Model-free Adaptive Switching Control of Time-Varying Plants

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Abstract

This paper addresses the problem of controlling an uncertain time-varying plant by means of a finite family of candidate controllers supervised by an appropriate switching logic. It is assumed that, at every time, the plant can be modeled by an uncertain single-input/single output (SISO) linear system. It is shown that global stability of the switched closed-loop system can be ensured provided that (i) at every time there is at least one candidate controller that would stabilize the current time-invariant "frozen" plant model, and (ii) the plant changes are either infrequent or satisfy a slow drift condition.

I. Introduction

ONE of the approaches for controlling uncertain or time-varying plants relies on the introduction of adaptation in the feedback loop. In recent years, *adaptive switching control* has emerged as an alternative to conventional continuous adaptation. In switching control, a so-called supervisor selects a specific feedback controller among a family of admissible ones, based on the measured data. The latter are processed so as to enable the supervisor to determine whether or not the current controller is adequate, and, in the negative, to replace it by a different candidate controller. For an early overview of the topic, the reader is referred to [1].

To date, most of the contributions on switching control have been basically of a two-fold nature. On one side the major emphasis in [2]–[8] has been on robust adaptive stabilization of

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time-invariant systems. Within these contributions, the *unfalsified control* approach developed by M.G. Safonov and co-workers [5], [6] provides guarantees of stability for plants subject to large parametric uncertainties, unmodeled dynamics and disturbances. Unfortunately, as will be discussed in detail later, this methodology is tailored to control time-invariant systems and has no direct generalization to time-varying systems. On the other side the main focus in [1], [9]–[14], has been on switching control schemes capable, at least in principle, of dealing with plant variations. However, these techniques can become ineffective if the available knowledge about the plant is limited so that the family of candidate plant models does not tightly approximate the process dynamics over the whole uncertainty set.

The main goal of this paper is to consider a novel switching supervisory scheme by which previous theoretical results on unfalsified control can be extended to time-varying plants. In particular, the proposed solution consists of a switching supervisory logic whereby a controller, selected from a finite family of pre-designed candidate controllers, is at any time switched-on in feedback with the plant, according to the information on the current and past closed-loop behavior. Unlike [4]–[6], [8], the supervisory scheme here proposed follows a fading memory paradigm and uses a specially devised mechanism to adaptively select when past recorded information can be discarded without destroying stability. As will be seen, this supervisory scheme not only retains the desirable robustness features of unfalsified control, but, being based on a fading memory paradigm, also provides stability guarantees for plants whose parameters are either slowly time-varying or subject to infrequent jumps. It will be also shown that the applicability of the proposed control scheme does not depend on the "complexity" of the plant uncertanty set, which need not be convex nor even connected.

A final point is worth-mentioning. Multiple schemes for switching adaptive control have been proposed in the literature, which differ on how they choose *when* to switch controllers, and *how* to select a new controller. Regarding the "how" question, we propose a supervisory scheme that is not pre-routed [3], [10] in that it decides which controller to place in the feedback loop based on an assessment of the potential performance of each candidate controller. This assessment is achieved by means of data-driven test functionals apt to quantify, from time to time, the suitability of each candidate controller to control the plant, and, hence, without actually placing each controller in the loop. Regarding the "when" question, we use a hysteresis scheme that switches to a new controller when a "significant" performance loss is detected. This type of supervisor does not directly enforce a dwell-time [9], [15] to slow down switching.

Notations. Throughout the paper, the prime denotes transpose, $|\cdot|$ Euclidean norm, and $\mathcal{S}$ the space of all real-valued vector sequences on the set $\mathbb{Z}_+$ of nonnegative integers. For any $s \in \mathcal{S}$, and $t_0, t \in \mathbb{Z}_+$, $t_0 \leq t$, we define $s|_{t_0}^t := \{s(t_0), \ldots, s(t)\}$. For simplicity, if $t_0 = 0$, $s|_{t_0}^t = s^t$. Given λ, $0 < \lambda \leq 1$, we denote the λ-exponentially weighted ℓ_2-norm of $s|_{t_0}^t$ by $\| s|_{t_0}^t \|_\lambda := \sqrt{\sum_{\tau=t_0}^t \lambda^{2(t-\tau)} |s(\tau)|^2}$ whenever $t \geq t_0$, or the zero number otherwise. If $\lambda = 1$, we let $\| s|_{t_0}^t \|$ denote the ℓ_2-norm of $s|_{t_0}^t$. The ℓ_∞-norm of $s|_{t_0}^t$ is defined as $\| s|_{t_0}^t \|_\infty := \max_{\tau \in \{t_0, \cdots, t\}} \max_i |s_i(\tau)|$ where s_i denotes the $i-$th component of s. The sequence $s \in \mathcal{S}$ is said to be *bounded* if its ℓ_∞-norm is finite.

II. Problem formulation and paper overview

A. Plant and control system

We consider the switched system depicted in Figure 1. The plant P consists of a discrete-time strictly causal SISO linear time-varying dynamic system described by

$$A_t(d)\, y(t) = B_t(d)\, (u(t) + n_u(t)) + A_t(d)\; n_y(t)\,, \qquad t \in \mathbb{Z}_+\,. \tag{1}$$

In (1), u is the input, y is the output, n_u is the input disturbance and n_t is the output disturbance. $A_t(d)$ and $B_t(d)$ denote time-varying polynomials in the unit backward shift operator d. It is assumed that at any time t the time-invariant system obtained by freezing the parameteres of (1) at their values at time t belongs to a set $\mathscr{P}$, which represents the set of all possible plant configurations. Further, let P_* denote a generic member of $\mathscr{P}$, and $B_*(d)/A_*(d)$ its corresponding transfer function. The set $\mathscr{P}$ will also account for both the range of parametric uncertainty and the unmodelled dynamics. Indeed, as we will see later on in this paper, no assumption on $\mathscr{P}$ will be required other than it is a compact set, *i.e.* for any P_* the polynomials A_* and B_* have (possibly unknown) bounded orders and their coefficients belong to a (possibly unknown) compact set.

