, if $r_\sigma(\hat{\Psi}(a)) \geq 1$ then $\exists_{a_1 \geq a} : r_\sigma(\hat{\Psi}(a_1)) = 1$. To see that a_1 is the unique real number that satisfies $r_\sigma(\hat{\Psi}(a_1)) = 1$, suppose that $r_\sigma(\hat{\Psi}(a_2)) = 1$ for another real number a_2. Due to Lemma 10, we must have $r_\sigma(\hat{\Psi}(y)) = 1$ in $y \in [\min\{a_1, a_2\}, \max\{a_1, a_2\}]$. Since, by the property (2.24), $r_\sigma(\hat{\Psi}(a_1)) = 1$ is an eigenvalue, this implies that $\det(I - \hat{\Psi}(y)) = 0$, in $y \in [\min\{a_1, a_2\}, \max\{a_1, a_2\}]$. But $\det(I - \hat{\Psi}(z)) = 0$ is analytic and the zeros of analytic functions (other than the identically zero function) are isolated (cf. [80, Th.10.18]), which leads to a contradiction.Since $\hat{\Psi}(a)$ is a positive operator due to Proposition 8, by the property (2.24) of positive operators w.r.t. solid cones, $r_\sigma(\hat{\Psi}(a_1)) = 1$ is an eigenvalue and this implies that $\det(I - \hat{\Psi}(a_1)) = 0$ (part (ii)). Finally, we argue by contradiction to prove (iii). If there exists w satisfying $\Re[w] > a_1$ and $\det(I - \hat{\Psi}(w)) = 0$ then $\hat{\Psi}(w)$ has an eigenvalue at 1 and therefore $r_\sigma(\hat{\Psi}(w)) \geq 1$. From Lemma 10, $r_\sigma(\hat{\Psi}(w)) \leq r_\sigma(\hat{\Psi}(\Re[w]))$ which implies that $r_\sigma(\hat{\Psi}(\Re[w])) \geq 1$. Now, $r_\sigma(\hat{\Psi}(\Re[w]))$ cannot be equal to one since a_1 is the unique real number that satisfies this property and $\Re[w] > a_1$. This leads to a contradiction since, again by Lemma 10, $r_\sigma(\hat{\Psi}(\Re[w])) \leq r_\sigma(\hat{\Psi}(a_1)) = 1$. □

Proof. (of the Lemma 15)

Theorems [39, Ch.7,Th.2.4,2.5] and the argument in [39, p.192] would allow us to conclude that (i) and (ii) hold if:

$$\inf_{z:\Re[z]=0} |\det(I - \hat{\Psi}(z))| > 0, \tag{2.58}$$

$$\lim_{|z|\to\infty, \Re[z]\geq 0} \inf |\det(I - \hat{\Psi}(z))| > 0, \tag{2.59}$$

$$g(t) \text{ is bounded and converges to zero.} \tag{2.60}$$

Note that (2.58) is restrictive since $\det(I - \hat{\Psi}(z))$ would not be allowed to have zeros or the imaginary axis (critical case). It is clear that (V3) implies (2.60). Note also that (V2) implies (2.59) and, since we assume (V2), the condition (2.58) can be simply stated as the characteristic equation $\det(I - \hat{\Psi}(z)) = 0$ has no zeros on the imaginary axis (non-critical case).

The proof of (i) under assumptions (V1), (V2) is a direct consequence of the arguments in [39, p.192] and [39, p.195] are is therefore omitted. To prove (ii) under assumptions (V1), (V2) and (V3) we argue as follows. Let $g_\delta(t)$ and Ψ_δ be as in Proposition 14. Due to (2.32) the zeros of $\det(I - \hat{\Psi}_\delta(z))$ are given by $z_i + \delta$, where z_i are the zeros of $\det(I - \hat{\Psi}(z))$. If $\det(I - \hat{\Psi}(z))$ has zeros on the imaginary axis (critical case), i.e., (2.58) does not hold, we can always choose (due to part (i) of the present theorem) a $\delta : 0 < \delta < \epsilon$, where ϵ is the constant in (V1), and such that the characteristic equation of the Volterra equation (2.31) does not have zeros on the imaginary axis. Thus, the Volterra equation (2.31) satisfies the conditions (2.58), (2.59), (2.60) required by Theorems [39, Ch.7,Th.2.4,2.5] to assure (i) and (ii). Thus from Theorems [39, Ch.7,Th.2.4,2.5] one can conclude that

$$y_\delta(t) = y_{s\delta}(t) + \sum_{i=1}^{n_z} \sum_{j=0}^{m_i - 1} s_{i,j} t^j e^{(z_i+\delta)t} + \sum_{i=1}^{n_w} \sum_{j=0}^{n_i - 1} q_{i,j} t^j e^{v_i t}, \tag{2.61}$$

where v_i are the zeros of $\det(I - \hat{\Psi}_\delta(z))$ in $0 < \Re[w_i] < \delta$ and we used the fact, due to (2.32), that the principal parts of $\hat{y}(z)$ about z_i and of $\hat{y}_\delta(z)$ about $z_i + \delta$ have the same coefficients $s_{i,j}$. Thus (2.33) follows from (2.61) due to $y(t) = y_\delta(t) e^{-\delta t}$.

□

2.9 Further Comments and References

Impulsive renewal systems are a special class of the piecewise deterministic processes considered in [25]. In this chapter, we restricted ourselves to linear dynamic and reset maps. Impulsive renewal systems with general non-linear dynamic and reset maps are considered in [44]. The paper [44] considers a more general class of impulsive systems in which the system evolves according to a stochastic differential equation between jumps (as opposed to a deterministic vector field). For this class of impulsive renewal systems, sufficient conditions for a form of exponential stability are provided.

Early work on systems with i.i.d. distributed parameters can be found in [4], [6], [59]. The stability problems considered in these early references are typically cast into a discrete-time setting, even if the motivating stability problem is in a continuous-time setting, as it is the case for randomly sampled systems [59]. Sufficient conditions for continuous-time mean square stability are provided in [69] for impulsive renewal systems that arise in model-based networked control systems with stochastically spaced transmissions. The approach followed in [4], [6], [59], [69] involves a stability test in terms of a matrix eigenvalue computation. An alternative approach that can be found in the literature to investigate the stability of such systems resorts to stochastic Lyapunov functions, leading to stability conditions formulated in terms of the existence of a Lyapunov function satisfying appropriate inequalities [32, 66]. We believe that the stability results (Theorems 3 and 4) that we obtain with our novel aproach cannot be obtained through any of the previous approaches [44], [4], [6], [69], [32], [66]. Our stability result (Theorem 3) obviates the conservativeness of the results in [44], [69] where only sufficient stability conditions are presented to the case where the probability distributions of the intervals between reset can have unbounded support. The stability tests in terms of an eigenvalue or an LMI computation resemble tests available for Markov linear systems [21], but, to the best of our knowledge, a Nyquist test has not appeared previously for this type of systems.

Volterra integral equations arise in many problems in physics, engineering and biology. The excellent textbook [39] discusses an extensive number of results for these equations. The Volterra equation considered here has the special property that its kernel is a positive operator in a sense to be defined below. It is important to note that the notion of positiveness needed in this chapter differs from the well-known notions of monotonicity (see [39, Ch.5]) and positiveness (see [39, Ch.16]).

The networked control setup is considered, e.g., in [44] and [87].

3

Network Features Modeled by Finite State Machines

In this chapter, we introduce Stochastic Hybrid Systems (SHSs) with renewal transitions, which are a generalization of impulsive renewal systems that allows one to considerably augment the number of networked control scenarios that one can model. In fact, with SHS with renewal transitions one can consider any scenario in which the network protocol, or more generally network features such as delays and packet drops, can be modeled by a finite state machine. In this chapter, we start by showing how one can model network features such as packet drops, round-robin protocols or delays, using finite state machines.

After presenting these networked control motivations, we formally define stochastic hybrid systems with renewal transitions. Stated briefly, stochastic hybrid systems with renewal transitions are systems with both continuous dynamics and discrete logic whose execution is specified by the dynamic equations of the continuous state, a set of rules governing the transitions between discrete modes, and reset maps determining jumps of the state at transition times.

Besides defining this class of systems, which is an important generalization of impulsive renewal systems, we provide results concerning the analyses of the transitory and asymptotic behavior of the statistical moments of a SHS with renewal transitions. First, we provide expressions for any moment of the SHS at a given time t in terms of a set of Volterra equations. Note that in Chapter 2 we considered only a special second degree moment for the special case of impulsive renewal systems. We discuss how by computing the moments one can provide information about the probability density function of the state of the SHS. For this latter problem, we highlight the advantages of our method when compared to other general methods in the literature. The second main result provides a method to obtain the Lyapunov exponents of even degree m, which are defined, in accordance to [66, p. 41], as the exponential decrease or increase rate at which the mth power of the norm of the state of the SHS, converges to zero or to infinity, respectively. We show that when m is even, the Lyapunov exponents can be efficiently determined by finding the zero of a monotonic function. As a corollary, we provide necessary and sufficient conditions for mean exponential stability, which is defined in terms of the exponential convergence to zero of the squared norm of the state of the SHS.

The remainder of the chapter is organized as follows. Examples of networked control scenarios that can be modeled by finite state machines are given in Section 3.1. Stochastic Hybrid Systems are formally defined in Section 3.2, and our main results are provided in Section 3.3. A numerical example is provided in Section 3.4. The proofs of the main results are given in Section 3.5. Section 3.6 provides further comments and references.

3.1 Modeling Several Network Features by Finite State Machines

We present next five networked control scenarios, where finite state machines are used to model (i) static and round-robin protocols; (ii) delays; (iii) packet drops; (iv) nodes independently accessing the same shared network; (v) nodes transmitting through more than one renewal network. We will present a detailed study of scenarios (i) and (ii) and provide numerical examples in Section 3.4. For the scenarios (iii), (iv), and (v), we only briefly sketch how they can be modeled by finite state machines. The finite state machines modeling NCS are loosely defined in this section by specifying the number and meaning of discrete modes and specifying transitions in terms of events that trigger these transitions. In the next section, we formally introduce stochastic hybrid systems, which allow to rigorously model these descriptions. The ideas provided here can be used to model combinations of these features (delays, packet drops, etc), when they occur simultaneously.

3.1.1 Round-Robin Protocols

Consider a networked control system for which a set of nodes consisting on sensors, actuators, and a controller, are connected through a communication network possibly shared by other users. The plant and controller are described by:

$$\text{Plant:} \quad \dot{x}_P = A_P x_P + B_P \hat{u}, \quad y = C_P x_P \tag{3.1}$$

$$\text{Controller:} \quad \dot{x}_C = A_C x_C + B_C \hat{y}, \quad u = C_C x_C + D_C \hat{y}. \tag{3.2}$$

where $\hat{u}$ is the input to the plant, $\hat{y}$ is the input to the controller, y is the output of the plant, and u is the output of the controller. As explained in Section 2.1, it is typically the case that one assumes that the controller has been designed to stabilize the closed-loop, when the process and the controller are directly connected, i.e., when $\hat{u}(t) = u(t)$, $\hat{y}(t) =$

$y(t)$, and one is interested in analyzing the effect of the network characteristics on the stability and performance of the closed loop. We can partition y as $y = (y_1, y_2, \ldots, y_{n_y})$, where $y_i \in \mathbb{R}^{q_i}, 1 \leq i \leq n_y$ is associated to one of n_y sensors, and we can partition u as $u = (u_1, u_2, \ldots, u_{n_u})$, where $u_j \in \mathbb{R}^{r_j}, 1 \leq j \leq n_u$ is associated to one of n_u actuators. We assume that the network is exclusive, i.e., only one node can transmit at a given time. The data generated by a given sensor is received by the controller shortly after it is sampled and transmitted by the sensor, and likewise a given actuator receives its data shortly after it is sent by the controller. We denote the times at which (at least) one of the nodes transmits by $\{t_k, k \in \mathbb{N}\}$. The time intervals $\{t_{k+1} - t_k, k \in \mathbb{N}\}$ are assumed to be independent and identically distributed following a given distribution μ_s, which characterizes the times nodes have to wait to access the network. The transmission delays are assumed to be negligible when compared to the time constants of the system dynamics. We consider here that there are no packet drops.

Between sampling times, $\hat{u}$ and $\hat{y}$ are held constant, i.e.,

$$\hat{u}(t) = \hat{u}(t_k), \;\; \hat{y}(t) = \hat{y}(t_k), t \in [t_k, t_{k+1}), \;\; k \in \mathbb{Z}_{\geq 0}.$$

Let

$$e(t) := (e_u(t), e_y(t)) = (\hat{u}(t) - u(t), \hat{y}(t) - y(t)),$$

which can be partitioned according to $e = (e_1, \ldots, e_{n_u+n_y})$ $e_i \in \mathbb{R}^{s_i}$, $s_i = r_i, 1 \leq i \leq n_u$ and $s_i = q_{i-n_u}, n_u+1 \leq i \leq n_y+n_u$ is the error associated with node $i \in \mathcal{M} := \{1, \ldots, n_y+n_u\}$. We assume that a static protocol is used by the nodes, which is defined as follows.

Static Protocol: The nodes transmit in a T_s-periodic sequence determined by a periodic function with period T_s, denoted by $\mathsf{s} : \mathbb{N} \mapsto \mathcal{M}$. In this case, the error e is updated at time t_k according to

$$e(t_k) = (I - \Lambda_{\mathsf{s}(k)})e(t_k^-). \tag{3.3}$$

where

$$\Lambda_j := \operatorname{diag}([0_{\sum_{i=1}^{j-1} s_i} \; I_{s_j} \; 0_{\sum_{i=j+1}^{m} s_i}]), \; j \in \mathcal{M}.$$

That is, only the components of $\hat{y}$ or $\hat{u}$ associated with the node that transmits are updated by the corresponding components of $y(t_k^-)$ or $u(t_k^-)$. We assume that s is onto, i.e., each node transmits at least once in a period.

When $T_s = n_u + n_y$, we call the static protocol a round-robin protocol, which by proper labeling the nodes can be defined as follows.

Round Robin Protocol: Static protocol where each node transmits exactly once in a period, i.e., $\mathsf{s}(k) = k$ if $1 \leq k \leq n_u + n_y$ and $\mathsf{s}(k) = \mathsf{s}(k - (n_y + n_u))$ if $k \geq n_y + n_u + 1$.

Defining $x := (x_A, e)$, where $x_A := (x_P, x_c)$, the following system describes (3.1),(3.2), and (3.3),

$$\begin{bmatrix} \dot{x}_A(t) \\ \dot{e}(t) \end{bmatrix} = \begin{bmatrix} I \\ A_e \end{bmatrix} \begin{bmatrix} A_{xx} & A_{xe} \end{bmatrix} \begin{bmatrix} x_A(t) \\ e(t) \end{bmatrix} \tag{3.4}$$

$$\begin{bmatrix} x_A(t_k) \\ e(t_k) \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & (I - \Lambda_{\mathsf{s}(k)}) \end{bmatrix} \begin{bmatrix} x_A(t_k^-) \\ e(t_k^-) \end{bmatrix}, \tag{3.5}$$

where

$$A_{xx} = \begin{bmatrix} A_P + B_P D_C C_P & B_P C_C \\ B_C C_P & A_C \end{bmatrix},$$

$$A_e = \begin{bmatrix} 0 & -C_C \\ -C_P & 0 \end{bmatrix}, \quad A_{xe} = \begin{bmatrix} B_P & B_P D_C \\ 0 & B_C \end{bmatrix}.$$

This networked control scenario with static protocols can be modeled by a finite state machine, specified as follows for the special case of a round-robin protocol.

Discrete modes. To model a NCS with a round-robin protocol as described above, we require $n_u + n_y$ discrete modes, where each of these modes corresponds to the node $1 \leq i \leq n_y + n_u$ having the transmission token, i.e., i is the node assigned to transmit when the network becomes available. At each discrete mode, x flows according to (3.4).

Transitions. The times between transitions equal the network access time, distributed according to μ_s. When a transition, corresponding to a transmission by node i is triggered, the discrete mode changes from i to $i+1$ for every $1 \leq i \leq n_u + n_y - 1$, and if $i = n_u + n_y$ the next state equals $i = 1$. At each transition, x is updated according to (3.5).

3.1.2 Delays

We start by modeling delays which may occur due to the times that users may have to wait to access the network, and we continue to assume as in Section 3.1.1 that transmission delays are negligible. Another model for delays, more suitable to model transmission delays, is presented in Section 5.1.1. For simplicity, we consider only a single renewal network, where only one node transmits, as in Section 2.1, but the same ideas carry through to the case where several nodes transmit through the network, as in Section 3.1.1.

Suppose that we wish to control a linear plant

$$\dot{x}_P(t) = A_P x_P(t) + B_P \hat{u}(t). \tag{3.6}$$

A state feedback controller taking the form $K_C x_P(t)$ is implemented digitally and the actuation is held constant $\hat{u}(t) = \hat{u}(s_\kappa), t \in [s_\kappa, s_{\kappa+1})$ between actuation update times denoted by $\{s_\kappa, \kappa \geq 0\}$.

The controller has direct access to the state measurements, but communicates with the plant actuators through a network possibly shared by other users. The controller attempts to do periodic transmissions of data, at a desired sampling period T_s but these regular transmissions may be perturbed by the medium access protocol. For example, users using CSMA for medium access, may be forced to back-off for a typically random amount of time until the network becomes available. We assume these random back-off times to be i.i.d. and denote by μ_s the associated probability measure.

We consider two different cases:

Case I: After waiting to obtain network access, the controller (re)samples the sensor, computes the control law and transmits this most recent data. This case is the most reasonable when transmitting dynamic data. Assuming that the transmission delays are negligible, and defining $x := (x_P, \hat{u})$, we have

$$\dot{x} = Ax, \quad A = \begin{bmatrix} A_P & B_P \\ 0 & 0 \end{bmatrix}, \tag{3.7}$$

$$x(s_k) = Jx(s_k^-), \quad J = \begin{bmatrix} I & 0 \\ K_C & 0 \end{bmatrix}. \tag{3.8}$$

Since the intervals $\{s_{k+1} - s_k,\, k \geq 0\}$ result from the controller waiting a fixed time T_s plus a random amount of time with a measure $\mu_s(s)$, these intervals are independent and identically distributed according to

$$\mu([0,\tau)) = \begin{cases} \mu_s(\tau - T_s), \text{ if } \tau \geq T_s \\ 0, \quad t \in [0, T_s). \end{cases} \tag{3.9}$$

This case is captured by the following simple finite state machine.

Discrete modes. A single discrete mode is required to model this scenario, in which x flows according to (3.7).

Transitions. The times between transitions equal the network access times, distributed according to (3.9). When a transition, corresponding to a transmission by node i is triggered, the discrete mode remains unchanged and x is updated according to (3.8).

Case II: After waiting to obtain access to the network, the controller transmits the data collected at the time it initially tried to transmit data. This case is more realistic than Case I since the controller typically sends the sensor data to the network adapter and does not have the option to update this data at the transmission times. Let $r_k = s_k + T_s$, $x := (x_P, \hat{u}, v)$ where $v(t) := u_\kappa, t \in [r_\kappa, r_{\kappa+1})$ is a variable that holds the last computed control value. We model this case by a finite state machine specified as follows

Discrete modes. Two modes are required to model this case.

- Mode 1- The controller waits for a fixed time T_s;
- Mode 2- The controller waits a random time to gain access to the network.

Transitions. When in mode 1 the system transits to mode 2 at times r_κ. The corresponding state jump models the update of the variable $v(r_\kappa) = u_\kappa$ that holds the last computed control value and is described by

$$x(r_\kappa) = J_{1,1}x(r_\kappa^-), \quad J_{1,1} = \begin{bmatrix} I & 0 & 0 \\ 0 & I & 0 \\ K_C & 0 & 0 \end{bmatrix}.$$

When in mode 2 the SHS transits to mode 1 at actuation update times s_κ. The state jump models the actuation update $\hat{u}(s_\kappa) = v(s_\kappa^-)$ and is described by

$$x(s_\kappa) = J_{2,1}x(s_\kappa^-), \quad J_{2,1} = \begin{bmatrix} I & 0 & 0 \\ 0 & 0 & I \\ 0 & 0 & I \end{bmatrix}.$$

The times spent in each mode can be modeled by reset-time measures given by

- Time spent in the discrete mode 1 follows a distribution $\mu_{1,1}(\tau) = \delta(\tau - T_s)$, where $\delta(\tau - T_s)$ is the dirac measure, i.e., a discrete measure that places all mass $w_i = 1$ at time T_s.
- Time spent in the discrete mode 2 follows a distribution $\mu_{2,1}(\tau) = \mu_s(\tau)$.

In both discrete modes, the continuous-time dynamics are described by $\dot{x} = A_i x$, $i \in \{1, 2\}$, $A_1 = A_2 = A$ where

$$A = \begin{bmatrix} A_P & B_P & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

3.1.3 Other Networked Control Features

In the networked control scenarios that we have presented so far, we can see a trend when modeling networked control systems using an emulation approach. The model includes differential equations modeling plant and controller dynamics, and reset maps are introduced to model resets of auxiliary modeling variables. For brevity, in the next networked control scenarios we address only how these auxiliary variables are reset. The full model for these networked control scenarios can be obtained by combining these ideas with the frameworks presented in Section 2.1, and Subsections 3.1 and 3.1.2.

3.1.3.1 Packet Drops

Consider, for simplicity a single renewal link and a single node transmitting over the network, which to fix ideas is considered to be the controller transmitting to the plant. The associated error variables are $e(t) = \hat{u}(t) - u(t)$, where $u(t)$ is the actual value of the actuation and $\hat{u}(t)$ is the perceived value of $u(t)$ by the plant, which is typically the last transmitted value, since this value is typically held constant at the plants' input. At transmission times $\{t_k, k \geq 0\}$ at which the actuation is successfully transmitted over the network, the error variables are set to zero $e(t_k) = 0$.

We consider two models for packet drops. In a first approximation we may assume that packet drops are statistically independent and that at each transmission times they occur with the same probability p_{drop}. This can be modeled by the following finite state machine.

Discrete modes. A single discrete mode is required to model independent and identically distributed packet drops.

Transitions. There are two possible transitions each time a transmission occurs. A transition may correspond to a successful transmission in which case the auxiliary variables are updated, i.e., $e(t_k) = 0$, or it may correspond to a packet drop in which case the auxiliary variables remain unchanged $e(t_k) = e(t_k^-)$. The times between transitions equal the network access times. When a transition corresponding to a transmission occurs, the discrete mode remains unchanged.

To take into account possible correlation between packet drops we can model packet drops occurences by a Markov chain. This is more appropriate when errors are likely to occur in bursts. A simple model to achieve this is the following.

Discrete modes. Two modes are required with the following meaning

- Mode 1. The last packet was successfully transmitted.
- Mode 2. The last packet was dropped.

Transitions. There are two possible transitions each time a transmission occurs for each of the two modes and this may me modeled by a Markov chain

$$\text{Prob}[\xi(k+1) = i | \xi(k) = j] = [P_{ij}]_{1 \leq i,j \leq 2} = \begin{bmatrix} p_1 & p_2 \\ 1 - p_1 & 1 - p_2 \end{bmatrix}$$

where $\xi(k) \in \{1, 2\}$ denotes the mode of the finite state machine and k indexes transmissions. If in mode 1, when a successful transmission occurs the discrete mode remains 1 and the auxiliary variables are updated, i.e., $e(t_k) = 0$, and when a transmission is dropped the discrete mode is reset to 2 and the auxiliary variables remain unchanged $e(t_k) = e(t_k^-)$. Typically, when the last packet was successfully transmitted, one associates a high probability of a successful transmission occurring next, i.e., p_1 is typically close to one. If in mode 2, when a successful transmission occurs the discrete mode is reset to 2 and the auxiliary variables are updated, i.e., $e(t_k) = 0.$, and when a transmission is dropped the discrete mode is reset to 2 and the the auxiliary variables remain unchanged $e(t_k) = e(t_k^-)$. Typically, when the last packet was dropped, one associates a high probability of a packet drop occurring next, since errors are likely to occur in bursts, i.e., p_2 is typically close to zero. The times between transitions equal the network access times.

3.1.3.2 Nodes Independently Accessing the Same Network

Consider a networked control system in which a remote controller communicates with a plant through a single shared communication network. Instead of assuming that plant and controller implement a high-level protocol as a round-robin protocol introduced in Subsection (3.1.1), suppose that plant and controller operate independently. Therefore they compete for the network in general with other network users. We assume that a carrier sense multiple access protocol for medium access is fair, in the sense that the probabilities of the plant transmitting to the controller and of the controller transmitting to the plant are the same. The associated auxiliary error variables are $e_u(t) = \hat{u}(t) - u(t)$, where $u(t)$ is the actual value of the actuation and $\hat{u}(t)$ is the perceived value of $u(t)$ by the plant, and $e_y(t) = \hat{y}(t) - y(t)$, where $y(t)$ is the actual value of the measurements and $\hat{u}(t)$ is the perceived value of $y(t)$ by the controller. Transmission times are denoted by $\{t_k, k \geq 0\}$ and the error variables are set to zero $e_u(t_k) = 0$, $e_y(t_k) = e_y(t_k^-)$ if the controller gains the access to the network at t_k and $e_u(t_k) = 0$, $e_u(t_k) = e_u(t_k^-)$ if the plant gains the access to the network at t_k. We can model this scenario by a finite state machine with stochastic transitions.

Discrete modes. A single discrete mode is required to model the case where plant and controller aim independently to transmit over the same network.

Transitions. There are two possible transitions each time a transmission occurs at t_k and both have the same probabilities of occurring. Whether the plant gain access to the controller and transmits to the plant, in which case $e_u(t_k) = 0$, $e_y(t_k) = e_y(t_k^-)$, or the controller gains access to the network and transmits to the plant, in which case $e_u(t_k) = 0$, $e_y(t_k) = e_y(t_k^-)$. The times between transitions equal the network access times.

3.1.3.3 Several Renewal Networks

Consider now a networked control system in which a remote controller communicates with a plant through two communication links; e.g., the actuation data is sent from the controller to the plant through a shared wired network and the sensor data is sent from the plant to the controller through a shared wireless network. The key point here is that these networks have in general access times with different distributions. Let e_y, e_u and t_k have the same meaning as in Subsection 3.1.3.2. If we assume that controller implement a (high-level) protocol with the plant, upon which they decide to transmit in a round-robin fashion, we can model this scenario by the following finite state machine.

Discrete modes. Two modes are required with the following meaning.

- -Mode 1. The plant has the transmission token.
- -Mode 2. The controller has the transmission token.

Transitions. When in mode 1, a transition occurs to mode 2. The time spent in mode 1 corresponds to the time the plant takes to access the network from which it transmits to the controller. When in mode 2, a transition occurs to mode 1. The time spent in mode 2 corresponds to the time the controller takes to access the network from which it transmits to the plant, which is generally differently distributed from the access times of the network from which the plant transmits to the controller.

We can generalize these ideas to the case where several sensors and actuators transmit through different, possibly many, networks. We shall consider the case where controller and plant transmit independently and asynchronously in Chapter 4, in which case it does not appear to be possible to capture this scenario by a finite state machine.

3.2 Stochastic Hybrid Systems with Renewal Transitions

A SHS with renewal transitions, is defined by (i) a linear differential equation

$$\dot{x}(t) = A_{q(t)}x(t), \quad x(0) = x_0,\, q(0) = q_0,\, t_0 = 0, \tag{3.10}$$

where $x(t) \in \mathbb{R}^n$ and $q(t) \in \mathcal{Q} := \{1, \dots, n_q\}$; (ii) a family of n_ℓ *discrete transition/reset maps*

$$(q(t_k), x(t_k)) = (\xi_\ell(q(t_k^-)), J_{q(t_k^-),\ell}\,x(t_k^-)), \quad \ell \in \mathcal{L} := \{1, \dots, n_\ell\}, \tag{3.11}$$

where ξ_ℓ is a map $\xi_\ell : \mathcal{Q} \mapsto \mathcal{Q}$ and the matrix $J_{q(t_k^-),\ell}$ belongs to a given set $\{J_{i,\ell} \in \mathbb{R}^{n\times n},\ i \in \mathcal{Q}, \ell \in \mathcal{L}\}$; and (iii) a family of *reset-time measures*

$$\mu_{i,\ell}, \qquad i \in \mathcal{Q},\ \ \ell \in \mathcal{L}. \tag{3.12}$$

Between *transition times* t_k, the discrete mode q remains constant whereas the continuous state x flows according to (3.10). At transition times, the continuous state and discrete mode of the SHS are reset according to (3.11). The intervals between transition times are independent random variables determined by the reset-time measures (3.12) as follows. A reset-time measure can be either a probability measure or identically zero. In the former case, $\mu_{i,\ell}$ is the probability measure of the random time that transition $\ell \in \mathcal{L}$ takes to trigger in the state $q(t) = i \in \mathcal{Q}$. The next transition time is determined by the minimum of the triggering times of the transitions associated with state $q(t) = i \in \mathcal{Q}$. When $\mu_{i,\ell}([0,s)) = 0, \forall_{s\geq 0}$, the transition ℓ does not trigger in the state $i \in \mathcal{Q}$, which allows for some reset maps not to be active in some states.

We assume that a transition probability measure $\mu_{i,\ell}$ can be decomposed as $\mu_{i,\ell} = \mu^c_{i,\ell} + \mu^d_{i,\ell}$, with $\mu^c_{i,\ell}([0,t)) = \int_0^t f_{i,\ell}(s)ds$, for some density function $f_{i,\ell}(s) \geq 0$, and $\mu^d_{i,\ell}$ is a discrete measure that captures possible point masses $\{b^r_{i,\ell} > 0, r \geq 1\}$ such that $\mu^d_{i,\ell}(\{b^r_{i,\ell}\}) = w^r_{i,\ell}$. The integral with respect to the measure $\mu_{i,\ell}$ is defined as

$$\int_0^t W(s)\mu_{i,\ell}(ds) = \int_0^t W(s)f_{i,\ell}(s)ds + \sum_{r:b^r_{i,\ell}\in[0,t]} w^r_{i,\ell}W(b^r_{i,\ell}). \tag{3.13}$$

When the discrete measures $\mu^d_{i,\ell}$ are different from the zero measure, different transitions may trigger at the same time with probability different from zero, leading to an ambiguity in choosing the next state. Thus, to guarantee that with probability one this does not occur, we assume the following.

Assumptions 17.

(i) For every $i \in \mathcal{Q}$, the reset-time measures $\mu_{i,\ell}$, $\ell \in \mathcal{L}$ for mode i have a finite number of point masses;

(ii) For every $i \in \mathcal{Q}$, no two measures for mode i have common point masses, i.e., $b^j_{i,l_1} \neq b^\kappa_{i,l_2}$ for $l_1 \neq l_2, l_1 \in \mathcal{L}$, $l_2 \in \mathcal{L}$, $\forall_{j\geq 1,\kappa\geq 1}$;

The construction of a sample path of the SHS with renewal transitions can be described as follows.

1. Set $k = 0$, $t_0 = 0$, $(q(t_k), x(t_k)) = (q_0, x_0)$.
2. For every $j \in \mathcal{L}$, obtain $\bar{h}^j_k$ as a realization of a random variable distributed according to $\mu_{q(t_k),j}$, if $\mu_{q(t_k),j}$ is not identically zero, and set $\bar{h}^j_k = \infty$ otherwise.

3. Take
$$h_k = \min\{\bar{h}_k^j, j \in \mathcal{L}\} \tag{3.14}$$
and set the next transition time to $t_{k+1} = t_k + h_k$. The state of the SHS in the interval $t \in [t_k, t_{k+1})$ is given by
$$(q(t), x(t)) = (q(t_k), e^{A_{q(t_k)}(t-t_k)}x(t_k)).$$

4. In the case where $t_{k+1} < \infty$, let l_k denote the index of the transition that achieves the minimum in step 3, i.e., $l_k = j : h_k = \bar{h}_k^j$ and update the state according to $(q(t_{k+1}), x(t_{k+1})) = (\xi_{l_k}(q(t_{k+1}^-)), J_{q(t_{k+1}^-),l_k}x(t_{k+1}^-))$. Set $k = k+1$ and repeat the construction from the step 2.

Due to the Assumption 17.(ii), there is zero probability that the minimum in step 3 is achieved by two or more indexes j.

3.2.1 Generalizations

Stochastic Hybrid Systems with renewal transitions are a special case of the stochastic hybrid systems model presented in [45]. Two of the three generalizations discussed in [45] still apply to SHS with renewal transitions. We cannot model the case where discrete transitions are triggered by deterministic conditions on the state, e.g., guards being crossed, but we can generalize SHS with renewal transitions to the case where the dynamics are driven by a Wiener process and the next state is chosen according to a given distribution, where the last generalization further requiring the transition distributions to be absolutely continuous. This latter generalization is especially useful to model the networked control scenarios described in Sections 3.1.3.1 and 3.1.3.2, and therefore it is described next with more detail.

Consider a dynamic system, which flows according to
$$\dot{x}(t) = A_1 x(t), \tag{3.15}$$
between transition times $\{t_k, k \geq 0\}$ at which the state is updated according to
$$x(t_k) = \bar{J}_{\sigma_k} x(t_k^-), \tag{3.16}$$
where $\{\sigma_k, k \geq 0\}$ are i.i.d. random variables with $\text{Prob}[\sigma_k = 1] = p$ and $\text{Prob}[\sigma_k = 2] = 1 - p$. The intervals between transmissions $\{t_{k+1} - t_k, k \geq 0\}$ are i.i.d. and are described by a probability density function f, i.e., $\text{Prob}[a \leq t_{k+1} - t_k \leq b] = \int_a^b f(s)ds$, for $b > a \geq 0$. An alternative characterization to f is given in terms of the survivor function of f, defined as
$$S(x) := \int_x^\infty f(s)ds,$$

or in terms of the hazard rate of f, defined as

$$\lambda_{\text{haz}}(x) := \frac{f(x)}{S(x)}. \tag{3.17}$$

One can confirm that this system indeed allows to model the scenarios described in Sections 3.1.3.1 and 3.1.3.2, where the state transitions are stochastic whether due to packet drops or due to the uncertainty on which node gains access to the network.

The system (3.18), (3.18) can be modeled by a SHS with a single discrete mode, the following dynamic equations associated with this discrete mode

$$\dot{x}(t) = A_1 x(t) \tag{3.18}$$

and two reset maps $x(t_k) = J_{1,1}x(t_k^-)$, where $J_{1,1} = \bar{J}_1$, and $x(t_k) = J_{1,2}x(t_k^-)$, where $J_{1,2} = \bar{J}_2$. The transition distributions associated with these two reset maps are set to $\mu_{1,1}([0,x]) = \int_0^x f_1(ds)$ and $\mu_{1,1}([0,x])) = \int_0^x f_2(ds)$, where $f_1(s)$ and $f_2(s)$ are probability density functions, completely characterized by the respective hazard rates $\lambda_{\text{haz},1}(x)$ and $\lambda_{\text{haz},2}(x)$, defined as in (3.17). If we let f_1 and f_2 be uniquely characterized by $\lambda_{\text{haz},1}(x) = p\lambda_{\text{haz}}(x)$ and $\lambda_{\text{haz},2}(x) = (1-p)\lambda_{\text{haz}}(x)$, then standard computation can show that the intervals between transitions, determined by (3.14), follow the distribution f and when these transitions are triggered the exists a probability p of transition 1 being triggered and a probability $1-p$ of transition 2 being triggered.

Besides, the generalization just mentioned, exogenous inputs described by linear systems can be easily incorporated in both dynamic equations and reset maps of the SHS. For example, if the dynamic equations for a given mode i are described by $\dot{x}_s(t) = D_i x_s(t) + B_i r(t)$, where the exogenous input $r(t)$ is described by $\dot{x}_r(t) = A_r x_r(t)$, $r(t) = C_r x_r(t)$, for some initial condition $x_r(0)$, then we can consider $x = (x_s\, x_r)$, and write these dynamic equations in the standard form (3.10), where

$$A_i = \begin{bmatrix} D_i & B_i C_r \\ 0 & A_r \end{bmatrix}.$$

3.3 Main Results

Consider a general $m-$th degree uncentered moment of the state of the SHS with renewal transitions, i.e.,

$$\mathbb{E}[x_1(t)^{i_1} x_2(t)^{i_2} \dots x_n(t)^{i_n}], \quad \sum_{j=1}^{n} i_j = m,\ i_j \geq 0. \tag{3.19}$$

We provide a method to compute (3.19) in subsection 3.3.1, we obtain the Lyapunov exponents of special moments of the SHS in Subsection 3.3.2, and we discuss how by computing the moments one can reconstruct probability density functions in Subsection 3.3.3. The proofs of the main results are deferred to Subsection 3.5.

3.3.1 Moment Computation

It is easy to see that there are

$$p := \frac{(m+n-1)!}{m!(n-1)!} \tag{3.20}$$

different monomials of degree m, and hence p different moments (3.19) of degree m. Let

$$\{\rho(\kappa) = [i_1(\kappa), \dots i_n(\kappa)], 1 \le \kappa \le p\} \tag{3.21}$$

be an enumeration of the indexes $i_1 \dots, i_n$ uniquely characterizing such monomials, e.g., for $m = n = 2$, one such enumeration is $\rho(1) = [1 \ \ 1]$, $\rho(2) = [2 \ \ 0]$, $\rho(3) = [0 \ \ 2]$. Then, we use the notation,

$$x^{[\kappa]} := x_1^{i_1} \dots x_n^{i_n}, \text{ for } \kappa : \rho(\kappa) = [i_1, \dots, i_n], \ \ 1 \le \kappa \le p,$$

and define the map $\Gamma^m : \mathbb{R}^{n\times n} \mapsto \mathbb{R}^{p\times p}$ as

$$\Gamma^m(A) = B, \tag{3.22}$$

where $B = [B_{ij}]$ is uniquely determined by

$$(Ax)^{[i]} = \sum_{j=1}^{p} B_{ij} x^{[j]}, 1 \le i, j \le p. \tag{3.23}$$

The following theorem provides a method to compute any moment of the state of the SHS. Let

$$\gamma_\kappa := \frac{m!}{(i_1! \dots i_n!)}, \text{ for } \kappa : \rho(\kappa) = [i_1, \dots, i_n], \sum_{j=1}^{n} i_j = m, \tag{3.24}$$

denote by e_i the canonical vector, i.e., the component j of e_i equals 1 if $j = i$ and 0 otherwise, and define the following operator

$$\Theta^m(u^\kappa(t)) := (\Theta^{m,1}(u^\kappa(t)), \dots, \Theta^{m,n_q}(u^\kappa(t))),$$

where $u^\kappa(t) := (u^{\kappa,1}(t), \dots, u^{\kappa,n_q}(t))$ and each $\Theta^{m,i}$ is a convolution operator defined by

$$\begin{aligned} &\Theta^{m,i}(u^\kappa(t)) := \\ &\sum_{\ell=1}^{n_\ell} \int_0^t \Gamma^m(e^{A_i^\intercal s} J_{i,\ell}^\intercal) u^{\kappa,\xi_\ell(i)}(t-s) \frac{r_i(s)}{r_{i,\ell}(s)} \mu_{i,\ell}(ds), \end{aligned} \tag{3.25}$$

where $i \in \mathcal{Q}$, $r_i(s) := \Pi_{\ell=1}^{n_\ell} r_{i,\ell}(s)$, $r_{i,\ell}(s) := \mu_{i,\ell}((s,\infty])$.

Theorem 18. A moment of degree m (3.19) of the SHS (3.10)- (3.12), indexed by κ : $\rho(\kappa) = [i_1 \dots i_n]$, can be computed as

$$\mathbb{E}[x(t)^{[\kappa]}] = \sum_{l=1}^{p} x_0^{[l]} u_l^{\kappa,q_0}(t), \tag{3.26}$$

where $u^\kappa = (u^{\kappa,1}, \dots, u^{\kappa,n_q})$ is uniquely determined by the Volterra equation

$$u^\kappa(t) = \int_0^t \Theta^m(u^\kappa(t)) + h^\kappa(t), \quad t \geq 0, \tag{3.27}$$

where $h^\kappa(t) := (h^{\kappa,1}(t), \dots, h^{\kappa,n_q}(t))$,

$$h^{\kappa,i}(t) = \Gamma^m(e^{A_i^\intercal t})\mathrm{e}_\kappa \frac{r_i(t)}{\gamma_\kappa}, \; i \in \mathcal{Q}.$$

□

An explicit solution to (3.27) takes the form

$$u^\kappa(t) = \sum_{k=0}^{\infty} I^k_{\Theta^m}(h^\kappa(t)) \tag{3.28}$$

where $I_{\Theta^m}(h^\kappa(t)) = \int_0^t \Theta^m(h^\kappa(s))ds$ and $I^k_{\Theta^m}(h^\kappa(t))$ denotes composition, i.e., e.g.,

$$I^2_{\Theta^m}(h^\kappa(t)) = I_{\Theta^m}\big(I_{\Theta^m}(h^\kappa(t))\big).$$

However, in practice a numerical method is used solve the Volterra equation (3.27). One such numerical method consists in choosing a set of integration nodes $\{a_l \in [0,t]\}$, and obtaining $u^\kappa(a_l)$ by iteratively replacing the integrals in (3.27) by quadrature formulas at the nodes $\{a_j : a_j \in [0, a_l]\}$. Note that in this procedure, we only need to compute $\Gamma^m(e^{A_i^\intercal a_j} J^\intercal_{i,\ell})$ and $\Gamma^m(e^{A_i^\intercal a_j})$, where Γ^m is specified by (3.23), and this can be done numerically efficiently by symbolically manipulating monomials.

In [66, Ch. 2], it is shown that the moments of a Markov Jump linear systems, which can be seen as a special case of a SHS with renewal transitions for which the duration of the times the system stays in each discrete mode are exponentially distributed, can be obtained by solving a linear differential equation. Theorem 18 shows that in the general case where these times can be arbitrarily distributed, one can still compute the moments by solving a Volterra equation instead of a differential equation.

A similar result to Theorem 18 was given in [AHS10b]. However, there exists redundancy both in the state of the Volterra equation proposed in [AHS10b] and in the state of the differential equation proposed in [66], that is eliminated by Theorem 18. This has a great impact on the achievable moments that one can compute. In fact, given a SHS with dimension n in [AHS10b] one needs to solve a Volterra equation with $n_q \times n^m$ unknown variables, to obtain a moment of degree m. This is also the number of unknown variables in the linear differential equation presented in [66]. The Volterra equation (3.27) has only $n_q \times p = n_q \times \frac{(m+n-1)\times\cdots\times(m+1)}{(n-1)!}$ unknowns, i.e., the number of variables in (3.27) grows polynomially with m, instead of exponentially.

3.3.2 Lyapunov Exponents

The following definition of Lyapunov exponent of a positive polynomial of degree m is adapted from [66, p. 41][1]. Recall the definition of L_p norm of a vector $x(t)$

$$\|x(t)\|_p^p := \sum_{i=1}^{n} |x_i(t)|^p.$$

Definition 19. Suppose that $\mathbb{E}[\|x(t)\|_m^m] \neq 0, \forall t_{\geq 0}$ and that for every $x_0 \neq 0$ the following limit exists

$$\lambda_L^m(x_0) := \lim_{t\to\infty} \frac{1}{t} \log \mathbb{E}[\|x(t)\|_m^m]. \tag{3.29}$$

Then the Lyapunov exponent λ_L^m of degree m for the SHS (3.10)- (3.12) is defined as

$$\lambda_L^m := \sup_{x_0 \in \mathbb{R}^n} \lambda_L^m(x_0).$$

Moreover, if $\exists_{b>0}\!:\!\mathbb{E}[\|x(t)\|_m^m]\!=\!0, \forall t > b$ then $\lambda_L^m\!:=\!-\infty$.

□

We assume in the next Theorem that all the distributions $\mu_{i,\ell}$ have bounded support. Let

$$\hat{\Theta}^m(z) := \begin{bmatrix} \hat{\Theta}^m_{1,1}(z) & \dots & \hat{\Theta}^m_{1,n_q}(z) \\ \vdots & \ddots & \vdots \\ \hat{\Theta}^m_{n_q,1}(z) & \dots & \hat{\Theta}^m_{n_q,n_q}(z) \end{bmatrix},$$

where

$$\hat{\Theta}^m_{i,j}(z) := \sum_{\ell=1}^{n_\ell} \int_0^\infty \Gamma^m(e^{A^\intercal s} J_{i,\ell}^\intercal) e^{-zs} \chi_{\xi_\ell(i)=j} \frac{r_i(s)}{r_{i,\ell}(s)} \mu_{i,\ell}(ds),$$

and $\chi_{x\in A}$ equals 1 if $x \in A$ and 0 otherwise.

Theorem 20. Suppose that all the distributions $\mu_{i,\ell}$ have finite support and let $b := \inf\{a : \hat{\Theta}^m(a)$ converges absolutely$\}$. Then, if m is even, the spectral radius $r_\sigma(\hat{\Theta}^m(a))$ of $\hat{\Theta}^m(a)$ is a monotone non-increasing function of a for $a > b$ and the Lyapunov exponent for the SHS (3.10)-(3.12) is given by

$$\lambda_L = \begin{cases} a \in \mathbb{R} : r_\sigma(\hat{\Theta}^m(a)) = 1, \text{ if such } a \text{ exists} \\ -\infty \text{ otherwise} \end{cases}. \tag{3.30}$$

□

[1] The definition in [66, p. 41] does not specify which norm to take for $\|x(t)\|$ (we choose to take $\|x(t)\|_m$). Note that we can in fact choose any norm due to the equivalence of norms in finite-dimensional vector spaces.

As we show next, Theorem 20 allows to establish necessary and sufficient stability conditions for the SHS with renewal transitions, which is defined in terms of a sum of second moments. We say the SHS with renewal transitions is mean exponentially stable if there exists constants $c > 0$ and $\alpha > 0$ such that for every (x_0, q_0),

$$\mathbb{E}[x(t)^\intercal x(t)] \leq ce^{-\alpha t} x_0^\intercal x_0, \forall_{t \geq 0},$$

Corollary 21. Suppose that all the distributions $\mu_{i,\ell}$ have bounded support. Then the SHS with renewal transitions is mean exponentially stable if and only if

$$r_\sigma(\hat{\Theta}^2(0)) < 1. \tag{3.31}$$

□

Using the results in Chapter (2), once can show that mean exponential stability can also be tested by asserting the feasibility of a set of linear matrix inequalities and by a test in terms of the Nyquist Criterion.

3.3.3 Probability Density Function

Let μ be the probability density function of the state of the SHS at time t, i.e., $\int_E \mu(ds) :=$ Prob$[x(t) \in E \subseteq \mathbb{R}^n]$. In this subsection we address the problem of obtaining information on μ, based on its moments (3.19). It is well known that in the case where all the moments are known, the measure μ can be uniquely determined [54], while in the case where only a finite number of moments are known, this reconstruction is naturally not unique. In this latter case, an elegant procedure to obtain an explicit expression for an approximating distribution to μ, is the following (cf. [54]). Assume, for simplicity, that μ has bounded support, which without loss of generality (by proper scaling) can be assumed to be contained in the interval $D := [0,1]^n = [0,1] \times \cdots \times [0,1]$ (see [54] for the case of unbounded support). Given a continuous function $f : D \mapsto \mathbb{R}$ define the higher dimensional Bernstein polynomials as:

$$B_r(f)(x) = \sum_{k:0 \leq k_i \leq r_i} f(\frac{k_1}{r_1}, \ldots, \frac{k_n}{r_n}) P_{r,k}(x)$$

where $r = (r_1, \ldots, r_n)$ is a vector of integers, and $P_{r,k}(x) := \Pi_{i=1}^n \binom{r_i}{k_i} x_i^{k_i}(1-x_i)^{r_i - k_i}$. It is shown in [54] that $B_r(f)(x)$ converges uniformly to $f(x)$ as $r_i \to \infty, \forall_{1 \leq i \leq n}$. Thus, if we let $I_{r,k} := \int_D P_{r,k}(x)\mu(ds)$, we have that

$$\sum_{0 \leq k_i \leq r_i} f(\frac{k_1}{r_1}, \ldots, \frac{k_n}{r_n}) I_{r,k} \tag{3.32}$$

is an approximation to $\mathbb{E}[f(x)] = \int_E f(s)\mu(ds)$ that converges to $\mathbb{E}[f(x)]$ as $r_i \to \infty, \forall_{1 \leq r_i \leq n}$, and it is shown in [54] that a measure μ_r defined by

$$\sum_{0 \leq k_i \leq r_i} I_{r,k} \delta([\frac{k_1}{r_1}, \ldots, \frac{k_n}{r_n}]).$$

is a discrete measure approximating μ in the sense that $\int_D f(s)\mu_r(ds) \to \int_D f(s)\mu(ds)$, for every continuous function $f : D \mapsto \mathbb{R}$, as $r_i \to \infty, \forall_{1\leq i\leq n}$,where $\delta(x)$ denotes the Dirac measure located on x.

It is important to mention that the problem of recovering a probability density function from its moments is regarded as a numerically ill-conditioned problem, meaning that small errors in the knowledge of the moments can produce large errors in the approximation for the probability density function. In the procedure just described, one can see that $I_{r,n}$ may have large errors for small error in the knowledge of the moments because of the large coefficients that arise from writing $P_{r,n}(x)$ as a sum of monomials. Thus, it is especially important to have a computationally efficient method to compute the moments, as provided by Theorem 18, so that one can obtain a good numerical precision in this procedure of obtaining an approximating distribution.