The switching controller C_σ is assumed to have the one-degree-of-freedom form $u(t) = C_{\sigma(t)}(r(t) - y(t))$, where r is the output reference, while the subscript $\sigma(t)$ identifies the specific candidate controller connected in feedback with the plant at time t. We shall assume that σ takes values in the (finite) set $\underline{N} := \{1, \cdots, N\}$. Then, $\mathscr{C} := \{C_i,\ i \in \underline{N}\}$ will denote the finite family of candidate controllers. In particular, each member of $\mathscr{C}$ is taken as a linear time-invariant (LTI) controller with transfer function $S_i(d)/R_i(d)$. Accordingly, the plant input u is given by

$$R_{\sigma(t)}(d)\, u(t) = S_{\sigma(t)}(d)\ (r(t) - y(t)) \tag{2}$$

Given a finite family $\mathscr{C}$ of candidate controllers, $\mathscr{C}_S(P_*)$ will denote the subset of $\mathscr{C}$ composed by all controllers which (internally) stabilize P_*.

Definition 1: The switched system (1)-(2) is said to be (globally) *stable* if, for all initial conditions, any bounded exogenous input (r, n_u, n_y) produces a bounded output (u, y). The problem is said to be feasible if $\mathscr{C}_S(P_*) \neq \emptyset$, $\forall\, P_* \in \mathscr{P}$. ▶

B. Supervisory control architecture

The supervisor processes the available plant I/O data to generate the sequence σ that specifies the switching controller C_σ. In particular, to decide when and how to change the controller, the supervisor embodies a family $\Pi := \{\Pi_i;\ i \in \underline{N}\}$ of test functionals that attempt to quantify the suitability of each candidate controller to control P. Given Π, the hysteresis switching logic considered hereafter is as follows: At each step, one computes the least index $i_*(t)$ in $\underline{N}$ such that $\Pi_{i_*(t)}(t) \leq \Pi_i(t)$, $\forall i \in \underline{N}$. Then, the switching index sequence σ is given by

$$\left.\begin{aligned} &\sigma(t+1) = l(\sigma(t), \Pi(t)), \qquad \sigma(0) = i_0 \in \underline{N} \\ &l(i, \Pi(t)) = \begin{cases} i, & \text{if} \quad \Pi_i(t) < \Pi_{i_*(t)}(t) + h \\ i_*(t), & \text{otherwise} \end{cases} \end{aligned}\right\} \tag{3}$$

where $h > 0$ is the hysteresis constant.

Many adaptive control schemes based on hysteresis switching have been considered for time-invariant plants, and shown to provide robustness against large uncertainties, unmodeled dynamics and disturbances. However, no similar results are available for plants subject to time variations.

To handle time variations in the plant dynamics, one needs to select Π with finite or fading memory. Unfortunately, this involves the potential risk that the switched system become unstable due to persistent switching. To the best of the authors' knowledge, efforts to ensure stability

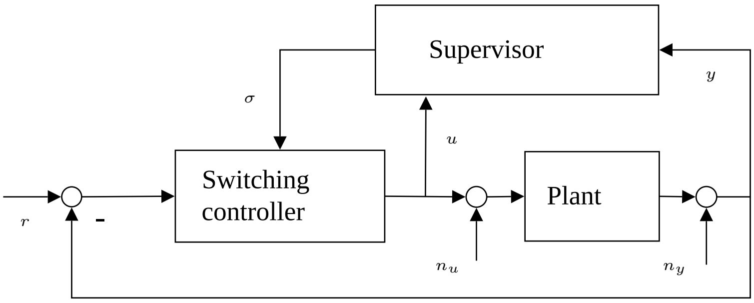

Fig. 1. Adaptive switching control scheme

without relying on a finite switching stopping time are restricted to [1], [9]–[14] However, these techniques can become ineffective if the available knowledge about the plant is limited, so that one cannot find a "tractable" parameterized set of plant models that tightly approximates the set of admissible plants. To prevent the type of instability induced by persistent switching and to provide robustness of the control scheme several techniques have been proposed [2], [5]–[8], [16]. These techniques combine (3) with test functionals of the form

$$\Pi_i(t) := \| \, \pi_i^t \, \|_\infty \tag{4}$$

where $\pi := \{\pi_i; \, i \in \underline{N}\}$ is an underlying family of test functionals. This type of supervisory schemes, including unfalsified control, provides a simple means for preventing the risk of instability caused by persistent switching. Indeed, the supremum norm in (4) ensures that all the test functionals admit a limit as t tends to infinity. Then, by the presence of the hysteresis in the logic (3), the switching stops whenever at least one of the π_i's is bounded ([17], Lemma 1). However, the same supremum typically leads to unbounded Π_i's in applications involving time-varying plants.

This paper aims at overcoming such limitations by considering novel hysteresis switching algorithms in which the test functionals have an *adaptive* memory. In particular, the scheme here proposed embeds in (3) a *resetting logic, viz.* a mechanism according to which the supervisor resets all the Π_i's to zero whenever suitable events (resetting conditions) occur. Specifically, we consider test functionals of the form

$$\begin{aligned} \Pi_i(t) := \| \, \pi_i|_{t_k}^t \, \|_\infty, \quad & t \in \mathbb{T}_k, \\ & \mathbb{T}_k := \{t_k, \ldots, t_{k+1}-1\} \end{aligned} \tag{5}$$

where $\{t_k\}_{k\in\mathbb{Z}_+}$ denotes a sequence of resetting instants to be specified. For clarity, we shall denote by HSL-∞ the Hysteresis Switching Logic defined by (3) and (4), and by HSL-R (Hysteresis Switching Logic with Resetting) the new switching logic defined by (3) and (5).

The remainder of the paper is as follows. In Sect. III we recall basic concepts underlying unfalsified control, and derive certain key properties characteristic of the HSL-R. The adaptive mechanism used by the supervisor to generate the resetting instants is analyzed in Sect. IV and Sect. V. It is shown that, for time-invariant systems, the proposed supervisory scheme can ensure stability without relying on a finite switching stopping time, thus widening the theoretical properties of unfalsified control to logics other than HSL-∞. In Sect. VI, we show that without any further modification, the same supervisory scheme can also ensure stability in the presence of time variations of the plant parameters, provided that the latter are infrequent or satisfy a slow drift condition. Finally, Sect. VII ends the paper with concluding remarks.

III. Model-free adaptive control

In unfalsified control, the feedback adaptation task of classic adaptive control is replaced by the so-called controller falsification. The basic concept behind this approach can be described as follows. At each time and for each index $i \in \underline{N}$ one computes in real-time time the solution v_i to the the difference equation (*cf. Remark 1*)

$$S_i(d)\ (v_i(t) - y(t)) = R_i(d)\, u(t) \tag{6}$$

As shown in Fig. 2, v_i equals the reference sequence which would reproduce the recorded I/O sequence (u, y) should the plant P be fed-back by the candidate controller C_i, irrespective of the way the plant input u is generated (this motivates for v_i the name *virtual* or *fictitious* reference [4]). The introduction of the virtual reference makes it possible to evaluate that would have been achieved by the feedback connection of the controller C_i with the real process (denoted by (P/C_i)) if the reference has been equal to v_i. In this respect, consider the time-varying feedback system (P/C_i) mapping the "input" $w_i := [\,v_i\ \ n_u\ \ n_y\,]'$ to the "output" $\zeta_i := [\,u\ \ (v_i - y)\,]'$. Accordingly, a possible related performance measure can be constructed as follows

$$\pi_i(t) := \frac{\|\,\zeta_i^t\,\|_\lambda}{\mu + \|\,v_i^t\,\|_\lambda}, \qquad t \in \mathbb{Z}_+ \tag{7}$$

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where μ is a positive constant. In case of noise-free LTI plant, π_i provides an estimate from below of the λ-weighted $\mathcal{H}_\infty$ mixed-sensitivity norm [18] of the loop (P/C_i) with virtual reference input v_i and output ζ_i containing the control input u and the virtual tracking error $v_i - y$, which would like to minimize. The test functional (7) can be viewed as a variant of the (unweighted) $\mathcal{H}_\infty$ mixed-sensitivity performance criterion often considered in unfalsified control [5].

Remark 1: On-line computation of (6) requires that all the candidate controllers be stably causally invertible (SCI), but suitable arrangements which remove this design constraint have been reported in the literature [19]. For simplicity of presentation, we will not pursue this issue further in this paper. ▼

A. Assumptions

We now introduce the assumptions needed to provide stability of the switched system. To this end, some preliminary definitions are needed.

Definition 2: A polynomial $p(d)$ is said to be a λ-*Hurwitz* polynomial (in the indeterminate d) if it has no root in the closed disk of radius λ^{-1} of the complex plane. ▶

Definition 3: The feedback loop (P_*/C_i) composed by the time-invariant plant P_* and the controller C_i is said to be λ-*stable* if its characteristic polynomial

$$\chi_{*/i}(d) := A_*(d)\, R_i(d) + B_*(d)\, S_i(d)$$

is a λ-*Hurwitz* polynomial. ▶

We make the following assumptions.

A1 The plant uncertainty set $\mathscr{P}$ is compact, in the sense that there exists an integer n^* (possibly unknown) such that for every $P_* \in \mathscr{P}$ the polynomials A_* and B_* have order smaller than n^* and their coefficients belong to a compact subset of $\mathbb{R}^{n^*+1}$. •

A2 For any $P_* \in \mathscr{P}$, there exists a candidate controller $C_i \in \mathscr{C}$ such that (P_*/C_i) is λ-*stable*. •

A3 For each candidate controller $C_i \in \mathscr{C}$, S_i is a λ-*Hurwitz* polynomial. •

A4 The exogenous inputs r, n_u, and n_y are bounded. •

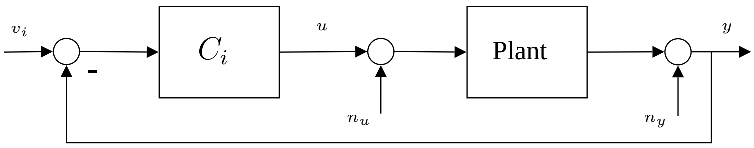

Fig. 2. The i-th virtual candidate loop.

Remark 2: The assumption A1 can be omitted when the plant is a time-invariant system. However, we introduce it now for simplicity of exposition. A2 implies feasibility, *viz.* $\mathscr{C}_S(P_*) \neq \emptyset$, $\forall\, P_* \in \mathscr{P}$. A3 requires that, in addition to the SCI condition, the inverse of each controller has a large enough stability margin under the SCI condition. Because $\mathscr{C}$ is a finite set, this requirement is no stronger than requiring all controllers to be Hurwitz (see Remark 1 above). ▼

B. Key lemmas

In this subsection, we introduce certain key properties upon which the stability analysis depends. To this end, some preliminary observations are needed.