One way to interpret the procedure just described is that each integral $I_{r,n}$ approximates the probability that x lies in a small neighborhood of $(\frac{k_1}{r_1}, \dots, \frac{k_r}{r_n})$. Instead of obtaining an approximation one can use similar ideas to obtain an upper bound on the probability that x belongs to a given region. That is, suppose, for example, that we wish to determine a bound on the probability that a random variable $y = g(x)$ is greater than a given value. Assume that y takes values in a compact set, which, without loss of generality, is assumed to be $[0, 1]$. Then we can choose a polinomial $p_n : [0, 1] \to \mathbb{R}_{\geq 0}$ such that $p_n(x) \geq \chi_{[a,1]}$, $a < 1$, and obtain the following bound

$$\text{Prob}[y \geq a] \leq \mathbb{E}[p_n(x)] \tag{3.33}$$

where the right-hand side of (3.33) can be computed using Theorem 18. Note that when $p_n(x) = \frac{x}{a}$ and $p_n(x) = \frac{x^2}{a^2}$, (3.33) correspond to the Markov and Chebychev inequalities, respectively.

Alternative approaches to estimate the probability density function of the state of the SHS include those based on the Focker-Plank equation and Dynkin's formula. The Focker-Plank equation to the SHS with renewal transitions can be obtained by specializing the expressions provided in [45], [41, Sec. 5.3], to this class of systems, and can be shown to be a integro-partial differential equation. Besides the numerical difficulty and computation burden associated with solving this equation, the derivation of these Focker-Plank equations requires the map Jx to be invertible. The approach based on the Dynkin's formula can be found in [25, Ch.3, Sec. 32.2] where a numerical method is provided to compute the expected value of a given function $\mathbb{E}[f(x)]$, and a fortiori estimating the probability density function from the relation $\mathbb{E}[\chi_E(x)] = \text{Prob}[x \in E]$. Note that the approximation (3.32), which can be obtained with the methods derived in the present chapter, provides an alternative to the method in [25, Ch.3, Sec. 32.2] to approximate the expected value of continuous functions of the state. Although we omit the derivations here, it is possible to prove that when specialized to SHSs with renewal transitions and to the case where f is a monomial, the recursive method provided in [25, Ch.3, Sec. 32.2] is equivalent to providing an approximation to (3.26) at each iteration ι of the recursive algorithm, taking the form,

$$\sum_{l=1}^{p} x_0^{[l]} v_l^{\kappa,q_0}(t), \tag{3.34}$$

with

$$v^{\kappa}(t) = \sum_{k=0}^{\iota} I_{\Theta^m}^{k}(h^{\kappa}(t)). \tag{3.35}$$

This last equation converges to (3.28), and hence (3.34) converges to (3.26), but this it is clearly an inefficient method to obtain the solution to the Volterra equation (3.27), when compared to the method described in Subsection 3.3.1 (cf. [64]). Thus, by exploiting linearity and the special structure of the SHS with renewal transitions, our approach provides an insight that allows to compute moments more efficiently, than when seeing the SHS with renewal transitions as a piecewise deterministic process and specializing the approach proposed in [25, Ch.3, Sec. 32.2].

3.4 Numerical Examples

3.4.1 Round-robin protocols and packet drops - Batch Reactor

We consider here a numerical example of the round-robin scenario, considered in Subsection 3.1.1. This example considers a linearized model of an open loop unstable batch reactor, described by (3.1), where

$$\begin{aligned} A_P &= \begin{bmatrix} 1.38 & -0.2077 & 6.715 & -5.676 \\ -0.5814 & -4.29 & 0 & 0.675 \\ 1.067 & 4.273 & -6.654 & 5.893 \\ 0.048 & 4.273 & 1.343 & -2.104 \end{bmatrix}, \\ B_P &= \begin{bmatrix} 0 & 0 \\ 5.679 & 0 \\ 1.136 & -3.146 \\ 1.136 & 0 \end{bmatrix}, \quad C_P = \begin{bmatrix} 1 & 0 & 1 & -1 \\ 0 & 1 & 0 & 0 \end{bmatrix}. \end{aligned} \tag{3.36}$$

We assume that only the outputs are sent through the network, using a round-robin protocol. The system is controller by a PI controller, described by (3.2), where

$$A_C = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, B_C = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, C_C = \begin{bmatrix} -2 & 0 \\ 0 & 8 \end{bmatrix}, D_C = \begin{bmatrix} 0 & -2 \\ 5 & 0 \end{bmatrix}. \tag{3.37}$$

This example appeared in [44]. To compare our results with the ones in [44] we consider uniformly and exponentially ($\mu([0,x)) = 1 - \exp(-\lambda_{\exp} x)$) distributed time intervals h_k.

This scenario can be modeled by a stochastic hybrid system with two discrete modes $n_q = 2$, and using the description given in Subsection 3.1.1, where transitions are defined in an obvious way. The results are summarized in Table 3.1, where one can see that our stability conditions eliminate the conservativeness of the stability conditions provided in [44].

Table 3.1: Stability conditions for the Batch Reactor Example

	Nec. & Suf. Cond.		Results taken from [44]	
	no drops	$p = 0.5$	no drops	$p = 0.5$
Maximum support τ of Uniform Distribution	0.112	0.0385	0.0517	0.0199
Max. expected value $1/\lambda_{\exp}$ of Exponential Distribution	0.0417	0.0188	0.0217	0.00924

3.4.2 Delays - Inverted Pendulum

We consider here a numerical example for the scenario considered in Subsection 3.1.2 where we illustrate the applicability of the results we present for stochastic hybrid systems. Suppose that the plant (3.6) is described by

$$A_P = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad B_P = \begin{bmatrix} 0 \\ 1 \end{bmatrix},$$

which by properly scaling the state and input can be viewed as a linearized model of a damp-free inverted pendulum. Moreover, suppose that the measure μ_s is uniform with support on the interval $[0, \tau]$, and fix $T_s = 0.1s$. A continuous-time state feedback controller is synthesized using LQR and is given by $\hat{u}(t) = K_C x(t), \quad K_C = -\begin{bmatrix} 1+\sqrt{2} & 1+\sqrt{2} \end{bmatrix}$, which is the solution to the problem $\min_{\hat{u}(t)} \int_0^\infty [x_P(t)^\intercal x_P(t) + \hat{u}(t)^2] dt$, yielding $\lambda_i(A_P + B_P K_C) = \{-1, -\sqrt{2}\}$. We wish to investigate the stability and performance of the closed-loop when instead of the ideal networked-free case we consider the scenarios of Cases I and II. To this effect we define the quantity

$$e(t) = x_P(t)^\intercal x_P(t) + \hat{u}(t)^2,$$

which can be written as $e(t) = x^\intercal P x$, where in the network-free case $P = I_2 + K_C^\intercal K_C$, and $x = x_P$; in case I, $P = I_3$ and $x = (x_P, \hat{u})$; and in case II, $P = \mathrm{diag}(I_2, 1, 0)$, and $x = (x_P, \hat{u}, v)$. Note that, in the network-free case, $e(t)$ is the quantity whose integral is minimized by LQR control synthesis and $e(t)$ decreases exponentially fast at a rate $\alpha = 2$, since the dominant closed-loop eigenvalue equals $\lambda_i(A_P + B_P K_C) = -1$. Note also that in cases I and II, $\mathbb{E}[e(t)]$ converging to zero is equivalent to MSS, which is equivalent to SS and MES since the reset-time measures have finite support (τ and τ_s are finite). Corollary 21 can be used to determine whether or not the closed-loop in cases I and II is MSS. Moreover, when the closed-loop is MSS, we can determine the exponential decay constant of $\mathbb{E}[e(t)]$ from the Theorem 20. The results are summarized in Table 3.2, for different values of the support τ of the uniform measure μ_s of the back-off time.

Table 3.2: Exponential decay rates $\mathbb{E}[e(t)] \leq ce^{-\alpha t}$

τ	0.4	0.6	0.8	1.0	1.2	> 1.211
α	2.000	2.000	1.969	0.477	7.63×10^{-5}	NOT MSS

(a) Case I

τ	0.1	0.2	0.3	0.4	0.5	> 0.521
α	2.000	2.000	2.000	0.849	0.118	NOT MSS

(b) Case II

The fact that closed-loop stability in preserved for larger values of τ in Case I, confirms what one would expect intuitively, i.e, Case I is more appropriate when transmitting dynamic data, since the most recent sampling information is sent through the network.

Using the state moment expressions provided by Theorems 18, we can perform a more detailed analysis by plotting the moments of $e(t)$, which can be expressed in terms of the moments of the state. For example, the two first moments take the form

$$\begin{aligned}\mathbb{E}[e(t)] &= \mathbb{E}[x(t)^\intercal P x(t)] = \mathbb{E}[(x(t)^\intercal)^{(2)}]\nu(P),\\ \mathbb{E}[e(t)^2] &= \mathbb{E}[(x(t)^\intercal P x(t))^2] = \mathbb{E}[(x(t)^\intercal)^{(4)}](\nu(P) \otimes \nu(P)).\end{aligned}$$

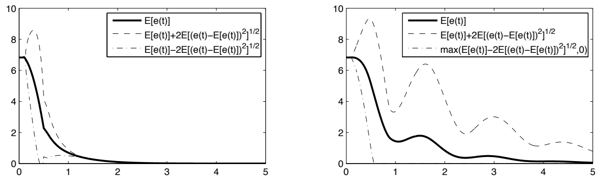
In Figure 3.1, we plot $\mathbb{E}[e(t)]$ and $\mathbb{E}[e(t)] \pm 2\mathbb{E}[(e(t) - \mathbb{E}[e(t)])^2]^{1/2}$ for a network measure support $\tau = 0.4$. Note that, from the Chebyshev inequality, we conclude that

$$\text{Prob}[|e(t) - \mathbb{E}[e(t)]| > a(t)] \leq \frac{\mathbb{E}[(e(t) - \mathbb{E}[e(t)])^2]}{a(t)^2},$$

and therefore one can guarantee that for a fixed t, $e(t)$ lies between the curves $\mathbb{E}[e(t)] \pm a(t)$, $a(t) = 2\mathbb{E}[(e(t) - \mathbb{E}[e(t)])^2]^{1/2}$ with a probability greater than $\frac{3}{4}$. The numerical method used to compute the solution of the Volterra-equation is based on a trapezoidal integration method. In case I, the expected value of the quadratic state function $e(t)$ tends to zero much faster, and with a much smaller variance than in case II , confirming once again that case I is more appropriate when transmitting dynamic data.

3.5 Proofs

We start by establishing some preliminary facts.



(a) Case I (b) Case II

Figure 3.1: Plot of $\mathbb{E}[e(t)]$, where $e(t)$ is quadratically state dependent. For a fixed t, $\mathbb{E}[e(t)]$ lies between the dashed curves with probability $> \frac{3}{4}$.

First, let $A^{(m)}$, $m > 0$, denote the m-th fold Kronecker product of a matrix or a vector A with itself, i.e.,

$$A^{(m)} := \underbrace{A \otimes A \cdots \otimes A}_{m},$$

and recall that

$$(A \otimes B)(C \otimes D) = (AC) \otimes (BD) \tag{3.38}$$

(cf. [46]). Using (3.38), we can write (3.19) as

$$\mathbb{E}[x_1(t)^{i_1} x_2(t)^{i_2} \dots x_n(t)^{i_n}] = \mathbb{E}[(x(t)^\intercal)^{(m)} c_\kappa], \tag{3.39}$$

where

$$c_\kappa := \mathrm{e}_1^{(i_1)} \otimes \cdots \otimes \mathrm{e}_n^{(i_n)}, \text{ for } \rho(\kappa) = [i_1, \dots, i_n]. \tag{3.40}$$

Second, let $\mathcal{T}(m, n)$ be the set of symmetric tensors, i.e., multilinear functions R on the m-fold $\mathbb{R}^n \times \cdots \times \dots \mathbb{R}^n$ (cf. [85, Ch. 4]) such that

$$R(x_1, x_2, \dots, x_m) = R(x_{\sigma(1)}, x_{\sigma(2)}, \dots, x_{\sigma(m)})$$

for every $\sigma : \{1, 2, \dots, m\} \mapsto \{1, 2, \dots, m\}$, which is an one-to-one function, i.e., defines a permutation of indexes. We note that there is a natural identification between monomials of degree m in $\mathbb{R}^n$ and $\mathcal{T}(m, n)$. In fact, to every $m-$degree monomial $x^{[\kappa]} = x_1^{i_1} x_2^{i_2} \dots x_n^{i_n}$, indexed by κ, as in (3.21), we can associate an element b^κ of the following orthogonal basis of symmetric tensors

$$b^\kappa = \sum_{j \in \mathcal{J}_\kappa} (\mathrm{e}_{j_1}^\intercal) \otimes (\mathrm{e}_{j_2}^\intercal) \otimes \dots \otimes (\mathrm{e}_{j_m}^\intercal), \; 1 \le \kappa \le p \tag{3.41}$$

where j belongs a set of γ_κ permutations of indexes defined as follows

$$\mathcal{J}_\kappa := \{j = (j_1, \dots, j_m) : x^{j_1} x^{j_2} \dots x^{j_n} = x^{[\kappa]}, \forall_{x \in \mathbb{R}^n}\} \tag{3.42}$$

where $\kappa : \rho(\kappa) = [i_1, \dots, i_n]$. Then one can obtain that

$$(x(t)^{(m)})^\intercal c_\kappa = (x(t)^{(m)})^\intercal d_\kappa \tag{3.43}$$

where $d_\kappa := \frac{(b^\kappa)^\intercal}{\gamma_\kappa}$. and we can write $x(t)^{(m)}$ in terms of the dual basis of b^κ, i.e.,

$$x(t)^{(m)} = \sum_{\kappa=1}^{p} x^{[\kappa]}(b^\kappa)^\intercal \tag{3.44}$$

where p is given by (3.20). Let $T_i(t)$ denote the transition matrix of the SHS starting at the discrete mode $q_0 = i$, i.e., $x(t) = T_{q_0}(t)x_0$ where

$$T_i(t) = e^{A_{q(t)}(t-t_r)} \dots J_{\xi_{l_0}(i),l_1} e^{A_{\xi_{l_0}(i)}} J_{i,l_0} e^{A_i t},$$

$r = \max\{k \in \mathbb{Z}_{\geq 0} : t_k \leq t\}$, and let

$$w^{\kappa,i}(t) := \mathbb{E}[(T_i(t)')^{(m)} d_\kappa], \quad i \in \mathcal{Q}. \tag{3.45}$$

We can express (3.19), or equivalently (3.39), in terms of $w^{\kappa,i}(t)$, $i \in \mathcal{Q}$. In fact, using (3.38), (3.39), and (3.43) we have that

$$\begin{aligned} \mathbb{E}[(x(t)^\intercal)^{(m)} c_\kappa] &= \mathbb{E}[(x(t)^\intercal)^{(m)} d_\kappa] \\ &= (x_0^\intercal)^{(m)} \mathbb{E}[(T_{q_0}(t)^\intercal)^{(m)} d_\kappa] \\ &= (x_0^\intercal)^{(m)} w^{\kappa,q_0}(t). \end{aligned} \tag{3.46}$$

Third, note that for a matrix $A \in \mathbb{R}^{n\times n}$, the map $T_A : \mathbb{R}^p \mapsto \mathbb{R}^p, y \mapsto v = T_A(y)$ defined by

$$\sum_{\kappa_2=1}^{p} v_{\kappa_2}(b^{\kappa_2})^\intercal = A^{(m)} \sum_{\kappa_2=1}^{p} y_{\kappa_2}(b^{\kappa_2})^\intercal, \tag{3.47}$$

can be simply described by

$$v = \Gamma^m(A)y \tag{3.48}$$

where Γ^m is given by (3.22). In fact, if $A = [A_{ij}] = [(a^1)^\intercal\,(a^2)^\intercal \dots (a^n)^\intercal]^\intercal$, then using (3.38), the definition (3.42), and (3.44), we have that, for $\kappa_1 : \rho(\kappa_1) = [i_1\, i_2, \dots\, i_n]$,

$$\begin{aligned} (Ax)^{[\kappa_1]} &= (a^1)^{(i_1)} \otimes (a^2)^{(i_2)} \otimes \dots \otimes (a^n)^{(i_n)} x^{(m)} \\ &= \frac{1}{\gamma_{\kappa_1}} \sum_{j\in\mathcal{J}_{\kappa_1}} a^{j_1} \otimes a^{j_2} \otimes \dots \otimes a^{j_m} x^{(m)} \\ &= \frac{1}{\gamma_{\kappa_1}} \sum_{j\in\mathcal{J}_{\kappa_1}} a^{j_1} \otimes a^{j_2} \otimes \dots \otimes a^{j_m} \sum_{\kappa_2=1}^{p} (b^{\kappa_2})^\intercal x^{[\kappa_2]} \\ &= \sum_{\kappa_2=1}^{p} B_{\kappa_1\kappa_2} x^{[\kappa_2]} \end{aligned}$$

where

$$B_{\kappa_1\kappa_2} = \frac{1}{\gamma_{\kappa_1}} \sum_{j\in\mathcal{J}_{\kappa_1}} \sum_{i\in\mathcal{J}_{\kappa_2}} A_{j_1 i_1} A_{j_2 i_2} \dots A_{j_n i_n}. \tag{3.49}$$

Applying the tensors $b^{\kappa_1}, 1 \leq \kappa_1 \leq p$ on both sides of (3.47) one also obtains that the linear map (3.47) is described by (3.49) in the basis $(b^j)^\intercal, 1 \leq j \leq p$.

Proof. (of Theorem 18) We start by showing that

$$w^\kappa(t) := (w^{\kappa,1}(t), \dots, w^{\kappa,n_q}(t)), \tag{3.50}$$

where the $w^{\kappa,i}$ are defined in (3.45), satisfies a Volterra equation.

Consider an initial condition $q_0 = i$, $i \in \mathcal{Q}$ and a given time t and partition the probability space into the events $[t_1 \leq t] \cup [t_1 > t]$. We can further partition $[t_1 \leq t]$ into $[t_1 \leq t] = \cup_{\ell=1}^{n_\ell} B_\ell(t) \cup B_0(t)$, where $B_0(t)$ is the event of two transitions triggering at the same time in the interval $[0, t]$, which has probability zero due to Assumption 17, and $B_\ell(t)$ is the event of the transition ℓ being the first to trigger in the interval $[0, t]$, i.e., $B_\ell(t) = [\min\{\bar{h}_0^j, j \in \mathcal{L}\} = \bar{h}_0^\ell = t_1 \leq t] \wedge [\bar{h}_0^j > \bar{h}_0^\ell,\ \ell \neq j]$. Notice that, since the initial state is $q_0 = i$, $\bar{h}_0^j$ is distributed according to $\mu_{i,j}$, for a given $j \in \mathcal{L}$ for which $\mu_{i,j}$ is the zero measure. When transition j does not trigger in state $q_0 = i$, the event $B_j(t)$ is empty in which case $\mu_{i,j}$ is the zero measure. Using this partition we can write

$$\begin{aligned} \mathbb{E}[(T_i(t)^\intercal)^{(m)} d_\kappa] = {} & \mathbb{E}[(T_i(t)^\intercal)^{(m)} d_\kappa \chi_{[t_1 > t]}] \\ & + \sum_{\ell=1}^{n_\ell} \mathbb{E}[(T_i(t)^\intercal)^{(m)} d_\kappa \chi_{B_\ell(t)}] \end{aligned} \tag{3.51}$$

where we denote by $\chi_{x \in A}$ the indicator function of a set A, i.e., $\chi_{x\in A}$ equals 1 if $x \in A$ and 0 otherwise. The first term on the right hand side of (3.51) is given by

$$\begin{aligned} \mathbb{E}[(T_i(t)^\intercal)^{(m)} d_\kappa \chi_{[t_1 > t]}] &= (e^{A_i^\intercal t})^{(m)} d_\kappa \ \mathbb{E}[\chi_{[t_1 > t]}] \\ &= (e^{A_i^\intercal t})^{(m)} d_\kappa\, r_i(t), \\ &= (e^{A_i^\intercal t})^{(m)} \frac{(b^\kappa)^\intercal}{\gamma_\kappa}\, r_i(t) \end{aligned} \tag{3.52}$$

where we used the fact that $\mathbb{E}[\chi_{[t_1>t]}] = \text{Prob}([t_1 > t]) = \Pi_{j=1}^{n_\ell} \text{Prob}[h_0^j > t] = \Pi_{j=1}^{n_\ell} r_{i,j}(t) = r_i(t)$. To obtain an expression for the second term on the right hand side of (3.51), notice first that for a function of the first jump time $G(t_1)$,

$$\begin{aligned} \mathbb{E}[G(t_1)\chi_{B_\ell(t)}] &= \int_0^t \mathbb{E}[G(s)\chi_{[h_0^j > s, j \neq \ell]} | \bar{h}_0^\ell = s] \mu_{i,\ell}(ds) \\ &= \int_0^t G(s) \Pi_{j=1, j\neq \ell}^{n_\ell} r_{i,j}(s) \mu_{i,\ell}(ds) = \int_0^t G(s) \frac{r_i(s)}{r_{i,\ell}(s)} \mu_{i,\ell}(ds). \end{aligned}$$

Notice also that $T_i(t) = \hat{T}_{\xi_\ell(i)}(t - t_1)(E_{i,\ell}(t_1))$ when the transition ℓ is first triggered, where $\hat{T}_{\xi_\ell(i)}(t - t_1)$ is the transition matrix of the SHS from t_1 to t starting the process at $q_1 = \xi_\ell(i)$. Each of the terms of the summation on the right hand side of (3.51) can then be expressed as

$$\begin{aligned} & \mathbb{E}[(T_i(t)^\intercal)^{(m)} d_\kappa \chi_{B_\ell(t)}] \\ & = \int_0^t (E_{i,\ell}(s)^\intercal)^{(m)} \mathbb{E}[(\hat{T}_{\xi_\ell(i)}(t-s)^\intercal)^{(m)} d_\kappa] \frac{r_i(s)}{r_{i,\ell}(s)} \mu_{i,\ell}(ds). \end{aligned} \tag{3.53}$$

where $E_{i,\ell}(s) := J_{i,\ell}e^{As}$. By construction of the process $\mathbb{E}[(\hat{T}_{\xi_\ell(i)}(t-s)^\intercal)^{(m)}] = \mathbb{E}[(T_{\xi_\ell(i)}(t-s)^\intercal)^{(m)}] = w^{\kappa,\xi_\ell(i)}(t-s)$. Replacing (3.52) and (3.53) in (3.51) and noticing that $q_0 = i \in \mathcal{Q}$ is arbitrary we obtain that

$$w^{\kappa,i}(t) = I_W^i(w^\kappa(t)) + (e^{A_i^\intercal t})^{(m)} \frac{(b^\kappa)^\intercal}{\gamma_\kappa} r_i(t), \quad i \in \mathcal{Q}. \tag{3.54}$$

where

$$I_W(w^\kappa(t)) := \sum_{\ell=1}^{n_\ell} \int_0^t (E_{i,\ell}(\tau)^\intercal)^{(m)} w^{\kappa,\xi_\ell(i)}(t-\tau) \frac{r_i(\tau)}{r_{i,\ell}(\tau)} \mu_{i,\ell}(d\tau)$$

An explicit solution to the set of equations (3.54) takes the form

$$w^\kappa(t) = \sum_{k=0}^{\infty} I_W^k((e^{A_i^\intercal t})^{(m)} \frac{(b^\kappa)^\intercal}{\gamma_\kappa} r_i(t)) \tag{3.55}$$

where I_W^k denotes composition, and from (3.55) it is possible to conclude that $w^{\kappa,i}(t)$ belongs to the dual vector space of the symmetric tensors for every $i \in \mathcal{Q}$, and therefore can be written as

$$w^{\kappa,i}(t) = \sum_{j=1}^{p} u_j^{\kappa,i}(t)(b^j)^\intercal \tag{3.56}$$

Replacing (3.56) in (3.54) we get

$$\begin{aligned} \sum_{j=1}^{p} u_j^{\kappa,i}(t)(b^j)^\intercal &= (e^{A_i^\intercal t})^{(m)} (b^\kappa)^\intercal \frac{1}{\gamma_\kappa} r_i(t) \\ &+ \sum_{\ell=1}^{n_\ell} \int_0^t (E_{i,\ell}(\tau)^\intercal)^{(m)} \sum_{j=1}^{p} u_j^{\kappa,\xi_\ell(i)}(t-\tau)(b^j)^\intercal \frac{r_i(\tau)}{r_{i,\ell}(\tau)} \mu_{i,\ell}(d\tau), \\ & \qquad i \in \mathcal{Q}. \end{aligned} \tag{3.57}$$

Applying the tensors $b^i, i \in \{1, \dots, n_\kappa\}$ in (3.57), and using the fact that the maps (3.47) and (3.48) are equivalent, we obtain the set of equations (3.27).

□

Proof. (of Theorem 20)

Similarly to (3.39) we can write

$$\mathbb{E}[\sum_{j=1}^{n} (x_j(t))^m] = \sum_{j=1}^{n} \mathbb{E}[(x(t)^\intercal)^{(m)} \mathrm{e}_j^{(m)}], \tag{3.58}$$

Each term of the summation $\mathbb{E}[(x(t)^\intercal)^{(m)} \mathrm{e}_j^{(m)}]$ can be obtained from Theorem 18, where the index κ in (3.26), (3.27), (18), should be taken as κ_j, given by

$$\kappa_j : \rho(\kappa_j) = \begin{bmatrix} j & j & \dots & j \end{bmatrix}, 1 \le j \le n.$$

Due to the linearity of the Volterra equation we can obtain

$$\sum_{j=1}^{n} \mathbb{E}[x_j(t)^p] = \sum_{l=1}^{p} x_0^{[l]} u_l^{m,q_0}(t), \tag{3.59}$$

where $u^m = (u^{m,1}, \ldots, u^{m,n_q})$ is uniquely determined by the Volterra equation

$$u^m(t) = \int_0^t \Theta^m(u^m(t)) + h^m(t), \quad t \geq 0, \tag{3.60}$$

where $h^m(t) := (h^{m,1}(t), \ldots, h^{m,n_q}(t))$,

$$h^{m,i}(t) = \Gamma^m(e^{A_i^\intercal t}) \sum_{j=1}^{n} \mathrm{e}_{\kappa_j} \frac{r_i(t)}{c}, \; i \in \mathcal{Q}.$$

It is then clear that the limit $\lambda_L(x_0) = \lim_{t\to\infty} \frac{1}{t} \log(\mathbb{E}[\|x(t)\|_m^m])$ exists for every x_0 if and only if the limit

$$\lambda_V := \lim_{t\to\infty} \frac{1}{t} \log(\|u^m(t)\|) \tag{3.61}$$

exists, and $\lambda_V = \sup_{x_0} \lambda_L(x_0) = \lambda_L$. Note also that $\mathbb{E}[\|x(t)\|_m^m] = 0, \forall_{x_0}, t > b > 0$ if and only if $u^m(t) = 0, t > b > 0$ in which case both λ_L and λ_V equal $-\infty$ according to our definitions.

Consider the sets

$$\mathcal{U} : \{T \in \mathcal{T}(m,n) : Tx^{(m)} \geq 0\}$$

and

$$\mathcal{K} : \{x \in \mathbb{R}^p : \sum_{i=1}^{p} x_i b^i \in \mathcal{U}\},$$

where b^i is a basis for $\mathcal{T}(m,n)$ described by (3.41). The set $\mathcal{K}$ is a cone, i.e., $\mathcal{K}$ is a closed convex set such that if $y, z \in \mathcal{K}$ then $\alpha_1 y + \alpha_2 z \in \mathcal{K}$, for $\alpha_1 \geq 0$, $\alpha_2 \geq 0$ and is such that the set $-\mathcal{K} := \{-y : y \in \mathcal{K}\}$ intersects $\mathcal{K}$ only at the zero vector. Moreover $\mathcal{K}$ is a solid cone, which in finite-dimensional spaces is equivalent to being reproducing, i.e., any element $y \in \mathbb{R}^p$ can be written as $y = y_1 - y_2$ where $y_1, y_2 \in \mathcal{K}$ (cf. [56, p. 10]). In fact, let $z \in \mathcal{K}$ be defined as $\sum_{i=1}^{p} \beta_i b^i$, where $\beta_i = \delta$ if $i : \rho(i) = [i \ldots i]$ and $\beta_i = 0$ otherwise. Then, from (3.41), $z = \delta \sum_{j=1}^{n} (e_j)^{(j)}$. Given any $y \in \mathbb{R}^n$, then $y = y_1 - y_2$, for $y_1 = z + y$ and $y_2 = z$, which belong to $\mathcal{K}$ for sufficiently large δ. Likewise one can also prove that $\mathcal{K}^{n_q} := \mathcal{K} \times \cdots \times \mathcal{K} \subset \mathbb{R}^p \times \mathbb{R}^p \times \cdots \times \mathbb{R}^p$, $y = (y_1, \ldots, y_{n_q}) \in \mathcal{K}^{n_q}$ if and only if $y_i \in \mathcal{K}$, $\forall_{1 \leq i \leq n_q}$ is a solid cone. We prove that the Volterra equation (3.60) has a positive kernel with respect to the solid cone $\mathcal{K}^{n_q}$, in the sense of Section 2.5, and therefore we can directly apply Theorem 13 to conclude that, $r_\sigma(\hat{\Theta}^m(a))$ is non-decreasing and that the root with largest real part of $\det(I - \hat{\Theta}^m(z))$ is real and coincide with the unique value a such that $r_\sigma(\hat{\Theta}(a)^m) = 1$. As stated in Theorem 13, the zero a equals λ_V, given by (3.61), which in turn equal the Lyapunov exponent λ_L, provided this zero is not a removable singularity of a given complex function. The proof of this latter statement follows similar steps to the ones provided in Theorem 4 and is therefore omitted.

To prove that the kernel of the Volterra equation (3.60) is a positive operator, we need to prove (cf. Section 2.5) that if $y = (y_1, \dots, y_{n_q}) \in \mathcal{K}^{n_q}$ then $M(s)y \in \mathcal{K}^{n_q}$, for every $s > 0$, where

$$M(s) := \begin{bmatrix} M_{1,1}(s) & \dots & M_{1,n_q}(s) \\ \vdots & \ddots & \vdots \\ M_{n_q,1}(s) & \dots & M_{n_q,n_q}(s) \end{bmatrix},$$

and

$$M_{i,j}(s) := \sum_{\ell=1}^{n_\ell} \Gamma^m(e^{A^\intercal s} J_{i,\ell}^\intercal) \chi_{\xi_\ell(i)=j} \frac{r_i(s)}{r_{i,\ell}(s)}.$$

Note that if $(z_1, z_2, \dots, z_{n_q}) = M(s)y$ then for a given s, each $z_i, 1 \leq i \leq n_q$ can be written as a sum of terms taking the form $\Gamma^m(e^{A^\intercal s} J_{i,\ell}^\intercal) y_j$ multiplied by positive scalars. Thus to prove that $z_i \in \mathcal{K}$ and hence $z \in \mathcal{K}^{n_q}$, it suffices to prove that $w = \Gamma^m(B)y \in \mathcal{K}$ for an arbitrary $1 \leq j \leq n_q$, $y \in \mathcal{K}$, and $B \in \mathbb{R}^{n \times n}$. To this effect, using the fact that the maps (3.47) and (3.48) are equivalent, we have that

$$w : \sum_{i=1}^{p} w_i (b^i)^\intercal = B^{(m)} \sum_{i=1}^{p} y_i (b^i)^\intercal$$

belongs to $\mathcal{K}$ because $\sum_{i=1}^{p} w_i(b^i)$ belongs to $\mathcal{U}$ since

$$\begin{aligned} \sum_{i=1}^{p} w_i(b^i) x^{(m)} &= \sum_{i=1}^{p} y_i(b^i)(B^\intercal)^{(m)} x^{(m)} \\ &= \sum_{i=1}^{p} y_i(b^i)(B^\intercal x)^{(m)} \geq 0 \end{aligned}$$

where in the last equality we used the fact that $\sum_{i=1}^{p} y_i(b^i) \in \mathcal{U}$. □

3.6 Further Comments and References

As surveyed in [76], various models of SHSs [47], [38], [25] have been proposed differing on which of these features randomness come into play. See also [45], [41], [44], [66].

The class of SHS with renewal transitions can be viewed as a special case of the SHS model in [45], which in turn is a special case of a piecewise deterministic process [25], and also a special case of a State-Dependent Jump-Diffusion [41, Sec. 5.3]. Stochastic Hybrid systems with renewal transitions can also be viewed as a generalization of a Markov Jump Linear System [66], in which the lengths of times that the system stays in each mode follow an exponential distribution, or as a generalization of an Impulsive Renewal System, in which there is only one discrete mode and one reset map.

4

Asynchronous Renewal Networks

In this chapter we consider networked control systems in which sensors, actuators, and controller transmit through asynchronous communication links, each introducing independent and identically distributed intervals between transmissions. We model these scenarios through impulsive systems with several reset maps triggered by independent renewal processes, i.e., the intervals between jumps associated with a given reset map are identically distributed and independent of the other jump intervals.

We provide stability results for impulsive systems triggered by superposed renewal processes, from which one can directly infer stability properties for the networked control systems just described. Our main result establishes that when the dynamic map a and the reset maps j_ℓ are linear, mean exponential stability is equivalent to the spectral radius of an integral operator being less than one, which can be efficiently tested numerically. To prove this result, we first derive conditions for mean exponential stability for impulsive systems with general non-linear dynamic and reset maps. When specialized to linear dynamic and reset maps, these stability conditions can be expressed in terms of the existence of a solution to an integro-differential equation, which, in turn, is related to the spectral radius of an integral operator. For the general nonlinear case, we show that the origin of the impulsive system is (locally) stable with probability one if the linearization of the impulsive system about zero equilibrium is mean exponentially stable, which justifies the importance of studying the linear case.

To illustrate the applicability of our results, we consider the linearized model of a batch-reactor that we considered in the previous chapter, where we assumed that the sensors transmit in a round-robin fashion through a single shared link. We can now test mean exponential stability in the case where the sensors transmit through two asynchronous links both introducing independent and identically distributed intervals between transmissions.

The remainder of the chapter is organized as follows. The connection between impulsive systems triggered by superposed renewal processes and networked control systems is given in Section 4.1. In Section 4.2 we formally defined an impulsive systems triggered by superposed renewal processes. In Section 4.3 we state and discuss our main results. An example is given in Section 4.4. Our main result concerning stability of linear impulsive systems, is proved in Section 4.5, whereas the proofs of auxiliary results are given in Section 4.6. Further comments and references are provided in Section 4.7.

4.1 Modeling Networked Control Systems with Impulsive Systems

Consider a continuous-time plant and a controller described by

$$\text{Plant:} \qquad \dot{x}_P = f_P(x_P, \hat{u}), \quad y = g(x_P) \tag{4.1}$$

$$\text{Controller:} \quad \dot{x}_C = f_C(x_C, \hat{y}), \quad u = h(x_C, \hat{y}). \tag{4.2}$$

The controller is assumed to yield the closed-loop stable when the plant and the controller are directly connected, i.e., $\hat{u}(t) = u(t)$, $\hat{y}(t) = y(t)$. However, sensors, actuators, and controller may be spatially distributed and linked via communication networks, in which case this ideal assumption is not valid. Suppose that there are $\tilde{n}_y$ sensors, among which, n_y are linked to the controller via n_y communication networks, i.e., each sensor transmits through a different network. Then we can partition y as $y = (y_D, y_{\bar{D}}) := (g_D(x_P), g_{\bar{D}}(x_P)) = g(x_P)$ where

$$y_D = (y_1, \dots, y_{n_y}),$$

comprises the measurement signals $y_i \in \mathbb{R}^{s_i}, 1 \leq i \leq n_y$ of the n_y sensors linked to the controller via a network, and $y_{\bar{D}}$ comprises the measurement signals of the sensors whose connection to the controller is ideal. Therefore, partitioning $\hat{y}$ as $\hat{y} = (\hat{y}_1, \dots, \hat{y}_{n_y}, \hat{y}_{\bar{D}})$, $\hat{y}_i \in \mathbb{R}^{s_i}$, we have $y_{\bar{D}}(t) = \hat{y}_{\bar{D}}(t)$. Likewise, assuming that there are $\tilde{n}_u$ actuators, among which, n_u communicate to the plant via a communication network, we can partition u as $u = (u_D, u_{\bar{D}}) := (h_D(x_C), h_{\bar{D}}(x_C)) = h(x_C)$, where

$$u_D = (u_1, \dots, u_{n_u}),$$

comprises the actuation signals $u_j \in \mathbb{R}^{r_j}, 1 \leq j \leq n_u$ of the n_u actuators linked to the controller via a network, and $u_{\bar{D}}$ comprises the actuation signals of the actuators whose connection to the controller is ideal. Also here, partitioning $\hat{u}$ as $\hat{u} = (\hat{u}_1, \dots, \hat{u}_{n_u}, \hat{u}_{\bar{D}})$, $\hat{u}_j \in \mathbb{R}^{r_j}$, we have $u_{\bar{D}}(t) = \hat{u}_{\bar{D}}(t)$. Let $n_\ell := n_y + n_u$, and for a given $\ell \in \{1, \dots, n_y\}$, let $\{t_k^\ell, k \geq 0\}$ denote the transmission times of the sensor y_ℓ and for a given $\ell \in \{n_y + 1, \dots, n_\ell\}$, let $\{t_k^\ell, k \geq 0\}$ denote the transmission times of the actuator $u_{\ell - n_y}$. Between transmission times we assume that $\hat{y}_i$ and $\hat{u}_j$ remain constant

$$\hat{y}_\ell(t) = \hat{y}_\ell(t_k^\ell), \;\; t \in [t_k^\ell, t_{k+1}^\ell), \;\; 1 \leq \ell \leq n_y, \tag{4.3}$$

and

$$\hat{u}_{\ell-n_y}(t) = \hat{u}_{\ell-n_y}(t_k^\ell), \;\; t \in [t_k^\ell, t_{k+1}^\ell), \;\; 1 \le \ell - n_y \le n_u, \tag{4.4}$$

while at transmission times we have the following update equations

$$\hat{y}_\ell(t_k^\ell) = y_\ell(t_k^{\ell-}), \;\; 1 \le \ell \le n_y, \tag{4.5}$$

and

$$\hat{u}_{\ell-n_y}(t_k^\ell) = u_{\ell-n_y}(t_k^{\ell-}), \;\; 1 \le \ell - n_y \le n_u. \tag{4.6}$$

We assume that in each of the n_ℓ networks that connect sensors and actuators to the controller, the intervals between transmissions are independent and identically distributed, i.e., $\{h_k^\ell := t_{k+1}^\ell - t_k^\ell\}$ are independent and identically distributed random variables, and also independent of the transmission intervals in the remaining networks. Defining,

$$e := (e_y, e_u) := (\hat{y}_D - y_D, \hat{u}_D - u_D), \tag{4.7}$$

and using the fact that we can write $\hat{y} = (e_y + g_D(x_P), g_{\bar{D}}(x_P))$, we can model the networked control system (4.1)-(4.7) as an impulsive system taking the form

$$\begin{aligned} \dot{x}(t) &= a(x(t)), \qquad t \ge 0, \;\; t \ne t_k^\ell, \;\; x(0) = x_0, \\ x(t_k^\ell) &= j_\ell(x(t_k^{\ell-})), \;\; k \ge 1, \;\; \ell \in \mathcal{L} := \{1, \dots, n_\ell\}, \end{aligned}$$

where $x = (x_P, x_C, e)$ is the state;

$$a(x) = \begin{bmatrix} f_P(\, x_P, \;\; (e_u + h_D(x_C, \hat{y}), \;\; h_{\bar{D}}(x_C, \hat{y})) \;) \\ f_C(\, x_C, \;\; (e_y + g_D(x_P), g_{\bar{D}}(x_P)) \;\;) \end{bmatrix} \tag{4.8}$$

models the plant, controller, and error dynamics; and

$$j_\ell(x) = (x_P, x_C, \hat{j}_\ell(e_1), \dots, \hat{j}_\ell(e_{n_\ell})), \tag{4.9}$$

models the transmissions at which the error associated with the transmitting sensor/actuator is reset to zero, i.e., $\hat{j}_\ell(e_i) = 0$, if $i = \ell$, and $\hat{j}_\ell(e_i) = e_i$, if $i \ne \ell$. The model (4.8) is termed in the next section an impulsive system triggered by superposed renewal processes.

4.2 Impulsive Systems Triggered by Superposed Renewal Processes

Impulsive systems triggered by superposed renewal processes are defined as follows

$$\begin{aligned} \dot{x}(t) &= a(x(t)), \qquad t \ge 0, \;\; t \ne t_k^\ell, \;\; x(0) = x_0, \\ x(t_k^\ell) &= j_\ell(x(t_k^{\ell-})), \;\; k \ge 1, \;\; \ell \in \mathcal{L} := \{1, \dots, n_\ell\}, \end{aligned} \tag{4.10}$$

where the duration of the intervals $\{h_k^\ell = t_{k+1}^\ell - t_k^\ell, k \geq 0\}$ between jumps times t_k associated with a given *reset map* j_ℓ, are independent and identically distributed and also independent of the jump intervals $\{h_k^j, j \neq \ell, j \in \mathcal{L}\}$. In (4.10), the notation $x(t_k^{\ell-})$ indicates the limit from the left of x at the point t_k^ℓ.

The maps a and $j_\ell, \ell \in \mathcal{L}$ are assumed to be differentiable and globally Lipschitz and the origin is an equilibrium point, i.e., $a(0) = 0$ and $j_\ell(0) = 0, \forall_{\ell \in \mathcal{L}}$. We denote by n the dimension of the state $x \in \mathbb{R}^n$. We assume that the intervals between jump times $\{h_k^\ell = t_{k+1}^\ell - t_k^\ell, \ell \in \mathcal{L}\}$, are described by a probability density function $f_\ell(t) \geq 0$, with finite support in the interval $[0, \gamma_\ell]$, $\gamma_\ell > 0$. We assume that the $f_\ell(t)$ are differentiable[1] on $(0, \gamma_\ell)$ and we denote the survivor function by

$$r_\ell(s) := \text{Prob}[h_k^\ell > s] = \int_s^{\gamma_\ell} f_\ell(r)dr, k \geq 1, s \in [0, \gamma_\ell],$$

and the hazard rates by

$$\lambda_\ell(\tau_\ell) := \frac{f_\ell(\tau_\ell)}{r_\ell(\tau_\ell)}, \quad \tau_\ell \in B_\ell. \tag{4.11}$$

The system (4.10) is initialized at $t = 0$, where it is subsumed that a time $\tau_\ell := -t_0^\ell$ has elapsed since the last jump associated with map ℓ. In other words, we consider that for each reset map $\ell \in \mathcal{L}$, the first jump times t_1^ℓ, satisfy

$$\text{Prob}([t_1^\ell > s]) = \frac{r_\ell(\tau_\ell + s)}{r_\ell(\tau_\ell)}, s \in [0, \gamma_\ell - \tau_\ell], \tag{4.12}$$

which is the probability that the next jump after $t = 0$ occurs after time s, given that at $t = 0$ a time τ_ℓ has elapsed since the map j was triggered.

We need to define the following auxiliary process

$$v(t) = (v_1(t), \dots, v_{n_\ell}(t)), \;\; v_\ell(t) := t - t_{k_\ell}^\ell, v(0) = \tau, \tag{4.13}$$

where $k_\ell := \max\{k \geq 0 : t_k^\ell \leq t\}$. The process $v(t)$ keeps track of the time elapsed since the last jump associated with each of the reset maps, and therefore at time $t = 0$, $v(0) = \tau = (\tau_1, \dots, \tau_{n_\ell}) = -(t_0^1, \dots, t_0^{n_\ell})$. Note that $v(t) \in B$, where

$$B := B_1 \times \cdots \times B_{n_\ell}. \tag{4.14}$$

and $B_\ell := [0, \gamma_\ell], \;\; \ell \in \mathcal{L}$. We also define

$$\mathbf{x}(t) := (x(t), v(t)), \;\; \mathbf{x}(0) := \mathsf{x} = (x_0, \tau), \tag{4.15}$$

which, as we shall see, is a Markov process, although, in general, $x(t)$ is not. In fact, (4.15) can be constructed as a piecewise deterministic process (cf. Theorem 35 in Section 4.6).

We consider the following definition of stability for (4.10).

[1] We assume differentiability on most functions of interest in the chapter to avoid complicating the proofs of our main results.

Definition 22. We say that (4.10) is *mean exponentially stable (MES)* if there exists constants $c > 0, \alpha > 0$ such that for any initial condition x_0, the following holds

$$\mathbb{E}[x(t)^\intercal x(t)] \leq ce^{-\alpha t} x_0^\intercal x_0, \forall_{t\geq 0}. \tag{4.16}$$

□

4.3 Main Results

We start by providing in Subsection 4.3.1 a stability result for (4.10) with general non-linear dynamic and reset maps. Building upon this result, we are able to establish our main result, presented in Subsection 4.3.2, which provides necessary and sufficient stability conditions when the dynamic and reset maps in (4.10) are linear. In Subsection 4.3.3 we relate the stability of the non-linear impulsive system with that of its linearization.

4.3.1 Non-linear Dynamic and Reset Maps

We start by providing an important auxiliary results, which results from that fact that (4.15) can be constructed as a piecewise deterministic process (cf. Theorem 35 in Section 4.6).

Theorem 23. If V is a differentiable function, such that

$$\mathbb{E}[\sum_{t_k^\ell \leq n} |V(\mathbf{x}(t_k^\ell)) - V(\mathbf{x}(t_k^{\ell-}))|] < \infty, \forall_{n\in\mathbb{N}} \tag{4.17}$$

where $\ell \in \mathcal{L}$, then

$$\mathbb{E}[V(\mathbf{x}(t))] = V(\mathbf{x}) + \mathbb{E}\int_0^t \mathfrak{A}V(\mathbf{x}(s))ds, \forall_{t\geq 0}, \tag{4.18}$$

where

$$\begin{aligned}\mathfrak{A}V(\mathbf{x}) := & \frac{\partial}{\partial\tau}V(\mathbf{x}) + \\ & \mathfrak{X}_x V(\mathbf{x}) + \sum_{\ell=1}^{n_\ell} \lambda_\ell(\tau_\ell)[V((j_\ell(x), \pi_\ell^0(\tau))) - V(\mathbf{x})],\end{aligned} \tag{4.19}$$

for $\mathbf{x} = (x, \tau) \in \mathbb{R}^n \times B$, and $\mathfrak{X}_x V(\mathbf{x}) := \sum_{i=1}^n \frac{\partial V(\mathbf{x})}{\partial x_i} a_i(x)$.

□

The following result establishes general conditions for (4.10) to be MES, providing a stochastic analog of a well-known result for deterministic non-linear systems (cf. [53, Th.4.10]). The proof can be found in the Section 4.6.

Theorem 24. The system (4.10) is MES if and only if there exists a differentiable positive function $V(\mathbf{x})$ which equals zero at zero, and positive constants c_1, c_2, r such that for every $\mathbf{x} = (x, \tau) \in \mathbb{R}^n \times B$,

$$c_1\|x\|^2 \leq V(\mathbf{x}) \leq c_2\|x\|^2 \tag{4.20}$$

$$\mathfrak{A}V(\mathbf{x}) \leq -r\|x\|^2 \tag{4.21}$$

□

4.3.2 Linear Dynamic and Reset Maps

In this subsection, we consider the following linear version of (4.10)

$$\begin{aligned} &\dot{x}(t) = Ax(t), \quad t \geq 0, \ \ t \neq t_k^\ell, \quad x(0) = x_0, \\ &x(t_k^\ell) = J_\ell x(t_k^{\ell-}), \ \ k \geq 1, \ \ \ell \in \mathcal{L}, \end{aligned} \tag{4.22}$$

where $\{h_k^\ell = t_{k+1}^\ell - t_k^\ell, k \geq 1\}$ and t_0^ℓ are as described for (4.10).