Let Σ_σ denote the switched system (1)-(2) mapping the "input" $w := [\,r \;\; n_u \;\; n_y\,]'$ to the "output" $\zeta := [\,u \;\; (r-y)\,]'$ under a given switching sequence σ. To avoid needless complications, we assume that the switching controller (2) as well as (6) are both initialized at time zero from zero initial conditions: By letting $m := \max\,\{\deg S_i,\, \deg R_i;\, i \in \underline{N}\}$ and assuming $\sigma(t) = i$ the control input is then given by

$$\begin{aligned} u(t) &= \sum_{k=0}^{m} s_{ik}\,(r(t-k) - y(t-k)) - \sum_{k=1}^{m} r_{ik}\,u(t-k) \\ r(k) &= u(k) = y(k) = 0, \quad k = -1,\, \ldots,\, -m \end{aligned}$$

where s_{ik} and r_{ik} denote the coefficients of the polynomials $S(d)$ and $R(d)$, respectively. An analogous initialization is made for (6). Regarding the initial condition of (1), denoting by $(u_P,\, y_P)$ the sequence of actual input/output pairs of the plant P, we have that

$$y_P(t) = \sum_{k=1}^{n^*} b_{k_t}\,u_P(t-k) - \sum_{k=1}^{n^*} a_{k_t}\,y_P(t-k)$$

where b_{k_t} and a_{k_t} denote the coefficients of the polynomials $B_t(d)$ and $A_t(d)$. Accordingly, $u_P(k)$ and $y_P(k)$, $k = -1,\, \ldots,\, -n^*$, denote the plant initial conditions. In view of the feedback configuration in Fig. 1, the sequence (u_P, y_P) satisfies $u_P = u + n_u$ and $y_P = y - n_y$ after the initial time $t = 0$, whereas no relationship generally holds between (u_P, y_P) and (u, y) before time zero. In order to simplify the notation and have the same symbols in the initialization of (1), (2) and (6), we let $u_P = u + n_u$ and $y_P = y - n_y$ hold true even for negative times. In practice, this is equivalent to the following initialization for (1): $u(k) = y(k) = 0$, $n_u(k) := u_P(k)$ and $n_y(k) := y_P(k)$, $k = -1, \ldots, -n^*$. With this in mind, we shall denote by $\xi_P := [\, u_P(-1) \cdots u_P(-n^*) \; y_P(-1) \cdots y_P(-n^*) \,]'$ the vector composed by the plant initial conditions.

Let now

$$\Pi^k := \min_{i \in \underline{N}} \{ \max_{t \in \mathbb{T}_k} \pi_i(t) \} + h, \qquad k \in \mathbb{Z}_+ \tag{8}$$

The following lemmas are the main results of this section and fundamental for the subsequent developments of the paper.

Lemma 1: Consider the HSL-R. Let $\mathfrak{N}_k$ denote the number of switchings over the interval $\mathbb{T}_k$, and $\lceil \alpha \rceil$ denote the smallest positive integer greater than or equal to $\alpha \in \mathbb{R}_+$. Then,

$$\pi_{\sigma(t+1)}(t) < \Pi^k, \qquad \forall\, t \in \mathbb{T}_k \tag{9}$$

$$\mathfrak{N}_k \leq N \left\lceil \Pi^k / h \right\rceil, \qquad \forall\, k \in \mathbb{Z}_+ \tag{10}$$

hold for any resetting sequence $\{t_k\}_{k \in \mathbb{Z}_+}$. ∎

The proof of Lemma 1 can be omitted since (9) follows directly from the hysteresis logic and (10) is a straighforward consequence of the fact the test functionals Π_i are monotone nondecreasing over each time interval $\mathbb{T}_k$.

Lemma 2: Consider a HSL-R switched system Σ_σ based on the test functionals (7). Let A1, A3, and A4 hold. Then, there exists a finite-valued function $\mathfrak{g}$ such that

$$\|\zeta^t\|_\lambda \leq \mathfrak{g}(\Pi^k)(\mu + |\xi_P|\, \lambda^{t+1} + \|w^t\|_\lambda + \|\zeta^{t_k - 1}\|_\lambda \lambda^{t - t_k + 1}), \qquad \forall\, t \in \mathbb{T}_k \tag{11}$$

holds for any resetting sequence $\{t_k\}_{k \in \mathbb{Z}_+}$.

Proof. See the appendix. ∎

Before concluding this section it is important to observe that in Lemma 2 the time variations of the plant parameters can be arbitrary and, at this time, we do not yet need assumption A2.

IV. Stability under admissible resetting

As seen from Lemma 2, the bound in (11) depend on both the sequences $\{\Pi^k\}_{k\in\mathbb{Z}_+}$ and $\{\zeta^{t_k-1}\}_{k\in\mathbb{Z}_+}$. In unfalsified control based on HSL-∞, the term $\|\zeta^{t_k-1}\|_\lambda$ is absent since there is only one reset time, $t_0 := 0$ at start-up, and $\mathbb{T}_0 = \mathbb{Z}_+$. In this case, stability depends only on boundedness of Π^0, and, hence, it can be guaranteed simply by ensuring boundedness of at least one of the test functionals (this nice property is precisely the motivation which led [5] to introduce the notion of *cost detectability* of the test functionals). Under resetting, the analysis becomes more complicated. Indeed, even if the plant is time-invariant but there are infinitely many resets, $t - t_k$ does not grow unbounded as $t \to \infty$, and, hence, the term $\|\zeta^{t_k-1}\|_\lambda \lambda^{t-t_k+1}$ need not vanish. This is consistent with the intuition that resetting destroys the monotonicity of the test functionals and therefore that boundedness of $\{\Pi^k\}_{k\in\mathbb{Z}_+}$ alone may not prevent instability due to persistent switching.

The remainder of this section is devoted to show that, nonetheless, adaptive resetting mechanisms do exist which preserve stability of the switched system.

A. Admissible resetting times

The notation for this subsection is as in Section III-B. Consider the switched system Σ_σ and define the following performance measure for the closed-loop,

$$\pi_*(t) := \frac{\| \zeta^t \|_\lambda}{\mu + \| r^t \|_\lambda}, \qquad t \in \mathbb{Z}_+ \,. \tag{12}$$

Consider now the following definition.

Definition 4: (Admissible Resetting Times). A sequence of reset times $\{t_k\}_{k\in\mathbb{Z}_+}$ is called *admissible* if, for every $k \in \mathbb{Z}_+$, we have that

$$\pi_*(t_k - 1) \;\leq\; \pi_{\sigma(t_k)}(t_k - 1) + \epsilon, \quad \epsilon > 0 \tag{13}$$

▸

In essence, (13) only allows the k-th reset to occur at the time t_k if (13) holds.

To understand the rationale for (13), notice that, when the plant is time-invariant, π_* can be viewed as an estimate of the *actual* reference-to-data induced gain, whereas π_σ can be viewed as an estimate of the *virtual* reference-to-data induced gain. To obtain stability it is important that these two estimates do not differ significantly. The inequality (13) guarantees that a reset occurs only if π_* is not much larger than π_σ, whereas, as will be seen soon, the selection of σ through HSL-R makes sure that π_σ remains bounded.

Consider an admissibile resetting sequence. Then, combining (12) with (13) and (9) we get $\|\zeta^{t_k-1}\|_\lambda \le (\Pi^{k-1}+\epsilon)(\mu+\|r^{t_k-1}\|_\lambda)$. Notice that the above inequality is well-defined for $k=0$ since by definition $\|\zeta^{-1}\|_\lambda = 0$ which leads to $\Pi^{-1}=0$. Further, $\|r^{t_k-1}\|_\lambda \lambda^{t-t_k+1} \le \|r^t\|_\lambda \le \|w^t\|_\lambda$. Then, under an admissible resetting sequence, Lemma 2 implies that

$$\|\zeta^t\|_\lambda \le \mathfrak{g}(\Pi^k)(|\,\xi_P\,|\,\lambda^{t+1} + (\Pi^{k-1}{+}\epsilon+1)(\,\mu+\|w^t\|_\lambda\,)),$$
$$\forall\, t\in\mathbb{T}_k \tag{14}$$

from which we have at once the following.

Theorem 1: Consider the same assumptions as in Lemma 2 and further assume that $\Pi^k \le \Pi^*$, $\forall\, k\in\mathbb{Z}_+$, for some finite constant Π^*. Then, the HSL-R switched system Σ_σ is stable for any admissible resetting sequence $\{t_k\}_{k\in\mathbb{Z}_+}$ and

$$\|\zeta^t\|_\lambda \;\le\; \mathfrak{g}\left(\Pi^*\right)|\,\xi_P\,|\,\lambda^{t+1}{+}\mathfrak{h}\left(\Pi^*\right)\left(\mu+\|w^t\|_\lambda\right),$$
$$\forall\, t\in\mathbb{Z}_+ \tag{15}$$

where $\mathfrak{h}\left(\Pi^*\right) := \mathfrak{g}\left(\Pi^*\right)\,\left(\Pi^*+\epsilon+1\right)$. ■

In essence, Theorem 1 indicates that, under the admissibility condition (13), stability of the switched system depends only on boundedness of $\{\Pi^k\}_{k\in\mathbb{Z}_+}$. This is precisely the point where assumption A2 becomes important. In particular, as discussed in next subsection, for time-invariant plants, A2 is sufficient to prove that, like HSL-∞, HSL-R leads to stability, as long as the resetting sequence is admissible in the sense of Definition 4.

B. Stability in the time-invariant case

In order to prove boundedness of $\{\Pi^k\}_{k\in\mathbb{Z}_+}$ when the plant is time-invariant, we use the following result.

Lemma 3: Let the HSL-R switched system Σ_σ be based on the test functionals (7). Let A1−A4 hold and further assume that on a given interval $\{\tau, \tau+1, ..., T\}$ the coefficients of the polynomials A_t and B_t in (1) remain constant. Then, there exist positive constants g_0, g_1, g_2 and g_3 such that, for any $P_* \in \mathscr{P}$,

$$\pi_s(t) \, \leq \, g_0 + g_1 \, | \, \xi_P \, | \, \lambda^{t+1} + g_2 \| \, w^t \, \|_\lambda + g_3 \| \, \zeta^{\tau-1} \, \|_\lambda \lambda^{t-\tau+1}, \quad \forall \, t \in \{\tau, \tau+1, \dots, T\} \tag{16}$$

holds true for some $s \in \underline{N}$.