Define a variable $\tau = (\tau_1, \ldots, \tau_{n_\ell}) \in B$ where $\tau_j \in B_j$. Furthermore, let $\hat{B}_j = B_1 \times \ldots B_{j-1} \times B_{j+1} \times \cdots \times B_{n_\ell}$ and define the two following maps: π_ℓ that removes the component ℓ from the vector τ, i.e., $\pi_\ell : B \mapsto \hat{B}_\ell$, $\pi_\ell(\tau) = (\tau_1, \ldots, \tau_{\ell-1}, \tau_{\ell+1}, \ldots, \tau_{n_\ell})$; and π_ℓ^0 that sets the component ℓ of the vector τ to zero, i.e.,

$$\pi_\ell^0 : B \mapsto B, \pi_\ell^0(\tau) = (\tau_1, \ldots, \tau_{\ell-1}, 0, \tau_{\ell+1}, \ldots, \tau_{n_\ell}). \tag{4.23}$$

Let $\mathcal{P}_a$ be the Cartesian product space of n_ℓ real symmetric matrix functions, each defined in $\hat{B}_\ell$, i.e., if $P = (P_1(\hat{\tau}_1), \ldots, P_{n_\ell}(\hat{\tau}_{n_\ell})) \in \mathcal{P}_a$, then $P_\ell(\hat{\tau}_\ell)^\intercal = P_\ell(\hat{\tau}_\ell), \forall \hat{\tau}_l \in \hat{B}_\ell$. Sum and multiplication by scalar in $\mathcal{P}_a$ are defined in a natural way and we consider the usual inner product $\langle Q, R\rangle = \sum_{\ell=1}^{n_\ell} \int_{\hat{B}_\ell} Q_\ell(\hat{\tau}_\ell)^\intercal R_\ell(\hat{\tau}_\ell) d\hat{\tau}_\ell$ for $Q, R \in \mathcal{P}$. Then we define the space $\mathcal{P} \subset \mathcal{P}_a$ as the space of elements $P \subset \mathcal{P}_a$ such that $\langle P, P\rangle < \infty$, which can be shown to be a Hilbert space. Let $\mathfrak{L} : \mathcal{P} \mapsto \mathcal{P}$ be the following integral operator

$$(Q_1, \ldots, Q_{n_\ell}) = \mathfrak{L}(P_1, \ldots, P_{n_\ell}) \tag{4.24}$$

where $Q_\ell(\pi_\ell(\tau)) := R(\pi_\ell^0(\tau))$,

$$R(\tau) := \sum_{\ell=1}^{n_\ell} \int_0^{\overline{\gamma}} (J_\ell e^{As})^\intercal P_\ell(\pi_\ell(\tau) + s 1_{n_\ell - 1}) J_\ell e^{As} \frac{\hat{r}_\ell(\tau, s)}{\bar{r}_\ell(\tau)} \frac{f_\ell(\tau_\ell + s)}{r_\ell(\tau_\ell)} ds, \tag{4.25}$$

$\hat{r}_\ell(\tau, s) := \Pi_{j=1, j\neq\ell}^{n_\ell} r_j(\tau_j + s)$, $\bar{r}_\ell(\tau) := \Pi_{j=1, j\neq\ell}^{n_\ell} r_j(\tau_j)$, $\overline{\gamma} := \min\{\gamma_\ell - \tau_\ell, \ell \in \mathcal{L}\}$ and $1_{n_\ell - 1}$ is a vector with $n_\ell - 1$ components set to one.

For example for $n_\ell = 1$, (4.24), (4.25) take the form

$$Q_1 = \int_0^{\gamma_1} (J_1 e^{As})^\intercal P_1 J_1 e^{As} f_1(s) ds, \tag{4.26}$$

where P_1 and Q_1 are symmetric matrices, and in this special case $\mathcal{P}$ is a finite dimensional space since its elements are matrices and not matrix-valued functions. For $n_\ell = 2$, we have

$$\begin{aligned}
Q_1(\tau_2) &= \int_0^{\overline{\gamma}_1} (J_1 e^{As})^\intercal P_1(\tau_2 + s) J_1 e^{As} \frac{r_2(\tau_2 + s)}{r_2(\tau_2)} f_1(s) ds \\
&\quad + \int_0^{\overline{\gamma}_1} (J_2 e^{As})^\intercal P_2(s) J_2 e^{As} r_1(s) \frac{f_2(\tau_2 + s)}{r_2(\tau_2)} ds, \\
&\qquad\qquad \tau_2 \in [0, \gamma_2], \\
Q_2(\tau_1) &= \int_0^{\overline{\gamma}_2} (J_2 e^{As})^\intercal P_2(\tau_1 + s) J_2 e^{As} \frac{r_1(\tau_1 + s)}{r_1(\tau_1)} f_2(s) ds \\
&\quad + \int_0^{\overline{\gamma}_2} (J_1 e^{As})^\intercal P_1(s) J_1 e^{As} r_2(s) \frac{f_1(\tau_1 + s)}{r_1(\tau_1)} ds. \\
&\qquad\qquad \tau_1 \in [0, \gamma_1],
\end{aligned} \tag{4.27}$$

where $\overline{\gamma}_1 = \min(\gamma_1, \gamma_2 - \tau_2)$, and $\overline{\gamma}_2 = \min(\gamma_1 - \tau_1, \gamma_2)$.

Since $\mathfrak{L}$ operates in a real space $\mathcal{P}$, to define its spectral radius we consider the complexification of $\mathcal{P}$ (cf. [56, p. 77]), i.e., the space $\tilde{\mathcal{P}} := \{Q = P + iR : P, R \in \mathcal{P}\}$. For $Q = P + iR \in \tilde{\mathcal{P}}$, one defines $\mathfrak{L}(Q) := \mathfrak{L}(P) + i\mathfrak{L}(R)$. The spectral radius is defined as follows:

$$r_\sigma(\mathfrak{L}) := \max\{|\lambda| : \lambda \in \sigma(\mathfrak{L})\}, \tag{4.28}$$

where $\sigma(\mathfrak{L}) := \{\lambda : \mathfrak{L} - \lambda \mathsf{I} \text{ is not invertible in } \tilde{\mathcal{P}}\}$ denotes the spectrum and I the identity. Note that, defining $\mathcal{P}$ as a real space, and defining the spectral radius of $\mathcal{L}$ acting on $\mathcal{P}$ as in (4.28) is generally different from considering $\mathcal{P}$ to be a complex space, where the matrix components P_ℓ of $P = (P_1, \ldots, P_{n_\ell}) \in \mathcal{P}$ are self-adjoint matrices, and defining the spectral radius of $\mathcal{L}$ as usual. We shall use the first construction since this will allow us to readily use the results for positive operators given in [56] to prove our results in Section 4.5.

The following is the main result of the chapter.

Theorem 25. The system (4.22) is MES if and only if $r_\sigma(\mathfrak{L}) < 1$.

□

The theorem is proved in Section 4.5. We discuss next how one can numerical compute $r_\sigma(\mathfrak{L})$, and some special cases of the impulsive system, for which one can provide alternative stability conditions to Theorem (25).

Computation of the Spectral Radius of $\mathfrak{L}$

One can show that $\mathfrak{L}$ is a compact operator (using, e.g., [14, p. 165, Th. 4.1]) and therefore its spectrum consists either of a finite set of eigenvalues $\lambda : \mathfrak{L}P = \lambda P$ for some $P \in \tilde{\mathcal{P}}$ or a countable set of eigenvalues with no accumulation point other than zero (cf., e.g., [14, p. 117, Th. 2.34]). For simplicity, consider first the case $n_\ell = 2$, where $\mathfrak{L}$ is described by (4.27). A numerical method to compute $r_\sigma(\mathfrak{L})$ is the following. Take a grid of points in the interval $\bar{\tau}_{1j} \in [0, \gamma_1]$, $1 \le j \le n_{d1}$, and $\bar{\tau}_{2j} \in [0, \gamma_2]$, $1 \le j \le n_{d2}$ and consider the map obtained by replacing $P(\tau_1), P(\tau_2)$ in (4.27) by interpolating piecewise polynomials at the points $\{P_1(\bar{\tau}_{2k}), P_2(\bar{\tau}_{1k})\}$, and evaluating the integrals (4.27) at the grid points to obtain $\{Q_1(\bar{\tau}_{2k}), Q_2(\bar{\tau}_{1k})\}$. This yields a finite rank operator, from the chosen space of piecewise polynomials in $\mathcal{P}$, to itself, and assuming $n_{d_1} = n_{d_2} = n_d$, its matrix representation has dimension $2n_d^2 n(n+1)/2$, since $P_i, Q_i, i \in \{2\}$ are symmetric. Denote by L_{n_d}, the finite rank operator obtained. The method just described is known as the collocation method, a special case of the projection method (cf. [14, p. 177]), and one can conclude from the results in [14] that $r_\sigma(L_{n_d}) \to r_\sigma(\mathfrak{L})$ as $n_d \to \infty$, for typical piecewise polynomial approximations, such as the trapezoidal approximation.[1] For general n_ℓ, this method involves computing the spectral radius of a $n_d^{n_\ell} n_\ell n(n+1)/2$ matrix, which means that computing $r_\sigma(\mathfrak{L})$ may require significant computational effort when the number of reset maps is large.

Special Cases

A first special case is when there is only one reset map, i.e., $n_\ell = 1$. In this case $\mathcal{P}$ is simply the finite dimensional space of symmetric matrices and $\mathcal{L}$ is the linear map $P_1 \mapsto Q_1$ between two finite dimensional space defined in (4.26). In this case $r_\sigma(\mathfrak{L}) < 1$ reduces to testing if the spectral radius of a matrix is less than one. This condition can be shown to be equivalent to the stability condition given in [AHS11d], where the case $n_\ell = 1$ is analyzed using a different approach, based on Volterra equations, which does not appear to generalize to the problem considered in this chapter.

A second special case is when the maps A, J_ℓ commute, i.e., $AJ_\ell = J_\ell A$, and $J_\ell J_r = J_r J_\ell$, $\forall_{\ell, r \in \mathcal{L}}$. In this case, the following result, proved in the Section 4.6, provides alternative stability conditions to Theorem 25.

Theorem 26. When the maps A and $J_\ell, \ell \in \mathcal{L}$ commute, the system (4.22) is MES if

$$2\bar{\lambda} + \sum_{\ell=1}^{n_\ell} \alpha_\ell < 0, \tag{4.29}$$

where $\bar{\lambda}$ is the maximum real part of the eigenvalues of A and the α_ℓ are given by

$$\alpha_\ell = \begin{cases} -\infty \text{ , if } r_\sigma(J_\ell^\intercal \otimes J_\ell^\intercal) = 0 \\ a \in \mathbb{R} : \displaystyle\int_0^{\gamma_\ell} e^{-as} f_\ell(s) ds = \frac{1}{r_\sigma(J_\ell^\intercal \otimes J_\ell^\intercal)}, \text{ otherwise} \end{cases} \tag{4.30}$$

□

[1] In fact, from the spectral characterization of compact operators described above, one can conclude from [14, p.232, Th. 5.5 and p.250, Example 5.14]), that the eigenvalues of the compact operators L_{n_d} converge to the eigenvalues of the compact operator $\mathfrak{L}$ which allows us to conclude that $r_\sigma(L_{n_d}) \to r_\sigma(\mathfrak{L})$.

It is important to emphasize, that even for the commuting case , the condition (4.29) is sufficient but not necessary, as shown in the following example.

Example 27. Suppose that $A = [0]_{2\times 2}$, $n_\ell = 1$, $J_1 = \begin{bmatrix} a_1 & 0 \\ 0 & 0 \end{bmatrix}$, and $J_2 = \begin{bmatrix} 0 & 0 \\ 0 & a_2 \end{bmatrix}$, where $a_1 > 1, a_2 > 1$. Then, $\bar{\lambda} = 0$ and, from (4.30), we can conclude that $\alpha_1 > 0$ and $\alpha_2 > 0$. Thus (4.29) does not hold. However, since the state remains constant between jump times $t_k^\ell, \ell \in \{1, 2\}$, and component i is reset to zero when the reset map i is triggered, $i \in \{1, 2\}$, we conclude that $\mathbb{E}[x(t)^\intercal x(t)] = 0, \;\; t > \max(\gamma_1, \gamma_2)$ and therefore (4.22) is MES.

□

A third special case is when the probability densities f_ℓ correspond to exponential distributions, i.e., $r_\ell(y) = e^{-\beta_\ell y}$. Since the support of the probability density functions is not bounded we assume the following

$$2\bar{\lambda} < \sum_{\ell=1}^{n_\ell} \beta_\ell, \tag{4.31}$$

where $\bar{\lambda}$ is the maximum real part of the eigenvalues of A. This assumption can be shown to assure that the expected value of a quadratic function of the state of the system does not go unbounded between jump times. This assertion can be obtained using a similar reasoning to [AHS11d, Th. 3]. The next theorem states that, in this case, we can provide stability conditions in the form of LMIs. The result is proved in Section 4.6.

Theorem 28. Suppose that $r_\ell(y) = e^{-\beta_\ell y}$ and that (4.31) holds. Then the system (4.22) is MES if and only if

$$\exists_{P>0} : A^\intercal P + PA + \sum_{\ell=1}^{n_\ell} \beta_\ell (J_\ell^\intercal P J_\ell - P) < 0 \tag{4.32}$$

□

4.3.3 Stability with Probability One

The following definition is adapted from [60].

Definition 29. We say that the origin of the system (4.10) is *(locally) stable with probability one* if for every $\rho > 0$ and $\epsilon > 0$ there is a $\delta(\rho, \epsilon) > 0$ such that, if $\|x_0\| < \delta(\rho, \epsilon)$ then

$$Prob\{\sup_{\infty > t \geq 0} \|x(t)\| \geq \epsilon\} \leq \rho. \tag{4.33}$$

The following result shows that one can assert stability with probability one of the origin of (4.10), by establishing mean exponential stability for its linearization, which can be tested by Theorem 25. The proof is provided in Section 4.6.

Theorem 30. If (4.22) is MES with $A = \frac{\partial}{\partial x}a(x)_{|x=0}$ and $J_\ell = \frac{\partial}{\partial x}j_\ell(x)_{|x=0}$, $\ell \in \mathcal{L}$ where a and j_ℓ are the non-linear maps in (4.10), then the origin of (4.10) is stable with probability one.

□

This theorem allow us to conclude a property analogous to the one proved in [62] stating that a standard periodic sampled-data connection of a non-linear plant and a non-linear controller is locally stable if the sampled-data connection of the linearization of the plant and of the linearization of the controller is stable. In fact, from Theorem 30, we can conclude that in the setup of Section 4.1, the non-linear networked control system described by (4.8), (4.9) is stable with probability one, if the networked control system obtained by replacing f_P, g, f_C, and h, by their local linearizations about the zero equilibrium is mean exponentially stable, which can be tested by Theorem 25.

4.4 Example - Batch Reactor

This example considers the control of a linearized model of an open loop unstable two-input two-output batch reactor, controlled by a PI controller. The plant and controller take the form (4.1), and (4.2), with $f_P(x_P, \hat{u}) = A_P x_P + B_P \hat{u}$, $g(x_P) = C_P x_P$, and $f_C(x_C, \hat{y}) = A_C x_C + B_C \hat{y}$, $h(x_C, \hat{y}) = C_C x_C + D_C \hat{y}$. The expressions for (A_P, B_P, C_P) and (A_C, B_C, C_C, D_C) are given in (3.36) and (3.37). The actuator is directly connect $\hat{u}(t) = u(t)$. However, the sensors are linked to the plant through communications networks.

In Section 3.4.1, it is assumed that the outputs are sent in a round-robin fashion through a single shared communication network. When the distribution of the intervals between consecutive transmissions is assumed to be, e.g., uniform with a support γ, we can use the results in Chapter 3 to study the stability of this system.

Suppose now that, instead of transmitting the two measurements in a round robin fashion through the same communication network, the two sensors transmit data through two independent communication links. We assume that both links are shared with other users and that the intervals between consecutive transmissions can be modeled by independent processes with support in the interval $[0, \gamma_\ell]$ for the link associated with the output y_ℓ, $l \in \{1, 2\}$. We can cast this system in the framework os Section 4.1, and use the techniques developed here to study the stability in this latter case.

When two links are used to transmit the measurements of the two sensors, we can use Theorem 25 to investigate the stability of the system as a function of the distributions for the intersampling times on each network. The results obtained are summarized in Figure 4.1 for the case of uniform distributions with different supports.

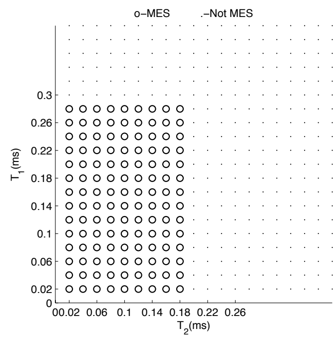


Figure 4.1: MES for various values of the support of a uniform distributions of the transmission intervals of two independent links.

For the case of two independent links we take a grid of values of γ_1 and γ_2, i.e., the support of the distributions between transmissions, and test mean exponential stability of the closed-loop using the numerical method described in Section 4.3 to compute the spectral radius of the operator defined in the Theorem 25. The results obtained are summarized in the Figure 4.1. If the distributions of the two links have the same support then stability is preserved for every $\gamma_1, \gamma_2 \in [0, \overline{\gamma}]$, with $\overline{\gamma} = 0.18$. It is interesting to compare this with the case of a round-robin single-link protocol where it was shown in Section 3.4.1 that the maximum support of a uniform distribution for which stability could be guaranteed was $\gamma = 0.11$. With a round-robin protocol, this would lead to a distribution between consecutive samples *for the same sensor* that is triangular with support 0.22. However, note that in this case the duration of the intervals between transmissions of the two outputs are not independent, and a different approach must be used to assert stability. If the two links have different supports one can conclude from the Figure 4.1 that the mean exponential stability of the closed-loop is lost for a lower value of the support of the distributions associated with the output y_2 than the value of the support associated with the output y_1.

4.5 Proof of Theorem 25

We prove the Theorem 25 through three steps: (i) we show that specializing the stability conditions of Theorem 24 to the system (4.22), yields mean square stability conditions for (4.22) in terms of the existence of a solution to an integro-differential equation; (ii) we establish that these conditions are equivalent to the existence of a solution to a Fredholm equation; (iii) we prove that (ii) is equivalent to the spectral radius of the integral operator of the Fredholm equation being less than one.

(i) MES for Linear Impulsive Systems

The Theorem 24 can be specialized to (4.22) as follows.

Theorem 31. The system (4.22) is MES if and only if for every differentiable functions $Y(\tau)$ and $Z(\tau)$ such that $a_1 I < Y(\tau) < a_2 I$, $\forall_{\tau\in B}$ and $b_1 I < Z(\tau) < b_2 I$, $\forall_{\tau\in B}$ there exists a differentiable function $X(\tau)$, such that $c_1 I < X(\tau) < c_2 I$, $\forall_{\tau\in B}$, and for every $\tau \in B$,

$$\begin{aligned} &\sum_{\ell=1}^{n_\ell} \frac{\partial}{\partial \tau_\ell} X(\tau) + A^\intercal X(\tau) + X(\tau)A + \\ &\sum_{\ell=1}^{n_\ell} \lambda_\ell(\tau_\ell)[J_\ell^\intercal X(\pi_\ell^0(\tau))J_\ell - X(\tau) + Z(\tau)] + Y(\tau) = 0, \end{aligned} \tag{4.34}$$

where π_ℓ^0 is defined by (4.23), and $a_i, b_i, c_i, i \in \{1,2\}$ are positive constants.

□

Proof. To prove sufficiency we use Theorem 24 and consider the function $V(\mathbf{x}(t)) = x^\intercal(t)X(v(t))x(t)$ where $\mathbf{x} = (x(t), v(t))$ and $X(\tau), \tau \in B$ satisfies (4.34) and the remaining conditions of the theorem. Then from (4.19) we have that

$$\begin{aligned} \mathfrak{A}(x^\intercal X(v)x) = x^\intercal[&\sum_{\ell=1}^{n_\ell} \frac{\partial}{\partial \tau_\ell} X(v) + A^\intercal X(v) + X(v)A + \\ &+ \sum_{\ell=1}^{n_\ell} \lambda_\ell(v_\ell)[J_\ell^\intercal X(\pi_\ell^0(v))J_\ell - X(v)]]x \end{aligned} \tag{4.35}$$

Using (4.34) we obtain

$$\mathfrak{A}(x^\intercal X(v)x) = -x^\intercal[(\sum_{\ell=1}^{n_\ell} \lambda_\ell(v_\ell)Z(v) + Y(v))]x \leq -a_1\|x\|^2,$$

and since $c_1 I \leq X(v) \leq c_2 I$, we have that $c_1\|x\|^2 \leq V(\mathbf{x}) \leq c_2\|x\|^2$. Using Theorem 24, applied to (4.22), we conclude that (4.22) is MES.

Necessity follows by using the same arguments as in the proof of Theorem 24 and noticing that the function (4.69) takes the form $V(\mathsf{x}) = x_0^\intercal X(\tau)x_0$, for $\mathsf{x} = (x_0, \tau) \in \mathbb{R}^n \times B$,

$$X(\tau) = X_1(\tau) + X_2(\tau), \tag{4.36}$$

where

$$\begin{aligned} X_1(\tau) &= \int_0^{+\infty} \mathbb{E}_\tau[\Phi(t)^\intercal Y(v(t))\Phi(t)]dt, \\ X_2(\tau) &= \sum_{k>0, \ell\in\mathcal{L}} \mathbb{E}_\tau[\Phi(t_k^\ell)^\intercal Z(v(t_k^\ell))\Phi(t_k^\ell)], \end{aligned} \tag{4.37}$$

$\Phi(t)$ is the transition matrix of the system (4.22), i.e.,

$$\Phi(t) = e^{A(t-t_r^{\ell_r})} J_{\ell_{r-1}} \dots J_{\ell_1} e^{Ah_1^{\ell_1}} J_{\ell_0} e^{Ah_0^{\ell_0}}, \tag{4.38}$$

where $\{\ell_j \in \mathcal{L}, j \geq 0\}$ is the triggered sequence of reset maps, $r = \max\{k : t_k \leq t\}$ and $\mathbb{E}_\tau$ emphasizes that expectation subsumes that the process $\Phi(t)$ depends on the initial conditions τ of the process $v(t)$. Since from Theorem 24, $c_1\|x_0\|^2 \leq V(\mathsf{x}) \leq c_2\|x_0\|^2$ it follows that $c_1 I \leq X(\tau) \leq c_2 I$. From [25, p.92, Th.(32.2)]) it follows that

$$\mathfrak{A}(x_0^\intercal X_1(\tau)x_0) = -x_0^\intercal Y(\tau)x_0$$

and from [25, p.90,91] we have that

$$\mathbb{E}[x_0^\intercal X_2(\tau)x_0] = \mathbb{E}[\int_0^\infty \lambda_T(v(t))x(t)^\intercal Z(v(t))x(t)],$$

where

$$\lambda_T(\tau) := \sum_{j=1}^{n_\ell} \lambda_j(\tau_j), \tag{4.39}$$

from which one can conclude again from [25, p.92, Th. (32.2)] that

$$\mathfrak{A}(x_0^\intercal X_2(\tau)x_0) = -\lambda_T(\tau)x_0^\intercal Z(\tau)x_0.$$

Thus, for every (x_0, τ), we have

$$\mathfrak{A}(x_0^\intercal X(\tau)x_0) = -x_0^\intercal(Y(\tau) + \lambda_T(\tau)Z(\tau))x_0 \tag{4.40}$$

Computing $\mathfrak{A}(x^\intercal X(v)x)$ from (4.19) we obtain (4.35) which must be equal to (4.40) when $\mathsf{x} = (x_0, \tau)$ is replaced by $\mathbf{x} = (x, v)$, from which we conclude (4.34).

■

□

(ii) Fredholm Equation (25)

Let $\mathcal{U}$ be the space of elements $(U_1(\hat{\tau}_1), \dots, U_{n_\ell}(\hat{\tau}_{n_\ell})) \in \mathcal{P}$ for which $U_\ell(\hat{\tau}_\ell) \geq 0, \forall_{\ell\in\mathfrak{L}}, \forall_{\hat{\tau}_\ell \in B_\ell}$. The space $\mathcal{V} \subset \mathcal{U}$ is defined similarly but requiring $U_\ell(\hat{\tau}_\ell) > 0, \forall_{\ell\in\mathfrak{L}}, \forall_{\hat{\tau}_\ell\in B_\ell}$.

Theorem 32. The system (4.10) is MES if and only if for every differentiable functions $Y(\tau)$ and $Z(\tau)$ such that $a_1 I < Y(\tau) < a_2 I$, $\forall_{\tau\in B}$ and $b_1 I < Z(\tau) < b_2 I$, $\forall_{\tau\in B}$, there exists a solution $P \in \mathcal{V}$ to the Fredholm equation

$$P = \mathfrak{L}(P) + U, \tag{4.41}$$

where $U = (U_1, \dots, U_{n_\ell})$,

$$U_\ell(\pi_\ell(\tau)) := W(\pi_\ell^0(\tau)), \tag{4.42}$$

and

$$\begin{aligned} W(\tau) :=& \sum_{\ell=1}^{n_\ell}\int_0^{\overline{\gamma}} e^{A^\intercal s} Z(\tau + s1_{n_\ell}) e^{As} \frac{\hat{r}_\ell(\tau,s)}{\bar{r}_\ell(\tau)} \frac{f_\ell(\tau_\ell+s)}{r_\ell(\tau_\ell)} ds + \\ & \int_0^{\overline{\gamma}} e^{A^\intercal s} Y(\tau + s1_{n_\ell}) e^{As} \Pi_{\ell=1}^{n_\ell} \frac{r_\ell(\tau_\ell+s)}{r_\ell(\tau_\ell)} ds \end{aligned}.$$

□

Proof. Suppose that (4.10) is MES and therefore there exists a solution X to (4.34) given by (4.36). Let

$$P(\pi_\ell(\tau)) := X(\pi_\ell^0(\tau)), \quad \pi_\ell(\tau) \in B_\ell. \tag{4.43}$$

We prove next that $X(\tau)$ satisfies

$$\begin{aligned} X(\tau) =& W(\tau) + \\ & \sum_{\ell=1}^{n_\ell}\int_0^{\gamma_\ell} (J_\ell e^{As})^\intercal X(\pi_\ell^0(\tau + s1_{n_\ell})) J_\ell e^{As} \frac{\hat{r}_\ell(\tau,s)}{\bar{r}_\ell(\tau)} \frac{f_\ell(\tau_\ell+s)}{r_\ell(\tau_\ell)} ds. \end{aligned} \tag{4.44}$$

Then sufficiency follows by directly using (4.44) in (4.43), and noticing that the fact that $X(\tau)$, given by (4.36), satisfies $X(\tau) > c_1 I$, $\forall_{\tau \in B}$, implies that $P \in \mathcal{V}$.

To this effect, we start by conditioning the integrand in (4.37) on the time of the first jump $t_1 := \min\{t_1^\ell, \ell \in \mathcal{L}\}$,

$$\begin{aligned} \mathbb{E}_\tau[\Phi(t)^\intercal Y(v(t))\Phi(t)] =& \mathbb{E}_\tau[(\Phi(t)^\intercal Y(v(t))\Phi(t))\mathbf{1}_{[t_1>t]}] \\ & + \sum_{\ell=1}^{n_\ell} \mathbb{E}_\tau[(\Phi(t)^\intercal Y(v(t))\Phi(t))\mathbf{1}_{C_\ell(t)}] \end{aligned} \tag{4.45}$$

where $C_\ell(t) = [\min\{t_1^j, j \in \mathcal{L}\} = t_1^\ell = t_1 \le t] \wedge [t_1^j > t_1,\ j \ne \ell]$, is the event that jump ℓ is the first to trigger, given that a trigger has occurred at time t. Using (4.12) the first term on the right hand side of (4.45) is given by $e^{A^\intercal t} Y(\tau + t1_{n_\ell}) e^{At} \Pi_{\ell=1}^{n_\ell} \frac{r_\ell(\tau_\ell+t)}{r_\ell(\tau_\ell)}$. Note that for a function $G(t_1)$,

$$\begin{aligned} \mathbb{E}_\tau[G(t_1)\mathbf{1}_{C_\ell(t)}] =& \int_0^t \mathbb{E}[G(s)\mathbf{1}_{[t_1^j>s, j\ne\ell]} | t_1^\ell = s] \frac{f_\ell(\tau_\ell+s)}{r_\ell(\tau_\ell)} ds \\ =& \int_0^t G(s) \frac{\hat{r}_\ell(\tau,s)}{\bar{r}_\ell(\tau)} \frac{f_\ell(\tau_\ell+s)}{r_\ell(\tau_\ell)} ds \end{aligned}$$

and that $\Phi(t) = \hat{\Phi}_\ell(t-t_1)(J_\ell e^{At_1})$ when the transition $\ell \in \mathcal{L}$ is first triggered, where $\hat{\Phi}_\ell(t-t_1)$ is the transition matrix from t_1 to t starting the process at $\pi_\ell^0(\tau + s1_{n_\ell})$ where π_ℓ^0 is defined by (4.23). Thus

$$\begin{aligned} & \mathbb{E}_\tau[\Phi(t)^\intercal Y(v(t))\Phi(t)\mathbf{1}_{C_\ell(t)}] = \int_0^t (J_\ell e^{As})^\intercal \dots \\ & \mathbb{E}_{\pi_\ell^0(\tau+s1_{n_\ell})}[\hat{\Phi}(t-s)^\intercal Y(v(t-s))\hat{\Phi}(t-s)] J_\ell e^{As} \alpha_\ell(\tau,s) ds, \end{aligned} \tag{4.46}$$

where $\alpha_\ell(\tau,s) := \frac{\hat{r}_\ell(\tau,s)}{\bar{r}_\ell(\tau)}\frac{f_\ell(\tau_\ell+s)}{r_\ell(\tau_\ell)}$. By construction of the process

$$\begin{aligned}\mathbb{E}_{\pi_\ell^0(\tau+s1_{n_\ell})}[(\hat{\Phi}_\ell(t-s)^\intercal Y(v(t-s))\hat{\Phi}_\ell(t-s)] = \\ \mathbb{E}_{\pi_\ell^0(\tau+s1_{n_\ell})}[\Phi(t-s)^\intercal Y(v(t-s))\Phi(t-s)].\end{aligned} \tag{4.47}$$

Replacing (4.47) in (4.46) and (4.46) and (4.45) we obtain

$$X_1(\tau) = \sum_{\ell=1}^{n_\ell} X_1^\ell(\tau) + \int_0^{\bar{\gamma}} e^{A^\intercal t} Y(\tau + t1_{n_\ell}) e^{At} \Pi_{\ell=1}^{n_\ell} \frac{r_\ell(\tau_\ell + t)}{r_\ell(\tau_\ell)} dt \tag{4.48}$$

where

$$\begin{aligned}X_1^\ell(\tau) = \int_0^\infty \int_0^t \mathbb{E}_{\pi_\ell^0(\tau+s1_{n_\ell})}[(J_\ell e^{As})^\intercal (\Phi(t-s)^\intercal Y(v(t-s)) \dots \\ \Phi(t-s)(J_\ell e^{As})\alpha(\tau,s)]dsdt\end{aligned}$$

Changing the order of integration in this last expression we have that (4.48) can be written as

$$\begin{aligned}X_1(\tau) = \int_0^{\overline{\gamma}} e^{A^\intercal t} Y(\tau + s1_{n_\ell}) e^{At} \Pi_{\ell=1}^{n_\ell} \frac{r_\ell(\tau_\ell + t)}{r_\ell(\tau_\ell)} dt + \\ \sum_{\ell=1}^{n_\ell} \int_0^{\overline{\gamma}} (J_\ell e^{As})^\intercal X_1(\pi_\ell^0(\tau+s1_{n_\ell})) J_\ell e^{As} \frac{\hat{r}_\ell(\tau,s)}{\bar{r}_\ell(\tau_\ell)} \frac{f_\ell(\tau_\ell+s)}{r_\ell(\tau_\ell)} ds\end{aligned}$$

With similar computations one can conclude that

$$\begin{aligned}X_2(\tau) = \sum_{\ell=1}^{n_\ell} \int_0^{\overline{\gamma}} e^{A^\intercal s} Z(\tau + s1_{n_\ell}) e^{As} \frac{\hat{r}_\ell(\tau,s)}{\bar{r}_\ell(\tau)} \frac{f_l(\tau_\ell + s)}{r_\ell(\tau_\ell)} ds + \\ \sum_{\ell=1}^{n_\ell} \int_0^{\overline{\gamma}} (J_\ell e^{As})^\intercal X_2(\pi_\ell^0(\tau+s1_{n_\ell})) J_\ell e^{As} \frac{\hat{r}_\ell(\tau,s)}{\bar{r}_\ell(\tau)} \frac{f_\ell(\tau_\ell+s)}{r_\ell(\tau_\ell)} ds\end{aligned}$$

Since $X(\tau) = X_1(\tau) + X_2(\tau)$ adding $X_1(\tau)$ and $X_2(\tau)$ we obtain (4.44).

Conversely, suppose that there exists a solution $P \in \mathcal{V}$ to (4.41). Then one can verify that

$$\begin{aligned}X(\tau) = W(\tau) + \\ \sum_{\ell=1}^{n_\ell} \int_0^{\gamma_\ell} (J_\ell e^{As})^\intercal P_\ell(\hat{\pi}_\ell(\tau) + s1_{n_\ell - 1}) J_\ell e^{As} \frac{\hat{r}_\ell(\tau,s)}{\bar{r}_\ell(\tau)} \frac{f_\ell(\tau_\ell + s)}{r_\ell(\tau_\ell)} ds\end{aligned} \tag{4.49}$$

satisfies all the assumptions of the function $X(\tau)$ of Theorem 31, and therefore (4.10) is MES. In fact, if there exists a solution $P \in \mathcal{V}$ to (4.41) one can obtain an explicit expression for the solution to (4.41) (cf. Theorem 33), which is given by

$$P = \sum_{i=0}^{\infty} \mathfrak{L}^i(U), \tag{4.50}$$

where $\mathfrak{L}^i$ denotes the composite operator obtained by applying i times $\mathfrak{L}$, e.g., $\mathfrak{L}^2(P) = \mathfrak{L}(\mathfrak{L}(P))$ and $\mathfrak{L}^0(P) := P$. From (4.50) we can conclude that P is bounded and differentiable with respect to τ, since we assume that the f_ℓ are differentiable. Then, it is clear that $X(\tau)$ is bounded, $X(\tau) \geq W(\tau) > c_1 I, \forall_{\tau \in B}$, for some $c_1 > 0$ and (4.34) can be obtained by direct computation.

■

□

(iii) Positive Solution of the Fredholm Equation

As a prelude to the next result, we note that $\mathcal{U}$ is a cone in the Hilbert space (and hence Banach space, with the usual norm inherit by the inner product) $\mathcal{P}$, in the sense of [56] since (i) it is closed; (ii) if $U, W \in \mathcal{U}$ then $\alpha_1 U + \alpha_2 W \in \mathcal{U}$ for $\alpha_1 \geq 0$ and $\alpha_2 \geq 0$; and (iii) the set[1] $-\mathcal{U} := \{-P : P \in \mathcal{U}\}$ intersects $\mathcal{U}$ only at the zero vector. Moreover, this cone is reproducing in $\mathcal{P}$, i.e., if $Z \in \mathcal{P}$, then there exists $U, W \in \mathcal{U}$ such that $Z = U - W$ (take for example, $U_i(\tau) = Z_i(\tau) + \epsilon I$ and $W_i(\tau) = \epsilon I$ for sufficiently large $\epsilon > 0$ such that $P_i(\tau) + \epsilon I > 0$ for all $i \in \{1, \dots, n_\ell\}, \tau \in B$). The operator $\mathfrak{L}$ is a positive operator with respect to $\mathcal{U}$, i.e., $\mathfrak{L}(U) \in \mathcal{U}$ if $U \in \mathcal{U}$.

Theorem 33. The equation (4.41) has a solution $P \in \mathcal{V}$ if and only if $r_\sigma(\mathfrak{L}) < 1$.

□

Note that the main result, Theorem 25, can be concluded from Theorems 32, 33.

Proof. Sufficiency follows from the fact that if $r_\sigma(\mathfrak{L}) < 1$ then $P = \sum_{i=0}^{\infty} \mathfrak{L}^i(U)$ exists which is the solution to $P = \mathfrak{L}(P) + U$. Since $\mathfrak{L}$ is a positive operator with respect to $\mathcal{U}$, P is a summation of $U \in \mathcal{V}$ plus elements in $\mathcal{U}$. Thus, taking into account the definitions of $\mathcal{U}$ and $\mathcal{V}$, we conclude that P belongs to $\mathcal{V}$.

To prove necessity, we start by noticing that it is possible to prove that the dual cone (cf. [11, Sec. 2.6][2]) of $\mathcal{U}$ can be identified with itself, i.e., using the nomenclature of [11, Sec. 2.6], $\mathcal{U}$ is self-dual. The proof follows similar arguments used to prove that the cone of positive semi-definite matrices is self-dual (cf. [11, p.52]), and is therefore omitted. From [56, p.22, Th. 2.5], we conclude that the adjoint operator $\mathfrak{L}^*$ is also a positive operator with respect to $\mathcal{U}$, and using [56, Th. 9.2] which states that a completely continuous positive operator with respect to a reproducing cone has an eigenvalue that equals the spectral radius and an eigenvector that belongs to the solid cone, we conclude that there exists $W \in \mathcal{U}$ (other than the zero element) such that

$$\mathfrak{L}^*(W) = r_\sigma(\mathfrak{L}^*)W. \tag{4.51}$$

[1] Recall that addition and multiplications by scalar are defined in a natural way in $\mathcal{P}$, e.g. if $P = (P_1, \dots, P_{n_\ell}) \in \mathcal{P}$ then $-P := (-P_1, \dots, -P_{n_\ell})\}$

[2] The nomenclature used in [56, Ch. 2] is adjoint cone instead of dual cone

In fact, $\mathfrak{L}^*$ is a compact operator, since $\mathfrak{L}$ is a compact operator (cf. [2, p.178]), which can be concluded from [14, p. 165, Th. 4.1], and a compact operator in a Banach space is completely continuous (cf. [2, p.177]). Suppose that $r_\sigma(\mathfrak{L}) \geq 1$ and (4.41) has a solution $P \in \mathcal{V}$. Then $r_\sigma(\mathfrak{L}^*) = r_\sigma(\mathfrak{L}) \geq 1$. Taking the inner product on both sides of (4.41) with $W \in \mathcal{U}$, such that (4.51) holds, yields

$$\begin{aligned}
&\langle W, P\rangle = \langle W, \mathfrak{L}(P)\rangle + \langle W, U\rangle \Leftrightarrow \\
&\langle W, P\rangle = \langle \mathfrak{L}^*(W), P\rangle + \langle W, U\rangle \Leftrightarrow \\
&\langle W, P\rangle\,(1 - r_\sigma(\mathfrak{L}^*)) = \langle W, U\rangle
\end{aligned} \tag{4.52}$$

Now $\langle W, P\rangle \geq 0$, since $W, P \in \mathcal{U}$. Moreover, one can conclude that $\langle W, U\rangle > 0$, since W is different from the zero element in $\mathcal{U}$ and one can conclude from (4.42) that $U \in \mathcal{V}$. Thus, from (4.52) we conclude that $r_\sigma(\mathfrak{L}^*) = r_\sigma(\mathfrak{L}) \geq 1$ leads to a contradiction.

■

□

4.6 Proofs of Auxiliary Results

Proof of Theorems 23) and 24

We start by describing a construction for the process $\mathbf{x}$, described by (4.15), which mimics the construction of a piecewise deterministic process, as described in [25, p. 59]. Let $\Omega := \{u_k^1, u_k^2, k \geq 0\}$ where $\{u_k^1\}$ and $\{u_k^2\}$ are mutually independent and identically distributed random variables uniformly distributed in the interval $[0,1]$. Let also $\phi_x(s, x(t_k))$, $\phi_v(s, v(t_k))$ be the flows at time s of the systems described by $\dot{x}(t) = a(x(t))$ and $\dot{v}(t) = 1$ with initial condition $x(t_k)$ and $v(t_k)$, respectively. Note that $\phi_v(s, v(t_k)) = v(t_k) + s1_{n_\ell}$. Set $k = 0$ and $t_0 = 0$, $\mathbf{x}(t_0) = (x_0, \tau)$, and consider the process $\mathbf{x}(t) = (x(t), v(t))$ obtained by iteratively repeating:

(I) Obtain h_k from

$$h_k = \inf\{t : e^{-\int_0^t \lambda_T(\phi_v(s, v(t_k)))ds} \leq u_k^1\}. \tag{4.53}$$

where λ_T is described by (4.39). Set $t_{k+1} = t_k + h_k$, and for $t \in [t_k, t_{k+1})$ make

$$\mathbf{x}(t) = (\phi_x(t - t_k, x(t_k)), \phi_v(t - t_k, v(t_k))) \tag{4.54}$$

(II) Make $\mathbf{x}(t_{k+1}) = \psi(u_k^2, \mathbf{x}(t_{k+1}^-))$, where

$$\psi(w,\ (x, \tau)\ ) = (j_\ell(x), \pi_\ell^0(\tau))\chi_{w \in (\sum_{j=1}^{\ell-1} \frac{\lambda_j(\tau_j)}{\lambda_T(\tau)}, \sum_{j=1}^{\ell} \frac{\lambda_j(\tau_j)}{\lambda_T(\tau)}]}, \tag{4.55}$$

and $\chi_{x\in A}$ denotes the characteristic function, i.e.,

$$\chi_{w\in A} = \begin{cases} 1 \text{ if } w \in A \\ 0, \text{ if } w \notin A \end{cases}. \tag{4.56}$$

□

Remark 34. Note that (4.53) simply states that

$$\text{Prob}[h_k > s|\mathbf{x}(t_k)] = \Pi_{j=1}^{n_\ell} r_j(s + v_j(t_k)), \ \forall_{k\geq 0}$$

and (4.55) simply states that

$$\text{Prob}[\mathbf{x}(t_k) = (j_\ell(x(t_k^-)), \pi_\ell^0(v(t_k^-)))|\mathbf{x}(t_k^-)] = \frac{\lambda_\ell(\tau_j(t_k^-))}{\lambda_T(\tau(t_k^-))}.$$

We choose to use the description (I) and (II) to mimic the piecewise deterministic process construction in [25, p. 59], which allows us to use the results from [25].

□

The next Theorem establishes the connection between (4.10), (4.13), and piecewise deterministic processes.

Theorem 35. The stochastic process $(x(t), v(t))$, described by (4.10) and (4.13), can be realized in the probability space Ω and constructed as the piecewise deterministic process defined by the steps (I) and (II).

□

Proof. For the process $(x(t), v(t))$, described by (4.10), (4.13), define $\{t_k \geq 0\}$ with $t_k < t_{k+1}, \forall_{k\geq 0}$ as a set containing the union of all the jump times in (4.10), i.e., $\{t_k \geq 0\} = \cup_{\ell=1}^{n_\ell}\{t_{r_\ell}^\ell, r_\ell \geq 0\}$, let $\{h_k := t_{k+1} - t_k, k \geq 0\}, h_{-1} := 0$, and consider the following discrete-time process

$$z_k := (h_{k-1}, x(t_k), v(t_k)). \tag{4.57}$$

There exist a one to one relation between z_k and $\mathbf{x}(t)$, described by (4.10), and (4.13). In fact, given a sample path $(x(t), v(t))$ one can identify the jump times t_k by the times at which $v_\ell(t_k) = 0$ for some ℓ, and from these construct h_k and hence z_k. Conversely, from z_k we can obtain h_k and hence t_k, and construct $(x(t), \tau(t))$ from $(x(t_k), (v(t_k))$ as

$$\begin{aligned}(x(t), v(t)) = (\phi_x(t - t_k, x(t_k)), \phi_v(t - t_k, v(t_k))), \\ t_k \leq t < t_{k+1}.\end{aligned} \tag{4.58}$$

Moreover, z_k is a discrete-time Markov process. To see this, it suffices to prove that

$$\mathrm{Prob}(z_{k+1}\in D|z_r, 0 \le r \le k) = \mathrm{Prob}(z_{k+1}\in D|z_k), \forall_{k\ge 0}, \tag{4.59}$$

where D is a measurable set. Since at t_k, a time $\tau_\ell(t_k)$ has elapsed since the last jump associated with the reset map $\ell \in \mathcal{L}$, the time t_{k+1} equals $t_{k+1} = t_k + h_k$, where

$$h_k := \min_{\ell\in\{1,\dots,n_\ell\}} \{w_k^\ell\},$$

and w_k^ℓ are random variables such that $\mathrm{Prob}(w_k^\ell > s) = \frac{r_\ell(v_\ell(t_k)+s)}{r_\ell(v_\ell(t_k))}$. Thus,

$$\mathrm{Prob}(h_{k+1} > s|z_r, 0 \le r \le k) = \Pi_{\ell=1}^{n_\ell} \frac{r_\ell(v_\ell(t_k) + s)}{r_\ell(v_\ell(t_k))}, \tag{4.60}$$

Let ξ_{k+1} denote which reset map triggers at t_{k+1}, i.e.,

$$\xi_{k+1} := \mathrm{argmin}_{\ell\in\{1,\dots,n_\ell\}} \{w_k^\ell\}.$$

Then,

$$\begin{aligned}&\mathrm{Prob}[\xi_{k+1} = \ell|h_k \in [s, s+\epsilon) \wedge z_r, 0 \le r \le k] = \\ &\quad\frac{\mathrm{Prob}[h_k \in [s, s+\epsilon) \wedge \xi_{k+1} = \ell|z_r, 0 \le r \le k]}{\sum_{j=1}^{n_\ell} \mathrm{Prob}[h_k \in [s, s+\epsilon) \wedge \xi_{k+1} = j|z_r, 0 \le r \le k]}.\end{aligned} \tag{4.61}$$

Now,

$$\begin{aligned}&\mathrm{Prob}[h_k \in [s, s+\epsilon) \wedge \xi_{k+1} = \ell|z_r, 0 \le r \le k] \\ &= \mathrm{Prob}[w_k^\ell \in [s, s+\epsilon) \wedge w_k^j > w_k^\ell, \forall_{j\neq\ell}|z_r, 0 \le r \le k] \\ &= \int_s^{s+\epsilon} \Pi_{j=1,j\neq\ell}^{n_\ell} \frac{r_j(v_j(t_k) + q)}{r_j(v_j(t_k))} \frac{f_\ell(v_\ell(t_k) + q)}{r_\ell(v_\ell(t_k))} dq\end{aligned} \tag{4.62}$$

Replacing (4.62) in (4.61), taking the limit as $\epsilon \to 0$, and dividing the numerator and denominator of the right hand side of (4.61) by $\Pi_{j=1}^{n_\ell} r_j(v_j(t_k + s))$, we obtain

$$\begin{aligned}&\mathrm{Prob}[\xi_{k+1} = \ell|h_k = s \wedge z_r, 0 \le r \le k] \\ &\qquad = \frac{\lambda_\ell(v_\ell(t_k) + s)}{\lambda_T(v(t_k) + s1_{n_\ell})} = \frac{\lambda_\ell(v_\ell(t_{k+1}^-))}{\lambda_T(v(t_{k+1}^-))}.\end{aligned}$$

where $\lambda_T(v(t_{k+1}^-)) := \sum_{\ell=1}^{n_\ell} \lambda_\ell(v_\ell(t_{k+1}^-))$. Thus, we conclude that

$$\begin{aligned}&\mathrm{Prob}[(x(t_k), v(t_k)) = (j_\ell(x(t_k^-)), \pi_\ell^0(v(t_k^-)))|h_k \wedge z_r, 0 \le r \le k] \\ &\qquad = \frac{\lambda_\ell(v_\ell(t_k^-))}{\lambda_T(v(t_k^-))}\end{aligned} \tag{4.63}$$

From (4.63), and (4.60), we conclude that

$$\begin{aligned}&\mathrm{Prob}[h_k \in [c, d], x(t_{k+1}) \in E_x, v(t_{k+1}) \in E_v|z_r, 0 \le r \le k] \\ &= \sum_{\ell=1}^{n_\ell} [\int_c^d \chi_{j_\ell(\phi_x(s,\mathbf{x}(t_k)))\in E_x \wedge \phi_v(s,\mathbf{x}(t_k))\in E_v} \cdots \\ &\qquad (\Pi_{j=1,j\neq\ell}^{n_\ell} \frac{r_j(v_j(t_k) + s)}{r_j(v_j(t_k))}) \frac{f_\ell(v_\ell(t_k) + s)}{r_\ell(v_\ell(t_k))}] ds,\end{aligned} \tag{4.64}$$

where $\chi_{(\dots)}$ denotes the characteristic function (4.56). Note that (4.64) depends only on $z_k = (h_{k-1}, x(t_k), v(t_k))$ and thus the Markov property (4.59) holds.

Consider now the piecewise deterministic process defined by steps (I) and (II) and let

$$y_k := (h_{k-1}, x(t_k), v(t_k))$$

where $h_{-1} = 0$, and h_{k-1}, t_k, $x(t_k)$, $v(t_k)$ are now the variables defined in steps (I) and (II). Then, by construction of the process and Remark 45, we immediately obtain that (4.60), and (4.63), also hold for this process, and this implies that (4.64) also holds for this process. Thus y_k is a Markov process with the same transition probability function, i.e., an alternative realization to z_k. Since there is a one-to-one relation between z_k and the process defined by (4.10), (4.13), and there is a one-to-one relation between y_k and the process defined by the piecewise deterministic process construction described by steps (I) and (II), and both processes are completed in the same way between jump times (see (4.54), (4.58)), we conclude that the process (4.10), (4.13) can be constructed as the piecewise deterministic process specified by steps (I) and (II).

■

□

Proof. (of Theorem 23) Theorem 35 allows us to apply the results available in [25]. In particular, Theorem 23 follows directly from [25, p.33, (14.17)], [25, p.66, Th. (26.14)] and [25, p. 70, Rem. (26.16)]), provided that we can prove that the assumption in [25, p.60, (24.4)] that the expected value of the number of jumps up to a given time t is bounded, which when specialized to the stochastic process $(x(t), v(t))$, described by (4.10), is equivalent to saying that $\mathbb{E}[\sum_{\ell=1}^{n_\ell} N_\ell(t)] < \infty$, where

$$N_\ell(t) := \max\{k \in \mathbb{N} : t_k^\ell \leq t\}. \tag{4.65}$$

This is in fact true, since each $N_\ell(t)$ is a renewal process [79] with intervals between renewal times following a probability density function f_ℓ with no atom points, and therefore $\mathbb{E}[N_\ell(t)] < \infty$ (cf. [79, p. 186]).

■

□

Proof. (of Theorem 24) To prove sufficiency, we use Theorem 23 applied to the function

$$W(\mathbf{x}(t), t) := e^{r_1 t} V(\mathbf{x})$$

where $V(\mathbf{x})$ satisfies (4.20) and (4.21), r_1 is a positive constant such that $r_1 > \frac{r}{c_2}$, and it is implicit that the process $(\mathbf{x}(t), t)$ is a piecewise deterministic process if $\mathbf{x}(t)$ is a piecewise deterministic process (cf. [25, p.84]). We need to show that W satisfies (4.17). Since we assume that a and j_ℓ are globally Lipschitz, we have $\|a(x)\| \leq L_1\|x\|$, $\forall_{x\in\mathbb{R}^n}$ and for every $\ell \in \mathcal{L}$, we have $\|j_\ell(x)\| \leq L_2\|x\|$, $\forall_{x\in\mathbb{R}^n}$. Thus, between jump times,

$$\|x(t)\|^2 \leq e^{2L_1(t-t_k^\ell)}\|x(t_k)\|^2, t \in [t_k^\ell, t_{k+1}^\ell),$$

(cf. [53, p.107, Exercise 3.17]) and at jump times, $\|x(t_k^\ell)\|^2 \leq (L_2)^2\|x(t_k^{\ell-})\|^2$. Thus,

$$\mathbb{E}[\|x(t)\|^2] \leq \mathbb{E}[e^{2L_1 t}\Pi_{\ell=1}^{n_\ell}(L_2)^{2N_\ell(t)}]\|x_0\|^2 \tag{4.66}$$

where $N_\ell(t)$ is the number of jumps associated with the reset map ℓ up to the time t, described by (4.65). We also have

$$\begin{aligned}\mathbb{E}[\sum_{t_k^\ell \leq n} &|W(\mathbf{x}(t_k^\ell)) - W(\mathbf{x}(t_k^{\ell-}))|] \\ &\leq \mathbb{E}[\sum_{t_k^\ell \leq n} c_2 e^{r_1 n}\|x(t_k^\ell))\|^2] \\ &\leq e^{2L_1 n}e^{r_1 n}\|x_0\|^2 \sum_{j=1}^{n_\ell}\sum_{k=0}^{\infty} c_3(L_2)^{2k}\mathbb{E}[\chi_{t_k^\ell \leq n}]\end{aligned} \tag{4.67}$$

where we used (4.20), and (4.66), and $\chi_{(\dots)}$ denotes the characteristic function (4.56). Note that $\mathbb{E}[\chi_{t_k^\ell \leq n}] = \text{Prob}[N_\ell(n) \geq k]$. The fact that the right-hand side of (4.67) is bounded is a direct application of [79, p.186, Th. 3.3.1], and therefore W satisfies (4.17).

From Theorem 23

$$\mathbb{E}(W(\mathbf{x}(t), t)) = W(\mathsf{x}, 0) + \mathbb{E}[\int_0^t \mathfrak{A}W(\mathbf{x}(s), s)ds]$$

for an initial condition $\mathsf{x} = (x_0, \tau)$. From [25, p. 84], we can conclude that

$$\mathfrak{A}W(\mathbf{x}(s), s) = r_1 W(\mathbf{x}(s), s) + e^{r_1 s}\mathfrak{A}V(\mathbf{x}(s))$$

and using (4.21) we obtain

$$\begin{aligned}\mathbb{E}(W(\mathbf{x}(t), t)) &\leq W(\mathsf{x}, 0) + \\ &\mathbb{E}[\int_0^t r_1 W(\mathbf{x}(s), s) - re^{r_1 s}\|x(s)\|ds]\end{aligned}$$

Using (4.20) and interchanging expectation with integral operations, we obtain

$$\mathbb{E}(W(\mathbf{x}(t), t)) \leq W(\mathsf{x}, 0) + (r_1 - \frac{r}{c_2})\int_0^t \mathbb{E}[W(\mathbf{x}(s), s)]ds.$$

which implies, from the integral form of the Gronwall's inequality [3, Lemma 1], that

$$\begin{aligned}\mathbb{E}[V(\mathbf{x}(t))e^{r_1 t}] &= \mathbb{E}[W(\mathbf{x}(t), t)] \\ &\leq \mathbb{E}[W(\mathsf{x}, 0)]e^{(r_1 - \frac{r}{c_2})t} = V(\mathsf{x})e^{(r_1 - \frac{r}{c_2})t}.\end{aligned} \tag{4.68}$$

Note that we can apply the Gronwall's inequality since $r_1 - \frac{r}{c_2} > 0$, and this is the reason to work with W, instead of directly using V. From (4.68), we conclude that

$$\mathbb{E}[V(\mathbf{x}(t))] \leq V(\mathsf{x})e^{-\frac{r}{c_2}t}$$

and using (4.20), we conclude that:

$$\begin{aligned}\mathbb{E}[\|x(t)\|^2] \quad &\leq \quad \frac{1}{c_1}\mathbb{E}[V(\mathbf{x}(t))] \leq \frac{1}{c_1}V(\mathsf{x})e^{-\frac{r}{c_2}t} \\ &\leq \frac{c_2}{c_1}\|x_0\|^2 e^{-\frac{r}{c_2}t}.\end{aligned}$$

Necessity is obtained by proving that

$$V(\mathsf{x}) := V_1(\mathsf{x}) + V_2(\mathsf{x}) \tag{4.69}$$

satisfies (4.20) and (4.21) where

$$\begin{aligned}V_1(\mathsf{x}) \quad &:= \quad \int_0^{+\infty} \mathbb{E}_{\mathsf{x}}[x(s)^{\intercal}Y(v(s))x(s)]ds \\ V_2(\mathsf{x}) \quad &:= \quad \sum_{\ell=1}^{n_\ell}\sum_{k>0} \mathbb{E}_{\mathsf{x}}[x(t_k^{\ell-})^{\intercal}Z(v(t_k^{\ell-}))x(t_k^{\ell-})]ds,\end{aligned}$$

$\mathbb{E}_{\mathsf{x}}$ emphasizes that expectation subsumes that the process starts at an initial condition $\mathsf{x} = (x_0, \tau)$, and $a_1 I < Y(\tau) < a_2 I, \forall \tau \in B$, and $b_1 I < Z(\tau) < b_2 I, \forall \tau \in B$, are differentiable functions.

To see that $V(\mathsf{x}) \geq c_1\|x_0\|^2$ denote by t_1 the time of the first jump, which can be from any of the n_ℓ reset maps, and note that $V(\mathsf{x}) \geq a_1 \int_0^{t_1} \mathbb{E}_{\mathsf{x}}[x(s)^{\intercal}x(s)]ds + b_1\mathbb{E}_{\mathsf{x}}[x(t_1^-)^{\intercal}x(t_1^-)]$ for any $t_1 \geq 0$. Let L_1 be a Lipchitz constant for a, $\|a(x)\| \leq L_1\|x\|$, which exists since we assume a is differentiable. Then we have $x(t)^{\intercal}x(t) \geq x_0^{\intercal}x_0 e^{-2L_1 t}$, $\forall_{t\geq 0}$ (cf. [53, p.107, Exercise 3.17]). Thus $V(\mathsf{x}) \geq c_1 x_0^{\intercal}x_0$ where $c_1 = a_1 \int_0^{t_1} e^{-2L1t}dt + b_1 e^{-2L1t_1} > 0$, $\forall_{t_1>0}$.

To see that $V(\mathsf{x}) \leq c_2\|x_0\|^2$, note that since (4.10) is MES, we have that $\mathbb{E}_{\mathsf{x}}[x(t)^{\intercal}x(t)] \leq a_2 c e^{-\alpha t}x_0^{\intercal}x_0$ for some constant $c > 0$. Thus $V_1(\mathsf{x}) \leq a_2\frac{c}{\alpha}x_0^{\intercal}x_0$, and $V_2(\mathsf{x}) \leq k_2 x_0^{\intercal}x_0$ where $k_2 := \sum_{\ell=1}^{n_\ell}\sum_{k=1}^{\infty} \mathbb{E}[b_2 c e^{-\alpha t_k^\ell}]$. Note that $\mathbb{E}[e^{-\alpha t_k^\ell}] = \eta_0\eta^{k-1}$, where $eta_0 = \mathbb{E}[e^{-\alpha h_k^\ell}]$, $\eta = \mathbb{E}[e^{-\alpha h_{k_1}^\ell}] < 1$, for some $1 \leq k_1 < k$, and therefore $k_2 < \infty$. Thus, $V(\mathsf{x}) \leq c_2\|x_0\|^2$ where $c_2 = a_2\frac{c}{\alpha} + k_2$.

It follows from [25, p. 92, Cor. (32.6)]) that $V_1(\mathsf{x})$ is differentiable since, as required in [25, p. 92, Cor. (32.6)]) $\lambda_T(\tau)$, is continuous. From [25, p.92, Th. (32.2)]

$$\mathfrak{A}V_1(\mathsf{x}) = -x_0^{\intercal}Y(\tau)x_0$$

and from [25, p.90,91] we have that

$$\mathbb{E}[V_2(\mathsf{x})] = \mathbb{E}_{\mathsf{x}}[\int_0^{\infty} \lambda_T(v(t))x(t)^{\intercal}Z(v(t))x(t)dt],$$

from which one can conclude again from [25, p.92, Th. (32.2)] that

$$\mathfrak{A}V_2(\mathsf{x}) = -\lambda_T(\tau)x_0^\intercal Z(\tau)x_0.$$

and that $V_2(\mathsf{x})$ is differentiable (again by [25, Cor.(32.6)]). Thus $V(\mathsf{x})$ is differentiable and $\mathfrak{A}V(\mathsf{x}) \leq -rx_0^\intercal x_0$ for $r = a_1$.

■

□

Proof of Theorems 26 and 28

Proof. (of the Theorem 26) From the explicit solution to (4.22), described by (4.38), and the commuting property, we obtain that

$$\mathbb{E}[x(t)^\intercal x(t)] = x_0^\intercal e^{A^\intercal t} e^{At} \Pi_{\ell=1}^{n_\ell} \mathbb{E}[(J_\ell^\intercal)^{N_\ell(t)} J_\ell^{N_\ell(t)}] x_0, \tag{4.70}$$

where $N_\ell(t)$ is described by (4.65). From [AHS11d, Th. 4], we can conclude for some symmetric matrix C and for α_ℓ described by (4.30), we have that $\mathbb{E}[(J_\ell^\intercal)^{N_\ell(t)} J_\ell^{N_\ell(t)}] \leq Ce^{\alpha_\ell t}$, if $r_\sigma(J_\ell^\intercal \otimes J_\ell^\intercal) \neq 0$. If $r_\sigma(J_\ell^\intercal \otimes J_\ell^\intercal) = 0$, the state is reset to zero after a finite number of jumps, from which the result follows.

■

□

Proof. (of the Theorem 28) Due to the memoryless property of the exponential distribution it is ready to establish that in this special case, the stochastic process defined by (4.22) can be constructed as a stochastic hybrid system with renewal transitions in the sense of [AHS09b] with one discrete mode and n_ℓ reset maps characterized by matrices J_ℓ and probability density functions f_ℓ. In [AHS09b] we provided necessary and sufficient stability conditions for the stability of stochastic hybrid systems with renewal transitions. Although we restricted ourselves to the case where the transition probability density functions have bounded support, the results provided in [AHS09b] can be generalized to the case of unbounded support using the same arguments provided in [AHS09b] for the special case where the stochastic hybrid system has only one discrete mode and only one reset map. Then, under the assumption (4.31), mean exponential stability can be shown to be equivalent to (special case of [AHS09b, Th.9.D])

$$\forall_{R>0} \exists_{P>0} : \sum_{\ell=1}^{n_\ell} Q_\ell(P) - P = -R \tag{4.71}$$

where the solution $P > 0$ is unique,

$$Q_\ell(P) := \int_0^\infty (J_\ell^\intercal e^{As})^\intercal P J_\ell^\intercal e^{As} e^{-\beta_T} \beta_\ell ds,$$

and $\beta_T := \sum_{j=1}^{n_\ell} \beta_j$. Note that $Q_\ell(P)$ is guaranteed to converge due to (4.31). It suffices then to prove that (4.32) is equivalent to (4.71). To this effect, we start by noticing that

$$(A - \frac{\beta_T}{2} I)^\intercal Q_\ell(P) + Q_\ell(P)(A - \frac{\beta_T}{2} I) = -\beta_\ell J_\ell^\intercal P J_\ell \tag{4.72}$$

from which we conclude

$$\begin{aligned}(A - \frac{\beta_T}{2} I)^\intercal (\sum_{\ell=1}^{n_\ell} Q_\ell(P)) + (\sum_{\ell=1}^{n_\ell} Q_\ell(P))(A - \frac{\beta_T}{2} I) \\ = -\sum_{\ell=1}^{n_\ell} \beta_\ell J_\ell^\intercal P J_\ell.\end{aligned} \tag{4.73}$$

Due to (4.31), $A - \frac{\beta_T}{2} I$ has all its eigenvalues in the left-half plane from which we conclude that for every $N > 0$ there exists a unique $M > 0$ such that

$$(A - \frac{\beta_T}{2} I)^\intercal M + M(A - \frac{\beta_T}{2} I) = -N. \tag{4.74}$$

To prove that (4.71) implies (4.32), choose $R = M$ in (4.71), where M satisfies (4.74), and replace $\sum_{\ell=1}^{n_\ell} Q_\ell(P) = P - M$ in (4.73). This yields

$$A^\intercal P + PA + \sum_{\ell=1}^{n_\ell} \beta_\ell (J_\ell^\intercal P J_\ell - P) = -N$$

which implies (4.32).

To prove that (4.32) implies (4.71), note that (4.32) can be written as

$$\exists_{P>0} : (A^\intercal - \frac{\lambda_T}{2})P + P(A - \frac{\lambda_T}{2}) = -\sum_{\ell=1}^{n_\ell} \beta_\ell J_\ell^\intercal P J_\ell - H \tag{4.75}$$

for some matrix $H > 0$. Substracting (4.75) and (4.73) one obtains

$$\begin{aligned}(A^\intercal - \frac{\lambda_T}{2} I)(P - \sum_{\ell=1}^{n_\ell} Q_\ell(P)) + \\ (P - \sum_{\ell=1}^{n_\ell} Q_\ell(P))(A - \frac{\lambda_T}{2} I) = -H.\end{aligned}$$

From (4.74) with N replaced by H, we obtain that $P - \sum_{\ell=1}^{n_\ell} Q_\ell(P) = M > 0$ from which we conclude (4.71).

■

□

Proof of the Main Result

Proof. (of the Theorem 30) We rewrite the dynamic and reset maps in (4.10) as

$$a(x) = Ax + f_e(x), \quad j_\ell(x) = J_\ell x + g_{e\ell}(x), \tag{4.76}$$

where $f_e(x) := a(x) - Ax$, and $g_{e\ell}(x) := j_\ell(x) - J_\ell x$ are differentiable functions such that

$$\frac{\|f_e(x)\|}{\|x\|} \to 0, \quad \text{and} \ \frac{\|g_{e\ell}(x)\|}{\|x\|} \to 0, \text{ as } \|x\| \to 0 \tag{4.77}$$

(cf. [53, p.138]). Let $V(\mathbf{x}(t)) = x(t)^\intercal X(v(t))x(t)$, where $\mathbf{x}(t)$ is described by (4.15) and $X(\tau), \tau \in B$ satisfies (4.34) and $c_1 I < X(\tau) < c_2 I$, $\forall \tau \in B$. Then, there exists $\nu > 0$ such that

$$\mathfrak{A}V(\mathbf{x}) = -r(\mathbf{x}) \leq -d_1\|x\|^2, \ \forall_{x:\|x\|\leq\nu} \tag{4.78}$$

for some $d_1 > 0$, where $\mathfrak{A}V(\mathbf{x})$ is given by (4.19). This expression (4.78) can be obtained by directly replacing (4.76) in (4.19), and using (4.77) in a similar fashion to the proof of an analogous result for deterministic non-linear systems (cf. [53, p.139, Th. 4.7]).

Using similar arguments to [60, Th. 1,Ch. 2], we consider the stopped process $\mathbf{x}_S(t) := \mathbf{x}(t \wedge \tau_m)$, where $t \wedge \tau_m := \min(t, \tau_m)$; $\tau_m = \inf\{t : \mathbf{x}(t) \notin B_m\}$ is the first exit time from the set $B_m := \{\mathbf{x} : V(\mathbf{x}) < m\}$; and $m \leq \frac{\nu^2}{c_1}$ is such that $B_m \subseteq \{(x, \tau) : \|x\| \leq \nu\}]$ and thus, from (4.78). It is easy to see that $\mathbf{x}(t \wedge \tau_m)$ is a piecewise deterministic process and from (4.19) and (4.78),

$$\mathfrak{A}V(\mathbf{x}(t \wedge \tau_m)) = \begin{cases} -r(\mathbf{x}) \text{ if } x \in B_m \\ 0 \text{ otherwise.} \end{cases} \tag{4.79}$$

Considering (4.18) for the process $\mathbf{x}_S$ and using (4.79), we obtain that $\mathbb{E}_\mathbf{x}[V(\mathbf{x}(t \wedge \tau_m))] \leq V(\mathbf{x})$, i.e., $V(\mathbf{x}(t \wedge \tau_m))$ is a super-martingale. From this latter fact, and using the fact that $\lim_{t\to 0} \mathbb{E}_\mathbf{x}[V(\mathbf{x}(t))] = V(\mathbf{x})$, (cf. [25, p.77, Th. (27.6)]), we can apply the super-martingale theorem [60, p.26, Eq. 7.4], and conclude that

$$\mathrm{Prob}_\mathsf{x}[\sup_{\infty > t \geq 0} V(\mathbf{x}(t \wedge \tau_m)) \geq m] \leq \frac{V(\mathsf{x})}{m} \tag{4.80}$$

where Prob_x denotes probability with respect to the Markov process $\mathbf{x}$ started at initial condition $\mathsf{x} = (x_0, \tau)$. Given ϵ, ρ, choose $m = \frac{\min(\nu,\epsilon)^2}{c_1}$, and $\delta = \sqrt{\frac{\rho m}{c_2}}$. Then, for any $\|x_0\| \leq \delta$,

$$\begin{aligned} &\mathrm{Prob}_\mathsf{x}[\sup_{\infty > t \geq 0} \|\mathbf{x}(t \wedge \tau_m))\| \geq \epsilon] \leq \\ &\mathrm{Prob}_\mathsf{x}[\sup_{\infty > t \geq 0} V(\mathbf{x}(t \wedge \tau_m)) \geq m] \leq \frac{V(\mathsf{x})}{m} \leq \frac{c_2\|x_0\|^2}{m} \leq \rho \end{aligned} \tag{4.81}$$

i.e., the origin of (4.10) is stable with probability one.

□

4.7 Further Comments and References

Several references to related work on networked control systems can be found in [43], including systems with asynchronous data transmissions, e.g., [97], and systems with independent and identically distributed intervals between transmission, e.g, [69]. See also [44], [89], [AHS11d], [27]. However, the networked control problem we consider here, and the associated class of impulsive systems, seem to have not been studied in the literature. Stability results for deterministic impulsive systems can be found in [77], [62]. The proof of our results builds upon results for piecewise deterministic systems [25] and the stochastic Lyapunov approach [60].

5

Dynamic Protocols

In this chapter we address the problem of designing a protocol for network access in a networked control system so that: (i) the range of parameters characterizing the networked control system that yield the networked control system stable is enhanced; (ii) optimal performance is achieved with respect to a cost function which is quadratic with respect to the states of the plant. As in the previous chapters, the network is assumed to introduce stochastic intervals between transmissions, delays and packet drops. However, the way we model delays in this chapter differs from previous chapters.

We consider two frameworks for the problem: (i) an emulation framework, in which only the protocol is to be designed; and (ii) a direct design framework, in which both the protocol and the controller are to be designed. In the emulation framework, we propose a dynamic protocol specified as follows. Associated to each node there is a set of quadratic state functions, which are evaluated at a given transmission time. The node allotted to transmit is the one corresponding to the least value of these quadratic state functions. These protocols are shown to be more general than quadratic protocols and Maximum Error First-Try Once Discard protocols, considered in the literature. We establish two stability results for the networked control system, both providing conditions in terms of linear matrix inequalities (LMIs) for investigating the stability in a mean exponential sense of given protocols, and conditions in terms of BMIs to design quadratic state functions, specifying the dynamic protocol, that yield the networked closed loop stable. The first stability result allows to prove that if the networked closed loop is stable for a static protocol then we can provide a dynamic protocol for which the networked closed loop is also stable. This is the main result obtained in this framework and gives an analytical justification on why one should utilize dynamic protocols rather than static. The second stability result allows for obtaining an observer-protocol pair that reconstructs the state of an LTI plant in a mean exponential sense to the case where transmission intervals are stochastic. We illustrate through benchmark examples, that the conditions in this chapter are significantly less conservative than other conditions that previously appeared in the literature.

A drawback of the emulation framework is that full state knowledge is assumed for protocol implementation, which restricts the range of applicability of these protocols. Considering a direct design framework, we tackle the problem of simultaneously designing the scheduling sequence of transmissions and the control law so as to optimize a quadratic objective. The plant is assumed to be disturbed by Gaussian noise, and for simplicity we restrict here the intervals between transmissions to be constant and consider no delays and no packet drops. Using the framework of dynamic programming, we propose a rollout strategy by which the scheduling and control decisions are determined at each transmission time as the ones that lead to optimal performance over a given horizon assuming that from then on controller and sensors transmit in a periodic order and the control law is a standard optimal law for periodic systems. We show that this rollout strategy results in a protocol where scheduling decisions are based on the state estimate and error covariance matrix of a Kalman estimator, and must be determined on-line. The resulting protocol obtained from the rollout algorithm can be implemented in a distributed way both in wireless and wired networks, based on previous data sent from sensors and actuators, as opposed to the requirements of the protocol obtained with the emulation framework where full-state is generally required. It follows by construction of rollout algorithms that our proposed scheduling method can outperform any periodic scheduling of transmissions. We illustrate this fact with an example.

The remainder of the Chapter is organized as follows. In Section 5.1 we address the emulation problem. We address the problem formulation in Subsection 5.1.1, establish our main results in Subsection 5.1.2, and provide an illustrative example in 5.1.3. In Section 5.2 we address the direct design problem. In Subsection 5.2.1 we address the problem formulation, in Subsection 5.2.2 we establish our main results, and in Subsection 5.2.3 we give an illustrative example. We provide the proof of the main results in Section 5.3 and we give further comments and references in Section 5.4.

5.1 Emulation

5.1.1 Problem Formulation

We start by introducing the networked control stability problem and then we show that it can be casted into analyzing the stability of an impulsive system.

5.1.1.1 Networked Control Set-up

We consider a networked control system in which sensors, actuators, and a controller, are connected through a communication network, possibly shared with other users. The plant and controller are described by the following state-space model:

$$\text{Plant:} \qquad \dot{x}_P = A_P x_P + B_P \hat{u}, \quad y = C_P x_P \tag{5.1}$$

$$\text{Controller:} \quad \dot{x}_C = A_C x_C + B_C \hat{y}, \quad u = C_C x_C + D_C \hat{y}. \tag{5.2}$$

Following an emulation approach, we assume that the controller has been designed to stabilize the closed loop, when the process and the controller are directly connected, i.e., $\hat{u}(t) = u(t), \ \hat{y}(t) = y(t)$, and we are interested in analyzing the effects of the network on the stability of the closed loop. We denote the times at which a node transmits a message by $\{t_k, k \in \mathbb{N}\}$, and assume that, $\hat{u}$ and $\hat{y}$ are held constant, between transmission times, i.e.,

$$\hat{u}(t) = \hat{u}(t_k), \ \ \hat{y}(t) = \hat{y}(t_k), t \in [t_k, t_{k+1}), \ \ k \geq 0. \tag{5.3}$$

We denote by e the error signal between process output and controller input $(\hat{y} - y)$ and between controller output and process input $(\hat{u} - u)$. In particular,

$$e(t) := (e_u(t), e_y(t)) := (\hat{u}(t) - u(t), \hat{y}(t) - y(t)). \tag{5.4}$$

We assume that while m nodes compete for the network, only one of them is allowed to transmit at each given transmission time. However, in our terminology a single transmitting node could be associated with several entries of the process output y or with several entries of the controller output u. For simplicity, we assume that the sensors and actuators have been ordered in such a way that we can partition the error vectors e_u and e_y as $e_u = (e_{u1}, \ldots, e_{un_u})$, where each $e_{ui}(t) \in \mathbb{R}^{a_i}$ is the error associated with an actuator node $i \in \{1, \ldots, n_u\}$, and $e_y = (e_{y1}, \ldots, e_{yn_y})$, where each $e_{yj}(t) \in \mathbb{R}^{b_j}$ is the error associated with a sensor node $j \in \{1, \ldots, n_y\}$. We let $\mathcal{M} := \{1, \ldots, m\}$, where $m = n_u + n_y$ is the number of nodes and we use the same index $i \in \mathcal{M}$ to label both actuator (for which $1 \leq i \leq n_u$) and sensor nodes (for wish $n_u + 1 \leq i \leq n_y + n_u$). The state of the networked control system is thus defined by the vector $x := (x_P, x_C, e)$, where $x_P \in \mathbb{R}^{n_P}$, $x_C \in \mathbb{R}^{n_C}$, $e \in \mathbb{R}^{n_e}$, and $x \in \mathbb{R}^{n_x}$. We are interested in scenarios for which the following assumptions hold:

(i) The time intervals $\{h_k := t_{k+1} - t_k\}$ are i.i.d. described by a probability measure μ with support on $[0, \gamma], \gamma \in \mathbb{R}_{\geq 0} \cup \{+\infty\}$, i.e., $\text{Prob}(a \leq h_k \leq b) = \int_a^b \mu(dr)$ for a, $b \in [0, \gamma]$.

(ii) Corresponding to a transmission at time t_k there is a transmission delay d_k no greater than $h_k = t_{k+1} - t_k$; A joint stationary probability density χ describes (h_k, d_k), in the sense that

$$\text{Prob}(a \leq h_k \leq b, c \leq d_k \leq d) = \int_a^b \int_c^d \chi(dr, ds) \tag{5.5}$$

where $a, b \in [0, \gamma]$ and $\text{Prob}(a \leq h_k \leq b, c \leq d_k \leq d) = 0$ if $c > b$. In view of (i) and (ii), we have that $\mu([a,\, b]) = \chi([a, b], [0, b])$.

(iii) At each transmission time there is a probability p_{drop} that a packet may not arrive at its destination or that it may arrive corrupted (packet drop).

(iv) The nodes implement one of the two protocols:
Dynamic protocol (DP): This protocol is specified by m_D symmetric matrices $\{R_i, i \in \mathcal{M}_D\}$, $\mathcal{M}_D := \{1, \ldots m_D\}$, $m_D \geq m$. A subset of these matrices $\{R_i, i \in I_j\}$ is associated with node $j \in \mathcal{M}$ where $I_j := \{i_1^j, i_2^j, \ldots, i_{r_j}^j\}$ is an index subset of $\mathcal{M}_D$. These subsets are assumed to be nonempty, i.e., $r_j \geq 1$, disjoint, and the r_j are such that $\sum_{j=1}^{m} r_j = m_D$. The node j allotted to transmit at t_k is determined by the map $\mathsf{d} : \mathbb{R}^{n_x} \mapsto \mathcal{M}$,

$$\mathsf{d}(x(t_k^-)) = \mathsf{d}_1 \circ \mathsf{d}_2(x(t_k^-)), \tag{5.6}$$

where $\mathsf{d}_2 : \mathbb{R}^{n_x} \mapsto \mathcal{M}_D$ is given by

$$\mathsf{d}_2(x(t_k^-)) := \text{argmin}_{i \in \mathcal{M}_D} x(t_k^-)^\intercal R_i x(t_k^-), \tag{5.7}$$

and $\mathsf{d}_1 : \mathcal{M}_D \mapsto \mathcal{M}$ is given by

$$\mathsf{d}_1(i) := \{j : i \in I_j\}. \tag{5.8}$$

In case the minimum in (5.7) is achieved simultaneously for several value of the index i, stability of the networked control system should be guaranteed regardless of the specific choice for the argmin. In view of (5.6), the error e is updated at time t_k according to

$$e(t_k) = (I_{n_e} - \Lambda_{\mathsf{d}(x(t_k^-))})e(t_k^-), \tag{5.9}$$

where

$$\Lambda_i := \text{diag}(\begin{bmatrix}\Omega_i & \Gamma_i\end{bmatrix}), \quad i \in \mathcal{M},$$

$$\Omega_i = \begin{cases} \text{diag}(0_{a_1}, \ldots, 0_{a_{i-1}}, I_{a_i}, 0_{a_{i+1}}, \ldots, 0_{a_{n_u}}), \\ \qquad \text{if } 1 \leq i \leq n_u \\ 0, \text{ if } n_u + 1 \leq i \leq n_u + n_y, \end{cases} \tag{5.10}$$

and

$$\Gamma_i = \begin{cases} 0, \text{ if } 1 \leq i \leq n_u \\ \text{diag}(0_{b_1}, \ldots, 0_{b_{i-1}}, I_{b_i}, 0_{b_{i+1}}, \ldots, 0_{b_{n_u}}), \\ \qquad \text{if } n_u + 1 \leq i \leq n_u + n_y. \end{cases} \tag{5.11}$$

That is, only the components of $\hat{y}$ or $\hat{u}$ associated with the node that transmits are updated by the corresponding components of $y(t_k^-)$ or $u(t_k^-)$.

Static Protocol (SP): The nodes transmit in a m_S-periodic sequence determined by a periodic function

$$\mathsf{s} : \mathbb{N} \mapsto \mathcal{M} \tag{5.12}$$

with period m_S. In this case, the error e is updated at time t_k according to

$$e(t_k) = (I_{n_e} - \Lambda_{\mathsf{s}(k)})e(t_k^-). \tag{5.13}$$

We assume that s is onto, i.e., each node transmits at least once in a period. When $m_S = m$, each node transmits exactly once in a period.

The class of dynamic protocols that we describe in (iv) allow a node to transmit if the state of the networked control system lies on a given region of the state space, partitioned according to quadratic restrictions. This class of protocols boils down to the quadratic protocols introduced in [27] when $m_D = m$. Thus, our definition allows for ampler partitions of the state-space than quadratic protocols, and as we shall see it also allows to obtain that dynamic protocols are in a sense better than static ones. If we make $m_D = m$ and chose $P > 0$ such that $R_i = P - \operatorname{diag}([0_{n_P+n_C}\ \Lambda_i]) > 0$, then (5.6) becomes the usual MEF-TOD protocol, where the node that transmits is the one with the maximum norm of the error $e_i(t)$ between its current value and its last transmitted value.

5.1.1.2 Impulsive systems

Suppose that there are no delays, i.e. $d_k = 0$, and no packet drops, i.e., $p_{\text{drop}} = 0$. Then we can write the networked control system (5.36), (5.2), (5.3), (5.4), in the form of the following impulsive system

$$\begin{aligned} \dot{x}(t) &= Ax(t),\ \ x(0) = x_0 \\ x(t_k) &= J_{\mathsf{p}(x(t_k^-),k)}x(t_k^-), \end{aligned} \tag{5.14}$$

where $x \in \mathbb{R}^{n_x}$ and $t_{k+1} - t_k$ are i.i.d. random variables characterized by the probability density μ, and the map p takes the following form for dynamic and static protocols

$$\text{DP:} \qquad \mathsf{p}(x(t_k^-), k) = \mathsf{d}(x(t_k^-)) \tag{5.15}$$

$$\text{SP:} \qquad \mathsf{p}(x(t_k^-), k) = \mathsf{s}(k). \tag{5.16}$$

For example, the following expressions for A and $\{J_i,\, i \in \mathcal{M}\}$, correspond to the case in which the controller and plant are directly connected and only the outputs are transmitted through the network, i.e., $\hat{u}(t) = u(t)$, $x = (x_P, x_C, \hat{y} - y)$.

$$\begin{aligned}
A &= \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \\
A_{11} &= \begin{bmatrix} A_P + B_P D_C C_P & B_P C_C \\ B_C C_P & A_C \end{bmatrix} \\
A_{12} &= \begin{bmatrix} B_P D_C \\ B_C \end{bmatrix} \\
A_{21} &= -\begin{bmatrix} C_P & 0 \end{bmatrix} A_{11} \\
A_{22} &= -\begin{bmatrix} C_P & 0 \end{bmatrix} A_{12} \\
J_i &= \operatorname{diag}([I_{n_P+n_C} \ \ I_{n_e} - \Lambda_i]), \ \ i \in \mathcal{M}.
\end{aligned} \tag{5.17}$$

This case will be considered in Section 5.1.3. Expressions for the general case considered in Section 5.1.1 can be easily obtained.

To take into account delays and packet drops modeled as described in Section 5.1.1, we consider the following impulsive system

$$\begin{aligned}
\dot{x}(t) &= Ax(t), \ \ x(0) = x_0. \\
x(t_k) &= K^{q_k}_{\mathsf{p}(x(t_k^-),k)} x(t_k^-), \\
x(s_k) &= Lx(s_k^-), \ \ t_k \le s_k \le t_{k+1},
\end{aligned} \tag{5.18}$$

where $\mathsf{p}(x_k, k)$ is defined as in (5.15) for dynamic protocols and as in (5.16) for static protocols. The random variables t_k and s_k are completely defined by the inter-sampling times $h_k := t_{k+1} - t_k$ and by the delays $d_k := s_k - t_k$. The (h_k, d_k) are i.i.d., and are as described by (5.5). The $q_k \in \{1, \dots, n_q\}$ are i.i.d., and such that $\text{Prob}[q_k = j] = w_j$ $\forall j \in \{1, \dots, n_q\}, k \ge 0$. We provide below expressions for A, L, w_i and K_i^j, $i \in \mathcal{M}$, $j \in \{1, \dots, n_q\}$ which model the case where the controller and the plant are directly connected and only the plant outputs are transmitted through the network, i.e., $\hat{u}(t) = u(t)$. The state is now considered to be $x = (x_P, x_C, \hat{y}, v) \in \mathbb{R}^{n_x}$ where $v \in \mathbb{R}^{n_e}$ is an auxiliary vector $(v_1, \dots v_m)$ that is updated with the sampled value $v_j = y_j(t_k)$ at each sampling time t_k at which node j is allowed to transmit. However, the update only takes place if a packet sent at t_k is not dropped and the sampled value v_j is only used to update the value of $\hat{y}_j$ after a transmission delay d_k, at the time $s_k = t_k + d_k$.

$$
\begin{aligned}
A &= \begin{bmatrix} A_{11} & A_{12} & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \\
A_{11} &= \begin{bmatrix} A_P & B_P C_C \\ 0 & A_C \end{bmatrix} \\
A_{12} &= \begin{bmatrix} B_P D_C \\ B_C \end{bmatrix} \\
n_q &= 2,\, w_1 = 1 - p_{\mathrm{drop}},\, w_2 = p_{\mathrm{drop}} \\
K_i^1 &= \begin{bmatrix} I_{n_P} & 0 & 0 & 0 \\ 0 & I_{n_C} & 0 & 0 \\ 0 & 0 & I_{n_e} & 0 \\ \Lambda_i C_P & 0 & 0 & I_{n_e} - \Lambda_i \end{bmatrix}, \\
K_i^2 &= I_{n_P+n_C+2n_e},\, i \in \mathcal{M} \\
L &= \begin{bmatrix} I_{n_P+n_C} & 0 & 0 \\ 0 & 0 & I_{n_e} \\ 0 & 0 & I_{n_e} \end{bmatrix}.
\end{aligned}
\tag{5.19}
$$

Again, the expressions for the general case considered in Section 5.1.1 can be easily obtained. It is also important to mention that there are other ways to model the setup with delays and packet drops described in Section 5.1.1. For example one can find a similar model to (5.18) but introduce the dependency on the variable q_k modeling the packet drops in the matrix L.

We say that (5.14) is *mean exponentially stable (MES)* if there exists constants $c > 0$ and $0 < \alpha < 1$ such that for any initial condition x_0, we have that

$$\mathbb{E}[x(t_k)^\intercal x(t_k)] \leq c\alpha^k x_0^\intercal x_0, \forall_{k \geq 0}. \tag{5.20}$$

The same definition of MES is used for the system (5.18). We assume that the following condition holds

$$e^{2\bar{\lambda}(A)t} r(t) < c e^{-\alpha_1 t} \text{ for some } c > 0,\ \alpha_1 > 0. \tag{5.21}$$

where $\bar{\lambda}(A)$ is the real part of the eigenvalues of A with largest real part and $r(t) := \mu((t, \gamma])$ denotes the survivor function. Assuming (5.21), we were able to prove in Chapters 2, 3, considering only static protocols, that (5.20) is equivalent to the more usual notion of mean exponential stability in continuous-time where one requires $\mathbb{E}[x(t)^\intercal x(t)]$ to decrease exponentially. In the present chapter we make no such assertion, although assuming (5.21) is still useful (e.g., (5.21) guarantees that (5.23) is bounded).

5.1.2 Main Results

For simplicity, we assume in Subsection 5.1.2.1 and 5.1.2.2, that there are no delays, i.e., $d_k = 0, \forall_k$, and no packet drops, i.e., $p_{\text{drop}} = 0$, and in Subsection 5.1.2.3 we consider the general case.

5.1.2.1 Stability Result I and dynamic vs. static protocols

The following is our first stability result for (5.14).

Theorem 36. The system (5.14) with dynamic protocol (5.15) is MES if there exist scalars $\{0 \leq p_{ji} \leq 1, j, i \in \mathcal{M}_D\}$, with $\sum_{j=1}^{m_D} p_{ji} = 1, \forall_{i \in \mathcal{M}_D}$, and $n_x \times n_x$ symmetric matrices $\{R_i > 0, i \in \mathcal{M}_D\}$ such that

$$J_{\mathsf{d}_1(i)}^\intercal (\sum_{j=1}^{m_D} p_{ji} E(R_j)) J_{\mathsf{d}_1(i)} - R_i < 0, \forall_{i \in \mathcal{M}_D}, \tag{5.22}$$

where

$$E(R_j) := \int_0^\gamma e^{A^\intercal h} R_j e^{Ah} \mu(dh). \tag{5.23}$$

$\square$

This result can be used to analyze if a given protocol yields the networked control system stable or to synthesize a protocol that achieves this.
Analysis: Note first that a given dynamic protocol specified by $R_i > 0, i \in \mathcal{M}_D$, is equivalent to a dynamic protocol specified by

$$\tilde{R}_i = P + R_i > 0, i \in \mathcal{M}_D, \tag{5.24}$$

where P can be any symmetric matrix such that $P + R_i > 0$. If we replace in (5.22) the matrices R_i by $\tilde{R}_i$, given by (5.24), we obtain that (5.22), (5.24) are LMIs in the variables P and p_{ji} (using the fact that $\sum_{j=1}^{m} p_{ji} = 1, \forall_{i \in \mathcal{M}_D}$).
Synthesis: If we allow R_i to be variables in (5.22), then (5.22) are in general BMIs. In fact, if we chose a basis B_l for the linear space of symmetric matrices, we have $R_i = \sum_{l=1}^{n_s} b_{il} B_l$ and (5.22) depends on the products $p_{ji} b_{il}$. In this case the dynamic protocol, determined by the matrices R_i comes out from the solution to (5.22).

To state the next theorem, we need the following result which can be found in [AHS09b]. Let $[i] := i$ if $i \in \{1, \dots, m_S - 1\}$ and $[i] = 1$ if $i = m_S$. Let $\mathcal{M}_S := \{1, \dots, m_S\}$.

Theorem 37. The system (5.14), with static protocol (5.16) is MES if and only if there exists $n_x \times n_x$ symmetric matrices $\{R_i > 0, i \in \mathcal{M}_S\}$ such that

$$J_{\mathsf{s}(i)}^\intercal E(R_{[i+1]}) J_{\mathsf{s}(i)} - R_i < 0, \forall_{i \in \mathcal{M}_S} \tag{5.25}$$

where $E(R_{[i+1]})$ is given as in (5.23).

From Theorems 36 and 37 we can conclude the following result.

Theorem 38. If the networked control system is MES for a static protocol with period m_S then there exists a dynamic protocol taking the form (5.6), with $m_D = m_S$, that yields the networked control system MES.

Proof. Since the stability conditions of Theorem 37 are necessary and sufficient, there exists a static protocol with period m_S that yields the networked control system MES if and only if there exists $\{R_i, i \in \mathcal{M}_S\}$ such that (5.25) holds for (5.14) with matrices defined by (5.17). This implies that if we consider a dynamic protocol with $m_D = m_S$, $I_j = \{k \in \mathcal{M}_S : \mathsf{s}(k) = j\}, j \in \mathcal{M}$, then $\mathsf{d}_1(i) = \mathsf{s}(i)$, for $i \in \mathcal{M}_S$ and (5.22) holds with

$$p_{ji} = \begin{cases} 1, \text{ if } i < m_D \text{ and } j = i+1, \\ 1, \text{ if } i = m_D \text{ and } j = 1, \\ 0 \text{ otherwise} \end{cases}$$

and with $\{R_i, i \in \mathcal{M}_S = \mathcal{M}_D\}$ taken to be the solution to (5.25). □

From the proof of Theorem 38 we see that the matrices $\{R_i, i \in \mathcal{M}_D\}$ that characterize the dynamic protocol mentioned in its statement can be taken to be the solution to (5.25). Note that in the special case where $m_D = m = m_S$, the Theorem 38 states that if there exists a round-robin protocol with period $m_S = m$, i.e., each node only transmits exactly once in a period, that yields the networked control system MES, then one can find a quadratic protocols as introduced in [27] that also yields the networked control system MES.

Remark 39. The fact that the stability conditions of Theorem 37 are necessary and sufficient is key to obtain Theorem 38. In the work [28] a similar reasoning to Theorem 38 can be used to prove that if the stability conditions provided there for quadratic protocols (cf. [28, Th. 3]) hold then so do the stability conditions for a static protocol in the special case where each node transmits only once in a period (cf. [27, Th. 3]). However, since the conditions provided in [28] are only sufficient for the RR protocol, it does not allow to conclude that if a stabilizing static protocol exists then so does a dynamic protocol, as stated in Theorem 38. Although [27] does not explicitly presents stability conditions for a static protocol, the same remarks should apply, since convex over-approximations introduce conservativeness.

5.1.2.2 Stability Result II and observer-protocol design

The following is our second stability result for (5.14).

Theorem 40. The system (5.14) with dynamic protocol (5.15) is MES if there exists an $n_x \times n_x$ symmetric matrix $W > 0$, scalars $\{c_{ij} > 0,\ i, j \in \mathcal{M}_D, i \neq j\}$ and $n_x \times n_x$ matrices R_i, $i \in \mathcal{M}_D$ such that

$$J_{\mathsf{d}(i)}^{\intercal} E(W) J_{\mathsf{d}(i)} + \sum_{j=1, j\neq i}^{m_D} c_{ij}(R_j - R_i) - W < 0, \forall_{i \in \mathcal{M}_D} \tag{5.26}$$

where $E(W) := \int_0^{\gamma} (e^{Ah})^{\intercal} W e^{Ah} \mu(dh)$.

□

Given a quadratic protocol, i.e., specific values for the matrices R_i, testing if (5.26) holds is an LMI feasibility problem. To design a protocol for which mean exponential stability of the networked control system is guaranteed, we can take the $\{R_i,\ i \in \mathcal{M}_D\}$ as additional unknowns and (5.26) should now be viewed as a BMI feasibility problem.

We show next that Theorem 40 allows to extend the observer-protocol design proposed in [24] to the case where the time intervals between transmissions are stochastic.

Observer Design

Suppose that we wish to estimate the state of the following plant

$$\dot{x}_P(t) = A_P x_P(t),\ \ y(t) = C_P x_P(t),\ \ x_P(0) = x_{P0}$$

where the m outputs $y(t) = (y_1, \ldots, y_m)$, $y_i \in \mathbb{R}^{s_i}$ are sent through a network that imposes i.i.d. intervals between transmissions to a remote observer. As in Section 5.1.1, we denote by μ the measure that defines the inter-transmissions times $h_k := t_{k+1} - t_k$. Let $\mathcal{M} := \{1 \ldots, m\}$ and $\Psi_j := \mathrm{diag}([0_{s_1}, \ldots, I_{s_j}, \ldots, 0_{s_m}])$, for $j \in \mathcal{M}$. A natural linear remote observer for this system is defined by

$$\dot{\hat{x}}(t) = A_P \hat{x}(t) + L_k \Psi_{\mathsf{c}(x_e(t_k^-))}(C_P \hat{x}(t_k^-) - y(t_k^-)), \tag{5.27}$$

where the observer gains L_k to be designed are allowed to depend on the index k and the map

$$\mathsf{c}(x_e(t_k^-)) := \mathrm{argmin}_{j \in \mathcal{M}} x_e(t_k^-)^{\intercal} C_P^{\intercal} S_j C_P x_e(t_k^-) \tag{5.28}$$

determines which node transmits at t_k based on the estimation error $x_e(t_k^-) := \hat{x}_P(t_k^-) - x_P(t_k^-)$, where $\{S_j, j \in \mathcal{M}\}$ is a set of m matrices. As argued in [24], the sensors should run a replica of the remote observer to access $\hat{x}(t)$, which allows each node to encode in the message arbitration field $x_e(t_k^-)^{\intercal} C_P^{\intercal} S_j C_P x_e(t_k^-)$, where $C_P x_e(t_k^-) = C_P \hat{x}(t) - y_j(t)$, $j \in \mathcal{M}$.

The resulting estimation error $x_e := \hat{x} - x_P$ evolves according to

$$\dot{x}_e(t) = A_P x_e(t) + L_k \Psi_{\mathsf{c}(x_e(t_k^-))} C_P x_e(t_k). \tag{5.29}$$

We can cast this problem into the framework of (5.14) with dynamic protocol (5.15) by adding an auxiliary variable v that holds the value of $x_e(t_k)$ between transmission times, considering $x = (x_e,\, v)$ and

$$\begin{aligned} A &= \begin{bmatrix} A_P & I_{n_P} \\ 0_{n_P} & 0_{n_P} \end{bmatrix}, \\ J_i &= \begin{bmatrix} I_{n_P} & 0_{n_P} \\ 0_{n_P} & 0_{n_P} \end{bmatrix} + \begin{bmatrix} 0 \\ L_k \end{bmatrix} \begin{bmatrix} \Psi_i C_P & 0 \end{bmatrix}. \\ R_i &= \begin{bmatrix} C_P^{\intercal} S_j C_P & 0 \\ 0 & 0_{n_P} \end{bmatrix} \end{aligned} \tag{5.30}$$

In the following theorem, we propose a method to obtain observer gains L_k that yield the networked control system MES. To state the result we need the following assumption:

$$H(s) := \int_0^s e^{A_P r} dr \text{ is invertible for every } s \in [0, \gamma] \tag{5.31}$$

While this assumption holds generically, it is possible to construct examples where it does not, as in the case where $\gamma > s = 2\pi$ and $A_P = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$, in which case $H(s) = 0$.

Theorem 41. Suppose that (5.31) hold. If there exists a $n_P \times n_P$ symmetric matrix $P > 0$, a $m \times m$ matrix Y, a $n_P \times m$ matrix M, $m \times m$ matrices $\{S_i, i \in \mathcal{M}\}$, and scalars $\{c_{ij} > 0, i, j \in \mathcal{M}\}$, such that

$$\begin{aligned} &F(P) + DM\Psi_i C_P + (DM\Psi_i C_P)^{\intercal} + C_P^{\intercal} Y C_P + \\ &\quad \sum_{j=1, j\neq i}^{m} c_{ij}(C_P^{\intercal} S_j C_P - C_P^{\intercal} S_i C_P) - P < 0, \ \forall_{i \in \mathcal{M}} \end{aligned} \tag{5.32}$$

$$\begin{bmatrix} P & M \\ M^{\intercal} & Y \end{bmatrix} > 0, \tag{5.33}$$

where $F(P) := \int_0^\gamma e^{A_P^{\intercal} r} P e^{A_P r} \mu(dr)$ and $D := \int_0^\gamma e^{A_P r} \mu(dr)$, then we have that the observer gain $L_k = H(h_k)^{-1} P^{-1} M$ yields (5.14) with matrices (5.30) MES.

□

Note that our proposed observer gain L_k depends on the length h_k of the time interval $\{t_{k+1} - t_k\}$, which is not known at time $t_k \le t \le t_{k+1}$ (5.27). In practice this results in a delay in constructing the state estimate that never needs to exceed h_k since the state of the remote observer (5.27) can only be updated with the measurement $y(t_k)$ at the time t_{k+1} at which h_k can be computed.

Similarly to the Theorem 40, the conditions of the Theorem 41 can be used to investigate the stability of a given protocol determined by matrices R_j, in which case the problem reduces to an LMI feasibility problem, or they can be used to design a protocol, in which case one needs to solve a BMI feasibility problem.

Remark 42. When the intervals between transmission are constant, one can show that the stability conditions (5.32) and (5.33) are equivalent to the ones given in [24], where such an assumption is made. In this case, the matrices L_k do not depend on k, and can therefore be computed off-line.