Proof. See the appendix. ■

From Lemma 3 one sees that when the plant is a time-invariant system ($\tau = 0$ and $T = \infty$), A2 is sufficient to ensure boundedness of $\{\Pi^k\}_{k \in \mathbb{Z}_+}$. Indeed, by letting $\kappa_0 := g_0$, $\kappa_1 := g_1$ and $\kappa_2 := (1-\lambda^2)^{-1/2} \, g_2$ and recalling that $\|\zeta^{-1}\|_\lambda = 0$, we have that (16) implies

$$\begin{aligned} \pi_s(t) + h \, \leq \, & \, \kappa_0 + \kappa_1 \, | \, \xi_P \, | + \kappa_2 \, \| \, w \, \|_\infty + h \\ =: & \, \, \Pi^*_{TI} \end{aligned} \tag{17}$$

for every $t \in \mathbb{Z}_+$.

Then, combining (17) with Theorem 1, it is immediate to derive the following result.

Theorem 2: Let the HSL-R switched system Σ_σ be based on the test functionals (7). Then, if the plant is time-invariant, under assumptions A1−A3, Σ_σ is stable for any admissible resetting sequence $\{t_k\}_{k \in \mathbb{Z}_+}$. ■

Remark 3: Notice that, for time-invariant systems, stability of switched systems based on unfalsified control can be proven using analysis tools quite simple compared to the ones given here (*cf.* [5]–[7]). On the other hand, the present analysis tools do not rely on switching stopping, a property that is crucial for the results in [5], [6] and [7]. ▼

V. Finite-time resetting

As described in the previous subsection, for time-invariant plants, HSL-R allows one to prove stability results similar to those available for HSL-∞. In this subsection, we show that the reset admissibility condition (13) is always attained in finite time, which ensures that past data records are periodically discarded whenever the plant dynamics remain constant over a large enough horizon, and it will become crucial in the presence of plant variations.

Taking (13) and Theorem 1 into account, consider the following *resetting rule*

$$\begin{aligned} & t_0 := 0 \\ & t_{k+1} := 1 + \min \big\{ t : \ t \geq t_k; \\ & \qquad\qquad \pi_*(t) \ \leq \ \pi_{\sigma(t+1)}(t) + \epsilon \big\} , \quad k \in \mathbb{Z}_+ \end{aligned} \tag{18}$$

which, by construction, always generates an admissible resetting sequence satisfying (13). Notice now that, under the same assumptions as in Theorem 1, the following upper bound on the plant data can be derived

$$\begin{aligned} \max_{t \in \mathbb{Z}_+} \|\zeta^t\|_\lambda \leq & \, \mathfrak{g}\left(\Pi^*\right) |\, \xi_P| + (1-\lambda^2)^{-\frac{1}{2}} \, \mathfrak{h}\left(\Pi^*\right) (\mu + \|w\|_\infty) \\ =: & \, Z\left(\Pi^*\right) , \end{aligned} \tag{19}$$

where $\|w\|_\infty$ is finite in view of assumption A4. Then, by letting

$$\mathfrak{N}_* := N \, \lceil \Pi^* / h \rceil \tag{20}$$

$$\Delta\left(\Pi^*\right) := (\mathfrak{N}_* + 1) \left\lceil \log_\lambda \frac{\epsilon \, \mu}{Z\left(\Pi^*\right)} \right\rceil , \tag{21}$$

the following result states that when $\{\Pi^k\}_{k \in \mathbb{Z}_+}$ is bounded, the HSL-R based on (18) always experiences at least one resetting.

Lemma 4: Consider the HSL-R based on (18). Then, under the same assumptions as in Theorem 1, one has

$$t_{k+1} - t_k \ \leq \ \Delta\left(\Pi^*\right) , \qquad \forall \, k \in \mathbb{Z}_+ \, . \tag{22}$$

Proof. See the appendix. ∎

From Lemma 4 it is also immediate to conclude that, when the plant is time-invariant, under assumptions A1$-$A4, a reset always occurs after at most $\Delta\left(\Pi^*_{TI}\right)$ time steps, where Π^*_{TI} is as in (17). Lemma 4, along with Theorem 2, completes the analysis for LTI plants. It is important to emphasize that the complexity of the control scheme proposed here does not depend on the "complexity" of the set $\mathscr{P}$ of frozen plant models, which could contain plants with very high order or be very non-convex.

A. Example 1

Although the introduction of HSL-R is mainly motivated by the goal of handling possible plant variations, there are good reasons to use it also in the time-invariant case. In fact, when the plant is time-invariant, equation (16) implies that

$$\pi_s(t) \, \leq \, g_0 + g_1 \, | \, \xi_P \, | \, \lambda^{t+1} + g_2 \| \, w|_{t_k}^{t} \, \|_\lambda + g_2 \| w^{t_k - 1} \|_\lambda \, , \qquad \forall \, t \in \mathbb{T}_k, \; k \in \mathbb{Z}_+ \tag{23}$$

holds for some $s \in \underline{N}$. This equation indicates that the effect of plant initial conditions and past disturbances on Π^k and, hence, on the controller selection, vanishes as k increases. In view of Lemma 4, this property, which cannot be exploited in HSL-∞ because of infinite memory, can provide definite improvements in performance for HSL-R.