5.1.2.3 Extensions to handle delays and packet drops

Theorems 40 and 36 can be extended to the case where the network introduces packet drops and delays modeled by (5.18) with matrices (5.19). We state these extensions next.

Theorem 43. The system (5.18) with dynamic protocol (5.15) is MES if there exist scalars $\{0 \leq p_{ji} \leq 1,\, j, i \in \mathcal{M}_D\}$, with $\sum_{j=1}^{m_D} p_{ji} = 1, \forall_{i \in \mathcal{M}_D}$, and $n_x \times n_x$ symmetric matrices $\{R_i > 0, i \in \mathcal{M}_D\}$ such that

$$\sum_{l=1}^{n_q} w_l \big(K^l_{\mathsf{d}_1(i)}{}^\intercal (\sum_{j=1}^{m_D} p_{ji} E(R_j)) K^l_{\mathsf{d}_1(i)}\big) - R_i < 0, \forall_{i \in \mathcal{M}_D},$$

where

$$E(R_j) := \int_0^\gamma \int_0^h (e^{Ah-s} L e^{As})^\intercal R_j e^{Ah-s} L e^{As} \chi(dh, ds). \tag{5.34}$$

□

Theorem 44. The system (5.18) with dynamic protocol (5.15) is MES if there exists an $n_x \times n_x$ symmetric matrix $W > 0$, scalars $\{c_{ij} > 0,\, i, j \in \mathcal{M}_D, i \neq j\}$ and $n_x \times n_x$ matrices R_i, $i \in \mathcal{M}_D$ such that

$$\sum_{l=1}^{n_q} w_l (K^l_{\mathsf{d}_1(i)})^\intercal E(W) K^l_{\mathsf{d}_1(i)} + \sum_{j=1, j \neq i}^{m} c_{ij}(R_j - R_i) - W < 0,$$

$\forall_{i \in \mathcal{M}_D}$, where $E(W)$ is defined as in (5.34).

5.1.3 Illustrative Example

In this section we show that Theorems 40 and 44 reduce the conservatism of the results in [44], [87] and [27]. These three works use the same benchmark problem for the control of a batch reactor, where the plant (5.36) and controller (5.2) matrices are given by

$$A_P = \begin{bmatrix} 1.38 & -0.2077 & 6.715 & -5.676 \\ -0.5814 & -4.29 & 0 & 0.675 \\ 1.067 & 4.273 & -6.654 & 5.893 \\ 0.048 & 4.273 & 1.343 & -2.104 \end{bmatrix},$$

$$B_P = \begin{bmatrix} 0 & 0 \\ 5.679 & 0 \\ 1.136 & -3.146 \\ 1.136 & 0 \end{bmatrix}, \quad C_P = \begin{bmatrix} 1 & 0 & 1 & -1 \\ 0 & 1 & 0 & 0 \end{bmatrix}.$$

$$A_C = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} B_C = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} C_C = \begin{bmatrix} -2 & 0 \\ 0 & 8 \end{bmatrix} D_C = \begin{bmatrix} 0 & -2 \\ 5 & 0 \end{bmatrix}.$$

Only the two outputs are sent through the network, i.e., $u(t) = \hat{u}(t)$. The network imposes i.i.d. intervals between transmissions, possibly packet drops and no delays. The networked control closed loop can be written as in (5.14), (5.17) in the absence of drops and as in (5.18)- (5.19) when drops occur. Thus, the stability of the networked control system can be tested by Theorems 36, 40, and 43, 44. The results are shown in the Table 5.1, considering two distributions μ for the inter-transmissions intervals h_k: uniform in the interval $[0, \gamma]$, and exponential with expected value $1/\lambda_{\text{exp}}$.

Table 5.1: Stability results for the batch reactor example-MEF-TOD and Round Robin protocol. NA stands for Not Available

	Dynamic Protocol		Static Protocol	
	no drops	$p = 0.5$	no drops	$p = 0.5$
Max. $\gamma : h_k \sim$ Uni.(γ)				
Results from [88]	NA	NA	NA	NA
Results from [44]	0.0372	0.0170	0.0517	0.0199
Results from [27]	0.11	NA	NA	NA
Ths. 40 and 44	0.0550	0.024	NA	NA
Ths. 36	0.111	NA	NA	NA
Th. 37	NA	NA	0.112	0.0385
Max. $1/\lambda_{\text{exp}}$: $h_k \sim$ Exp.(λ_{exp})				
Results from [88]	0.0095	0.0046	NA	NA
Results from [44]	0.0158	0.00795	0.0217	0.00924
Results from [27]	NA	NA	NA	NA
Ths. 40 and 44	0.0226	0.01124	NA	NA
Ths. 36	0.0357	NA	NA	NA
Th. 37	NA	NA	0.0417	0.0188

From Table 5.1 we can conclude that our results allow to significantly reduce the conservatism of the conditions in [44] and [88] for the same benchmark examples. The results in [27] are very close to the ones obtained with Theorem 36 and both outperform the results obtained with Theorem 40.

In Table 5.2, we show the results obtained by allowing R_i in Theorem 36 to be additional variables, i.e., the protocol is to be designed. Note that Theorem 38 assures that the values obtained with Theorem 36 for the maximum support of a uniform distribution that preserves stability when a dynamic protocol (obtained from solving (5.22)) is utilized, are larger than the ones obtained with the necessary and sufficient conditions provided by Theorem 37 for the static protocol, which matches well with the results in Table 5.2.

Table 5.2: Stability results for the batch reactor example-Protocol design, no packet drops

	Dynamic Prot. Design (Th.36)	Static Prot. (Th.37)
Max.$\gamma : h_k \sim$Uni.(γ)	0.140	0.112

5.2 Direct Design

5.2.1 Problem Formulation

For the direct design problem [1] we assume, for simplicity, evenly spaced times between transmission, i.e., $t_{k+1} - t_k = \tau_s$, $\forall_{k \geq 0}$, for some sampling period τ_s, and we assume that there are no packet drops and no delays. We introduce the scheduling sequence

$$\sigma(k) \in \mathcal{M}, \text{ for } k \in \mathcal{K}, \tag{5.35}$$

which indicates that at the time t_k, $\sigma(k)$ is the node that transmits, where $\mathcal{K} := \{0, \dots, k_F\}$, with $k_F \in \mathbb{N} \cup \{\infty\}$, denotes the set of the time instants of interest. In this case, if we let $x[k] = x_P(t_k^-)$, the discretization of the model for the plant 5.36 at times t_k is given by

$$\begin{aligned} x[k+1] &= Ax[k] + B\tilde{u}[k] \\ \tilde{y}[k] &= Cx[k] \end{aligned} \tag{5.36}$$

where $A = e^{A_P \tau_s}$, $B = \int_0^{\tau_s} e^{A_P s} ds B_P$ and $C = C_P$, and $\tilde{y}[k] = y(t_k^-)$, $\tilde{u}[k] = \hat{u}(t_k^-)$. However, to directly take into account the communication constraints imposed by the network, and to consider external disturbances acting on the plant we consider

$$\begin{aligned} x[k+1] &= Ax[k] + B\Omega_{\sigma(k)} u[k] + w_x[k], \quad k \geq 0 \\ y[k] &= \Gamma_{\sigma(k)} Cx[k] + w_y[k] \end{aligned} \tag{5.37}$$

where Ω_j and Γ_j, $j \in \mathcal{M}$ are given in (5.11) and (5.11). The vectors $w_x[k]$ and $v_y[k]$ are zero mean independent Gaussian processes characterized by the covariance matrices $E[w_x[k]w_x[k]^\intercal] = \Phi_x$ and $E[w_y[k]w_y[k]^\intercal] = \Phi_y$. The initial state $x[0]$ is assumed to be a Gaussian variable, with $\mathbb{E}[x[0]] = \bar{x}_0$, and $\mathbb{E}[x[0]x[0]^\intercal] = \Phi_0$. Note that (5.37) indeed captures the fact that $\sigma(k)$ selects at each time step k which components of the input vector $u[k]$ can influence the plant model, and which components of the output vector $y[k]$

[1] This Section was revised taking into account a follow-up work to the thesis [AHHS11a] and [AHHS11b]

are available for feedback. Moreover, we assume that when the actuation associated with actuator $1 \leq j \leq n_u$ is not available it is set to zero throughout the interval $[t_k, t_{k+1})$, which is already captured in the model (5.37). Note that this model differs from the model considered for there emulation approach, where we considered the input of the plant to be held constant.

We are interested in the problem

$$\min_{\sigma(k),u_k} \mathbb{E}[\sum_{k=0}^{k_F-1} x[k]^\intercal Qx[k] + u[k]^\intercal Ru[k] + x[k_F]^\intercal Qx[k_F]] \tag{5.38}$$

where the matrices Q and R are assumed to be positive definite. The set of admissible control laws $u[k]$ is the set of functions of the information available to the controller from listening the network up to time t_k, i.e., $\{y_\ell, u_\ell, \ell < k\}$.

Remark 45. Note that we do not need to impose the restrictions that $u_j[k] = 0$ at times at which the actuator j does not receive an update, since its value at these times does not affect (5.37). In fact, since $u_j[k]$ is always weighted in the performance cost it is automatically set to zero at these times by the optimal and suboptimal controllers that we propose in the sequel (cf. Remark 42). Likewise, we do not need to assume that the controller does not know $y_i[k]$ at times at which sensor i does not send data to the controller. In fact, $y_i[k]$ is purely noise at these times, and it is automatically not taken into account by the optimal and suboptimal controllers that we propose in the sequel (cf. Remark 42).

5.2.2 Main Results

In this section, we start by showing how to compute the cost (5.38) for a periodic base policy and then we present the scheduling protocol that a rollout policy with this periodic base policy leads to.

5.2.2.1 Base Policy

To defined a periodic scheduling, we consider a set of h consecutive schedules for $\sigma(k)$ denoted by

$$(v_0, \dots, v_{h-1}), \tag{5.39}$$

where $v_\ell \in \mathcal{M}$, $0 \leq \ell \leq h-1$, which are periodically repeated as explained next. If we let $\lfloor k \rfloor_h$ denote the remainder after division of k by h, we have

$$\sigma(k) = \theta_k^\kappa, \quad k \in \{0, \dots, k_F\}, \tag{5.40}$$

where

$$\theta_k^\kappa := v_{\lfloor k+\kappa \rfloor_h},\, k \geq 0, \tag{5.41}$$

for some $0 \leq \kappa \leq h-1$ that characterizes the initial condition of the periodic scheduling θ_k^κ.

Once the periodic scheduling is fixed, we can obtain the optimal standard solution to the problem (5.37), (5.38), which we use as the base policy for the control law. This optimal solution, which can be found e.g., in [1] is summarized in the next Proposition. Let

$$\begin{aligned}
F_j(P) &:= A^\intercal PA + Q \\
&\quad - A^\intercal PB\Omega_j(R + \Omega_j B^\intercal PB\Omega_j)^{-1}\Omega_j B^\intercal PA \\
G_j(P) &:= -(R + \Omega_j B^\intercal PB\Omega_j)^{-1}\Omega_j B^\intercal PA \\
H_j(N) &:= ANA^\intercal + \Phi_x \\
&\quad - ANC^\intercal\Gamma_j(\Phi_y + \Gamma_j CNC^\intercal\Gamma_j)^{-1}\Gamma_j CNA^\intercal \\
M_j(N) &:= -ANC^\intercal\Gamma_j(\Phi_y + \Gamma_j CNC^\intercal\Gamma_j)^{-1}
\end{aligned} \tag{5.42}$$

Proposition 46. The solution to the problem (5.37), (5.38) in the case where the scheduling is periodic and described by (5.40) is given by

$$u[k] = K_k\hat{x}[k] \tag{5.43}$$

where $\hat{x}[k]$ is obtained from

$$\begin{aligned}
\hat{x}[k+1] &= (A + B\Omega_{\theta_k^\kappa}K_k + L_k\Lambda_{\theta_k^\kappa}C)\hat{x}[k] - L_k y[k] \\
u[k] &= K_k\hat{x}[k], \qquad \hat{x}[0] = \bar{x}_0
\end{aligned} \tag{5.44}$$

and the gains K_k and L_k, $k \in \mathcal{K}$ are obtained from

$$\begin{aligned}
P_K &= Q, \quad P_{k-1} = F_{\theta_k^\kappa}(P_k), \quad k_F - 1 \geq k \geq 1 \\
K_{k-1} &= G_{\theta_{k-1}^\kappa}(P_k)
\end{aligned} \tag{5.45}$$

and

$$\begin{aligned}
N_0 &= \Phi_0, \quad N_{k+1} = H_{\theta_k^\kappa}(N_k), \quad 0 \leq k \leq H-1 \\
L_k &= M_{\theta_k^\kappa}(Q_k)
\end{aligned} \tag{5.46}$$

Moreover, the optimal cost (5.38) is given by $J^{\text{base}}(\hat{x}[0], \kappa)$, where

$$\begin{aligned}
&J^{\text{base}}(\hat{x}, \Phi_0, \kappa) = \hat{x}^\intercal P_0\hat{x} + \operatorname{tr}(P_0\Phi_0) + \sum_{k=0}^{k_F-1}\operatorname{tr}(P_{k+1}\Phi_x) + \\
&\sum_{k=0}^{k_F-1}\operatorname{tr}(N_k A^\intercal P_{k+1}B\Omega_{\theta_k^\kappa}(R + \Omega_{\theta_k^\kappa}B^\intercal P_{k+1}B\Omega_{\theta_k^\kappa})^{-1}\Omega_{\theta_k^\kappa}B^\intercal P_{k+1}A).
\end{aligned} \tag{5.47}$$

□

5.2.2.2 Rollout Policy

We propose to choose at each iteration the control law and the node to transmit as the ones that leads to optimal performance over a fixed lookahead horizon, assuming that from then on a periodic base policy is used. In other words, at each iteration ℓ, $0 \leq \ell \leq k_F - 1$, the schedules

$$\sigma(\ell), \sigma(\ell+1), \dots, \sigma(\ell+H-1)$$

and the controls

$$u[\ell], u_[\ell+1], \dots, u[\ell+H-1]$$

are assumed to be free variables, where H denotes the length of the lookahead horizon, while

$$\sigma(\ell+H), \sigma(\ell+H+1), \dots$$

and

$$u(\ell+H), u(\ell+H+1), \dots$$

are fixed and follow a periodic policy as in (5.40), (5.41), and (5.56), (5.44). The free scheduling variables are denoted by $\nu = (\nu_0, \dots, \nu_{H-1})$, i.e.,

$$\sigma(k) = \nu_{k-\ell}, \text{ for } k \in \{\ell, \dots, \ell+H-1\} \tag{5.48}$$

and the fixed scheduling variables can be written as

$$\sigma(k) = \theta^{\kappa}_{k-(\ell+H)}, \text{ for } k \in \{\ell+H, \dots, k_F - 1\}. \tag{5.49}$$

Note that at time $\ell + H$ the base policy is assumed to start at an initial schedule v_κ, determined by κ. We consider that $\kappa \in \{0, \dots, h-1\}$ is also a decision variable, and the decision set is denoted by $\mathcal{I} := \mathcal{M}^H \times \{0, \dots, h-1\}$, i.e, $(\nu, \kappa) \in \mathcal{I}$. The length of the lookahead horizon is a fixed constant, but naturally needs to be adapted when the iteration step is close to the terminal step time k_F, i.e,

$$H(\ell) := \min(H_c, k_F - 1 - \ell), \tag{5.50}$$

where $1 \leq H_c \leq k_F - 1$ is a constant. The dependency of H on ℓ is omitted hereafter. The process is restarted at each step.

As we shall see in the sequel this procedure boils down to the following protocol.

Protocol 1. Given the data $\{y_k, u_k, k < \ell\}$ obtained by listening to the network, at each iteration ℓ choose

$$\sigma(\ell) = \bar{\nu}_{\ell 0}, \tag{5.51}$$

where $\bar{\nu}_{\ell 0}$ is the first entry of the vector $\bar{\nu}_\ell = (\bar{\nu}_{\ell 0}, \dots, \bar{\nu}_{\ell H-1})$ obtained from

$$(\bar{\nu}_\ell, \bar{\kappa}_\ell) = \text{argmin}_{(\nu,\kappa)\in\mathcal{I}} \hat{x}_\ell^\intercal P_{\nu,\kappa,\ell} \hat{x}_\ell + \delta(\nu, \kappa, \ell) + \beta(\nu, \kappa, \ell, N_\ell), \tag{5.52}$$

and the quantities in (5.52) are determined as follows.

- At iteration ℓ, compute $P_{\nu,\kappa,\ell} = \tilde{P}_0$, from

$$\tilde{P}_H = \bar{P}_H, \quad \tilde{P}_j = F_{\nu_j}(\tilde{P}_{j+1}), \quad H-1 \geq j \geq 0, \tag{5.53}$$

where $\bar{P}_H$ is obtained from

$$\bar{P}^{k_F} = Q, \quad \bar{P}_j = F_{\theta_j^\kappa}(\bar{P}_{j+1}), \quad k_F - 1 \geq j \geq H. \tag{5.54}$$

Moreover, compute

$$\delta(\nu,\kappa,\ell) = \sum_{j=0}^{H-1} \operatorname{tr}(\tilde{P}_j)\Phi_x) + \sum_{j=H}^{k_F-1} \operatorname{tr}(\bar{P}_j\Phi_x).$$

- The N_ℓ and $\hat{x}_\ell$ are obtained recursively by at each iteration $k \leq \ell$ updating the following iterations based on past schedules

$$N_0 = \Phi_0, \quad N_{k+1} = H_{\sigma(k)}(N_k), \quad k < \ell, \tag{5.55}$$

and

$$\hat{x}[k+1] = (A + B\Omega_{\sigma(k)}K_k + L_k\Lambda_{\sigma(k)}C)\hat{x}[k] - L_k y_k \tag{5.56}$$

where $L_k = I_{\sigma(k)}(N_k)$, $K_k = G_{\sigma(k)}(\tilde{P}_1)$, and $\tilde{P}_1$ is the matrix obtained by running (5.53), (5.54) at iteration $k < \ell$.

- The function β can be computed by

$$\begin{aligned}\beta(\nu,\kappa,\ell,\ell) = \operatorname{tr}(N_\ell\tilde{P}_0)&\\ + \sum_{j=0}^{H-1}&\operatorname{tr}(\tilde{N}_j(A^\intercal\tilde{P}_{j+1}B\Omega_{\nu_j}(R+\Omega_{\nu_j}B\tilde{P}_{j+1}B\Omega_{\nu_j})^{-1}\Omega_{\nu_j}B^\intercal\tilde{P}_{j+1}A)\\ + \sum_{j=H}^{k_F-1}&\operatorname{tr}(\bar{N}_j(A^\intercal\bar{P}_{j+1}B\Omega_{\theta_j^\kappa}(R+\Omega_{\theta_j^\kappa}B\bar{P}_{j+1}B\Omega_{\theta_j^\kappa})^{-1}\Omega_{\theta_j^\kappa}B^\intercal\bar{P}_{j+1}A).\end{aligned}$$

where $\bar{N}_k$ and $\tilde{N}_k$ are obtained at iteration ℓ from

$$\begin{aligned}\tilde{N}_{j+1} &= H_{\nu_j}(\tilde{N}_j), 0 \leq j \leq H-1, \quad \tilde{N}_0 = N_\ell\\ \bar{N}_{j+1} &= H_{\theta_j^\kappa}(\bar{N}_j), H \leq j \leq k_F - 1, \quad \bar{N}_H = \tilde{N}_H\end{aligned}$$

Moreover, the control law u_ℓ is given by

$$u_\ell = K_\ell\hat{x}_\ell, \tag{5.57}$$

where $K_\ell = G_{\sigma(\ell)}(\tilde{P}_1)$, $\tilde{P}_1$ is the matrix obtained by running (5.53), (5.54) at iteration ℓ, and $\hat{x}[\ell]$ is obtained from (5.56).

□

As stated next, this protocol does in fact correspond to the rollout algorithm described above, and it always outperforms the corresponding periodic base policy.

Theorem 47. The rollout scheduling algorithm with the base policy described in Subsection 5.2.2.1 is determined by the Protocol 1. Moreover

$$J^{\text{dyn}}(\hat{x}, \Phi) \leq \min_{0 \leq \kappa \leq k-1} J^{\text{base}}(\hat{x}, \Phi, \kappa) \tag{5.58}$$

where $J^{\text{dyn}}(x)$ is the cost (5.38) when the scheduling sequence $\sigma(k)$ and the control law u_k for the system (5.37) are as described by Protocol 1, starting with an initial state estimate $x_0 = \hat{x}$ and associated covariance matrix $\Phi_0 = \Phi$.

□

The Protocol 1 can be implemented in the nodes of the network in a distributed way. In fact every node can listen to the network at each transmission time, and therefore compute $\hat{x}_\ell$, and also Q_ℓ, at time t_ℓ. In fact, knowing the data sent over the network at times $\{t_k < t_\ell, k \geq 0\}$, nodes can locally iterate (5.56) and (5.55). Moreover, since the knowledge of $\hat{x}_\ell$, and also Q_ℓ, at time t_ℓ is all one needs to run the Protocol (1), each node can implement the same rollout algorithm independently and therefore determine which node is the next to transmit. At each transmission step, the node that gains arbitration simply transmits and the other nodes do not. This implementation does not assume anything from the network, e.g., whether it is wired or wireless, besides from the fact that the nodes have the ability to listen to other nodes messages. This contrasts to, e.g., the solution proposed in [91] for CAN-BUS networks where the assumption that we have available an arbitration field that is used in the network messages plays an essential role.

Another important observation, is that every information that the nodes require to compute the arbitration function (5.52) is available at time $t_{\ell-1}$ and therefore computations can be done between time $t_{\ell-1}$ and t_ℓ, in a way that there are no delays at time t_ℓ. The feasibility of this implementation depends on the computational capacity of the nodes, i.e., whether they can perform these calculations in a time interval no larger than one sampling interval h.

Remark 48. The equations (5.56) and (5.57). confer what was stated intuitively in Remark 42. In fact, if an actuator $1 \leq i \leq m$ does not receive an update at time t_ℓ then the corresponding entries of the matrix $\Omega_{\sigma(\ell)}$ is zero, and from the structure of the gains $K_\ell = G_{\sigma(\ell)}(\tilde{P}_1)$ where G is defined in (5.42) we also have that $u_j[k] = 0$. Likewise, if a sensor $1 \leq i \leq n_y$ does not transmit at time t_ℓ then the corresponding entries of the matrix $\Gamma_{\sigma(\ell)}$ is zero, and from the structure of the gains $L_\ell = I_{\sigma(\ell)}(N_\ell)$ where I is defined in (5.42) we have that $y_j = 0$ is not taken into account in (5.56).

5.2.3 Illustrative Example

Consider the problem of controlling an inverted pendulum by a remote controller, which is connected to the pendulum system though a communication network as the CAN-BUS, Wireless 802.11, or the Ethernet. Thus, the controller has to choose at each transmission time either to send a control action (torque) for the pendulum or obtain a measurement, which is assumed to be a measurement of the displacement angle. Control decisions at time t_k correspond to $\sigma(k) = 1$, and measurement decision to $\sigma(k) = 2$. We assume that transmissions occur with a sampling period of $\tau_s = 0.1$. A discretized model of a linearization of the inverted pendulum about the unstable equilibrum point is given by (5.37), with

$$A = e^{A_P \tau_s}, \quad B = \int_0^{\tau_s} e^{A_P s} B_P ds, \quad C = [1 \;\; 0],$$

where

$$A_P = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad B_P = \begin{bmatrix} 0 \\ 1 \end{bmatrix},$$

and $\Gamma_1 = 0$, $\Gamma_2 = 0$, $\Omega_1 = 1$, $\Omega_2 = 0$. The noise signals in (5.37) are characterized by

$$\Phi_0 = 0.1 I_2, \quad , \Phi_x = 0.001 I_2, \quad \Phi_y = 0.001$$

The performance criterium is given,

$$\mathbb{E}[\sum_{k=0}^{k_F - 1} (Cx[k])^2 + u[k]^2 dt + (Cx[k_F])^2] \tag{5.59}$$

with $k_F = 100$ and the controller aims at finding simultaneously the scheduling sequence $\sigma(k)$ and the control law u_k that minimize (5.59).

Consider for the base policy a periodic scheduling law that alternates between sampling and control, i.e, with period 2. We can compute from Proposition 46 the cost (5.59) when this periodic scheduling and the optimal control law given in Proposition 46 are applied to (5.37) with the matrices described above. However, we have no closed-form solution to compute the cost (5.59) when the scheduling is obtained by running the Protocol 1. Thus, we estimate it by running Monte Carlo simulations of the protocol. The results for the base and rollout policies are shown in Table 5.3. The cost for the base policy can also be confirmed by running Monte Carlo simulations. The values in Table 5.3 corroborate Theorem 47, i.e., the cost obtained with the rollout policy is less than the one obtained with the base policy. In fact, a significant improvement of about 22 percent of the base policy cost is achieved.

Periodic	Rollout
88.4	68.7

Table 5.3: Performance of base (periodic) and rollout policies for the pendulum example

5.3 Proofs

Proof. (of the Theorem 5.22) Consider the discrete-time process $u_k := x(t_k^-)$, where $x(t_k^-)$ is the state of (5.14), which is easily shown to be Markov due to the i.i.d. assumption on h_k. In particular,

$$\mathbb{E}_{u_k}[\mathbb{E}_{u_{k+l}}[V(u_{k+l+m})]] = \mathbb{E}_{u_k}[V(u_{k+l+m})] \tag{5.60}$$

for any bounded measurable function V, where $\mathbb{E}_{u_k}$ denotes expectation given u_k, i.e., $\mathbb{E}_{u_k}[.] := \mathbb{E}[.|u_k]$. If one can find a function $V(u)$ such that

$$c_1\|u\|^2 \leq V(u) \leq c_2\|u\|^2, \, \forall u \in \mathbb{R}^{n_x} \tag{5.61}$$

and

$$\mathbb{E}_{u_k}[V(u_{k+1})] - V(u_k) \leq -c_3\|u_k\|^2, \, \forall u_k \in \mathbb{R}^{n_x} \tag{5.62}$$

then we can prove that

$$\mathbb{E}[u_k^\intercal u_k] \leq c\alpha^k u_0^\intercal u_0, \forall_{k\geq 0} \text{ for some } 0 < \alpha < 1, c > 0. \tag{5.63}$$

which implies MES for (5.14) according to the definition (5.20) since $x(t_k) = J_i u_k$ for some $i \in \mathcal{M}$. In fact if (5.61) and (5.62) hold then

$$E_{u_k}[V(u_{k+1})] \leq \alpha V(u_k) \tag{5.64}$$

where $0 < \alpha = 1 - \frac{c_3}{c_2} < 1$ must be greater than zero since V is positive. From (5.82) and (5.64) we can conclude that

$$E_{u_0}[V(u_k)] \leq \alpha^k V(u_0). \tag{5.65}$$

From (5.61) and (5.65) we obtain

$$E_{u_0}[\|u_k\|^2] \leq \frac{c_2}{c_1}\alpha^k\|u_0\|^2, \, k \geq 0.$$

Take $V(u_k) := \min_{i\in\mathcal{M}_D} u_k^\intercal R_i u_k$, which satisfies (5.61) since $R_i > 0 \; \forall_{i\in\mathcal{Q}}$. Suppose that u_k is such that $i = \mathsf{d}_2(u_k) = \operatorname{argmin}_{j\in\mathcal{M}_D} u_k^\intercal R_j u_k$, i.e., $V(u_k) = \min u_k^\intercal R_i u_k$. Note that, in case the minimum is achieved simultaneously for several value of the index i, any of these indexes can be chosen without affecting the present proof. Then, for any $p_{ji} \geq 0 : \sum_{j=1}^{m_D} p_{ji} = 1$, we have that

$$\begin{aligned}
&\mathbb{E}_{u_k}[V(u_{k+1})] = \\
&\qquad \mathbb{E}_{u_k}[\min_{j\in\mathcal{M}_D} u_k^\intercal J_{\mathsf{d}_1(i)}^\intercal e^{A^\intercal h} R_j e^{Ah} J_{\mathsf{d}_1(i)} u_k] \\
&\leq \quad \mathbb{E}_{u_k}[\sum_{j=1}^{m_D} p_{ji} u_k^\intercal J_{\mathsf{d}_1(i)}^\intercal e^{A^\intercal h} R_j e^{Ah} J_{\mathsf{d}_1(i)} u_k] \\
&= \quad u_k^\intercal J_{\mathsf{d}_1(i)}^\intercal \sum_{j=1}^{m_D} p_{ji} E(R_j) J_{\mathsf{d}_1(i)} u_k
\end{aligned} \tag{5.66}$$

Suppose we choose p_{ji} such that (5.22) holds, i.e.,

$$J_{\mathsf{d}_1(i)}^\intercal \sum_{j=1}^{m_D} p_{ji} E(R_j) J_{\mathsf{d}_1(i)} - R_i = -Q_i$$

for some $Q_i > 0$. Then from (5.66) we conclude that $\mathbb{E}_{u_k}[V(u_{k+1})] - V(u_k) \leq -u_k^\intercal Q_i u_k$, $i = \mathsf{d}_2(u_k)$, which implies (5.62) and concludes the proof. □

Proof. (of the Theorem 40) As in the proof of the Theorem 36 it suffices to find a function $V(u_k)$ such that (5.61), (5.62) hold. Take $V(u_k) = u_k^\intercal W u_k$, where W is the solution to (5.26) and suppose that $i = \mathrm{argmin}_{j\in\mathcal{M}} u_k^\intercal R_j u_k$. Then

$$\begin{aligned}
\mathbb{E}_{u_k}[V(u_{k+1})] - V(u_k) &= u_k^\intercal [J_{\mathsf{d}_1(i)}^\intercal E(W) J_{\mathsf{d}_1(i)} - W] u_k, \\
&= -u_k^\intercal [\sum_{j=1, j\neq i}^{m_D} c_{ij}(R_j - R_i) + Q_i] u_k
\end{aligned} \tag{5.67}$$

where $Q_i > 0$. Since $i = \mathrm{argmin}_{j\in\mathcal{M}} u_k^\intercal R_j u_k$ we have that $u_k^\intercal [\sum_{j=1, j\neq i}^{m_D} c_{ij}(R_j - R_i)] u_k \geq 0$ for every $u_k \in \Re^{n_x}$ and therefore from (5.67) we conclude that V satisfies (5.62). It is also clear that V satisfies (5.61), which concludes the proof. □

Proof. (of the Theorem 41) We start by noticing that if there exists $P > 0$ such that

$$\begin{aligned}
&\int_0^\gamma [e^{A_P h} + H(h) L_k \Psi_i C_P)^\intercal P (e^{A_P h} + H(h) L_k \Psi_i C_P] \mu(dh) \\
&\qquad + \sum_{j=1, j\neq i}^{m} c_{ij}(C_P^\intercal S_j CP - C_P^\intercal S_i C_P) - P < 0
\end{aligned} \tag{5.68}$$

then (5.26) holds for (5.14) with matrices (5.30). In fact, since $m = m_{DP}$, we can assume that $\mathsf{d}_1(i) = i$, $\forall_{i\in\mathcal{M}}$ since if this is not the case we can relabel the sensor nodes in such a way this holds. Noticing that for A, J_i given by (5.30) we have that

$$e^{Ah} J_i = \begin{bmatrix} e^{A_P h} + H(h) L_k \Psi_i C & 0 \\ L_k \Psi_i C & 0 \end{bmatrix}.$$

We can prove that (5.26) holds for $W = \mathrm{diag}([P\,\epsilon I_{n_P}])$ since in this case (5.26) takes the form

$$\int_0^\gamma \begin{bmatrix} B(h) & 0 \\ 0 & 0 \end{bmatrix} + \sum_{j=1, j\neq i}^{m} c_{ij} \begin{bmatrix} C_P^\intercal S_j CP - C_P^\intercal S_i C_P & 0 \\ 0 & 0_{n_P} \end{bmatrix} - \begin{bmatrix} P & 0 \\ 0 & \epsilon I_{n_P} \end{bmatrix} < 0, \forall_{i\in\mathcal{M}} \tag{5.69}$$

where

$$B(h) = (e^{A_P h} + H(h)L_k\Psi_i C_P)^\intercal P(e^{A_P h} + H(h)L_k\Psi_i C_P) \tag{5.70}$$

$$+\epsilon(L_k\Psi_i C_P)^\intercal(L_k\Psi_i C_P) \tag{5.71}$$

Thus if (5.68) holds then (5.69) holds for sufficiently small ϵ. Set $L_k = H(h_k)^{-1}P^{-1}M$ for a $n_P \times m$ matrix. Then (5.68) can be written as

$$\begin{aligned} F(P) + DM\Psi_i C_P + (DM\Psi_i C_P)^\intercal + C_P^\intercal M P^{-1} M C_P + \\ \sum_{j=1, j\neq i}^{m} c_{ij}(C_P^\intercal S_j C_P - C_P^\intercal S_i C_P) - P < 0,\ \forall_{i\in\mathcal{M}} \end{aligned} \tag{5.72}$$

If we let $Y > 0$ be such that $M^\intercal P^{-1} M < Y$, which applying the Shur complement can be seen to be equivalent to (5.33), we see that if (5.32) holds then (5.72) holds, which concludes the proof.

□

5.3.1 Proof of Theorem 5.3.1

For simplicity we restrict the proof to the case where μ is absolutely continuous, i.e., $\mu([0,t)) = \int_0^t f(s)ds$ where f is a probability density function.

Let $\tau := t - t_k$ be a variable that keeps track of the time elapsed since the last transmission time t_k. Then the pair $\mathbf{x} = (x, \tau)$ is a Piecewise Deterministic Process (PDP) in the sense of [25], where x is the state of (5.14). A PDP is a process where randomness comes into play at fixed or random *jump times* but there is no source of uncertainty between these times. In the present case these times are t_k. A PDP can be described by three ingredients: (i) a deterministic flow, which describes the evolution of the state between jump times. In the present case this is the linear dynamics of (5.14); (ii) a possibly state-dependent transition jump intensity which determines the occurrences of the jump times. In the present case the transitions intervals are independent and identically distributed, and therefore, at time t, the transition intensity depends on the time of the last jump before t and is given by the hazard function of f, $\lambda(\tau) := g(\tau)/s(\tau)$ where $s(\tau)$ is the survivor function $s(\tau) = \int_\tau^\gamma f(r)dr$ [25]; (iii) a transition probability measure $Q(\mathbf{x}, dy)$ over the state-space that characterizes the probability of a jump from a state to another state at jump times. In the present case the transitions are state-dependent but deterministic and can be described by the characteristic distribution

$$\mathrm{Prob}[x(t_k) \in A|E] = \int_A Q(x(t_k^-), dy) \tag{5.73}$$

$$= \begin{cases} 1 \text{ if } J_j x(t_k^-) \in A \text{ and} \\ \qquad\qquad\qquad\qquad x(t_k^-)^\intercal R_j x(t_k^-) \geq x(t_k^-)^\intercal R_i x(t_k^-),\ i \in \mathcal{M}\ , \\ 0 \text{ otherwise} \end{cases} \tag{5.74}$$

where E is the event that a jump occurred at time t_k and that the state is $x(t_k^-)$.

In [25], an expression is given for the extended generator $\mathfrak{A}$ of a PDP and a characterization is given for its domain, which consists of the class of functions for which

$$\mathbb{E}_{\mathbf{x_0}}(V(\mathbf{x}(t))) = V(\mathbf{x}(0)) + \mathbb{E}_{\mathbf{x_0}} \int_0^t \mathfrak{A}V(\mathbf{x}(s))ds, \forall_{t\geq 0}, \tag{5.75}$$

where $\mathbb{E}_{\mathbf{x_0}}$ denotes expectation given that the process started at the initial state $\mathbf{x_0} = \mathbf{x}(0)$. We see from (5.75) that the extended generator for PDPs plays the role of a Lyapunov derivative for deterministic systems. The expression for the extended generator $\mathfrak{A}$ takes the form

$$\begin{gathered}\mathfrak{A}V(\mathbf{x}) := \mathfrak{X}V(\mathbf{x}) + \lambda(\tau)[V(J_j x, 0) - V(\mathbf{x})], \\ \text{if } x^\intercal R_j x \leq x^\intercal R_i x,\ i \in \mathcal{M}\end{gathered} \tag{5.76}$$

where $\mathfrak{X}V(\mathbf{x}) := \sum_i \frac{\partial V(\mathbf{x})}{\partial \mathbf{x}_i} \dot{x}_i$.

The following result will be key to prove Theorem 40. The proof is analogous to the proof of the sufficiency part of the Theorem 24, and therefore it is omitted.

Theorem 49. If there exists a differentiable positive function $V(\mathbf{x})$ which equals zero at zero, and positive constants c_1, c_2, k such that for every $\mathbf{x} = (x, \tau)$,

$$c_1\|x\|^2 \leq V(\mathbf{x}) \leq c_2\|x\|^2 \tag{5.77}$$

$$\mathfrak{A}V(\mathbf{x}) \leq -k\|x\|^2 \tag{5.78}$$

where $\mathfrak{A}V(\mathbf{x})$ is given by (5.76) then the system (5.14) is MES.

□

Proof. (of Theorem 40) We start by noticing that (5.26) is equivalent to existing $P > 0$, $U > 0$, $c_{ij} > 0$ and R_i, $i, j \in \mathcal{M}$ such that

$$E(W) - U < 0 \tag{5.79}$$

$$J_i^\intercal U J_i + \sum_{j=1, j\neq i}^{m} c_{ij}(R_i - R_j) - W < 0, \forall_{i\in\mathcal{M}}. \tag{5.80}$$

To see that (5.79) and (5.80) imply (5.26) note that (5.79) implies that $J_i^\intercal(E(W)-U)J_i < 0$ for every $i \in \mathcal{M}$ and adding this last equation to (5.80) for each i one obtains (5.26). Conversely, if (5.26) holds, $U = E(W) + \epsilon I$ satisfies (5.79) and (5.80) for sufficiently small ϵ.

We consider a function $V(\mathbf{x}) = x^\intercal P(\tau)x$, for a $P(\tau)$ to be introduced shortly, and prove that $V(\mathbf{x})$ satisfies the conditions of Theorem 49 provided that (5.79), (5.80) are satisfied.

Taking the generator of $V(x)$ using (5.76) we obtain

$$\begin{aligned}\mathfrak{A}V(\mathbf{x}) = x^\intercal[\dot{P}(\tau) + A^\intercal P(\tau) + P(\tau)A +\\ \lambda(\tau)(J_j^\intercal P(0)J_j - P(\tau))]x\\ \text{if } x^\intercal R_j x \geq x^\intercal R_i x,\ i \in \{1,\dots,m\}\end{aligned}$$

we shall show that there exists $P(\tau)$ such that $c_1 I < P(\tau) < c_2 I$ and

$$\dot{P}(\tau) + A^\intercal P(\tau) + P(\tau)A + \lambda(\tau)[J_i^\intercal P(0)J_i - P(\tau)) \tag{5.81}$$
$$+\textstyle\sum_{j=1}^m c_{ij}(R_i - R_j) + Z_i] + Y = 0$$

for some matrices $d_1 I < Z_i < d_2 I$, $i \in \mathcal{M}$, $e_1 I < Y < e_2 I$, to be introduced where $c_l, d_l, e_l, l \in \{1,2\}$ are positive constants. This implies that if we take such $P(\tau)$ for $V(\mathbf{x}) = x^\intercal P(\tau)x$ then (5.61) takes the form

$$\begin{aligned}\mathfrak{A}V(\mathbf{x}) = -x^\intercal[Y + \lambda(\tau)(Z_j + (\sum_{j=1}^m c_{ij}(R_i - R_j)))]x\\ \text{if } x(t_k^-)^\intercal R_j x(t_k^-) \geq x(t_k^-)^\intercal R_i x(t_k^-),\ i \in \{1,\dots,m\}.\end{aligned}$$

which implies that $\mathfrak{A}V(\mathbf{x}) = -x^\intercal Yx,\ \forall_{x\in\mathbb{R}^{n_x}}$

Note that $\lambda(\tau)$ is a positive function. Thus, $V(x)$ satisfies (5.77) and (5.78). If suffices to prove that (5.79)-(5.80) implies that there exists $c_1 I < P(\tau) < c_2 I$, $d_1 I < Z_i < d_2 I$, $e_1 I < Y < e_2 I$ such that (5.81) holds.

Let $P(0) = U$, $Z_i = W - (\sum_{j=1}^m c_{ij}(R_i - R_j) + J_i^\intercal P(0)J_i) > 0$, where U, W, c_{ij} are the solution to (5.79), (5.80). Moreover since from (5.79), $E(W) - U = -Q$ for some $Q > 0$ we can chose Y such that $\int_0^\gamma e^{A^\intercal h} Y e^{Ah} f(h)dh = Q$. Then the unique solution to (5.81) is given by

$$\begin{aligned}P(\tau) = \textstyle\int_0^{\gamma-\tau} e^{A^\intercal r} W e^{Ar} \frac{f(\tau+r)}{s(\tau)} dr\\ + \textstyle\int_0^{\gamma-\tau} e^{A^\intercal s} Y e^{As} \frac{s(\tau+s)}{s(\tau)} ds\end{aligned} \tag{5.82}$$

which can be confirmed by direct replacement. It is clear that $P(\tau)$ is bounded and positive definite, and this concludes the proof. □

5.4 Further Comments and References

Walsh and co-authors [90] made strides in the analysis of control systems closed via a local area network, such as the controller-area network, the ethernet and wireless 802.11 networks. The key assumptions in [90] are that there exists a bound on the interval between transmissions denoted by *maximum allowable transfer interval* (MATI), and that transmission delays and packet drops are negligible. In [90] an emulation set-up is considered in the sense that the controller for the networked control system is obtained from a previously designed stabilizing continuous-time controller. Two basic types of protocols have been proposed: static protocols, such as the round-robin (RR) protocol where nodes take turns transmitting data in a periodic fashion; and dynamic protocols such as the maximum error first-try once discard (MEF-TOD) protocol, where the node that has the top priority in using the communication medium is the one whose current value to transmit differs the most from the last transmitted value. Under this setup, one can attempt to provide an upper bound on the MATI for which stability can be guaranteed. Since these protocols have been proposed in the papers referenced above, MATI bounds have been improved [73], [88], [28]. Although, [90] illustrate through simulations that using the MEF-TOD protocol allows for preserving stability of the networked closed loop for a larger MATI than that obtained when using the RR protocol, and similar conclusions are obtained in [73]- [28] from sufficient stability conditions, no analytical result has been established proving that this holds in general.

As mentioned in [90], the occurrence of transmission events on the network is often more appropriately modeled as a random process. This feature is taken into account in [44], which considers networked control systems with MEF-TOD and RR protocols, and independent and identically distributed (i.i.d.) intervals between transmissions. It is shown that stability can be guaranteed for distributions of the inter-transmission intervals that have a support larger than previously derived deterministic upper bounds for the MATI [90], [73], [88]. The conservativeness of these results for linear networked control systems using the RR protocol was eliminated in [AHS09b]. Recently, [27] addresses a model of networked control systems with i.i.d. intervals between transmissions and stochastic delays for a class of quadratic protocols that is more general than MEF-TOD. Through a convex over-approximation approach, sufficient conditions are given for mean exponential stability. In [24] a method is proposed to design an observer-protocol pair to asymptotically reconstruct the states of an LTI plant where the plants outputs are sent through a network with constant intervals between transmissions. The protocol to be designed can be viewed as a weighted version of the MEF-TOD.

6

Output Regulation for Multi-rate Systems

One of the celebrated problems in automatic control, commonly known as output regulation, is that of controlling the output of a linear time-invariant system so as to achieve asymptotic tracking of an exogenous signal generated by the free motion of a linear time-invariant system, so-called exosystem, while guaranteeing closed loop stability. A solution to this problem is required to yield closed-loop stability and should be such that output regulation is achieved even in the presence of small plant uncertainties and exogenous disturbances also generated by the exosystem.

In this chapter we address the output regulation problem for multi-rate systems. Multi-rate systems can arise in the scope of networked control system in cases where several nodes negotiate fixed, but different, bandwidths with the possibly several networks connecting that are part of the networked control system. Networks that guarantee a fixed bandwith, are, for example, circuit switchign networks [58]. It can also simply represent a sensor, which outputs measurements at a fixed sampling rate, or an actuator which allows actuation updates at a fixed rate. Typical cases of the use of multi-rate control systems design arise from hardware restrictions, for example, due to the fact that the discrete-analog (D/A) converters are generally faster than the analog-discrete (A/D) converters or from the use of camera-based sensors, which might require larger sampling periods than other sensors and actuators ([35]). Another important application arises in the design of integrated guidance and control systems for vehicular applications ([ASC07], [ASC08b]), where the linear positioning sensors (e.g. Global Positioning System (GPS)) are usually available at a lower rate than the remaining sensors.

We propose a controller that achieves stability for the closed loop and output regulation for a number of regulated outputs equal to the number of inputs, while taking advantages of the remaining outputs for feedback. Besides incorporating an internal model of the exosystem, the key feature of our proposed controller is that it includes a system that blocks signals generated by the exosystem arriving to the controller from the non-regulated outputs. The concept of a system that blocks signals is made precise, by introducing the notion of blocking zero with respect to a matrix, both for LTI and periodic systems, which generalizes the standard notion of blocking zero for LTI systems (see, e.g., [98]). We show that there exists a stabilizing controller with the proposed structure, i.e., incorporating an internal model of the exosystem and a system that blocks signals generated by the exosystem, that achieves output regulation even in the presence of plant uncertainties and disturbances generated by the exosystem.

The remainder of the chapter is organized as follows. In Section 6.1, we provide the problem setup and state the output regulation problem for multi-rate systems. In Section 6.2 we propose a solution to the problem statement summarized in our main result. An illustrative example showing that our solution allows to solve problems not possible to solve through previous works is provided in Section 6.3. In Section 6.4 we introduce the notion of blocking zero with respect to a matrix, which is a key concept to prove the main results which is established in Section 6.5. Further comments and references are given in Section 6.6.

6.1 Problem Formulation

We describe first the multi-rate set-up, and the exosystem. Then we state the output regulation problem.

6.1.1 Multi-Rate Set-up

We consider a continuous-time plant

$$\begin{bmatrix} \dot{x}_P(t) \\ y_P(t) \end{bmatrix} = \begin{bmatrix} A_P & B_P \\ C_P & 0 \end{bmatrix} \begin{bmatrix} x_P(t) \\ u_P(t) \end{bmatrix} + \begin{bmatrix} E_P \\ 0 \end{bmatrix} v_P(t), \ \ t \geq 0, \tag{6.1}$$

where $x_P(t) \in \mathbb{R}^n$ is the state, $u_P(t) \in \mathbb{R}^m$ is the input, and $v_P(t) \in \mathbb{R}^{n_v}$ is a disturbance vector generated by the following system

$$\begin{aligned} \dot{w}_P(t) &= S_V w_P(t), \ \ t \geq 0, \\ v_P(t) &= E_V w_P(t). \end{aligned} \tag{6.2}$$

The output vector $y_P(t) \in \mathbb{R}^p$ can be partitioned into

$$y_P(t) = (y_P^1(t), \dots, y_P^{n_y}(t))$$

where $y_P^i(t) \in \mathbb{R}^{p_i}$, is associated with sensor $i \in \{1, \dots, n_y\}$, and $\sum_{i=1}^{n_y} p_i = p$. The sensors are assumed to operate at different sampling rates, with periods that are rationally related.

This model for the measurements can be described by

$$y[k] := (y^1[k], \dots, y^{n_y}[k]),$$

where

$$y^i[k] := \Gamma_k^i y_P^i(t_k), \;\; k \geq 0, \;\; 1 \leq i \leq n_y, \tag{6.3}$$

$t_k := kt_s$, for some $t_s > 0$, $\Gamma_k^i = \gamma_k^i I_{p_i}$, and

$$\gamma_k^i := \begin{cases} 1, \text{ if sensor } i \text{ is sampled at } t_k \\ 0, \text{ otherwise} \end{cases}. \tag{6.4}$$

Let

$$\Gamma_k := \text{bdiag}(\Gamma_k^1, \dots, \Gamma_k^{n_y}).$$

Note that, due to our assumption that the periodicities of the sensors sampling are rationally related, Γ_k is a periodic function of k, i.e., there exists h such that $\Gamma_k = \Gamma_{k+h}, \forall_k$. We can assume that each diagonal entry of Γ_k is non-zero at least once in a period since otherwise a given sensor component would never be sampled and could be disregarded.

The actuator mechanism is assumed to be a standard sample and hold device, and it is assumed to be available for update at every sampling time t_k, i.e.,

$$u(t) = u[k], t \in [t_k, t_{k+1}), \tag{6.5}$$

where $u[k]$ is the actuation update at time t_k.

Denote the sampled state at times t_k by $x[k] := x(t_k)$. Then we can write (6.1), (6.3) and (6.5) at times t_k as

$$\mathsf{P} := \left\{ \quad \begin{bmatrix} x[k+1] \\ y[k] \end{bmatrix} = \begin{bmatrix} A & B \\ \Gamma_k C & 0 \end{bmatrix} \begin{bmatrix} x[k] \\ u[k] \end{bmatrix} + \begin{bmatrix} B_V \\ 0 \end{bmatrix} v[k] \right. \tag{6.6}$$

where $A = e^{A_P h}$, $B = \int_0^{t_s} e^{A_P s} ds B_P$, $C = C_P$, $B_V = \int_0^{t_s} e^{A(t_s - s)} E_P E_V e^{S_V s} ds$, and $v[k] = v_P(t_k)$ is generated by the discretization of (6.2), which is given by

$$\begin{aligned} w_V[k+1] &= e^{S_V t_s} w_V[k], \;\; k \geq 0, \\ v[k] &= E_V w_V[k]. \end{aligned} \tag{6.7}$$

6.1.2 Exosystem

Suppose that we further partition the output vector $y[k]$ according to

$$y[k] = \begin{bmatrix} y_m[k] \\ y_r[k] \end{bmatrix} = \begin{bmatrix} \Gamma_{mk} C_m \\ \Gamma_{rk} C_r \end{bmatrix} x[k] \tag{6.8}$$

where Γ_{mk} and Γ_{rk} are $n_m \times n_m$ and $m \times m$ matrices, respectively, such that $\Gamma_k = \text{bdiag}(\Gamma_{mk}, \Gamma_{rk})$. Note that $n_m = p - m$. The component $y_r[k] \in \mathbb{R}^m$ is a set of outputs that we wish to asymptotically track a reference signal $r[k]$, and $y_m[k] \in \mathbb{R}^{p_m}$ is an additional set of measurements available for feedback. Subsumed in this partition is that $p > m$. A solution to the output regulation problem in the case where $p \leq m$ can be found in [82].

The reference signal $r[k] \in \mathbb{R}^m$ and the disturbance signal $v[k] \in \mathbb{R}^{n_v}$ are assumed to be generated by the following model, which we denote by *exosystem*,

$$\begin{aligned} w[k+1] &= Sw[k] \\ r[k] &= C_R w[k] \\ v[k] &= C_V w[k], \end{aligned} \tag{6.9}$$

where $w[k] \in \mathbb{R}^{n_w}$. The matrices S and C_V must be compatible with (6.7), i.e., the same signal $v[k]$ should be generated by (6.9) and (6.7). Consider the Jordan canonical form of S i.e.,

$$S = V\text{bdiag}(S_1, \dots, S_{n_s})V^{-1} \tag{6.10}$$

where V is an invertible matrix and the matrices S_j take the form

$$S_j = \begin{bmatrix} \mu_j & 1 & 0 & \dots \\ 0 & \mu_j & 1 & \dots \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \mu_j \end{bmatrix} \in \mathbb{C}^{\kappa_j \times \kappa_j}, \;\; 1 \leq j \leq n_s \tag{6.11}$$

where $n_s \leq n_w$, and $\sum_{j=1}^{n_s} \kappa_j = n_w$. We assume that:

(S1) $\|\mu_i\| \geq 1$, $\forall_{1 \leq i \leq n_s}$.

(S2) $\mu_i \neq \mu_j$ for $i \neq j$.

To see that (S1) and (S2) introduce no loss of generality, note first that the exosystem can generated signals $r[k]$ and $v[k]$ taking the form

$$\xi[k] = \sum_{j=1}^{n_s} \sum_{l=0}^{\kappa_j - 1} b_{jl} \binom{k}{l} \mu_j^{k-l}, \tag{6.12}$$

where b_{jl} can be made arbitrarily by properly choosing C_R, C_V, and $w[0]$. Since, as we shall see shortly, output regulation is an asymptotic property, if (S1) would not hold then the disturbance and reference terms in (6.12) corresponding to stable eigenvalues of S would play no role. If (S2) would not hold, the exosystem would still only be able to generate the same class of reference and disturbance signals (6.12). Another way of stating (S2) is to say that the characteristic polynomial of S coincides with the minimal polynomial of S, which is a statement more commonly seen in related works addressing the internal model principle (see, e.g., [13]).

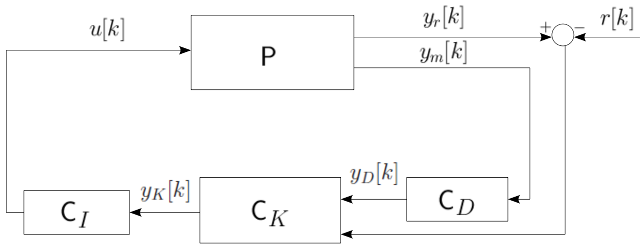


Figure 6.1: Proposed controller structure to achieve output regulation; Plant:P; Controller: C_I-internal model, C_D blocking system, C_K stabilizer

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6.1.3 Problem Statement

Consider a linear controller for the system (6.6), i.e., a map $y[k] \mapsto u[k]$ with state $x_c[k] \in \mathbb{R}^{n_c}$. We say that the closed-loop is stable if $(x[k], x_c[k]) \to 0$ as $k \to \infty$, when $r[k] = 0$, $\forall_{k \geq 0}$. Moreover, we say that output regulation is achieved if $(C_r x[k] - r[k]) \to 0$ as $k \to \infty$.

The problem we are interested in this chapter can be stated as follows.

Problem 50. Find a linear controller for the system (6.6) such that the closed loop is stable and output regulation is achieved.

6.2 Output Regulator for Multi-Rate Systems

The structure of the controller that we propose to solve the Problem 50 is shown in Figure 6.1.

The purpose of the systems C_I, C_D and C_K and the rationale behind this structure are briefly explained as follows. Since the controller must provide the adequate input value $u[k]$ such that the output $y_r[k]$ of the linear plant tracks the desired reference signal $r[k]$, the system C_I is such that it is capable of providing such an input to the plant, when the input to C_I is identically zero. Denote by *steady state* the state of the plant at which output regulation is achieved. At steady state, due to the linearity of the plant, the non-regulated output $y_m[k]$ consists of a signal with the same frequency content of the input to the plant. The system C_D has the purpose of yielding a zero output when the steady state signal $y_m[k]$ is applied to its input, while assuring that other signals of interest, do not yield a zero output and thus can still be utilized to control the plant.

Closed loop can be guaranteed if the following condition is met

(G1) The system ”seen” by the controller, i.e., obtained by computing the series of C_I, P and C_D, with input $y_K[k]$ and output $(y_D[k], y_r[k] - r[k])$, is detectable and stabilizable.

In fact, from standard results for linear systems, if (G1) holds one can compute a stabilizing controller C_K. Note that, when output regulation is achieved, both the input to the controller and its output are zero.

Before we state our main result, we provide possible choices for C_I and C_D. We shall impose in Section 6.5 some requirements that C_I and C_D must meet to prove our main output regulation result. We shall see that the systems proposed below meet these requirements. However other choices may exist that satisfy these requirements.

6.2.1 System C_I

The system C_I is an LTI system described by

$$\mathsf{C}_I := \left\{ \quad \begin{bmatrix} x_I[k+1] \\ y_I[k] \end{bmatrix} = \begin{bmatrix} A_I & B_I \\ C_I & 0 \end{bmatrix} \begin{bmatrix} x_I[k] \\ u_I[k] \end{bmatrix} \right. \tag{6.13}$$

where $x_I[k] \in \mathbb{R}^{n_I}$, $u_I[k] \in \mathbb{R}^m$, and $y_I[k] \in \mathbb{R}^m$. The system C_I should be capable of providing the adequate input to the plant such that output regulation is achieved. Such an input takes the general form (6.12). One realization for (6.13) that achieves this is given by

$$\begin{aligned} A_I &= \mathrm{bdiag}(S, \ldots, S) \in \mathbb{R}^{(mn_w)\times(mn_w)}. \\ B_I &= \mathrm{bdiag}(B_J, \ldots, B_J) \in \mathbb{R}^{(mn_w)\times m} \\ C_I &= \mathrm{bdiag}(C_J, \ldots, C_J) \in \mathbb{R}^{m\times(mn_w)} \end{aligned} \tag{6.14}$$

where $B_J \in \mathbb{R}^{n_I\times 1}$ is such that (S, B_J) is stabilizable, and $C_J \in \mathbb{R}^{1\times n_w}$ is such that (C_J, S) is detectable. It is straightforward to verify that this implies that (A_I, B_I) is observable, and (C_I, A_I) is detectable, respectively (cf. Proposition 62 below). Note that, the system (6.13) with matrices (6.14) incorporates an $m-$fold reduplication of the exosystem (6.9), in the sense of [34].

6.2.2 System C_D

The system C_D , is a linear periodically time-varying system, described by

$$\mathsf{C}_D := \left\{ \quad \begin{bmatrix} x_D[k+1] \\ y_D[k] \end{bmatrix} = \begin{bmatrix} A_{Dk} & B_{Dk} \\ C_{Dk} & D_{Dk} \end{bmatrix} \begin{bmatrix} x_D[k] \\ u_D[k] \end{bmatrix} \right. \tag{6.15}$$

where $x_D[k] \in \mathbb{R}^{n_D}$, $u_D[k] \in \mathbb{R}^{n_m}$, and $y_D[k] \in R^{n_m}$, and A_{Dk}, B_{Dk}, C_{Dk} and D_{Dk} are $h-$periodic, i.e., e.g., $A_{Dk} = A_{Dk+h}$, $\forall_{k\geq 0}$.

The system C_D has the purpose of blocking the signals that can be generated by the exosystem. By this we mean that the output of (6.15) is zero for every input generated by the exosystem, which takes the general form (6.12). The notion is made precise in the Appendix (cf. Definition 58).

One realization for (6.15) that achieves this is given by

$$\begin{aligned} A_{Dk} &= \text{bdiag}(A_k^1, \ldots, A_k^{n_m}) \in \mathbb{R}^{n_m n_w \times n_m n_w}. \\ B_{Dk} &= \text{bdiag}(B_k^1, \ldots, B_k^{n_m}) \in \mathbb{R}^{(n_m n_w) \times n_m} \\ C_{Dk} &= \text{bdiag}(C_k^1, \ldots, C_k^{n_m}) \in \mathbb{R}^{n_m \times (n_m n_w)} \\ D_{Dk} &= \text{bdiag}(D_k^1, \ldots, D_k^{n_m}) \in \mathbb{R}^{n_m \times n_m}, \quad k \geq 0 \end{aligned} \tag{6.16}$$

where, corresponding to each output $i \in \{1, \ldots, \}$, the matrices are given by

$$A_k^i = \begin{bmatrix} 0 & 0 \\ I_{n_w-1} & 0 \end{bmatrix}, \; B_k^i = \begin{bmatrix} 1 \\ 0_{n_w-1} \end{bmatrix}, \; C_k^i = (c^{ik})^\intercal, \; D_k^i = 1, \tag{6.17}$$

if the output i is sampled at t_k, and

$$A_k^i = I_{n_w}, \; B_k^i = 0_{n_w \times 1}, \; C_k^i = 0_{1 \times n_w} \; D_k^i = 0, \tag{6.18}$$

otherwise, where c^{ik} is a $h-$periodic vector, i.e., $c^{ik} = c^{i(k+h)}$ which is described shortly. The matrices A_k^i and B_k^i, which correspond to the ith component of the output vector $y_m[k]$, are such that the system C_D holds the last n_w sampled values of the output i when k is sufficiently large. A condition for k that assures this is $k \geq n_w h$. In fact, from (6.17) we see that if the output is sampled at t_k then the new measurement is introduced in the state while the least recent is dropped. From (6.18) we see that if the output is not sampled at t_k the system C_D holds the previous state. The matrices C_k^i and D_k^i are such that the output is zero when steady state is achieved, in which case $y_m[k]$ is a signal taking the form (6.12). This will be shown in Proposition 63. The n_w dimensional periodic vector c^{ik} can be determined as follows. Since the c^{ik} are $h-$periodic we need only to specify c^{ik} along a period, i.e., e.g., for $k \in \{1, \ldots, h\}$, and for values for which $\gamma_k^i = 1$ (otherwise $C_k^i = 0$), where γ_k^i is given by (6.4). Let $\lfloor k \rfloor$ be the remainder of the division of k by h if $k \geq 1$, i.e., e.g., $\lfloor k+1 \rfloor = k+1$ if $1 \leq k \leq h-1$, and $\lfloor k+1 \rfloor = 1$ if $k = h$. Moreover if $k \leq 0$ use the same notation to denote $\lfloor k \rfloor := \lfloor k + rh \rfloor$ for some $r \in \mathbb{N}$ such that $k + rh \geq 1$. For each $k \in \{1, \ldots, h\}$ such that $\gamma^{ik} = 1$, define a set of $n_w + 1$ indexes $\{\tau_l^{ik}, 0 \leq l \leq n_w\}$ by

$$\tau_l^{ik} = \begin{cases} 0, \text{ if } l = 0, \\ \tau_{l-1}^{ik} + \min\{k_1 > 0 : \gamma^i_{\lfloor k - \tau_{l-1}^{ik} - k_1 \rfloor} = 1\}, \\ \hfill \text{if } 1 \leq l \leq n_w. \end{cases}$$

Define the following set of matrices

$$M_k^i = \begin{bmatrix} N_k^i(\mu_1) \\ \vdots \\ N_k^i(\mu_{n_s}) \end{bmatrix} \in \mathbb{C}^{n_w \times n_w}$$

where $k \in \{1, \dots, h\}$, and

$$N_k^i(\mu_j) := \begin{bmatrix} \mu_j^{-\tau_1^{ik}} & \mu_j^{-\tau_2^{ik}} & \dots & \mu_j^{-\tau_{n_w}^{ik}} \\ \tau_1^{ik}\mu_j^{-\tau_1^{ik}} & \tau_2^{ik}\mu_j^{-\tau_2^{ik}} & \dots & \tau_{n_w}^{ik}\mu_j^{-\tau_{n_w}^{ik}} \\ \vdots & \vdots & \vdots & \vdots \\ (\tau_1^{ik})^{\kappa_j}\mu_j^{-\tau_1^{ik}} & (\tau_2^{ik})^{\kappa_j}\mu_j^{-\tau_2^{ik}} & \dots & (\tau_{n_w}^{ik})^{\kappa_j}\mu_j^{-\tau_{n_w}^{ik}} \end{bmatrix}.$$

Define also the set of vectors

$$b_k^i = \begin{bmatrix} d_1 \\ \vdots \\ d_{n_s} \end{bmatrix} \in \mathbb{R}^{n_w \times 1}$$

where $k \in \{1, \dots, h\}$ and

$$d_j := \begin{bmatrix} 1 & 0_{1\times(\kappa_j - 1)} \end{bmatrix}^\intercal, 1 \leq j \leq n_s.$$

We make the following assumption:

$$M_k^i \text{ is invertible for every } k \in \{1, \dots, h\},\ 1 \leq i \leq n_y. \tag{6.19}$$

Take $c^{ik} = [c_1^{ik} \ \dots \ c_{n_w}^{ik}]$ as the solution to

$$M_k^i c^{ik} = -b_k, \tag{6.20}$$

which is unique due to (6.19) and it is real. To see that it is real note that if μ_i is a complex eigenvalue of S then so is its conjugate since S is real. Note also that to the eigenvalue μ_i and to its conjugate, correspond complex conjugate rows of M_k^i and therefore both c^{ik} and its conjugate satisfy (6.20).

The assumption holds in the special case where the sensors are sampled at a single-rate. In fact, in this case we have that $\tau_l^{ik} = l, \forall_{1\leq k\leq h, 1\leq i\leq n_y, 1\leq l\leq n_w}$, and the rows of M_i^k correspond to linear independent functions $l^r \mu_j^{-l}$ where $1 \leq r \leq \kappa_j$. This is also in general true in the multi-rate case where the τ_l^{ik} are in general not equal to l. However, the assumption may fail in some pathological cases as we illustrate in the next example.

Example 51. Suppose that $y_m[k]$ is one dimensional and corresponds to a sensor which is sampled once every five times in a period, i.e., $h = 5$, $\gamma_k^1 = 1$, if $k = 1$ and, $\gamma_k^1 = 0$ if $k \in \{2, 3, 4, 5\}$. If $n_w = n_s = 3$, then we can obtain that $\tau_1^{1k} = 5$, $\tau_2^{1k} = 10$, and $\tau_3^{3k} = 15$

if $k = 1$ and τ_l^{ik} do not need to be specified if $k \in \{2,3,4,5\}$. Suppose that $\mu_1 = e^{i2\pi/5}$, $\mu_2 = e^{-i2\pi/5}$, $\mu_3 = 1$. Then the matrices M_k^1, which needs only be specified for $k = 1$, is given by

$$M_1^1 = \begin{bmatrix} (e^{i2\pi/5})^{-5} & (e^{i2\pi/5})^{-10} & (e^{i2\pi/5})^{-15} \\ (e^{-i2\pi/5})^{-5} & (e^{-i2\pi/5})^{-10} & (e^{-i2\pi/5})^{-15} \\ 1 & 1 & 1 \end{bmatrix},$$

which is singular and therefore the Assumption 6.19 does not hold.

6.2.3 Main Result

We make the following assumptions on the multi-rate discrete-time plant (6.6):

(P1) (A, B) is stabilizable and (C, A) is detectable.

(P2) There are no invariant zeros from the input of the plant to the regulated output that coincide with the eigenvalues of S, i.e.,

$$\begin{bmatrix} A - \mu_j I_n & B \\ C_r & 0 \end{bmatrix} \text{ is invertible } \forall_{1 \leq j \leq n_s}.$$

(P3) Consider the following systems

$$\begin{aligned} \hat{x}[k+1] &= A\hat{x}[k] & \hat{w}[k+1] &= S\hat{w}[k] \\ \hat{y}_r[k] &= \Gamma_{rk} C_r \hat{x}[k] & \hat{r}[k] &= \Gamma_{mk} C_M \hat{w}[k] \\ \hat{y}_m[k] &= \Gamma_{mk} C_m \hat{x}[k] \end{aligned} \tag{6.21}$$

with initial conditions $x[0] = x_0$ and $w[0] = w_0$. Then, there does not exist x_0 different from the zero vector, such that there exists a w_0 and C_M for the free motion of (6.21), such that $\hat{y}_r[k] = 0, \forall_{k \geq 0}$ and $\hat{y}_m[k] = \hat{r}[k], \forall_{k \geq 0}$.

We assume (P1)-(P3) to obtain (G1). The assumptions (P1) and (P2) are typical in related problems (cf. [34]). The assumption (P3) is closely related to the following assumption, which is easier to test:

(P3') A and S do not share an eigenvalue, or if A and S share an eigenvalue $\mu_j, 1 \leq j \leq n_s$, then the corresponding eigenvector of A is not in the kernel of $\Gamma_{rk} C_r$ for every $k \in \{1, \ldots, h\}$.

While (P3) and (P3') are not equivalent, it is straightforward to show that (P3) implies (P3'). However if (P3') holds then (P3) may not hold in pathological cases.

Example 52. Suppose that $\Gamma_{rk} = 1, \forall_{k\geq 0}$ and $\Gamma_{mk} = 1$ if $k = 0$, $\Gamma_{mk} = 0$ if $k \in \{1, 2, 3, 4\}$ and $\Gamma_{mk} = \Gamma_{m(k+5)}$, $\forall_{k\geq 0}$. Let $S = 1$, $C_m = [1\,0\,0]$, $C_r = [0\,0\,1]$, and

$$A = \begin{bmatrix} \cos(\frac{2\pi}{5}) & -\sin(\frac{2\pi}{5}) & 0 \\ \sin(\frac{2\pi}{5}) & \cos(\frac{2\pi}{5}) & 0 \\ 0 & 0 & \frac{1}{2} \end{bmatrix}.$$

Although A does not have eigenvalues that coincide with the eigenvalues of S, and therefore (P3') holds, if we make $x_0 = [0\,1\,0]^\intercal$ and $w_0 = 1$ then we have that $y_r[k] = 0, \forall_{k\geq 0}$, and $\hat{r}[k] = y_m[k] = 1, \forall_{k\geq 0}$, which means that (P3) does not hold.

The following is the main result of the chapter. We denote by plant uncertainties the fact that the matrices A, B, C in (6.6) might not be known exactly, i.e., although the controller of Fig. 6.1 is designed for the model (6.6), the actual plant is described by the matrices $\tilde{A}$, $\tilde{B}$, $\tilde{C}$ and is given by

$$\tilde{\mathsf{P}} := \left\{ \begin{bmatrix} x[k+1] \\ y[k] \end{bmatrix} = \begin{bmatrix} \tilde{A} & \tilde{B} \\ \Gamma_k \tilde{C} & 0 \end{bmatrix} \begin{bmatrix} x[k] \\ u[k] \end{bmatrix} + \begin{bmatrix} B_V \\ 0 \end{bmatrix} v[k] \right. \tag{6.22}$$

Since asymptotic stability is a robust property, if $\tilde{A}$, $\tilde{B}$, and $\tilde{C}$ are sufficiently close to A, B, and C, respectively, and if the controller of Fig. 6.1 designed for (6.42) asymptotic stabilizes the closed-loop, then asymptotic stability is preserved when P is replaced by the actual plant $\tilde{\mathsf{P}}$.

Theorem 53. Suppose that (P1)-(P3) hold for the plant P, and that C_I is given by (6.13), and C_D is given by (6.16), (6.17), and (6.18). Then there exists matrices B_J, C_J for C_I, and C_K such that the closed-loop in Figure 6.1 is stable. Moreover, output regulation for $y_m[k]$ is achieved even in the presence of plant uncertainties that do not destroy closed-loop stability.

□

6.3 Example

Example 54. The following continuous-time linear system is considered in [ASC08a].

$$\mathsf{P}_C = \begin{cases} \dot{x}_1(t) = -x_1(t) - x_2(t) \\ \dot{x}_2(t) = -x_2(t) + u(t) \\ \dot{x}_3(t) = -x_2(t) + 0.5x_3(t) + u(t) \end{cases} \tag{6.23}$$

A sensor measuring $x_1(t)$ works at a fixed sampling period of $t_{s1} = 0.25$, while the actuator update mechanism can be done at a sampling period of $t_u = 0.05$. We wish that $x_1(t)$ tracks a prescribed reference signal. However, one can verify that P_C is not detectable from $x_1(t)$. Therefore we cannot use the solution provided in [82] for output regulation of square multi-rate systems, since this solution would not guarantee closed-loop stability. We consider that $x_3(t)$ is also available for feedback at a sampling period of $t_2 = 0.1$. According to the framework of Section 6.1 we have that $t_s = 0.05$, $h = 10$. The system is now detectable from $(x_1(t), x_3(t))$, and since the sampling period $t_s = 0.05$ is not pathological (see, e.g., [15], the discretization of (6.23) is also detectable with respect to the multi-rate outputs $y_r[k] := \Gamma_{rk} x_1(kt_s)$, and $y_m[k] := \Gamma_{mk} x_3(kt_s)$ where the h-periodic matrices Γ_{mk}, Γ_{rk}, Ω_k are determined by

$$\Gamma_{mk} = \begin{cases} 1 & k \text{ odd}, \\ 0 & \text{otherwise} \end{cases}, \qquad \Gamma_{rk} = \begin{cases} 1 & k = 1, 6 \\ 0 & \text{otherwise} \end{cases}.$$

and we also used the Proposition 67 to conclude the detectability of the multi-rate discretization. Consider the problem of designing a linear controller for P_C that achieves closed loop stability and such that the output $y_r[k]$ tracks the reference $\Gamma_r r[k]$ with zero steady-state error, where $r[k]$ is described by

$$\begin{aligned} w[k+1] &= Sw[k], \;\; w[0] = w_0, \\ r[k] &= C_R w[k] \end{aligned}$$

where

$$S = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \;\; C_R = \begin{bmatrix} 1 & 0 \end{bmatrix}$$

Thus, the reference $r[k]$ takes the form

$$r[k] = c_1 + c_2 k \tag{6.24}$$

where c_1, c_2 can be made arbitrary. Such non-constant references, prevent the use of the results from [ASC08a]. Contrarily to [82], this problem can be solved with the solution we provide in the present chapter. In fact, the discretization of P can be verified to satisfy (P1)-(P3) and therefore a linear controller with the structure depicted in Figure 6.1 can be synthesized for this system. The stabilizing controller C_K in the Figure 6.1 can be obtained, e.g., from the solution provided in [18] and the gains c_k^{1k} determining the system C_D are in this special case time-invariant and given by

$$c^{1k} = [-2 \;\; 1]^\intercal, \forall_{k \text{ odd}}$$

and do not need to be specified for k even since $y_m[k]$ is only sampled for odd k. In Figure 6.2, we show the response of the output $x_1[k]$ when a reference signal $r[k]$ consisting of a concatenation of signals taking the general form (6.24) is applied to the closed-loop system. We see that zero steady-state error is achieved after a transitory period, as desired.

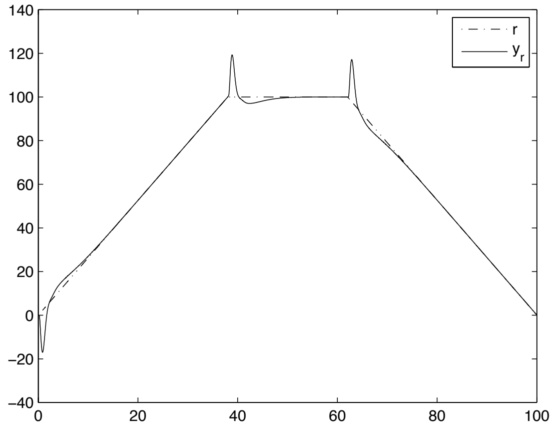


Figure 6.2: $y_r[k]$ and $r[k]$.

In Figure 6.3 we show several signals of the closed-loop for a short period of time where a transition of the reference signal occurs. Note that before the transition, in steady state, the output $y_D[k]$ of the blocking system C_D is zero as desired, and that at steady state $u[k]$ has the desired value to be applied to the plant so that output regulation can be achieved.

6.4 Blocking Zeros with respect to a Matrix

Consider an LTI system

$$\begin{aligned} x[k+1] &= Ax[k] + Bu[k] \\ y[k] &= Cx[k] + Du[k], \quad k \geq 0, \end{aligned} \tag{6.25}$$

and a periodic linear system

$$\begin{aligned} x[k+1] &= A_k x[k] + B_k u[k] \\ y[k] &= C_k x[k] + D_k u[k], \quad k \geq 0, \end{aligned} \tag{6.26}$$

where both (6.25) and (6.26) have the same number of inputs and outputs, and $x[k] \in \mathbb{R}^{n_A}$, $u[k] \in \mathbb{R}^{n_B}$, $y[k] \in \mathbb{R}^{n_B}$.

We generalize next the definition of blocking zero for LTI systems (see, e.g. [98]) and subsequently introduce the notion for periodic systems. This generalization lies at the heart of the solution we propose in Figure 6.1 since it is the property that the key system C_D is required to satisfy (cf. Subsection 6.5.3).

Definition 55. We say that (6.25) has a *blocking zero with respect to a matrix* $R \in \mathbb{R}^{n_R}$,

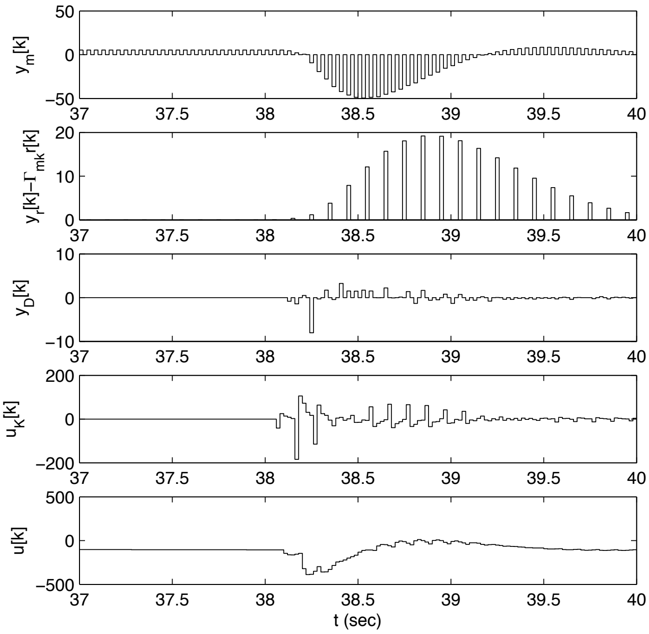


Figure 6.3: Various closed-loop signals

if there exists $\Pi \in \mathbb{R}^{n_A \times n_R}$ such that for every $E \in \mathbb{R}^{n_B \times n_R}$ we have that

$$\Pi R = A\Pi + BE \tag{6.27}$$

$$0 = C\Pi + DE \tag{6.28}$$

□

To interpret the nomenclature blocking zero used in the previous definition, we need the following proposition. Consider the system

$$\begin{aligned} w_R[k+1] &= Rw_R[k], \quad k \geq 0, \\ u[k] &= C_U w_R[k]. \end{aligned} \tag{6.29}$$

Proposition 56. Suppose that the input of the system (6.25), is generated by (6.29). Then there exists a solution $\Pi \in \mathbb{R}^{n_A \times n_R}$ to

$$\Pi R = A\Pi + BC_U \tag{6.30}$$

if and only if the solution to (6.26) satisfies $x[k] = \Pi w[k]$ when $x[0] = \Pi w[0]$ for an arbitrary $w[0] \in \mathbb{R}^{n_R}$. Moreover, if A has all its eigenvalues inside the open unit dist and R has all its eigenvalues outside the open unit disk then the solution to (6.30) is unique, and $x[k] \to \Pi w[k]$ as $k \to \infty$ for every initial condition $x[0]$, $w[0]$.

□

Proof. If there exists Π such that (6.30) has a solution, then we can argue by induction that $x[k] = \Pi w[k]$. In fact this is valid by hypothesis for $k = 0$, and if it is valid for $k = r$, then

$$\begin{aligned} x[r+1] &- \Pi w[r+1] && (6.31) \\ &= Ax[r] + BC_U w[r] - \Pi R w[r] \\ &= (A\Pi + BC_U - \Pi R) w[r] + A(x[r] - \Pi w[r]) \\ &= A(x[r] - \Pi w[r]) && (6.32) \\ &= 0. \end{aligned}$$

Conversely if $x[k] = \Pi w[k]$ when $x[0] = \Pi w[0]$ for an arbitrary $w[0] \in \mathbb{R}^{n_R}$, then $0 = x[1] - \Pi w[1] = (A\Pi + BC_U - \Pi R)w[0]$, which implies (6.30) since $w[0]$ is arbitrary.

Now suppose that A has all its eigenvalues inside the open unit disk and R has all its eigenvalues outside the open unit disk, and that there exists two solutions Π and $\tilde{\Pi}$, which satisfy (6.30). Let $E := \Pi - \tilde{\Pi}$. Then $ER = AE$, which implies that $ER^k = A^k E$ for every $k \geq 0$ and this implies that $E = 0$. Moreover from (6.31), (6.32) and the stability of A we conclude that for every $x[0], w[0]$, we have that $x[k] \to \Pi w[k]$ when $k \to \infty$. □

From Proposition 56, we can conclude that according to the Definition 55, the system (6.25) has a blocking zero with respect to R if the following hold. If the input of (6.25) is generated by (6.29) for an arbitrary matrix $C_R = E$ and the initial condition of (6.25) satisfies $x[0] = \Pi w[0]$, then the output of (6.25) is identically zero. This suggests the nomenclature blocking zero with respect to the matrix R.

According to [98, Def. 3.14] the system (6.25) has a blocking zero at a complex number $z_0 \in \mathbb{C}$ that does not belong to the spectrum of A, if

$$C(z_0 I - A)^{-1} B + D = 0. \tag{6.33}$$

As stated in the next proposition, in the case where R is scalar our definition coincides with the one from [98, Def. 3.14],

Proposition 57. Suppose that $R = z_0 \in \mathbb{C}$ is a complex number that does not belong to the spectrum of A. Then, the system (6.25) has a blocking zero according to the Definition 55 if and only if (6.33) holds.

□

Proof. To prove sufficiency it suffices to multiply (6.27) by $C(z_0 I - A)^{-1}$ and sum the result to (6.28). One obtains $(C(z_0 I - A)^{-1} B + D)E = 0$ for every E, which implies (6.33). To prove necessity take $\Pi = (z_0 I - A)^{-1} BE$, and see that (6.27)-(6.28) holds if (6.33) holds. □

Note that if the matrix R has only simple eigenvalues, then having a blocking zero with respect to R, is equivalent to having n_R zeros with respect to all the eigenvalues of R. Our definition of a blocking zero is broader since it allows to consider a matrix $R \in \mathbb{R}^{n_R}$ with a Jordan block structure

$$R = \begin{bmatrix} \lambda & 1 & 0 & \dots & 0 \\ 0 & \lambda & 1 & 0 & \dots \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \dots & 0 & \lambda \end{bmatrix} \tag{6.34}$$

and conclude if (6.25) has a blocking zero with respect to R, then every signal taking the form

$$u[k] = \sum_{l=0}^{n_R-1} c_l \binom{k}{l} \lambda^{k-l}, \tag{6.35}$$

is blocked (yields a zero output) by (6.25).

We extend the definition of blocking zero to periodic systems as follows. Recall that $\lfloor r \rfloor$ denotes the remainder of the division by h.

Definition 58. We say that (6.26) has a *blocking zero with respect to* $R \in \mathbb{R}^{n_R}$, if there exists $\Pi_r \in \mathbb{R}^{n_A \times n_R}$ such that for every $E_r \in \mathbb{R}^{n_B \times n_R}$, $1 \leq r \leq h$, we have that

$$\begin{aligned} \Pi_{\lfloor r+1 \rfloor} R &= A_r \Pi_r + B_r E_r \\ 0 &= C_r \Pi_r + D_r E_r, \ \ 1 \leq r \leq h. \end{aligned} \tag{6.36}$$

Moreover, if (6.36) hold only when $E_r = E$ for an arbitrary matrix $E \in \mathbb{R}^{n_B \times n_R}$ we say that (6.26) has a *time-invariant blocking zero with respect to* R

□

To interpret the nomenclature used in the previous definition, we need the following proposition. Consider the system

$$\begin{aligned} w_R[k+1] &= R w_R[k], \ \ k \geq 0, \\ u[k] &= C_{Uk} w_R[k]. \end{aligned} \tag{6.37}$$

where $C_{Uk} = C_{U(k+h)}, \forall_{k \geq 0}$ and R is now assumed to be invertible.

Proposition 59. Suppose that the input of the system (6.26), is generated by (6.37). Then, there exists a solution $\Pi_r \in \mathbb{R}^{n_A \times n_R}$, $1 \leq r \leq h$ to

$$\Pi_{\lfloor r+1 \rfloor} R = A_r \Pi_r + B_r C_{Ur}, \ \ 1 \leq r \leq h \tag{6.38}$$

if and only if the solution to (6.26) satisfies $x_k = \Pi_{\lfloor k \rfloor} w_k$, $k \geq 1$ when $x[0] = \Pi_h w[0]$ for an arbitrary $w[0] \in \mathbb{R}^{n_R}$. Moreover, if (6.26) is stable, and R has all its eigenvalues outside the open unit disk, then the solution to (6.38) is unique, and $x[k] \to \Pi_{\lfloor k \rfloor} w[k]$ as $k \to \infty$ for every initial condition $x[0]$, $w[0]$.

□

Proof. If there exists Π_r such that (6.38) has a solution, then we can argue by induction that $x[k] = \Pi_{\lfloor k \rfloor} w[k]$. In fact, this is valid by hypothesis for $k = 0$, and if it is valid for $k = l$, then

$$
\begin{aligned}
x[l+1] &- \Pi_{\lfloor l+1 \rfloor} w[l+1] && (6.39)\\
&= A_l x[l] + B_l C_{Ul} w[l] - \Pi_{\lfloor l+1 \rfloor} R w[r]\\
&= (A_l \Pi_{\lfloor l \rfloor} + B_l C_{Ul} - \Pi_{\lfloor l+1 \rfloor} R) w[l] + A(x[l] - \Pi_{\lfloor l \rfloor} w[l])\\
&= A_l (x[l] - \Pi_{\lfloor l \rfloor} w[l]) && (6.40)\\
&= 0.
\end{aligned}
$$

Conversely if $x[k] = \Pi_{\lfloor k \rfloor} w[k]$ when $x[0] = \Pi_T w[0]$ for an arbitrary $w[0] \in \mathbb{R}^{n_R}$, then $0 = x[r] - \Pi_r w[r] = (A\Pi_r + BC_U - \Pi_r) R w[r] = (A\Pi_r + BC_U - \Pi_r R) R^r w[0]$, which implies (6.30) since $w[0]$ is arbitrary and R is invertible.

Now suppose that (6.26) is stable, R has all its eigenvalues outside the open unit disk, and that there exists two solutions Π_r and $\tilde{\Pi}_r$, which satisfy (6.30). Let $E_r := \Pi_r - \tilde{\Pi}_r$ for $1 \leq r \leq h$. Then $E_{\lfloor k \rfloor} R = A_k E_{\lfloor k \rfloor}$, which implies that $E_r = 0$ for every $1 \leq r \leq h$. Moreover from (6.39), (6.40) and the stability of (6.26) we conclude that for every $x[0], w[0]$, we have that $x[k] \to \Pi_{\lfloor k \rfloor} w[k]$ when $k \to \infty$. □

From Proposition 59, we can conclude that according to Definition 58, the system (6.26) has a blocking zero with respect to R if the following hold. If the input of (6.26) is generated by (6.37) for an arbitrary matrices $C_{Uk} = E_k$, then the output of (6.25) is identically zero. If this hold when (6.37) is time-invariant, then (6.26) with $n_A = n_B$ has a time-invariant blocking zero with respect to R.

The relation between the blocking zeros of a periodic system and the blocking zeros of its lift is provided in the next result.

Proposition 60. The periodic system (6.26) has a blocking zero with respect to R if and only if its LTI lifted system has a blocking zero with respect to R^h.

□

Proof. From Proposition 59 we conclude that (6.26) has a blocking zero with respect to R, if and only if for every signal generated by (6.37), the output is zero. This holds if and only if (6.43) has zero output when the input is generated by

$$
\begin{aligned}
\hat{w}[l+1] &= R^h \hat{w}[l]\\
\hat{u}[l] &= F w[k],
\end{aligned}
\tag{6.41}
$$

where

$$F = \begin{bmatrix} C_{U0} \\ C_{U1}R \\ \dots \\ C_{U(h-1)}R^{h-1} \end{bmatrix}$$

Since the $C_{U\kappa}, \kappa \in \{1, \dots, h\}$ are arbitrary and R is invertible, we see that F can be made arbitrary, and using Proposition 56 we conclude that (6.43) has a blocking zero with respect to the matrix R^h.

□

6.5 Proof of the Main Result

We start by reviewing some general definitions for periodically time-varying systems. Then we state the assumptions that we make on C_I, C_D that lead to establishing that there exists a system C_K that yields the closed-loop of Figure 6.1 stable. After establishing the existence of such a system C_K we prove the main result. In the Appendix we introduce the notion of blocking zero with respect to a matrix, which is key to understand the assumptions on the block C_D, and to prove some of the results in the present Section.

6.5.1 Periodic Systems

Consider a discrete-time linear periodic system

$$\mathsf{R} = \begin{cases} x[k+1] = A_k x[k] + B_k u[k] \\ \quad y[k] = C_k x[k] + D_k u[k], \quad k \geq 0, \end{cases} \tag{6.42}$$

where $x[k] \in \mathbb{R}^{n_A}$, $u[k] \in \mathbb{R}^{n_B}$, $y[k] \in \mathbb{R}^{n_C}$, and A_k, B_k, C_k, D_k are h-periodic matrices, e.g., $A_k = A_{k+h}$. Many system analytical notions for (6.42) are defined by considering the *lifted* time-invariant system $\bar{\mathsf{R}}$ associated with R, which is defined as

$$\bar{\mathsf{R}} = \begin{cases} \bar{x}[l+1] = \bar{A}\bar{x}[l] + \bar{B}\bar{u}[l] \\ \quad \bar{y}[l] = \bar{C}\bar{x}[l] + \bar{D}\bar{u}[l], \quad l \geq 0, \end{cases} \tag{6.43}$$

where $\bar{x}[l] = x[lT]$,

$$\bar{u}[l] := (u[lh], u[lh+1], \dots, u[lh+h-1]),$$
$$\bar{y}[l] := (y[lT], y[lT+1], \dots, y[lT+h-1]),$$

and the system matrices in (6.43) are given by $\bar{A} := \Phi(h,0)$,

$$
\begin{aligned}
\bar{B} &:= \left[\Phi(h,1)B_0 \;\; \Phi(h,2)B_1 \ldots B_{h-1}\right],\\
\bar{C} &:= (C_0, C_1\Phi(1,0), \ldots, C_{h-1}\Phi(h-1,0)),\\
\bar{D} &:= \begin{bmatrix} E_{11} & \ldots & E_{1T} \\ \vdots & \vdots & \vdots \\ E_{T1} & \ldots & E_{Th} \end{bmatrix},\\
E_{ij} &:= \begin{cases} C_{i-1}\Phi(i-1,j)B_{j-1} & i > j \\ D_i & i = j \\ 0 & i < j \end{cases},
\end{aligned}
$$

where $\Phi(i,j) := A_{i-1}A_{i-2}\ldots A_j$, for $i > j$ and $\Phi(i,i) := I$.

The system R is stable, stabilizable and detectable if and only if $\bar{\mathsf{R}}$ is stable, stabilizable or detectable, respectively. Equivalently, stability of (6.42) is characterized by all the eigenvalues of the matrix $\bar{A}$ having norm less that one, i.e., $\|\lambda_i(\bar{A})\| < 1, \;\; \forall i$, stabilizability of (6.42), denoted by (A_k, B_k) is stabilizable, is characterized by there exists a set of periodic matrices F_k, $F_k = F_{k+h}$, such that $x[k+1] = (A_k + B_kF_k)x[k]$ is stable, and detectability of (6.42), denoted by (C_k, A_k) is detectable, is characterized by there exists a set of periodic matrices G_k, $G_k = G_{k+h}$, such that $x[k+1] = (A_k + G_kC_k)x[k]$ is stable (cf. [7]).

6.5.2 Assumptions on C_I:

The system C_I, described by (6.13), must be such that:

(I1) (A_I, B_I) is stabilizable and (C_I, A_I) is detectable,

(I2) C_I does not have invariant zeros at the unstable eigenvalues of the plant P, i.e.,

$$
\begin{bmatrix} A_I - \rho I_{n_I} & B_I \\ C_I & 0 \end{bmatrix} \text{ is invertible} \tag{6.44}
$$

for $\rho \in \{\lambda_i(A) : \|\lambda_i(A)\| \geq 1\}$.

(I3) For every $Z \in \mathbb{R}^{m \times n_w}$, the following equation

$$
\begin{aligned}
A_I\Pi_I &= \Pi_I S \\
C_I\Pi_I &= Z
\end{aligned} \tag{6.45}
$$

has a solution $\Pi_I \in \mathbb{R}^{n_I \times n_w}$. Moreover such solution is unique.

(I4) The eigenvalues of A_I belong to the set of eigenvalues of S.

We assume (I1)-(I2) to obtain (G1). The assumption (I4) is required to limit the set of possible systems C_I. As stated in the next proposition the assumption (I3) is closely related to the purpose of the system C_I, i.e., to provide the adequate input to the plant so that it tracks the reference signal. Note that due to the linearity of the plant, an adequate input takes the same form of the reference signal one wishes to follow, i.e., it is generated by

$$\begin{aligned} w[k+1] &= Sw[k], \quad k \geq 0, \\ u[k] &= C_U w[k]. \end{aligned} \tag{6.46}$$

Proposition 61. If (I3) holds then for any signal $u[k]$ generated by (6.46) there exists an initial condition x_0 such that the free motion of

$$\begin{aligned} x_I[k+1] &= A_I x_I[k], \quad x_I[0] = x_0 \\ y_I[k] &= C_I x_I[k] \end{aligned}$$

is such that $y_I[k] = u[k]$.

□

Proof. (of proposition 61) If (I3) holds then for an initial condition $x_I[0] = \Pi_I w[0]$, we have that

$$y_I[k] = C_I A_I^k \Pi_I w[0] = C_I \Pi_I S^k w[0] = Z S^k w[0].$$

The result follows by making $Z = C_U$. □

Due to the following proposition it is always possible to find B_J and C_J such that (I1)-(I3) hold for the system (6.13) with matrices (6.14).

Proposition 62. The set of matrices $(B_J, C_J) \in \mathbb{R}^{n_w \times 1} \times \mathbb{R}^{1 \times n_w}$ for which (I1)-(I3) do not hold for the system (6.13) with matrices (6.14) is a set of measure zero in $\mathbb{R}^{n_w \times 1} \times \mathbb{R}^{1 \times n_w}$.

□

Proof. Since the union of sets of measure zero has measure zero it suffices to prove that for each assumption (I1), (I2), and (I3) the set of matrices $(B_J, C_J) \in \mathbb{R}^{n_w \times 1} \times \mathbb{R}^{1 \times n_w}$ which do not satisfy each of these assumption has measure zero. We start by showing this for (I1). If (S, B_J) is not stabilizable, there exists a left eigenvalue of S, say w_i, for some $1 \leq i \leq n_s$, such that $w_i^\intercal B_J = 0$. Since the set $\{B_J : w_i^\intercal B_J = 0, \text{ for every } 1 \leq i \leq n_s\}$ where (S, B_J) is not stabilizable has measure zero, we establish that (S, B_J) is stabilizable except in a set of measure zero in $B_J \in \mathbb{R}^{n_w \times 1}$. A similar reasoning allows to conclude that (C_J, S) is detectable except in a set of measure zero in $C_J \in \mathbb{R}^{1 \times n_w}$. Since stabilizability of (S, B_J) implies that of (A_I, B_I) and detectability of (C_J, S) implies that of (C_I, A_I), we see that (I1) holds almost everywhere for the matrices (6.14). Next we consider (I2). Note that due to the structure of the matrices (6.14), the condition (6.44) is equivalent to the following condition

$$\begin{bmatrix} S-\rho I_{n_w} & B_J \\ C_J & 0 \end{bmatrix} \text{ is invertible }, \tag{6.47}$$

for $\rho \in \{\lambda_i(A) : \|\lambda_i(A)\| \geq 1\}$. For each $\rho \in \{\lambda_i(A) : \|\lambda_i(A)\| \geq 1\}$, the set

$$\{(B_J, C_J) : \det(\begin{bmatrix} S-\rho I_{n_w} & B_J \\ C_J & 0 \end{bmatrix}) = 0\}$$

is at most a manifold of dimension one in $\in \mathbb{R}^{n_w \times 1} \times \mathbb{R}^{1 \times n_w}$, from which we conclude that (6.47) holds except in a set of measure zero.

Finally, note that (I3) holds for the system (6.13) if and only if

$$\begin{aligned} S\Theta &= \Theta S \\ C_J\Theta &= Y \end{aligned} \tag{6.48}$$

has a solution $\Theta \in \mathbb{R}^{n_w \times n_w}$ for every $Y \in \mathbb{R}^{1 \times n_w}$. Considering the Jordan canonical decomposition of $S = VS_JV^{-1}$, where $S_J = \text{bdiag}(S_1, \ldots, S_{n_w})$, and making $\Theta = V^{-1}[\Pi_1, \ldots, \Pi_{n_s}]V$ we can conclude that (6.48) is equivalent to existing $\Pi_i \in \mathbb{R}^{n_w \times \kappa_i}$, $1 \leq i \leq n_s$ such that for every $W_i \in \mathbb{R}^{1 \times \kappa_i}$ we have that

$$\begin{aligned} S_J\Pi_i &= \Pi_i S_i \\ C_J\Pi_i &= W_i. \end{aligned} \tag{6.49}$$

It suffices to consider (6.49) for $i = 1$, i.e., prove that the set of C_J such that (6.49) does not hold for $i = 1$ has measure zero.

We can decompose Π_1 into $\Pi_1 = [v_1\, v_2\, \ldots\, v_{\kappa_1}]$ where v_i are such that there exists α_i such that

$$S_J(\alpha_1 v_1) = \mu_1(\alpha_1 v_1)$$

and, for $i = 2, \ldots, \kappa_1$,

$$v_i = w_i + \alpha_i v_1$$

for arbitrary α_i, where w_i are such that

$$(S_J - \mu_1 I)(w_i + \alpha_i v_1) = v_{i-1}.$$

Then

$$C_J\Pi_1 = [\alpha_1 C_J v_1 \;\; C_J w_2 + \alpha_2 C_J v_1 \;\; \ldots C_J w_{\kappa_1} + \alpha_{\kappa_1} C_J v_1]$$

If $C_J v_1 \neq 0$ then we can choose $\alpha_1, \ldots, \alpha_{\kappa_1}$ such that $C_J\Theta = W_1$ for arbitrary W_1. Since, the set of C_J such that $C_J v_1 = 0$ is a set of measure zero, we have that (I3) does not hold except in a set of measure zero. $\square$

6.5.3 Assumptions on C_D:

The system C_D must be such that:

(D1) (A_{Dk}, B_{Dk}) is stabilizable and (C_{Dk}, A_{Dk}) is detectable.

(D2) The following equations

$$\begin{aligned} A_{Dr}\Pi_{D\lfloor r+1\rfloor} + B_{Dr}\Gamma_{mr}Y &= \Pi_{Dr}S \\ C_{Dr}\Pi_{Dr} + D_{Dr}\Gamma_{mr}Y &= 0, \qquad 1 \le r \le h \end{aligned} \tag{6.50}$$

have a solution $\Pi_{Dr} \in \mathbb{R}^{n_D \times n_w}$, $1 \le r \le h$ such that (6.50) holds for every $Y \in \mathbb{R}^{n_m \times n_w}$.