As a simple illustration, consider the continuous-time LTI unstable plant with transfer function $P(s) = K/(s - 0.4)$, $K \in [0.1, 1]$, controlled by feeding its input via a zero-order holder and sampling its output every $0.2s$. Two proportional-integral (PI) controllers, $C_1(s) = (3.485 - 3.265\, d)/(1 - d)$ and $C_2(s) = (27.36 - 25.63\, d)/(1 - d)$, have been designed so as to provide good performance over the uncertainty set. We compare HSL-∞ and HSL-R both with hysteresis $h = 1$ and with $\lambda := 0.99$ and $\mu = 1$ in the test functionals (7). In particular, λ has been selected large enough so as to satisfy assumptions A2 and A3 for the uncertain discrete-time plant and the candidate controllers. For HSL-R, we have adopted the reset rule (18) with $\epsilon := 0.01$. Let $K = 0.16$ so that both the controllers stabilize the plant but only C_2 performs satisfactorily and further assume that n_y and n_u are zero everywhere except on the interval $[400, 800]$ where a burst in noise is modeled by taking n_y to be uniformly distributed on $[-0.5, 0.5]$. Fig. 3-4 show the plant output response for zero plant initial conditions and a square-wave reference of period $500\, s$ and amplitude 2.5. In both HSL-∞ and HSL-R, the output noise causes C_2 to be switched-off. However, due to its memory feature, the HSL-∞ does not allow C_2 to be switched-on again. On the contrary, with HSL-R, C_2 is promptly re-selected right after resetting, as shown in Fig. 5.

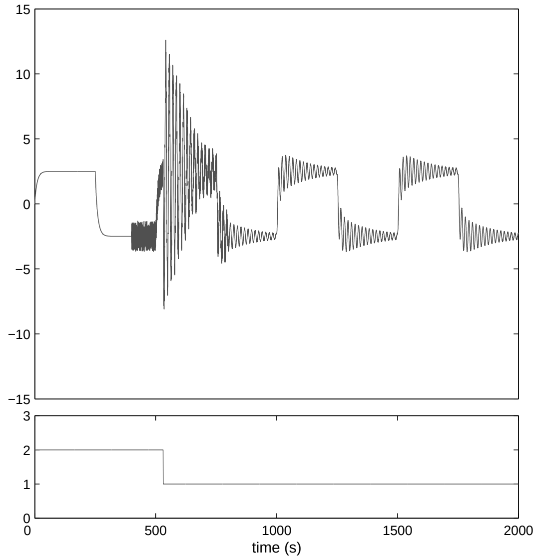

Fig. 3. Supervision based on HSL-∞; Top: Plant Output; Bottom: Switching sequence.

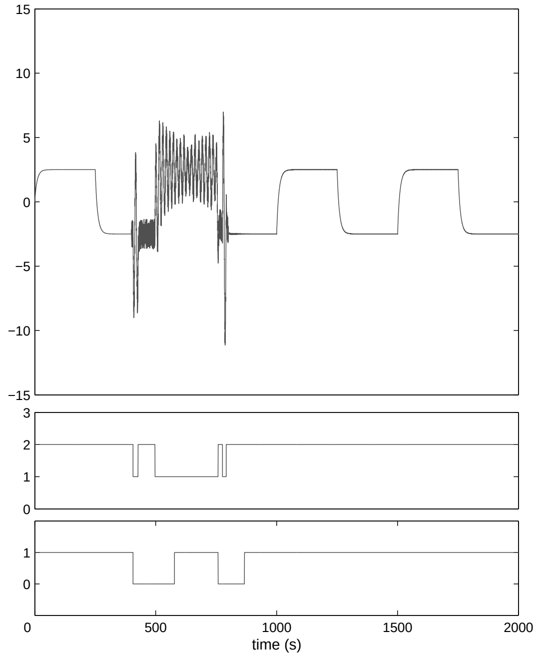

Fig. 4. Supervision based on HSL-R; From top to bottom: Plant Output; Switching sequence; Resetting sequence [1 stands for resetting]

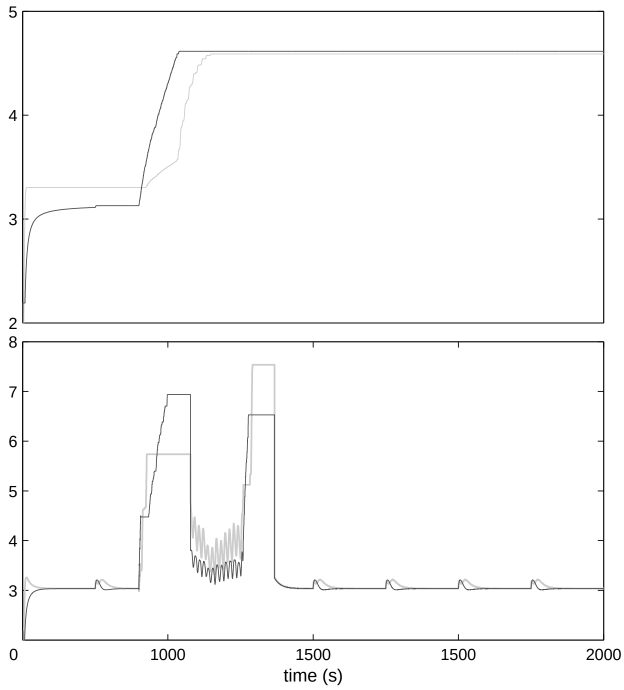

Fig. 5. Test functionals. Top: Supervision based on HSL-∞; Bottom: Supervision based on HSL-R (Π_1 grey line, Π_2 black line).

VI. Stability under time variations of the plant parameters

This section shows how HSL-R makes it possible to achieve stability properties similar to those derived for LTI plants in the presence of time variations of the plant parameters. Recall from Theorem 1 that the switched system is stable if there exists a finite constant Π^* such that $\Pi^k \leq \Pi^*$ for every $k \in \mathbb{Z}_+$. Notice that here, unlike the time-invariant case, the existence of an upper bound Π^* requires that *none* of the test functionals can grow without bound whether or not the corresponding candidate controller is stabilizing. Closer to what one could expect, even if a particular controller C_i does not stabilize the plant, the corresponding test functional π_i will not grow unbounded. Indeed, one sees from (7) that

$$\pi_i(t) \leq 1 + \mu^{-1} \| \zeta^t \|_\lambda + \mu^{-1} \| w^t \|_\lambda \tag{24}$$

for every $i \in \underline{N}$ and every $t \in \mathbb{Z}_+$. As discussed hereafter, stability may be obtained when the plant variations are infrequent or satisfy a slow drift condition.