(D3) Consider the system

$$\begin{aligned} \hat{w}[k] &= \hat{S}\hat{w}[k], \;\; \hat{w}[0] = \hat{w}_0 \\ \hat{r}[k] &= C_R\hat{w}[k] \end{aligned} \tag{6.51}$$

and suppose that $\hat{S}$ is such that there exists $\hat{w}_0$ such that $\hat{r}[k] \neq r[k]$ for any arbitrary w_0, C_R in (6.9). Then (D2) does not hold when S is replaced by $\hat{S}$.

We assume (D1)-(D3) to obtain (G1). From the Definition 58 of a time-invariant blocking zero given in the Appendix, we see that (D2) is equivalent to the system obtained by computing the series between C_D and Γ_m having a time-invariant blocking zero with respect to S. From the interpretation of blocking zeros given in Proposition 59, we see that the assumption (D2) is closely related to the purpose of the system C_D, i.e., to block the signals that are generated from the exosystem (6.9). Moreover, using again the same interpretation of the Proposition 59, the assumption (D3) states that C_D does not block any signal other than the ones generated by the exosystem.

Proposition 63. The system (6.15) with matrices (6.17) satisfies (D1)-(D3).

□

Proof. Note that (6.15) with matrices (6.17) is a stable system. In fact, one can check that all the eigenvalues of $\hat{A}_D = A_{D(h-1)} \dots A_{D2}A_{D1}A_{D0}$ are equal to zero, and therefore not only it is stable, but also the corresponding lifted system is a deadbeat system. Thus, its lifted system is stabilizable and detectable, and therefore so is (6.15) with matrices (6.17).

To prove (D2) it suffices, from the Proposition 59 in the Appendix, to prove that there exists an initial condition for C_D such that C_D has zero output for every signal generated by the following system

$$\begin{aligned} w[k+1] &= Sw[k] \\ r[k] &= \Gamma_{mk}C_Rw[k] \end{aligned} \tag{6.52}$$

Denote the n_w basis functions that generate (6.12) by

$$f_\iota[k] = \binom{k}{l}\mu_j^{k-l}, \text{ if } \iota \in (\sum_{q=1}^{j-1}\kappa_q, \sum_{q=1}^{j}\kappa_q], \tag{6.53}$$

where $l = \iota - \sum_{q=1}^{j-1}\kappa_q - 1$, $1 \leq \iota \leq n_w$. Recall that at a given time $k \geq hn_w$ the state of C_D holds the last n_w sampled values corresponding to each output component of $y_r[k]$. Then, also by construction of the C_{Dr} and D_{Dr} of the system C_D, proving that for any input signal taking the form (6.12), there exists and initial condition $x_D[0]$, such that the output of C_D is zero, is equivalent to proving that the following holds

$$\begin{bmatrix} f_1[k] \\ \vdots \\ f_{n_w}[k] \end{bmatrix} + \begin{bmatrix} f_1[k-\tau_1^{ik}] & \dots & f_1[k-\tau_{n_w}^{ik}] \\ \vdots & \vdots & \vdots \\ f_{n_w}[k-\tau_1^{ik}] & \dots & f_{n_w}[k-\tau_{n_w}^{ik}] \end{bmatrix} \begin{bmatrix} c_1^{ik} \\ \vdots \\ c_{n_w}^{ik} \end{bmatrix} = 0, \tag{6.54}$$

for every index i corresponding to sensor i, $1 \leq i \leq n_s$, and for every $k \geq 0$, where (6.54) only needs to be verified for $k \in \{1, \dots, h\}$ such that sensor i is sampled at t_k. The c^{ik} are obtained from (6.20). We establish this by proving that each row of (6.54) imposes the same restriction as each row of (6.20). It is easy to see that to do so, it suffices to consider, without loss of generality, a single sensor ($i = n_s = 1$) and a single set of κ_1 rows corresponding to the eigenvalue μ_1, i.e., prove that (6.54) with $n_w = \kappa_1$, imposes the same restrictions on c^{1k} as the following set of equations

$$\begin{bmatrix} \mu_1^{-\tau_1^{1k}} & \mu_1^{-\tau_2^{1k}} & \dots & \mu_1^{-\tau_{n_w}^{1k}} \\ \tau_1^{1k}\mu_1^{-\tau_1^{1k}} & \tau_2^{1k}\mu_1^{-\tau_2^{1k}} & \dots & \tau_{n_w}^{1k}\mu_1^{-\tau_{n_w}^{1k}} \\ \dots & \dots & \dots & \dots \\ (\tau_1^{1k})^{\kappa_1}\mu_1^{-\tau_1^{1k}} & (\tau_2^{1k})^{\kappa_1}\mu_1^{-\tau_2^{1k}} & \dots & (\tau_{n_w}^{1k})^{\kappa_1}\mu_1^{-\tau_{n_w}^{1k}} \end{bmatrix} \begin{bmatrix} c_1^{1k} \\ \vdots \\ c_{n_w}^{1k} \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ \dots \\ 0 \end{bmatrix} \tag{6.55}$$

We argue by induction. The first row of (6.55) imposes the restriction

$$\begin{bmatrix} \mu_1^{-\tau_1^{1k}} & \mu_1^{-\tau_2^{1k}} & \dots & \mu_1^{-\tau_{n_w}^{1k}} \end{bmatrix} c^{1k} = 1 \tag{6.56}$$

while the first row of (6.54) imposes the restriction

$$\begin{bmatrix} \mu_1^{k-\tau_1^{1k}} & \mu_1^{k-\tau_2^{1k}} & \dots & \mu_1^{k-\tau_{n_w}^{1k}} \end{bmatrix} c^{1k} = \mu_1^k \tag{6.57}$$

which are obviously equivalent since $\mu_1 \neq 0$. It is also insightful to see that the second row of (6.55), given by,

$$\begin{bmatrix} \tau_1^{1k}\mu_1^{-\tau_1^{1k}} & \tau_2^{1k}\mu_1^{-\tau_2^{1k}} & \dots & \tau_{n_w}^{1k}\mu_1^{-\tau_{n_w}^{1k}} \end{bmatrix} c^{1k} = 0 \tag{6.58}$$

imposes the same restriction as the second row of (6.54), given by

$$\begin{aligned} &\begin{bmatrix} (k-\tau_1^{1k})\mu_1^{k-\tau_1^{1k}} & \dots & (k-\tau_{n_w}^{1k})\mu_1^{k-\tau_{n_w}^{1k}} \end{bmatrix} c^{1k} = k\mu_1^k \\ \Leftrightarrow &\begin{bmatrix} -\tau_1^{1k}\mu_1^{-\tau_1^{1k}} & \dots & -\tau_{n_w}^{1k}\mu_1^{-\tau_{n_w}^{1k}} \end{bmatrix} c^{1k} = 0 \end{aligned} \tag{6.59}$$

where we used (6.56).

Now assuming that the first $1 \leq r-1 < n_w - 1$ rows of (6.54) impose the same restriction as the first $1 \leq r-1 < n_w - 1$ rows (6.55), we prove that the same is true for the row r, i.e.,

$$\begin{bmatrix} f_r[k-\tau_1^{1k}] & f_r[k-\tau_2^{1k}] \dots & f_r[k-\tau_{nw}^{1k}] \end{bmatrix} c^{1k} = f_r[k] \tag{6.60}$$

To this effect, note that we can write (6.53) as

$$\begin{aligned} f_r[k-\tau] &= \binom{k-\tau}{r} \mu_1^{k-r-\tau} \\ &= (\sum_{m=0}^{r} a_m k^{(r-m)} \tau^m) \mu_1^{k-r-\tau} \\ &= (\sum_{m=0}^{r-1} a_m k^{(r-m)} \tau^m + a_r \tau^r) \mu_1^{k-r-\tau} \end{aligned} \tag{6.61}$$

for some coefficients a_m implicitly defined from

$$\begin{aligned} \binom{k-\tau}{r} &= (k-\tau)(k-\tau-1)\dots(k-\tau-r) \\ &= \sum_{m=0}^{r} a_m k^{(r-m)} \tau^m. \end{aligned}$$

Note that $a_r = (-1)^{r+1}$. If we replace (6.61) in (6.60) and use the fact that the first $r-1$ restrictions of (6.55) hold, we obtain

$$a_r \begin{bmatrix} (\tau_1^{1k})^r \mu_j^{k-r-\tau_1^{1k}} & \dots & (\tau_{n_w}^{1k})^r \mu_j^{k-r-\tau_{n_w}^{1k}} \end{bmatrix} c_k^1 = 0$$

which is equivalent to the restriction associated with row r of (6.55) since $\mu_1 \neq 0$ and $a_r \neq 0$.

To prove that (6.15) with matrices (6.17) satisfies (D3), note that using the interpretation of a blocking zero given in the Proposition 59, the assumption (D3) states that C_D does not block any other signal other than the ones generated by the exosystem. To see that this is true, suppose that there exists a signal $g[k]$ such that there exists an initial condition to the system C_D such that the output of the system C_D is identically zero when $g[k]$ is applied to its input. Then, by construction of C_D, i.e., by the fact that for $k \geq n_w h$, $x_D[k]$ will hold the value of the last n_w samples of every sensor $1 \leq i \leq n_s$, $g[k]$ must satisfy

$$g[k] + \begin{bmatrix} g[k-\tau_1^{ik}] & g[k-\tau_2^{ik}] & \dots & g[k-\tau_{n_w}^{ik}] \end{bmatrix} c^{ik} = 0$$

for every, $1 \leq i \leq n_s$ and $k \in \{1, \dots, h\}$ such that sensor i is sampled at t_k. From (6.54) and uniqueness of the c^{ik}(cf. (6.19), (6.20)) we conclude that $g[k]$ must be a linear combination of the $f_\iota[k]$, $1 \leq \iota \leq n_w$ and therefore $g[k]$ can be generated by the exosystem.

□

6.5.4 System C_K:

The system C_K takes the form

$$\mathsf{C}_K := \begin{cases} x_K[k+1] = A_{Kk}x_K[k] + B_{Kk}u_K[k] \\ \quad y_K[k] = C_{Kk}x_K[k] \end{cases}. \tag{6.62}$$

where the matrices A_{Kk}, B_{Kk}, C_{Kk}, and D_{Kk} are $h-$periodic, i.e., e.g., $A_{Kk} = A_{K(k+h)}$ and is such that the closed loop of Figure 6.1 is stable. The next Lemma shows that such controller always exists.

Lemma 64. Suppose that P, C_I, C_D are such that (P1)-(P5), (I1)-(I4), (D1)-(D4) hold. Then there exists a stabilizing controller C_K for the closed loop system of the Fig. 6.1 taking the form (6.62).

□

To prove the Lemma 64 we need the following two results. For two dimensionally compatible LTI systems described by

$$\mathsf{C}_1 := \left\{ \quad \begin{bmatrix} x_1[k+1] \\ y_1[k] \end{bmatrix} = \begin{bmatrix} A_1 & B_1 \\ C_1 & 0 \end{bmatrix} \begin{bmatrix} x_1[k] \\ u_1[k] \end{bmatrix} \right. \tag{6.63}$$

and

$$\mathsf{C}_2 := \left\{ \quad \begin{bmatrix} x_2[k+1] \\ y_2[k] \end{bmatrix} = \begin{bmatrix} A_2 & B_2 \\ C_2 & D_2 \end{bmatrix} \begin{bmatrix} x_2[k] \\ u_2[k] \end{bmatrix} \right. \tag{6.64}$$

The series of the system C_2 and C_1 obtained by making $u_1[k] = y_2[k]$, is defined by:

$$\mathsf{C}_3 := \left\{ \quad \begin{bmatrix} x_3[k+1] \\ y_1[k] \end{bmatrix} = \begin{bmatrix} A_3 & B_3 \\ C_3 & 0 \end{bmatrix} \begin{bmatrix} x_3[k] \\ u_2[k] \end{bmatrix} \right. \tag{6.65}$$

where

$$A_3 = \begin{bmatrix} A_1 & B_1C_2 \\ 0 & A_2 \end{bmatrix}, B_3 = \begin{bmatrix} 0 \\ B_2 \end{bmatrix}, C_3 = \begin{bmatrix} C_1 & D_1C_2 \end{bmatrix}.$$

Recall that for an LTI system, say (6.63), an eigenvalue λ_1 of A_1, i.e., there exists v_1 and w_1 such that $A_1v_1 = \lambda v_1$ and $w_1^\intercal A_1 = \lambda w_1^\intercal$, is observable if $C_1v_1 \neq 0$, and controlladble if $w_1^\intercal B_1 \neq 0$. The pair (A_1, C_1) is detectable if all the unstable eigenvalues of A_1 are observable, and the pair (A_1, B_1) is stabilizable if all the unstable eigenvalues of A_1 are controllable. One can find these definitions in [98] for continuous-time systems, which have an obvious counterpart for discrete-time systems. Denote the set of eigenvalues of A by

$$\Lambda_A := \{\lambda : Av = \lambda v, \text{ for some } v\},$$

where A can be replaced by A_1, A_2, A_3. Also denote the set of eigenvalues of A which are not eigenvalues of B by

$$\Lambda_{A/B} := \{\lambda : \lambda \in \Lambda_A \text{ and } \lambda \notin \Lambda_B\},$$

where A, B can be replace by A_1 and A_2.

Proposition 65. Consider C_1, C_2 and the series C_3. Then

(i) $\Lambda_{A_3} = \Lambda_{A_1} \cup \Lambda_{A_2}$

(ii) If $\lambda \in \Lambda_{A_1/A_2}$ then λ is an observable eigenvalue of C_3 if λ is an observable eigenvalue of C_1.

(iii) If $\lambda \in \Lambda_{A_2}$ then λ is an observable eigenvalue of C_3 if λ is an observable eigenvalue of C_2 and C_1 has no invariant zeros at λ.

(iv) If $\lambda \in \Lambda_{A_2/A_1}$ then λ is a controllable eigenvalue of C_3 if λ is a controllable eigenvalue of C_2.

(v) If $\lambda \in \Lambda_{A_1}$ then λ is a controllable eigenvalue of C_3 if λ is a controllable eigenvalue of C_1 and C_2 has no invariant zeros at λ.

□

Proof. (i) We can conclude from $\det(\lambda I - A_3) = \det(\lambda I - A_1)\det(\lambda I - A_2)$, which holds due to the structure of A_3.
We prove only (ii) and (iii) since (iv) and (v) can be obtained from the fact that detectability and stabilizability are dual notions. To this effect consider the following equation for the eigenvalues of A_3

$$\begin{bmatrix} A_1 & B_1C_2 \\ 0 & A_2 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \lambda \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$$

If $\lambda \in \Lambda_{A_1/A_2}$ then $v_1 : A_1v_1 = \lambda v_1$ and $v_2 = 0$. Thus $C_3[v_1^\intercal\ v_2^\intercal]^\intercal = C_1v_1 \neq 0$, which means that λ is an observable eigenvalue of C_3 if it is an observable eigenvalue of C_1, which is (ii).

If $\lambda \in \Lambda_{A_2}$ then $v_2 : A_2v_2 = \lambda v_2$ and $v_1 : A_1v_1 + B_1v_2 = \lambda v_1$. If λ was not an observable eigenvalue of C_3 then we would have $\mathsf{C}_3[v_1^\intercal\ v_2^\intercal]^\intercal = 0$ which would imply that

$$\begin{bmatrix} A_1 - \lambda_1 I & B_1 \\ C_1 & D_1 \end{bmatrix} \begin{bmatrix} v_1 \\ C_2v_2 \end{bmatrix} = 0 \tag{6.66}$$

where $C_2v_2 \neq 0$ since λ is an observable eigenvalue of C_2. The equation (6.66) contradicts the assumption that C_1 has no invariant zeros at λ.

□

The series of two periodically time-varying systems is defined similarly to the series of two LTI systems.

Proposition 66. The lift of the series of two periodic systems is the series of the LTI lift systems of each periodic system.

□

Proof. Obtained by direct replacement. □

We shall need the following proposition.

Proposition 67. Consider the system

$$\begin{aligned} x[k+1] &= Ax[k] \\ y[k] &= \mathrm{bdiag}(\gamma_k^1, \dots, \gamma_k^{n_y})Cx[k] \end{aligned} \tag{6.67}$$

where $y[k] \in \mathbb{R}^{n_y}$. The $\gamma_k^i \in \{0,1\}, 1 \leq i \leq n_y$ are $h-$ periodic, i.e., $\gamma_k^i = \gamma_{k+h}^i$, and for each i are equal to 1 at least once in a period. The (6.67) is detectable if and only if the pair (A, C) is detectable.

□

Proof. See [18] □

We shall also need to take into account that if we apply the lift for an LTI system we obtain the following eigenvalues of the lifted system.

$$\Lambda_{A^h} := \{\lambda^h, \lambda \in \Lambda_A\} \tag{6.68}$$

The Lemma 64 is proved next.

Proof. In [16], it is shown that a periodic system R taking the form (6.42) is detectable and stabilizable if and only if there exists a periodic linear controller, taking the form (6.62) such that the closed loop system is asymptotically stable. Thus it suffices to prove the stabilizability and detectability of the periodic system obtained by computing the series of C_I, P, and C_D, which is a periodic system that we shall term C_A. This can proved by establishing stabilizability and detectability of the lifted LTI system, denoted by $\bar{\mathsf{C}}_A$. From Proposition 66 the lift of C_A is the series of the lift of each individual system C_I, P, and C_D, which are denoted by $\bar{\mathsf{C}}_I$, $\bar{\mathsf{P}}$, and $\bar{\mathsf{C}}_D$, respectively. Thus, we only have to prove the observability and controllability of the unstable eigenvalues of $\bar{\mathsf{C}}_A$, corresponding to the unstable eigenvalues of $\bar{\mathsf{C}}_I$ and $\bar{\mathsf{P}}$, since $\bar{\mathsf{C}}_D$ is a deadbeat system and thus stable.

We start by establishing stabilizability. From Proposition 65(iv) the controllability of the unstable eigenvalues of $\bar{\mathsf{C}}_A$ associated with $\bar{\mathsf{P}}$ or with $\bar{\mathsf{C}}_I$, do not depend of $\bar{\mathsf{C}}_D$

(which has only stable eigenvalues), and since C_I, P are LTI systems, this amounts to an LTI test. In fact, for an LTI system, considered to be a special case of a periodic system with period h, stabilizability of the lift is equivalent to stabilizability of the original LTI system, which can be concluded from the definition of stabilizability for periodic systems. Thus, it suffices to prove that the LTI series of C_I, P, which we denote by C_{IP}, is stabilizable. The eigenvalues of C_{IP} belonging to $\Lambda_{\mathsf{C}_I/\mathsf{C}_P}$ are controllable due to (I1) and Proposition 65(iv). The eigenvalues of C_{IP} belonging to Λ_{C_P} are controllable due to (P1), (I2), and Proposition 65(v).

We prove next detectability. To prove that observability of the unstable eigenvalues of $\bar{\mathsf{C}}_A$ belonging to $\Lambda_{\bar{\mathsf{C}}_I}$, which in turn correspond to unstable eigenvalues of C_I, due to (6.68), it suffices to prove that these eigenvalues are detectable from the plant output $y_r[k]$, i.e., if we consider the system obtained by the series of C_I and P_r where P_r is the system obtained by considering the plant with $y_r[k]$ has the only output. Again this amounts to an LTI test, since the series of C_I and P_r is an LTI system with multi-rate measurements as in Proposition 67, where we state that for such system detectability can be proved by an LTI test. The desired conclusion follows then from Assumptions (I1), (I4), (P2), and Proposition 65(iii).

To see the observability of the unstable eigenvalues of $\bar{\mathsf{C}}_A$ that correspond to those of P it suffices to consider those not observable through $y_r[k]$. By this we mean, the eigenvalues of the plant P that are not an observable eigenvalue of the pair (A, C_r). The existence of such eigenvalue, is equivalent to existing an initial condition for the plant $x[0]$ such that

$$\begin{aligned} x[k+1] &= Ax[k] \\ y_r[k] &= \Gamma_r C_r x[k] \end{aligned} \tag{6.69}$$

has zero output, i.e., $y_r[k] = 0$, $\forall_{k\geq 0}$. From assumption (P3) there cannot exist an initial condition for the plant such that (6.69) has a zero output $y_r[k] = 0$ and $y_m[k]$ can be generated by the exosystem. Since by assumption (D3), the system C_D does not cancel signals other than those generated by the exosystem, we obtain that this yields a non-zero output for C_D and thus these eigenvalues of A are also observable for the system $\bar{\mathsf{C}}_A$.

□

6.5.5 Output Regulation

The following result and Lemma 64 allows to conclude the main result of the chapter, i.e., Theorem 53.

Theorem 68. Suppose that (P1)-(P3) hold for the plant P. Then one can find C_I such that (I1)-(I4) holds, C_D such that (D1)-(D4) hold, and C_K such that the closed-loop in Figure 6.1 is stable. Moreover, output regulation is achieved even in the presence of plant uncertainties that do not destroy closed-loop stability.

□

Proof. The first part of the Theorem has been established in Proposition 62 and 63 and Lemma 64. Thus, it suffices to prove that output regulation is achieved. We start by writing the equations for the series connection of C_I, P, and C_D, which are given by

$$\begin{aligned} x_A[k+1] &= A_{Ak}x_A[k] + B_{Ak}u_A[k] + B_{Awk}w[k] \\ y_A[k] &= C_{Ak}x_A[k] + D_{Awk}w[k] \end{aligned}$$

where

$$\begin{aligned} A_{Ak} &= \begin{bmatrix} A & BC_I & 0 \\ 0 & A_I & 0 \\ B_{Dk}\Gamma_{mk}C_m & 0 & A_{Dk} \end{bmatrix} \\ B_{Ak} &= \begin{bmatrix} 0 \\ B_I \\ 0 \end{bmatrix} \\ C_{Ak} &= \begin{bmatrix} D_{Dk}\Gamma_{mk}C_m & 0 & C_{Dk} \\ \Gamma_{rk}C_r & 0 & 0 \end{bmatrix} \\ B_{Awk} &= \begin{bmatrix} B_V C_V \\ 0 \\ 0 \end{bmatrix} \\ D_{Awk} &= \begin{bmatrix} 0 \\ -\Gamma_{rk}C_R \end{bmatrix} \end{aligned}$$

Let C_K be a stabilizing controller described by (6.62), whose existence is established in Lemma 64. Then the closed-loop is described by

$$\begin{bmatrix} x_A[k+1] \\ x_K[k+1] \end{bmatrix} = \begin{bmatrix} A_{Ak} & B_{Ak}C_{Dk} \\ B_{Kk}C_{Ak} & A_{Kk} \end{bmatrix} \begin{bmatrix} x_A[k] \\ x_K[k] \end{bmatrix} + \begin{bmatrix} B_{Awk} \\ B_{Kk}D_{Awk} \end{bmatrix} w[k]. \tag{6.70}$$

where $w[k]$ is described by (6.9). Using the fact that the periodic system (6.70) is stable, since C_K is a stabilizing controller, from Proposition 59 in the Appendix, we conclude that there exists unique $\Pi_k, k \in \{1, \ldots h\}$, such that

$$\begin{bmatrix} A_{Ak} & B_{Ak}C_{Kk} \\ B_{Kk}C_{Ak} & A_{Kk} \end{bmatrix} \Pi_k + \begin{bmatrix} B_{Awk} \\ B_{Kk}D_{Awk} \end{bmatrix} = \Pi_{\lfloor k+1 \rfloor} S \tag{6.71}$$

and for any initial condition $(x_A[0], x_K[0])$ the state of the system tends asymptotically to

$$\begin{bmatrix} x_A[k] \\ x_K[k] \end{bmatrix} = \Pi_k w[k] \tag{6.72}$$

We provide a solution to (6.71), which is unique as mentioned above, and see that corresponding to such solution, we have that (6.72) is such that output regulation is achieved, i.e.,

$$y_{rk} = r[k]$$

or equivalently, using (6.72),

$$[C_r \;\; 0 \;\; 0]\Pi_k w[k] = C_R w[k]. \tag{6.73}$$

Such solution Π_k is obtained as follows. Make

$$\Pi_k = \begin{bmatrix} \Pi_{Ak} \\ 0 \end{bmatrix} \tag{6.74}$$

where

$$\Pi_{Ak} = \begin{bmatrix} \Pi_P \\ \Pi_I \\ \Pi_{Dk} \end{bmatrix}. \tag{6.75}$$

Then, from (6.71) we obtain that (6.75) must be such that

$$\begin{aligned} \begin{bmatrix} \Pi_P \\ \Pi_I \\ \Pi_{D\lfloor k+1 \rfloor} \\ 0 \\ 0 \end{bmatrix} S = & \begin{bmatrix} A & BC_I & 0 \\ 0 & A_I & 0 \\ B_{Dk}\Gamma_{mk}C_m & 0 & A_{Dk} \\ B_{K1k}D_{Dk}\Gamma_{mk}C_m & 0 & B_{K1k}C_{Dk} \\ B_{K2k}\Gamma_{rk}C_r & 0 & 0 \end{bmatrix} \begin{bmatrix} \Pi_P \\ \Pi_I \\ \Pi_{Dk} \end{bmatrix} \\ & + \begin{bmatrix} B_V C_V \\ 0 \\ 0 \\ 0 \\ -B_{K2k}\Gamma_{rk}C_R \end{bmatrix} \end{aligned} \tag{6.76}$$

where B_{K1k} and B_{K2k} are appropriate partitions of $B_{Kk} = [B_{K1k}\, B_{K2k}]$.

The matrices Π_P, Π_I, Π_{Dk} are obtained as follows.

(i) Take Π_P to be the solution, along with $E \in \mathbb{R}^{m \times n_w}$, to

$$\begin{aligned} \Pi_P S &= A\Pi_P + BE + B_V C_V \\ C_r \Pi_P &= C_R \end{aligned} \tag{6.77}$$

which as explained in [13] exists if and only if (P2) holds.

(ii) Take Π_I to be the unique solution to

$$\begin{aligned} A_I \Pi_I &= \Pi_I S \\ C_I \Pi_I &= E \end{aligned}$$

where E is, along with Π_P, the solution to (6.77). Note that such solution Π_I exists due to the Assumption (I3).

(iii) Take Π_{Dk} to be the solution to

$$\begin{aligned} A_{Dk}\Pi_{Dk} + B_{Dk}\Gamma_{mk}C_m\Pi_P &= \Pi_{D\lfloor k+1 \rfloor}S \\ C_{Dk}\Pi_{Dk} + D_{Dk}\Gamma_{mk}C_m\Pi_P &= 0 \end{aligned} \tag{6.78}$$

which exists due to the Assumption (D3) and is unique due to Proposition 59 and the fact that C_D is a stable system.

By construction, we conclude that Π_k given by (6.74), where Π_{Ak} is given by (6.75), and Π_P, Π_I and Π_{Dk} are described by (i), (ii) and (iii), respectively, satisfies (6.71) and (6.73) and therefore output regulation is achieved.

Note that in the proof we only required the controller C_K to stabilize the closed-loop. If the plant describing matrices are not known but C_K still stabilizes the closed-loop the proof remains unchanged. ☐

Proof. (of Theorem 53) Follows as a Corollary of Lemma 64 and Theorem 68, provided that we notice that the matrices B_J and C_J can be chosen such that C_I, given by (6.13), satisfies (I1)-(I4) (cf.Proposition 62) and that we notice that C_D, given by (6.16), (6.17), (6.18) satisfies (D1)-(D3) (cf.Proposition (63)). ☐

6.6 Further Comments and References

Several solutions were proposed in the seventies using different approaches (see, e.g., [26], [95], [33], [34]), and all incorporate an internal model of the exosystem, as well as a controller that stabilizes the closed loop. The necessity of incorporating a model of the exosystem in the controller was proved in [34], and is thereafter known as the internal model principle.

The blossom of the research in multi-rate systems took place in the early nineties, where many standard problems for LTI systems have been extended to multi-rate systems. For example, structural properties for multi-rate systems including stabilizability and detectability notions are provided in [65], a solution to the output pole placement problem can be found in [18], and the linear quadratic gaussian problem for multi-rate systems is solved in [20]. The close relation between multi-rate and periodic systems (see, e.g., [9]), entails that these definitions and problem solutions are often based on the concept of lifting (see [68]), i.e., considering the LTI system obtained by writing the equations of the periodic system along a period.

The output regulation problem for multi-rate systems was addressed in [82]. See also [19] for the special case where the exogenous references are constant signals. However, both in [82] and [19] the analysis is restricted to square systems, i.e., systems with the same number of inputs and measured outputs, which are also the outputs to be regulated. The solution in [82] and [19] consists of designing a stabilizing controller for the system obtained by adding an internal model of the exosystem in *series* with the input of the plant. The peculiarities of the multi-rate systems that prevented the authors from generalizing the solution to non-square systems, are discussed in [82].

The present chapter may be viewed as a follow up work of [ASC08a], and [ASC10], where the output regulation problem was considered for non-square multi-rate systems in the special case where the exogenous reference signals are constant signals. For constant reference signals, the proposed controller structure, is shown in [ASC08a], and [ASC10] not only to be suited for output regulation but also that it can be exploited to implement nonlinear gain-scheduled controllers in such a way that a fundamental property known as linearization property is satisfied.

7

Gain-Scheduled Controllers for Multi-Rate Systems

In this chapter we present a new methodology for the design and implementation of gain-scheduled controllers for multi-rate systems.

The standard procedure for designing a gain-schedulling controller involves the following steps (cf. [81]): (i) the selection of scheduling variables or parameters; (ii) linearization of the non-linear plant about the equilibrium manifold; (iii) synthesis of controllers for the family of plant linearizations, which typically involves linear controller design for a given set of equilibrium points; and (iv) implementation of the controller.

Since the controllers are designed for time-frozen parameters and when implemented the parameters are allowed to vary, the linearization of the nonlinear gain-scheduled controller about a given equilibrium point does not in general match the designed time-frozen linear controller. This mismatch is commonly known as the hidden coupling [81] and might lead to performance degradation or even instability. Therefore, for a correct implementation of the controllers, the following property is required to hold:

Linearization property: At each equilibrium point, the nonlinear gain-scheduled controller must linearize to the linear controller designed for that equilibrium.

A technique known as the velocity implementation, presented in [51], [52] and discussed in [53], [81] which is related with the work presented herein, provides a simple solution for the implementation of controllers with integral action that satisfies the linearization property.

Our proposed implementation satisfies the linearization property and boils down to the velocity implementation for single-rate systems. The key to our solution is a regulator structure which can be obtained with the tools developed in the previous Chapter. Building upon the work presented in [81] for the continuous time case, and in [62, 63] for the single-rate sampled-data case, we guarantee *i)* local stability of the feedback interconnection of the sampled-data multi-rate system and gain-scheduled controller about individual equilibria, and *ii)* ultimate boundedness of a conveniently defined closed-loop error in response to slowly varying exogenous inputs.

We illustrate the applicability of our results in the problem of steering an autonomous rotorcraft along a predefined trajectory. Simulation results obtained with a full non-linear rotorcraft dynamic model are presented and discussed.

The remainder of the Chapter is organized as follows. Section 7.1 describes the problem formulation and Sections 7.2 and 7.3 present the main results. In Section 7.4 we apply the proposed methodology to the design of an integrated guidance and control system that addresses the trajectory tracking problem for a small-scale rotorcraft, and evaluate the performance of the resulting nonlinear feedback system in simulation. Section 7.5 gives further comments and references.

7.1 Problem Formulation

Consider the nonlinear system

$$G := \begin{cases} \dot{x}(t) = f(x(t), u(t), w(t)) \\ y(t) = h(x(t), w(t)) \end{cases} \tag{7.1}$$

where f and h are twice continuously differentiable functions, $x(t) \in \mathbb{R}^n$ is the state, $u(t) \in \mathbb{R}^m$ is the control input, and the vector $w(t) \in \mathbb{R}^{n_w}$ contains references and possibly other exogenous inputs. The vector $y(t) \in \mathbb{R}^p$ can be decomposed as $y(t) = [y_m(t)^T \;\; y_r(t)^T]^T = [h_m(x(t), w(t))^T \;\; h_r(x(t), w(t))^T]^T$ where $y_m(t) \in \mathbb{R}^{n_{ym}}$ is a vector of measured outputs available for feedback and $y_r(t) \in \mathbb{R}^{n_{yr}}$ is a vector of tracking outputs, which we assume to have the same dimensions as the control input, $n_{y_r} = m$. This vector is required to track the reference $r(t)$ with zero steady state error, i.e., the error vector defined as $e(t) := y_r(t) - r(t)$ must satisfy $e(t) = 0$ at steady-state. Some of the components of $y_r(t)$ may be included in $y_m(t)$ as well.

7.1.1 Linearization Family

We assume that there exists a unique family of equilibrium points for G of the form

$$\Sigma := \{(x_0, u_0, w_0) : f(x_0, u_0, w_0) = 0,\; y_{r0} = h_r(x_0, w_0) = r_0\}$$

which can be parameterized by a vector $\alpha_0 \in \Xi \in \mathbb{R}^s$, s.t.

$$\Sigma = \{(x_0, u_0, w_0) = a(\alpha_0), \alpha_0 \in \Xi\} \tag{7.2}$$

where a is a continuously differentiable function. We further assume that there exists a continuously differentiable function v such that $\alpha_0 = v(y_0, w_0)$. By applying the function v to the measured values of y and w, we obtain the variable

$$\alpha = v(y, w), \tag{7.3}$$

which is usually referred to as the scheduling variable.

Linearizing the nonlinear system G about the equilibrium manifold Σ parameterized by α_0 yields the family of linear systems

$$G_l(\alpha_0) := \begin{bmatrix} \dot{x}_\delta(t) \\ y_\delta(t) \end{bmatrix} = \begin{bmatrix} A(\alpha_0) & B_1(\alpha_0) & B_2(\alpha_0) \\ C_2(\alpha_0) & D_{21}(\alpha_0) & 0 \end{bmatrix} \begin{bmatrix} x_\delta(t) \\ w_\delta(t) \\ u_\delta(t) \end{bmatrix} \tag{7.4}$$

where, e.g. $A(\alpha_0) = \frac{\partial f}{\partial x}(a(\alpha_0))$ and $x_\delta(t) = x(t) - x_0$.

7.1.2 Multi-Rate Sensors and Actuators

We consider that the sample and hold devices that interface the discrete-time controller and the continuous-time plant operate at different rates. Associated with each sampler S_i, corresponding to the ith component of $y(t)$, there is a sequence of sampling times $\{\sigma_1^i, \sigma_2^i, ..\}$ that verify $0 < \sigma_j^i < \sigma_{j+1}^i$. Similarly, associated with each holder H_i, corresponding to the ith component of $u(t)$, there is a sequence of hold times $\{\tau_1^i, \tau_2^i, ..\}$ that verify $0 < \tau_j^i < \tau_{j+1}^i$. We assume that the sample and hold operations are periodic and that their periods are related by rational numbers. Thus we can define a sequence of equally spaced time instants $\{t_0, t_1, ...\}$, $t_{k+1} - t_k = t_s, k \in \mathbb{Z}^+$, such that for every sampling time σ_j^i and hold time τ_j^i there exists a k_1 and a k_2 for which $\sigma_j^i = t_{k_1}$ and $\tau_j^i = t_{k_2}$. In addition, we introduce the matrix $\Gamma_k = \mathrm{diag}(g_1(k), ..., g_p(k))$, where $g_i(k) = 1$ if $\sigma_j^i = t_k$ for some j and $g_i(k) = 0$ otherwise, and the matrix $\Omega_k := \mathrm{diag}(r_1(k), ..., r_m(k))$, where $r_i(k) = 1$ if $\tau_j^i = t_k$ for some j and $r_i(k) = 0$ otherwise. Due to the periodic nature of the sample and hold devices we have $\Gamma_k = \Gamma_{k+h}$ and $\Omega_k = \Omega_{k+h}$, for some positive integer h which denotes the period. We further assume that every output is sampled and every input is updated at least once in a period.

The multi-rate sample and hold operators can then be written as

$$\begin{aligned} &S : \mathcal{L}(\mathbb{R}^+) \mapsto l(\mathbb{Z}^+) && H : l(\mathbb{Z}^+) \mapsto \mathcal{L}(\mathbb{R}^+) \\ &S = \Gamma_d S_{t_s} && H = H_{t_s}\Omega_d \end{aligned} \tag{7.5}$$

where the operators $\Omega_d : l(\mathbb{Z}^+) \mapsto l(\mathbb{Z}^+)$, $H_{t_s} : l(\mathbb{Z}^+) \mapsto \mathcal{L}(\mathbb{R}^+)$, $S_{t_s} : \mathcal{L}(\mathbb{R}^+) \mapsto l(\mathbb{Z}^+)$ and $\Gamma_d : l(\mathbb{Z}^+) \mapsto l(\mathbb{Z}^+)$ are given by

$$\begin{aligned} \Omega_d : &\xi_{k+1} = (I - \Omega_k)\xi_k + \Omega_k u_k,\ \xi_0 = 0 \qquad & H_{t_s}\, u(t) &= \tilde{u}_k \\ &\tilde{u}_k = (I - \Omega_k)\xi_k + \Omega_k u_k & t &\in [t_k,\, t_{k+1}[\end{aligned}$$

$$S_{t_s}: \tilde{y}_k = \begin{bmatrix} \tilde{y}_{mk} \\ \tilde{y}_{rk} \end{bmatrix} = y(t_k) \qquad \Gamma_d: y_k = \begin{bmatrix} \Gamma_{mk} & 0 \\ 0 & \Gamma_{rk} \end{bmatrix} \begin{bmatrix} \tilde{y}_{mk} \\ \tilde{y}_{rk} \end{bmatrix}$$

and $\Gamma_k = \begin{bmatrix} \Gamma_{mk} & 0 \\ 0 & \Gamma_{rk} \end{bmatrix}$ is partitioned according to the output decomposition $y^T = [y_m^T \; y_r^T]$. Similarly to $y(t)$, the values of $r(t)$ sampled at the instants t_k are denoted by $\tilde{r}_k$ and to take into account the multi-rate nature of the outputs we define $r_k = \Gamma_{rk}\tilde{r}_k$. We can then introduce the error variables $\tilde{e}_k = \tilde{y}_{rk} - \tilde{r}_k$ and $e_k = y_{rk} - r_k$.

7.1.3 Problem Statement

Given this setup the problem addressed in this paper can be stated as follows:

Problem 69. 1) For a fixed operating point α_0, find a possibly time-varying discrete-time linear controller $C(\alpha_0) : [y_{\delta mk}, e_k] \mapsto u_{\delta k}$

$$C(\alpha_0) = \left\{ \begin{bmatrix} x^c_{\delta k+1} \\ u_{\delta k} \end{bmatrix} = \begin{bmatrix} A^c_k(\alpha_0) & B^c_{1k}(\alpha_0) & B^c_{2k}(\alpha_0) \\ C^c_k(\alpha_0) & D^c_{1k}(\alpha_0) & D^c_{2k}(\alpha_0) \end{bmatrix} \begin{bmatrix} x^c_{\delta k} \\ y_{\delta mk} \\ e_k \end{bmatrix} \right. \tag{7.6}$$

for the linearization of the nonlinear plant (7.4) with multi-rate interface (7.5) that stabilizes the closed loop system and achieves zero steady-state error for e_k, where $u_{\delta k} = u_k - u_0$, $y_{\delta mk} = y_{mk} - \Gamma_{mk} y_{m0}$, and $y_{m0} = h_m(x_0, w_0)$.

2) Based on the family of linear controllers $C(\alpha_0)$, implement a discrete-time controller K, possibly nonlinear and time-varying, that verifies the linearization property, which will be defined shortly, and takes the form

$$K = \begin{cases} x^c_{k+1} = & f_c(x^c_k, y_{mk}, e_k, \alpha_k, k) \\ u_k = & h_c(x^c_k, y_{mk}, e_k, \alpha_k, k) \end{cases}, \tag{7.7}$$

where $x^c_k \in \mathbb{R}^{n_K}$ and given its dependence on the scheduling variable α_k sampled at time t_k, K is referred to as a gain-scheduled controller. By linearization property we formally mean that if we consider a family of equilibrium points Σ^c for the controller compatible with the family of equilibrium points Σ defined in (7.2), such that

$$\Sigma^c := \{x^c_0 : x^c_0 = f_c(x^c_0, y_{m0}, 0, \alpha_0, k), u_0 = h_c(x^c_0, y_{m0}, 0, \alpha_0, k), \\ (x_0, u_0, w_0) = a(\alpha_0),\ \alpha_0 \in \Xi\}$$

the controller K linearizes to $C(\alpha_0)$, at each equilibrium point α_0, that is, for example, $\frac{\partial f_c}{\partial x^c}(\alpha_0, k) = A^c_k(\alpha_0)$ and $\frac{\partial h_c}{\partial y_m}(\alpha_0, k) = D^c_{1k}(\alpha_0)$.

The proposed solution for part 2) has an a gain-scheduled structure and we will show that under mild assumptions the linearization property just described is sufficient to guarantee local stability about each equilibrium point of the feedback interconnection of the non-linear plant and a gain-scheduled controller with multi-rate interface.

7.2 Proposed Solution

7.2.1 Regulator Structure

In this section we focus on solving part 1) of the problem statement. To this end, we consider a simple linear system of the form

$$\dot{x}(t) = Ax(t) + B_2 u(t) \tag{7.8}$$

$$y(t) = \begin{bmatrix} y_m(t) \\ y_r(t) \end{bmatrix} = C_2 x(t) = \begin{bmatrix} C_m \\ C_r \end{bmatrix} x(t), \tag{7.9}$$

where $u(t)$ and $y_r(t) \in \mathbb{R}^m$.

We propose the controller structure depicted in Fig. 7.1 for the regulation of non-square systems, which can be shown to be a special case of the structure proposed in the previous case where the class of references and disturbance signals was not restricted to constant signals. In the figure C_I and C_D correspond to discrete time linear periodic integrators and differentiators, respectively, that can be written as

$$C_I = \begin{cases} x^I_{k+1} = x^I_k + \Omega_k u^I_k \\ y^I_k = x^I_k + \Omega_k u^I_k \end{cases} \tag{7.10}$$

$$C_D = \begin{cases} x^D_{k+1} = (I - \Gamma_{mk}) x^D_k + \Gamma_{mk} u^D_k \\ y^D_k = -\Gamma_{mk} x^D_k + \Gamma_{mk} u^D_k \end{cases} \tag{7.11}$$

At equilibrium, the constant values of the non regulated outputs are differentiated thereby obviating the need to feedforward these values, whose accurate determination is in general precluded by the presence of model uncertainties. It is straightforward to derive expressions for $\Gamma_D = C_D \Gamma_d$, $\Omega_I = \Omega_d C_I$ and G_d which results from discretizing G_L and is

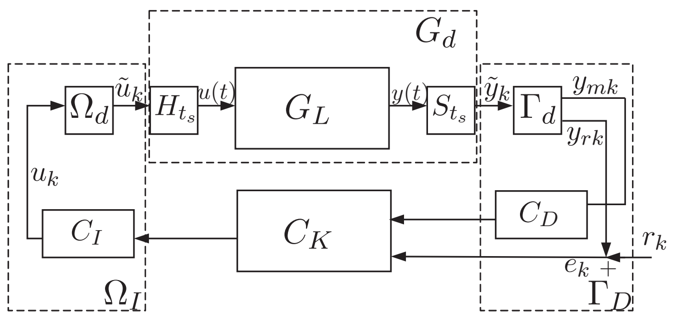


Figure 7.1: Regulator structure for non-square systems

given by

$$G_d = \begin{cases} x_{k+1} = A_d x_k + B_d \tilde{u}_k \\ \tilde{y}_k = \begin{bmatrix} \tilde{y}_{mk} \\ \tilde{y}_{rk} \end{bmatrix} = C_d x_k = \begin{bmatrix} C_{dm} \\ C_{dr} \end{bmatrix} x_k \end{cases}, \tag{7.12}$$

where $A_d = e^{At_s}$, $B_d = \int_0^{t_s} e^{A\tau} d\tau B_2$ and $C_d = C_2$.

Going back to the original problem stated in Section 7.1.3, we can conclude that a stabilizing controller of the form

$$C(\alpha_0) = \begin{cases} \begin{bmatrix} x^K_{\delta k+1} \\ y^K_{\delta k} \end{bmatrix} = \begin{bmatrix} A^K_k(\alpha_0) & B^K_{1k}(\alpha_0) & B^K_{2k}(\alpha_0) \\ C^K_k(\alpha_0) & D^K_{1k}(\alpha_0) & D^K_{2k}(\alpha_0) \end{bmatrix} \begin{bmatrix} x^K_{\delta k} \\ y^D_{\delta k} \\ e_{\delta k} \end{bmatrix} \\ \begin{bmatrix} x^D_{\delta k+1} \\ y^D_{\delta k} \end{bmatrix} = \begin{bmatrix} I - \Gamma_{mk} & \Gamma_{mk} \\ -\Gamma_{mk} & \Gamma_{mk} \end{bmatrix} \begin{bmatrix} x^D_{\delta k} \\ y_{\delta mk} \end{bmatrix} \\ \begin{bmatrix} x^I_{\delta k+1} \\ u_{\delta k} \end{bmatrix} = \begin{bmatrix} I & \Omega_k \\ I & \Omega_k \end{bmatrix} \begin{bmatrix} x^I_{\delta k} \\ y^K_{\delta k} \end{bmatrix} \end{cases} \tag{7.13}$$

where the first subsystem is a realization for $C_K(\alpha_0)$, provides a solution for part 1) of the problem statement, for fixed α_0. We assume that the design phase produces a family of controllers $C(\alpha_0)$ such that its parameters are continuously differentiable functions of α_0. The next example illustrates some of the concepts introduced so far.

Example 70. Suppose the nonlinear system (7.1) is given by

$$G = \begin{cases} \dot{x}_1(t) = -x_1(t) - x_2(t) \\ \dot{x}_2(t) = -x_2(t) + u(t) \\ \dot{x}_3(t) = -x_2(t) + x_2^3(t) + 0.5x_3(t) + u(t) \end{cases}. \tag{7.14}$$

The output $y_r(t) = x_1(t)$ is required to track the reference $r(t)$ with zero steady-state error. Considering $\alpha_0 = r_0$, the equilibrium manifold (7.2) is given by

$$\Sigma = \{x_{10} = -x_{20} = -u_0 = \alpha_0,\ \ x_{30} = 2\alpha_0^3,\ \ \alpha_0 \in \mathbb{R}\}.$$

The linearization family $G_l(\alpha_0)$, described by (7.4), can also be easily obtained, where for example

$$A(\alpha_0) = \begin{bmatrix} -1 & -1 & 0 \\ 0 & -1 & 0 \\ 0 & -1 + 3\alpha_0^2 & 0.5 \end{bmatrix}.$$

Consider the problem of designing a linear controller for $G_l(\alpha_0)$ with $\alpha_0 = 0$. Notice that if we consider the output to be $y_r = x_1$ this linear system is non-minimum phase. Furthermore, it is straightforward to check that $G_l(0)$ is not detectable from the regulated output $y_r = x_1$. Hence we assume that x_3 is also available for feedback and set $y_m =$

$[x_1 \ \ x_3]^T$, which implies that the system becomes non-square. The sampling and updating periods for y_{m1}, y_{m2}, and u are set to $t_{sy_1} = 0.25$, $t_{sy_2} = 0.1$ and $t_{su} = 0.05$, respectively. According to the framework of Section 7.1.2 we have $t_s = 0.05$, $h = 10$, and the h-periodic matrices Γ_{mk}, Γ_{rk}, Ω_k are determined by

$$\Gamma_{mk} = \begin{bmatrix} \sigma_k^1 & 0 \\ 0 & \sigma_k^2 \end{bmatrix}, \ \Gamma_{rk} = \sigma_k^3, \ \Omega_k = \tau_k^1 = 1, \forall_k$$
$$\sigma_k^2 = \begin{cases} 1 & k \text{ odd}, \\ 0 & \text{otherwise} \end{cases}, \quad \sigma_k^1 = \sigma_k^3 = \begin{cases} 1 & k = 1, 6 \\ 0 & \text{otherwise} \end{cases}.$$

A linear controller with the structure (7.13) can be synthesized for this system. The stabilizing controller C_K in (7.13) is obtained using the standard H_2 output-feedback synthesis solution for periodic systems. The performance of this controller will be evaluated in simulation and compared with that of a gain-scheduled controller and we will show that zero-steady state error is obtained for $y_r = x_1$.

7.2.2 Gain-Scheduled Implementation

Having designed the parameterized family of linear controllers $C(\alpha_0)$ as described in (7.13), suppose we implement the gain-scheduled nonlinear controller K as follows

$$K = \begin{cases} \begin{bmatrix} x_{k+1}^K \\ y_k^K \end{bmatrix} = \begin{bmatrix} A_k^K(\alpha_k) & B_{1k}^K(\alpha_k) & B_{2k}^K(\alpha_k) \\ C_k^K(\alpha_k) & D_{1k}^K(\alpha_k) & D_{2k}^K(\alpha_k) \end{bmatrix} \begin{bmatrix} x_k^K \\ y_k^D \\ e_k \end{bmatrix} \\ \begin{bmatrix} x_{k+1}^D \\ y_k^D \end{bmatrix} = \begin{bmatrix} I - \Gamma_{mk} & \Gamma_{mk} \\ -\Gamma_{mk} & \Gamma_{mk} \end{bmatrix} \begin{bmatrix} x_k^D \\ y_{mk} \end{bmatrix} \\ \begin{bmatrix} x_{k+1}^I \\ u_k \end{bmatrix} = \begin{bmatrix} I & \Omega_k \\ I & \Omega_k \end{bmatrix} \begin{bmatrix} x_k^I \\ y_k^K \end{bmatrix} \\ \alpha_k = g(\tilde{y}_k, w_k) \\ \begin{bmatrix} x_{k+1}^Y \\ \tilde{y}_k \end{bmatrix} = \begin{bmatrix} I - \Gamma_k & \Gamma_k \\ I - \Gamma_k & \Gamma_k \end{bmatrix} \begin{bmatrix} x_k^Y \\ y_k \end{bmatrix} \qquad (7.15.\text{A}) \end{cases} \tag{7.15}$$

which is a gain-scheduled controller. Notice that α_k, which was considered to be a constant design parameter during the design process, now becomes a scheduling variable computed on-line from the plant outputs and exogenous variables. Due to the multi-rate nature of the output, the system described by (7.15.A) is used to perform a hold operation on the output y_k so that the scheduling variable α_k is computed, at each iteration, according to the last sampled value of the output. The exogenous vector is assumed to be available at each sampling instant, so that $w_k = w(t_k)$. Notice that the non-linear controller proposed in (7.15) conforms to the general description of K given in (7.7) with $x_k^c = \left[(x_{k+1}^K)^T \ (x_{k+1}^D)^T \ (x_{k+1}^I)^T\right]^T$. Moreover as we will show in Section 7.3.1, it verifies the linearization property and therefore constitutes a solution to part 2) of the problem statement.

As a final remark, when the multi-rate set-up particularizes to the single-rate case, there is a close relation between the method presented herein and the velocity implementation ([52]), which is a method to implement gain-scheduled controllers for the single-rate case.

7.3 Stability Properties

In this section we show that the linearization property holds for the gain-scheduled implementation (7.15) and establish the results of local stability and ultimate boundedness in response to exogenous inputs for the feedback interconnection of the nonlinear multi-rate system and the proposed controller. These two last results can be obtained using the theoretical framework of jump systems and building upon the work presented in [62, 63] for the single-rate sampled-data case.

In what follows, the feedback interconnection of the nonlinear system G, described by (7.1), and gain-scheduled controller K, described by (7.15), with multi-rate sampled-data interface (7.5) is denoted by $F_{nl} := \mathcal{F}(SGH, K)$. Similarly, for each α_0, the feedback interconnection of the linearized system $G_l(\alpha_0)$, described by (7.4), and the designed controller $C(\alpha_0)$, described by (7.13), with multi-rate sampled-data interface (7.5) is denoted by $F_l(\alpha_0) := \mathcal{F}(SG_l(\alpha_0)H, C(\alpha_0))$.

7.3.1 Linearization Property

The required linearization property for the gain-scheduled controller implementation (7.15) is stated in the next result.

Theorem 1. Suppose for each parameter vector $\alpha_0 \in \Xi$, $F_l(\alpha_0)$ is asymptotically stable. Then F_{nl} admits an unique equilibrium point associated with α_0 and the linearization of the gain-scheduled controller K about this equilibrium coincides with the designed controller $C(\alpha_0)$.

7.3.2 Local Stability at each Operating Point

In order to establish local stability for F_{nl} about each equilibrium point, we start by considering the generic feedback interconnection of a continuous-time plant and a discrete-time periodically time-varying controller

$$\begin{aligned} \dot{x}(t) &= f(x(t), u(t)) & z_{k+1} &= a(z_k, y_k, k) \\ y(t) &= h(x(t)), \ \ t \geq 0 & u_k &= c(z_k, y_k, k), \ k \geq 0 \end{aligned} \tag{7.16}$$

with standard sample and hold interface

$$u(t)=u_k,\ t\in[t_k,t_{k+1}[,\ t_{k+1}-t_k=t_s \quad y_k=y(t_k) \tag{7.17}$$

where functions f and h are continously differentiable, and a and c are continously differentiable in z and y and h-periodic in k, e.g., $a(.,.,k)=a(.,.,k+h)$. Assuming that the closed-loop system has a single equilibrium point at the origin $f(0,0)=0$, $h(0)=0$, $a(0,0,k)=0$, $c(0,0,k)=0$ and introducing the vector

$$x_s(t)=[x(t)^T\ \ z_{k+1}^T\ \ u_k^T]^T,\ \ t\in[t_k,t_{k+1}[,$$

local exponential stability for this interconnection about the origin equilibrium can be defined as

$$\exists_{r_0,C,\gamma}:\forall_{t_0\geq 0, x_s(t_0)\in B_{r_0}(0)}\|x_s(t)\|\leq Ce^{-\gamma(t-t_0)}\|x_s(t_0)\|,\\ t\geq t_0.$$

Consider also the feedback interconnection of the linearized system and linearized controller

$$\begin{aligned}\dot{x}_\delta(t)&=Ax_\delta(t)+Bu_\delta(t) & z_{\delta k+1}&=N_k z_{\delta k}+M_k y_{\delta k}\\ y_\delta(t)&=Cx_\delta(t),\quad t\geq 0 & u_{\delta k}&=L_k z_{\delta k}+K_k y_{\delta k},\ k\geq 0\end{aligned} \tag{7.18}$$

with sample and hold interface similar to that defined in (7.17), and where, for example, $B=\frac{\partial f}{\partial u}(0,0)$, $N_k=\frac{\partial a}{\partial z}(0,0,k)$ and $K_k=\frac{\partial c}{\partial y}(0,0,k)$. Notice that N_k, M_k, L_k, K_k are h-periodic matrices. The next theorem establishes the stability relation between systems (7.16) and (7.18).

Theorem 2. The following statements are equivalent:
i) The system described by (7.16) is exponentially stable about the origin.
ii) The linearized system (7.18) is exponentially stable.
iii) For $A_d=e^{At_s}$ and $B_d=\int_0^{t_s}e^{A\tau}d\tau B$,

$$\sigma(\prod_{k=1}^{h}\begin{bmatrix}A_d+B_dK_kC & B_dL_k\\ M_kC & N_k\end{bmatrix})<1 \tag{7.19}$$

The local stability property for F_{nl} is given by the following corollary.

Corollary 71. Suppose for each parameter vector $\alpha_0\in\Xi$, $F_l(\alpha_0)$ is asymptotically (exponentially) stable. Then F_{nl} is locally exponentially stable about each equilibrium point associated with α_0.

Proof. About each equilibrium point, characterized by constant α_0 and $w(t)=w_0$, the nonlinear system G can be rewritten as $\dot{x}_\delta(t)=\bar{f}(x_\delta(t),u_\delta(t))$, $y_\delta(t)=\bar{h}(x_\delta(t))$, where $\bar{f}(x_\delta(t),u_\delta(t)):=f(x_0+x_\delta(t),u_0+x_\delta(t),w_0)$, and $\bar{h}(x_\delta(t)):=h(x_0+x_\delta(t))$ and a similar redefinition can be applied to K. Moreover, simple block manipulations show that

$F_{nl} = \mathcal{F}(SGH, K) = \mathcal{F}(\Gamma_d S_{t_s} G H_{t_s} \Omega_d, K) = \mathcal{F}(S_{t_s} G H_{t_s}, \Omega_d K \Gamma_d)$. Hence, F_{nl} conforms to (7.16)-(7.17), where G is the continuous-time system and $\Omega_d K \Gamma_d$ is the discrete-time controller connected by standard sample and hold interface. According to (7.4) and the key linearization property of K, the linearizations of G and $\Omega_d K \Gamma_d$ are given by $G_l(\alpha_0)$ and $\Omega_d C(\alpha_0)\Gamma_d$, respectively. By Theorem 2 we can conclude that the asymptotic stability of $F_l(\alpha_0) = \mathcal{F}(S_{t_s} G_l(\alpha_0) H_{t_s}, \Omega_d C(\alpha_0)\Gamma_d)$ implies local exponential stability of F_{nl} about each equilibrium point. □

7.3.3 Ultimate Boundedness for Slowly Varying Inputs

In this section we restrict the scheduling variable to depend solely on the exogenous inputs $\alpha(t) = w(t)$ so that we can impose a bound on its time-derivative. Then, building upon the work presented in [62, 63] for single-rate sampled-data systems, it is possible to address the multi-rate case and show that for any initial condition starting near the equilibrium manifold described by Σ and Σ^c, the error between the state of F_{nl} and the corresponding equilibrium value parameterized by $w(t)$ is ultimately bounded, with an ultimate bound that depends on the time-derivative of $w(t)$. Defining

$$x_s(t) := [x^T(t)\, (x^c_{k+1})^T\, u_k^T]^T,\ t \in [t_k, t_{k+1}[,$$

as the state of F_{nl} and

$$x_{s0}(\alpha_0) := [x_0^T(\alpha_0)\ \ (x_0^c(\alpha_0))^T\ \ u_0^T(\alpha_0)]^T,$$

as the corresponding parameterized equilibrium points, we can establish the following result.

Theorem 3. Suppose for each parameter vector $\alpha_0 \in \Xi$, $F_l(\alpha_0)$ is asymptotically stable. Then there exist positive constants δ_1, δ_2, k and γ, and a class $\mathcal{K}$ function $b(.)$ such that the following property holds. If, for any $t_0 \geq 0$, a continuously differentiable exogenous input $w(t) = \alpha(t)$ satisfies $w(t) \in \Xi$, $t \geq t_0$

$$\|x_s(t_0) - x_{s0}(w(t_0))\| < \delta_1 \ \text{ and } \ \nu := \sup_{t \geq t_0} \|\dot{w}(t)\| < \delta_2$$

then there exists a $t_1 \geq t_0$ such that

$$\|x_s(t) - x_{s0}(w(t))\| \leq k e^{-\gamma(t-t_0)} \|x_s(t_0) - x_{s0}(w(t_0))\|, \qquad t_0 \leq t < t_1$$

$$\|x_s(t) - x_{s0}(w(t))\| \leq b(\nu),\ \ t \geq t_1.$$

We return to Example 70 to illustrate these stability properties.

Example 1. (cont.) We design a gain-scheduling controller for the multi-rate system considered in Example 70, using the following methodology: *i)* the parameter space is discretized according to $\alpha_{0l} = -0.9+0.1l,\ \ l \in \{0, ..., 18\}$; *ii)* for each value α_{0l}, a controller with the structure (7.13) is computed using the standard H_2 output-feedback synthesis solution for periodic systems, yielding a finite set of controllers; and *iii)* the controller coefficients are interpolated by quadratic parameter dependent functions. The scheduling variable is set to $\alpha = r$. Figure 7.2 shows the response of the closed-loop system output $y_r(t)$ to an input $r(t)$ consisting of a sequence of steps, obtained with both the gain-scheduled and linear controllers. This simulation justifies the use of parameter varying controllers

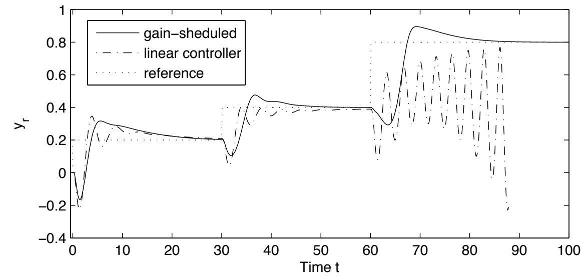


Figure 7.2: Gain-scheduled and linear controller

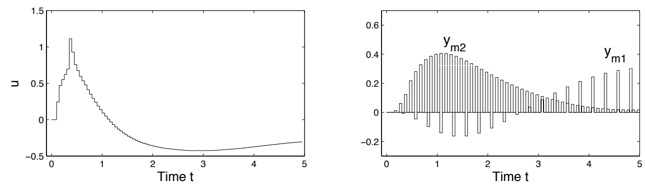


Figure 7.3: Actuation and multi-rate inputs y_{m1} and y_{m2}

since the linear controller leads to instability. The actuation and the multi-rate inputs y_{m1} and y_{m2} of the gain-scheduled controller are shown in Fig. 7.3 for a short period of time. Notice that zero steady-state error is obtained for y_r. In Fig. 7.4, the responses to a slow and a fast ramp input in $r(t)$ obtained with the gain-scheduled controller are shown. One can see that the deviation between the output and the corresponding equilibrium value (which coincides with the value of the reference) depends on the rate of variation of the reference which is in agreement with Theorem 3.

7.4 Trajectory Tracking Control for Autonomous Rotorcraft

In this section the proposed method is applied to the control of an autonomous rotorcraft. To this effect, the dynamics model of a small-scale helicopter, parameterized for the Vario X-

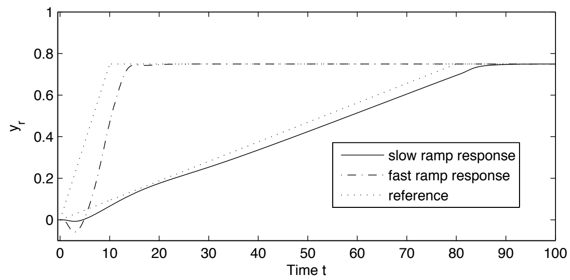


Figure 7.4: Slow and fast ramp response

treme R/C helicopter [23], is used. We address a trajectory tracking control problem, which can be described as the problem of steering a vehicle along a predefined trajectory defined in terms of space and time coordinates. The solution presented relies on the definition of an adequate error space to express the model of the vehicle [84]. The following section briefly describes the helicopter dynamic model.

7.4.1 Vehicle Dynamic Model

The helicopter dynamics are described using the conventional six degree of freedom rigid body equations

$$\begin{cases} \dot{v} = f\,(v,\,\omega,\,u) + \mathcal{R}^{-1}(\lambda)[0\ 0\ g]^{\intercal} \\ \dot{\omega} = n\,(v,\,\omega,\,u) \\ \dot{p} = \mathcal{R}(\lambda)v \\ \dot{\lambda} = \mathcal{Q}(\phi_B, \theta_B)\,\omega \end{cases}, \tag{7.20}$$

where $(p,\ \mathcal{R}) \in SE(3) \triangleq \mathbb{R}^3 \times SO(3)$ denotes the configuration of the body frame $\{B\}$ attached to the vehicle's center of mass with respect to the inertial frame $\{I\}$ and the rotation matrix $\mathcal{R} = \mathcal{R}(\lambda)$ can be parameterized by the Z-Y-X Euler angles $\lambda = [\phi_B\ \ \theta_B\ \ \psi_B]^{\intercal}$, $\theta_B \in\ ]-\pi/2, \pi/2[,\ \phi_B,\, \psi_B \in \mathbb{R},\ \mathcal{R} = R_Z(\psi_B)R_Y(\theta_B)R_X(\phi_B)$. The inertial frame $\{I\}$ is such that its $z-$axis is aligned with the gravity vector. The linear and angular body velocities are denoted by $v = [u_B\ \ v_B\ \ w_B]^T \in \mathbb{R}^3$ and $\omega = [p_B\ \ q_B\ \ r_B]^T \in \mathbb{R}^3$, respectively. Notice that the gravitational term $f_g(\phi_B, \theta_B) = \mathcal{R}^{-1}(\lambda)[0\ 0\ g]^{\intercal}$ depends only on the roll and pitch angles and that the euler angles rates $\dot{\lambda}$ and the angular velocities are related by the well-known transformation matrix $\mathcal{Q}(\phi_B, \theta_B)$.

The actuation $u = [\theta_0\ \ \theta_{1s}\ \ \theta_{1s}\ \ \theta_{0t}]$ comprises the main rotor collective input θ_0, the main rotor cyclic inputs, θ_{1s} and θ_{1c}, and the tail rotor collective input θ_{0t}. The dynamic equations for the helicopter are highly non-linear and its derivation is only accomplishable assuming several simplifications. For a detailed explanation of the modeling of the small-scale helicopter used in this chapter the reader is referred to [23].

7.4.2 Generalized Error Dynamics

The integrated guidance and control strategy proposed in [84] for the trajectory tracking problem, consists in defining a convenient non-linear transformation to be applied to the vehicle dynamic and kinematic model. In the new error space the trajectory tracking problem is reduced to that of regulating the error variables to zero using the fact that the linearization of the error dynamics is time-invariant about any trimming trajectories. It is well-known that these comprise helix and straight lines, parameterized by the vehicle linear speed, flight path angle, yaw rate and yaw angle with respect to the path [84]. Thus, the control design for the trajectory tracking problem can be solved by using tools that borrow from gain scheduling control theory, where the aforementioned parameters play the role of scheduling variables that interpolate the parameters of linear controllers designed for a finite number of representative trimming trajectories.

In order to define the non-linear transformation we start by formally introducing the trimming trajectories. Consider the helicopter equations of motion presented in (7.20), and let v_C, ω_C, p_C, $\lambda_C = [\phi_C \;\; \theta_C \;\; \psi_C]^T$ and u_C denote the trimming values of the state and input vectors, respectively. At trimming, these vectors satisfy $\dot{v}_C = 0$ and $\dot{\omega}_C = 0$, implying that $\dot{u}_C = 0$, $\dot{\phi}_C = 0$ and $\dot{\theta}_C = 0$. Given the dependence of the gravitational terms on the roll and pitch angles, only the yaw angle can change without violating the equilibrium condition. However, ψ_C satisfies

$$\begin{bmatrix} 0 \\ 0 \\ \dot{\psi}_C \end{bmatrix} = \mathcal{Q}(\phi_C, \theta_C)\omega_C$$

and thus the yaw rate $\dot{\psi}_C$ is constant. From this analysis it is easy to show [84] that trimming trajectories correspond to helices (which degenerate into straight lines when $\dot{\psi}_C = 0$) that can be described by

$$\lambda_C = \begin{bmatrix} \phi_C \\ \theta_C \\ \dot{\psi}_C t + \psi_0 + \psi_{CT} \end{bmatrix}, \quad p_C = \begin{bmatrix} \frac{V_C}{\dot{\psi}_C}\cos(\gamma_C)\sin(\dot{\psi}_C t + \psi_0) \\ -\frac{V_C}{\dot{\psi}_C}\cos(\gamma_C)\cos(\dot{\psi}_C t + \psi_0) \\ -V_C \sin(\gamma_C) t \end{bmatrix} + \begin{bmatrix} x_0 \\ y_0 \\ z_0 \end{bmatrix},$$

where $V_C = \|v_C\|$ is the linear body speed and γ_C is the flight path angle. Therefore, apart from a z-rotation and a translation, the parameter vector $(V_C, \gamma_C, \dot{\psi}_C, \phi_C, \theta_C, \psi_{CT},)$ completely characterizes a trimming trajectory. As explained in [22], due to the tail rotor actuation, helicopters can describe trimming trajectories with arbitrary but constant yaw angle relative to the path, that is, with arbitrary ψ_{CT}. However the roll and pitch angles ϕ_C and θ_C are automatically constrained by this choice. The trimming trajectory can thus be described by the following parameterization

$$\xi_0 = \begin{bmatrix} V_C & \gamma_C & \dot{\psi}_C & \psi_{CT} \end{bmatrix}^T$$

Moreover, introducing $\xi = (V, \gamma, \dot{\psi}, \psi)$

$$V = \|v\| \quad \gamma = \arctan(-\frac{w'_B}{u'_B}) \quad \dot{\psi} = \dot{\psi}_B \quad \psi = \psi_B - \psi_0 \quad , \qquad \begin{bmatrix} u'_B \\ v'_B \\ w'_B \end{bmatrix} = \mathcal{R} \begin{bmatrix} u_B \\ v_B \\ w_B \end{bmatrix} \tag{7.21}$$

we have at equilibrium $\xi = \xi_0$. For the sake of simplicity, in this particular case study, the vector ξ_0 is partitioned according to $\xi_0 = (\varphi_0, \alpha_0)$, where $\varphi_0 := (V_C, \dot{\psi}_C, \psi_{CT})$ is constant and $\alpha_0 := \gamma_C$ is allowed to vary. That is, we consider paths formed by the concatenation of trimming trajectories that can be parameterized by α_0 only. By imposing this restriction on the reference trajectories and considering a similar partition for the scheduling variable $\xi = (\varphi, \alpha)$, where $\varphi = (V, \dot{\psi}, \psi)$ and $\alpha = \gamma$, we assume that φ does not change significantly between trajectories. Notice that α conforms with the general description of scheduling variable given in (7.3).

The generalized error vector relating the vehicle state and the commanded trajectory parameterized by ξ_0, or equivalently α_0 since φ_0 is constant, can be defined using the nonlinear transformation

$$x_e = \begin{bmatrix} v_e \\ \omega_e \\ p_e \\ \lambda_e \end{bmatrix} = \begin{bmatrix} v - v_C \\ \omega - \omega_C \\ \mathcal{R}^{-1}(p_B - p_C) \\ \mathcal{Q}^{-1}(\lambda_B - \lambda_C) \end{bmatrix} \tag{7.22}$$

Since $\dot{v}_C = 0$ and $\dot{\omega}_C = 0$ for any trimming trajectory, the nonlinear error dynamics can be written as [84]

$$G = \begin{cases} \dot{v}_e = \dot{v} \\ \dot{\omega}_e = \dot{\omega} \\ \dot{p}_e = v_B - \mathcal{R}_e^{-1} v_C - \mathcal{S}(\omega) p_e \\ \dot{\lambda}_e = \omega - \mathcal{Q}^{-1}\mathcal{Q}_C \omega_C - (\frac{d}{dt}\mathcal{Q}^{-1})\mathcal{Q}\lambda_e \end{cases} \tag{7.23}$$

where $\mathcal{R}_e^{-1} = \mathcal{R}^{-1}\mathcal{R}(\phi_C, \theta_C)$, $\mathcal{Q}_C = \mathcal{Q}(\phi_C, \theta_C)$ and $S(\omega)$ is a skew-symmetric matrix defined such that $\mathcal{S}(\omega) = [\omega \times]$.

The output vector for which we require steady-state tracking is defined as

$$y_{er} = \begin{bmatrix} p_e \\ \psi_e \end{bmatrix} \quad , \quad \psi_e = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix} \lambda_e$$

It is straightforward to obtain that $y_{er} = 0$ implies $x_e = 0$ and that the vehicle will follow the path with the desired linear speed and orientation if and only if $x_e = 0$. As mentioned in the next section, the state is assumed available, although at different rates, and thus we have

$$y_{em} = x_e$$

The linearization of (7.23) about the equilibrium ($x_e = 0, u = u_C$), or equivalently, the linearization of (7.20) about α_0, can be written in compact form as

$$G_l(\alpha_0) = \begin{cases} \dot{x}_{e\delta} = A_e(\alpha_0)x_{e\delta} + B_e(\alpha_0)u_\delta \\ y_{em\delta} = x_{e\delta} \\ y_{er\delta} = C_{er}(\alpha_0)x_{e\delta} \end{cases}, \tag{7.24}$$

where $x_{e\delta} = x_e$, $u_\delta = u - u_C$, $y_{er\delta} = y_{er}$, $y_{em\delta} = y_{em}$, and $C_{er}(\alpha_0) = \begin{bmatrix} 0 & I_3 & 0 \\ 0 & 0 & [0\,0\,1] \end{bmatrix}$. For details on the matrices of the linearization see [84]. Note that, $G_l(\alpha_0)$ is the parameterized local system model, for which the controller is to be designed.

7.4.3 Multi-rate Characteristics of the Sensors

As already mentioned, for an autonomous vehicle the dynamic and kinematic state-variables comprise linear and angular velocities, position and orientation which are usually available at different rates (for example the position is typically measured by a GPS receiver which imposes a slow rate). In the present chapter we assume that the state variables of the vehicle corresponding to linear and angular velocities and orientation are measured at a sampling rate of 50 Hz, which corresponds to a sampling period $t_s = 0.02$ s, whereas the position variables are measured at a sampling rate of 2.5 Hz, corresponding to a sampling rate of $t_{sp} = 0.4$ s. The actuators are assumed synchronous and updated at a rate of 50 Hz. According to these sampling rates, the h-periodic matrices Γ_{mk}, Γ_{rk} and Ω_k that characterize the multi-rate setup, are given by

$$\begin{aligned} \Omega_k &= I_4 \\ \Gamma_{mk} &= \begin{cases} I_{12} & k = 0 \\ \mathrm{diag}([1_3 \;\; 1_3 \;\; 0_3 \;\; 1_3]) & \text{otherwise} \end{cases} \\ \Gamma_{rk} &= \begin{cases} I_4 & k = 0 \\ \mathrm{diag}([0_3 \;\; 1]) & \text{otherwise} \end{cases}, \qquad k = 0, .., h-1. \end{aligned}$$

where $h = 20$.

7.4.4 Controller Synthesis and Implementation

The commonly used method for the design of a family of controllers for the parameterized family of models, which is also adopted in the present chapter, comprises the following steps:

i) obtain a finite set of parameter values from the discretization of the continuous parameter space, *ii)* synthesize a linear controller for each linear plant, described by (7.24), obtained from the linearization of the nonlinear plant for each value of the schedule parameter, *iii)* interpolate the coefficients of the linear controllers to obtain a continuously parameter-varying controller.

We have restricted the parameter space to $\alpha = \gamma$, which means that the synthesized controllers are valid in operating conditions where the remaining parameters $\varphi = (V, \dot{\psi}, \psi)$ do not change significantly about nominal values $(V_C, \dot{\psi}_C, \psi_C)$. The finite set of values for the discretization of this one-dimensional parameter space was chosen to be

$$\bar{\alpha}_0 = \{\bar{\alpha}_{0i}\} = [\text{-50 -40 -30 -20 -10 0 10 20 30 40 50}]\,\frac{\pi}{180} \text{ rad.}$$

so that the conditions under which the vehicle is expected to operate include straight lines $\dot{\psi}_C = 0$ and z-aligned helices $\dot{\psi}_C \neq 0$, with a flight path angle between -50ï£ · and 50ï£ · degrees, that the vehicle is required to follow with constant velocity, yaw rate and orientation with respect to the path $((V, \dot{\psi}, \psi) = (V_C, \dot{\psi}_C, \psi_C))$.

For each fixed value of α_0, the standard output feedback $\mathcal{H}_2$ formulation for periodic system was used to synthesize a controller C_K for the augmented system G_a where A_d, B_d, C_d are the discretizations of system matrices of (7.24). Notice that G_a has a number of states equal to $n + m + n_{y_r} = 12 + 4 + 12 = 28$ and a number of outputs equal to $n_{y_m} + n_{y_r} = 12 + 4 = 16$. For an h-periodic system G of the form

$$G := \begin{cases} x_{k+1} = A_k x_k + B_{1k} w_k + B_{2k} u_k \\ z_k = C_{1k} x_k + D_{11k} w_k + D_{12k} u_k \\ y_k = C_{2k} x_k + D_{21k} w_k + D_{22k} u_k \end{cases} \tag{7.25}$$

with $D_{11k} = 0$, the $\mathcal{H}_2$ control problem can be interpreted as the problem of finding a linear h-periodic controller $K : y_k \mapsto u_k$ that minimizes the average over one period of the norms of the impulse responses of the closed-loop system,

$$G_{cl} := \begin{cases} \underline{\mathrm{x}}_{k+1} = \underline{\mathrm{A}}_k \, \underline{\mathrm{x}}_k + \underline{\mathrm{B}}_{1k} w_k \\ z_k = \underline{\mathrm{C}}_{1k} \underline{\mathrm{x}}_k \end{cases}$$

which can be written as

$$\|G_{cl}\|_2 = (\frac{1}{h} \sum_{j=0}^{h-1} \sum_{i=1}^{n_w} \|G_{cl}\delta(k-j)e_i\|_2)^{\frac{1}{2}}. \tag{7.26}$$

where $G_{cl}\delta(k-j)e_i$ denotes the output response to an impulse applied at input i and time j. The solution to this problem can be obtained by solving two periodic Riccati equations [8] or by the solution of two LMI optimization problems [93]. The matrices associated with the performance channel $w_k \mapsto z_k$ are typically used as tuning knobs to improve the performance during extensive simulations. For the results presented in the next subsection, these performance matrices were chosen to be independent of k and α_{0i}.

The resulting finite set of synthesized controller coefficients, for example $\{A^c_k(\bar{\alpha}_{0i})\}$, were interpolated using least squares fitting yielding a family of continuously parameter dependent controllers whose describing matrices are quadratically parameter dependent, for example

$$A^c_k(\alpha_0) = A^{c1}_k + \alpha_0 A^{c2}_k + \alpha_0^2 A^{c3}_k$$

The disadvantage of this technique is that by the interpolation process there is no guarantees that, even for fixed parameter values, the controller obtained by interpolation stabilizes the closed loop system. An a posteriori analysis showed that for a dense grid of fixed values of α_0 the closed loop system is stabilized. In this particular case, this would not happen if we had used a simpler linear interpolation instead of a quadratic interpolation. Note also that using a piecewise linear interpolation would not comply with the assumption that the scheduling controller is a continuously differentiable function of the scheduling variable.

Having designed the family of controllers $C(\alpha_0)$, the non-linear controller $K(\alpha_k)$ can be implemented with $\alpha_k = \gamma_k$ becomes a time-varying parameter computed according to (7.21). The final implementation scheme is shown in Figure 7.5. Notice that system C_I in this particular case degenerates into simple integrators $C_I = \frac{1}{1-z^{-1}} I_4$ since the actuators were considered synchronous and updated at a sampling rate of 50 Hz.

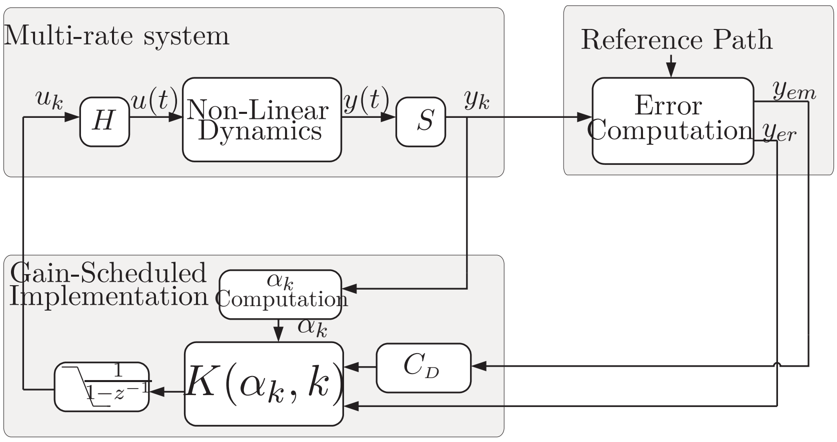


Figure 7.5: Block diagram with the final implementation scheme

Some additional features of this implementation are worthwhile emphasizing. The placement of the integrators at the plant's input, has the following advantages: *i)* the implementation of anti-windup schemes is straightforward *ii)* auto-trimming property- the controller automatically generates adequate trimming values for the actuation signals.

Though in the present case simple integrators are being used, the same advantages would hold if the inputs were updated at different rates and the integrators took the form of Ω_I.

7.4.5 Simulation Results

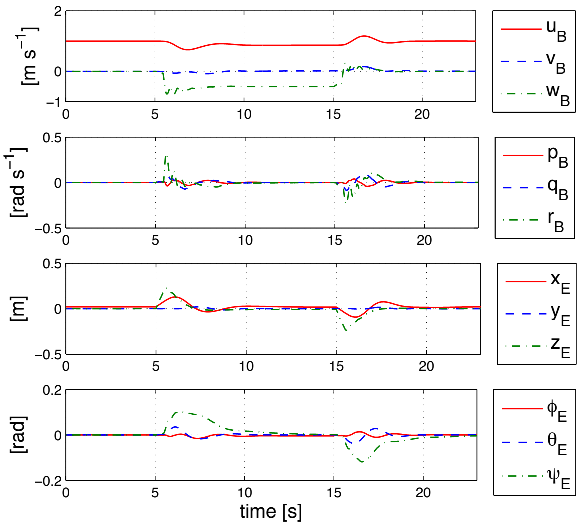
We start with a simple example that compares the performance of a multi-rate guidance and control law with that obtained using a standard single-rate $\mathcal{H}_2$ compensator designed using equivalent weighting matrices. First, we compare the values of the closed-loop $\mathcal{H}_2$ norms while changing the rate of the linear position measurement and keeping the sampling periods of the other outputs and of the input at $t_s = 0.02$ s. The periodic $\mathcal{H}_2$ controllers were synthesized for a single operating condition, characterized by $\xi_0 = \begin{bmatrix} V_C & \gamma_C & \dot{\psi}_C & \psi_{CT} \end{bmatrix}^T = \begin{bmatrix} 1\,\mathrm{m.s}^{-1} & 0\,\mathrm{rad} & 0\,\mathrm{rad.s}^{-1} & 0\ \mathrm{rad} \end{bmatrix}^T$. The results are shown in Table 7.1.

Notice that when the sampling rate of the linear position measurement equals the sampling rate of the other outputs ($t_s = 0.02$ s), the multi-rate set-up particularizes to the single-rate case and the periodic $\mathcal{H}_2$ controller is equivalent to a standard discrete-time $\mathcal{H}_2$ controller. As expected the performance of the closed-loop system, given in terms of the $\mathcal{H}_2$ norm, deteriorates as the sampling period for the linear position increases. In order to assess the impact of this performance loss on the tracking response of the vehicle, a simple maneuver task is set up for the rotorcraft. During the maneuver the vehicle is required to follow a path with constant velocity $V_C = 1$ m.s^{-1} and constant orientation with respect to the path, consisting of: *i)* a level flight segment along the x axis, *ii)* a climbing ramp with a flight path angle of $\gamma_C = \frac{\pi}{6}$ rad, *iii)* a level flight segment along the x axis. A gain-scheduled controller was synthesized following the procedure given in Section 7.4 for constant parameters $\varphi_0 = (V_C, \dot{\psi}_C, \psi_{CT}) = (1\,\mathrm{m.s}^{-1}, 0\,\mathrm{rad.s}^{-1}, 0\,\mathrm{rad})$. For the single-rate case, all the outputs are measured at a sampling rate of $t_s = 0.02$ s, and for the multi-rate case, the sampling rates are given according to Section 7.4.7.4.3. The results comparing these two cases, which include the temporal evolution of the actuation, dynamic variables v and ω, and kinematic errors $p_E = p_B - p_C = [x_E\ y_E\ z_E]^T$ and $\lambda_E = \lambda_B - \lambda_C = [\phi_E\ \theta_E\ \psi_E]^T$, are shown in Figures 7.6 and 7.7.

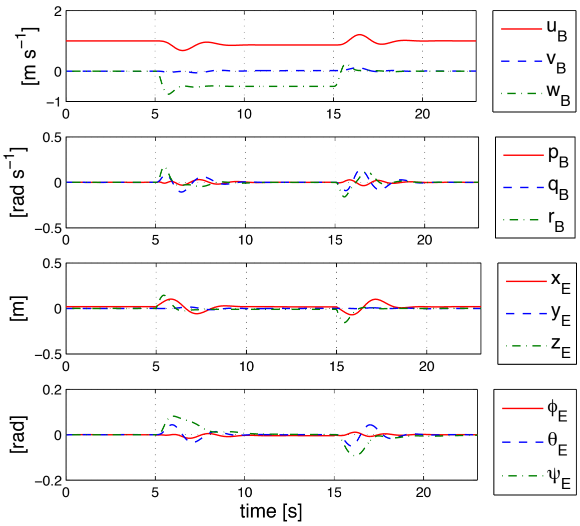
As expected, we can notice some degradation in the actuation and error responses for the multi-rate case in comparison with the single-rate case. Nevertheless, while the proposed method takes into account the multi-rate characteristics of the sensors, it also achieves good tracking performance and displays a smooth behavior throughout the differ-

Table 7.1: $\mathcal{H}_2$ closed-loop values for different position sampling rates

Linear position sampling period t_{sp}(s)	0.02	0.04	0.1	0.2	0.4
Closed loop $\mathcal{H}_2$ norm $\|G_{cl}\|_2$	10.40	11.08	12.18	13.15	14.27



(a) Errors multi-rate

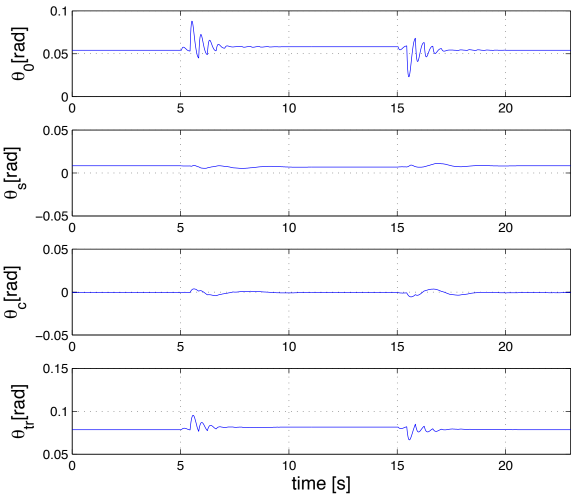


(b) Errors single-rate

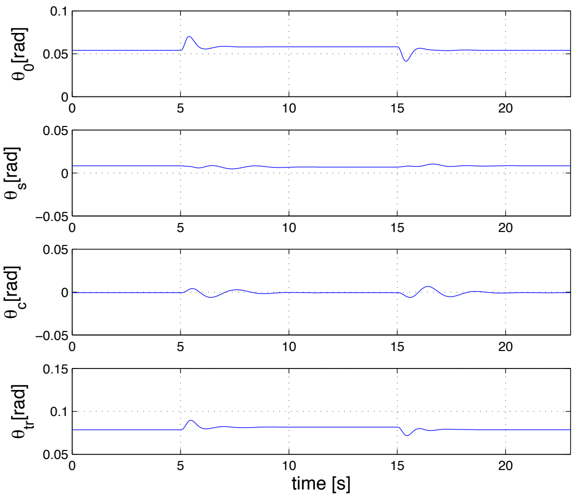
Figure 7.6: Results for the first simulation- Errors

ent stages of the trajectory. Notice also that, as desired, steady-state is achieved after the transitions between trimming trajectories, with zero-steady state value for $y_{er} = [p_e \;\; \psi_e]$, and that the trimming values for the actuation are naturally acquired.

Another simulation is presented in order to test the proposed methodology in other flight conditions as well as analyze the impact of noise in the sensors. In this simulation



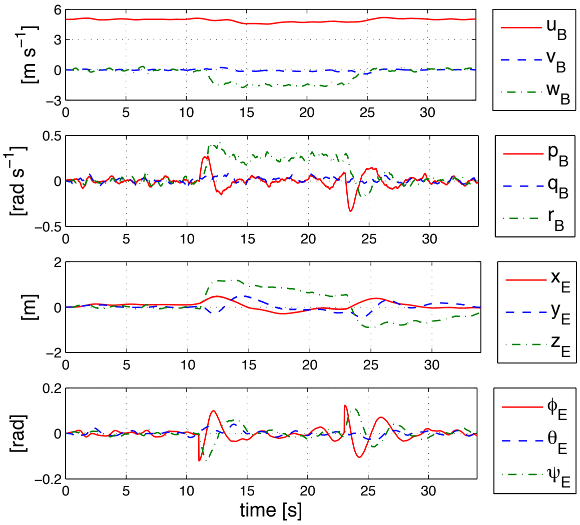
(a) Actuation multi-rate



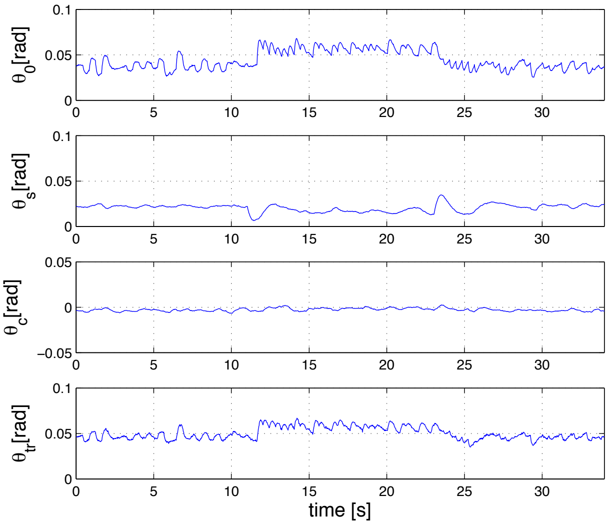
(b) Actuation single-rate

Figure 7.7: Results for the first simulation- Actuation

the vehicle is required to follow a path with constant velocity $V_C = 5\text{m.s}^{-1}$ and constant orientation with respect to the path, consisting of *i)* a level flight segment along the x axis, *ii)* a climbing helix with a flight path angle of $\gamma_C = 0.29$ rad and yaw rate $\dot{\psi}_C = 0.26$ rad.s^{-1}, *iii)* a level flight segment along the x axis. The controller is synthesized according to the methodology presented in Section 7.4 for constant parameters $\varphi_0 = (V_C, \dot{\psi}_C, \psi_{CT}) = (5\,\text{m.s}^{-1}, 0\,\text{rad.s}^{-1}, 0\,\text{rad})$ and the weights of the $\mathcal{H}_2$ controller synthesis problem, described in (7.25) are changed to accommodate for the different operating conditions. We have considered additive white gaussion noise for all the measurements with autocorrelations matrices $(R_{vv}(\tau), R_{\omega\omega}(\tau), R_{pp}(\tau), R_{\lambda\lambda}(\tau) = (\sigma_v^2 I\delta(\tau), \sigma_\omega^2 I\delta(\tau), \sigma_p^2 I\delta(\tau), \sigma_\lambda^2 I\delta(\tau))$, where $(\sigma_v, \sigma_\omega, \sigma_p, \sigma_\lambda) = (0.02 \text{ m.s}^{-1}, 0.3^o.\text{s}^{-1}, 0.1 \text{ m}, 0.5^o)$. The results presented in Figure 7.8 show that good sensor noise rejection is achieved and that the helicopter performs the required task keeping the actuation within the limits of operation. A 3-D view of both maneuvers is shown in Figure 7.9.

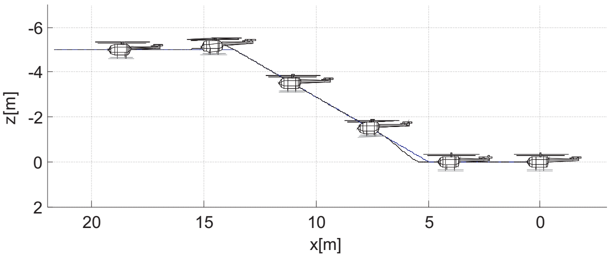


(a) Errors

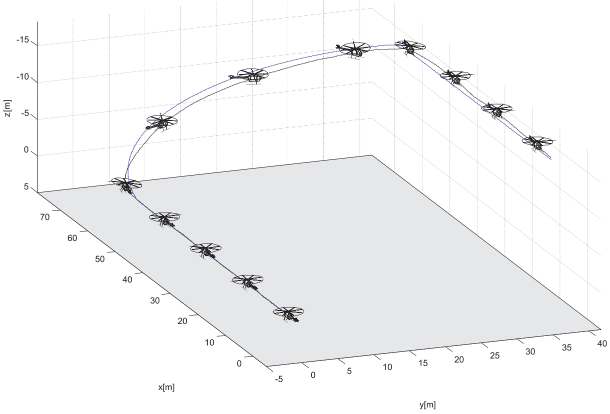


(b) Actuation

Figure 7.8: Results for the second simulation



(a) First Simulation



(b) Second Simulation

Figure 7.9: Rotorcraft Maneuvers

7.5 Further Comments and References

Over the last few decades there has been a surge of interest in the development of efficient and reliable guidance and control algorithms for unmanned vehicles. An endless number of applications are reported in the literature, ranging from underwater geological surveillance [75] to spacecraft missions [93].

Traditionally, the guidance and control problem for autonomous vehicles involves the design of an inner and an outer loop [86]. The inner loop is designed to stabilize the vehicle dynamics and usually requires a high sampling rate, whereas the outer loop design relies essentially on the vehicle's kinematic model, converting tracking errors into inner loop commands, and is amenable to a lower sampling rate. As explained in [84], since the two systems, kinematics and dynamics, are effectively coupled, stability and adequate performance of the combined systems are not guaranteed. An interesting feature of this technique is that, since the kinematic variables are commonly available at lower rates (consider for example the case of a GPS receiver), the design of the different loops at different rates often handles naturally the multi-rate characteristic of the sensors.

Another line of work, presented in [22], [50], [84] proposes and integrated solution for the guidance and control problem. This solution amounts to applying a conveniently defined path-dependent transformation to the vehicle's equations of motion, which involves both kinematic and dynamic variables of the vehicle. In this new space, referred to as error-space, the problems of trajectory tracking or path-following reduce to the problem of driving this newly-defined error to zero. The family of error transformations presented in [22], [50], [84] have the notable property of guaranteeing that the linearization of the error dynamics is time-invariant along trimming trajectories, which comprise arbitrary straight lines and z-aligned helices. Since the vehicle's dynamic behavior changes considerably throughout its flight envelope, gain-scheduling control laws, which have become a standard solution in flight control systems, are typically used. In this setting, the multi-rate characteristics of the sensors have been traditionally handled by the navigation system design [75] without guaranteeing the performance (or even the stability) of the overall closed loop system, although the problem can be formulated as a control design problem.

This chapter follows the line of work reported in [22], [50], [84], adopting the integrated guidance and control approach and proposes a novel method to take into account the multi-rate characteristics of the sensors in the controller design. To this effect, theoretical results are first derived which make use of the existing background for multi-rate systems and gain-scheduling theory. The theory for multi-rate systems is intimately related with the theory of periodically time-varying systems. See, for example, [78] and the references therein for early work on the subject and [61] for more recent developments. Noteworthy is the bulk of work in this field by Bittanti, Colaneri and co-workers. With particular interest to the present chapter are the definitions of stabilizability and detectability for periodic systems [7], and the solutions of the problems of regulation for square multi-rate systems [19], output stabilization for periodic systems [16], and LQG optimal control for multi-rate systems [17]. In the field of gain-scheduling an excellent survey can be found in [81].

The velocity implementation, was presented in [51], [52] and is also discussed in [53, Ch. 12], [81] which is related with the work presented herein, provides a simple solution for the implementation of controllers with integral action that satisfies the linearization property.

8

Conclusions and Future Work

In this thesis, we addressed several scenarios in Networked Control Systems. We modeled these scenarios under the framework of impulsive systems and stochastic hybrid systems and we derived several theoretical results for such systems, which directly entails properties for the networked control system. The results were shown to be of interest in other research areas. In a nutshell, the main conclusions are as follows:

- The framework of Volterra renewal type equations is a useful framework for the analysis of linear networked control systems with independent and identically distributed time between transmissions, and with several other features amenable with a finite-state machine model. This approach leads to interesting results, such as providing a stability condition in terms of the Nyquist criterion for the analysis of these networked control systems.

- Networked Control Systems with asynchronous renewal transmissions can be modeled by piecewise deterministic processes. Using tools that borrow from this latter class of systems, one can provide a necessary and sufficient stability test in terms of the computation of the spectral radius of an integral operator.

- One can construct a class of dynamic protocols that outperform static protocols by construction. In an optimal control framework where the intent is to simultaneously design the control law and the protocols, our approach, based on a rollout strategy, leads to a distributed algorithm which nodes can run independently.

- There is a solution to the output regulation problem for multi-rate systems, even in the case where the number of outputs is larger than the number of inputs. In the special case of constant references, this same solution is useful to tackle the gain-scheduling problem for non-linear systems.

There are several directions for future work. We include here the following:

- *Direct design.* The problem of directly designing a controller in the setting of Chapters 2, Chapter 3, and Chapter 4 should be looked into with more detail. We have shown in [AHS09a] how to optimally design a state feedback controller for simple networked control set-ups as in the framework of Chapter 2. An interesting direction for future work is to extend these results for output feedback. In particular one could ask if the separation or equivalence principle holds, i.e., if the optimal output feedback controller could be obtained by separately computing the optimal controller and the optimal state observer. In the case of networked control systems with asynchronous renewal networks, as in Chapter 4, the problem was not addressed, and therefore remains open.

- *Dynamic Protocols.* The rollout strategy that we propose to solve the simultaneous controller and protocol design may be in general computationally demanding to be implemented on-line. Therefore, a direction for future work is to propose heuristics that approximate this policy, while being within reasonable computational burdens for on-line implementation.

- *Gain-Scheduling structure.* The gain-scheduling structure that we propose for multi-rate systems is inspired on the velocity implementation. However, this implementation involves a differentiation, which may introduce significant noise at the input of the gain-scheduled controller, and in general this can have malicious effects for the closed-loop. An interesting direction for future work would be to obtain a structure that also satisfies the linearization property, but does not require differentiation.

- *Output regulator Controller Structure.* It would also be interesting to obtain a controller structure that does not need to differentiate the inputs in the output regulation problem, since differentiation may lead to large transitory behavior.

- *Nonlinear Networked Control Systems.* The Theorem 24 provides an analog result to a well-known stability results for non-linear stochastic hybrid systems. One can explore this fact by posing several problems for non-linear closed-loops with independent and indentically distributed intervals between transmissions, which parallel problems for non-linear deterministic systems.

- *Assumptions on the intervals between transmissions.* We argued that in the Ethernet and Wireless 802.11 networks the times between transmissions are independent and identically distributed, following similar reasonings to other works in the literature [44]. A direction for future work is to confirm this assumption experimentally.

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