A. Infrequent plant changes

Let $\{\ell_c\}_{c\in\mathbb{Z}_+}$ denote the sequence of time instants at which a plant variation occurs, with $\ell_0 := 0$ by convention. Accordingly, $\mathbb{L}_c := \{\ell_c, \dots, \ell_{c+1} - 1\}$, $c \in \mathbb{Z}_+$, will denote the c-th time interval over which the plant is constant. Although we can no longer use Π^*_{TI} in (17) to deduce that the switched system is stable, Lemma 3 ensures that for every $c \in \mathbb{Z}_+$ there exists a candidate index $s \in \underline{N}$ such that

$$\pi_s(t) + h \leq \Pi^*_{TI} + g_3 \left\|\zeta^{\ell_c - 1}\right\|_\lambda \lambda^{t-\ell_c+1}, \;\; \forall\, t \in \mathbb{L}_c \tag{25}$$

where g_3 is as in (17). Thus, for any given accuracy ν and provided that $\mathbb{L}_c$ be large enough, the right hand side of (25) will eventually become smaller than $\Pi^*_{TI} + \nu$. In this respect, let

$$\mathbb{L}^\nu_c := \left\{\, t \in \mathbb{L}_c;\;\, g_3\, \|\zeta^{\ell_c-1}\|_\lambda \lambda^{t-\ell_c+1} \leq \nu \right\}$$

Then, if at least two resets occur over $\mathbb{L}^\nu_c$, *i.e.* there is at least one k such that $\mathbb{T}_k \subseteq \mathbb{L}^\nu_c$ one can use equation (14) to conclude that at time ℓ_{c+1} the following upper bound holds

$$\left\|\zeta^{\ell_{c+1}-1}\right\|_\lambda \leq Z\left(\Pi^*_{TI} + \nu\right) \tag{26}$$

with $Z(\cdot)$ as in (19). Notice that one single reset would not be sufficient since the bound in (14) depends on both Π^k and Π^{k-1}. At the present stage, Theorem 1 cannot be invoked to conclude stability of the switched system since the existence of a finite upper bound for $\{\Pi^k\}_{k\in\mathbb{Z}_+}$ is not evident from boundedness of $\{\zeta^{\ell_c-1}\}_{k\in\mathbb{Z}_+}$. Nonetheless, this implication actually holds as we show below.

Lemma 5: Let the HSL-R switched system Σ_σ be based on the test functionals (7) and reset rule (18). Let A1$-$A4 hold. Further assume that

$$\forall\, c \in \mathbb{Z}_+ \quad \exists\, k \in \mathbb{Z}_+ \quad \text{such that} \;\; \mathbb{T}_k \subseteq \mathbb{L}^\nu_c\,. \tag{27}$$

Then, $\Pi^k \leq \Pi^*_{TV}$ for every $k \in \mathbb{Z}_+$, with

$$\Pi^*_{TV} := \Pi^*_{TI} + g\, Z\left(\Pi^*_{TI} + \nu\right) \tag{28}$$

where $g := \max\{g_3, 2\mu^{-1}\}$. Hence, Σ_σ is stable.

Proof. See the appendix. ∎

In the light of Lemma 5 one sees that a sufficient condition for stability of the switched system is that the minimum interval between two consecutive plant variations (or plant *dwell-time*) be large enough to allow the fulfillment of the condition (27). In this respect, let ℓ_c^ν denote the first time instant of $\mathbb{L}_c^\nu$, *i.e.* $\ell_c^\nu := \min\{t : t \in \mathbb{L}_c^\nu\}$. Thus, condition (27) amounts to requiring that, for any $c \in \mathbb{Z}_+$, ℓ_{c+1} is always greater or equal to ℓ_c^ν plus the time needed for two resetting to occur. To this end, notice that using (26) in the definition of $\mathbb{L}_c^\nu$ one obtains

$$\ell_c^\nu - \ell_c \le \left\lceil \log_\lambda \frac{\nu}{g_3\, Z\left(\Pi_{TI}^* + \nu\right)} \right\rceil$$

Moreover, by simple induction argument, if condition (27) is satisfied up to a certain ℓ_c then Π_{TV}^* is an upper bound on the smallest test functional over $\mathbb{L}_c$. In turns, in agreement with Lemma 4, this implies that after at most $2\Delta(\Pi_{TV}^*)$ steps subsequent to ℓ_c^ν the two required reset times occur. Hence, the following result can be claimed.

Theorem 3: Let the HSL-R switched system Σ_σ be based on the test functionals (7) and the resetting rule (18). Let A1$-$A4 hold. Then, Σ_σ is stable provided that

$$\ell_{c+1} - \ell_c \ \ge\ \left\lceil \log_\lambda \frac{\nu}{g_3\, Z\left(\Pi_{TI}^* + \nu\right)} \right\rceil + 2\Delta\left(\Pi_{TV}^*\right)$$

holds for every $c \in \mathbb{Z}_+$. ■

B. Slow parameter drift

Let $\theta(t) \in \mathbb{R}^{2n^*}$, $t \in \mathbb{Z}_+$, denote the vector of time-varying parameters composed by the coefficients of $A_t(d)$ and $B_t(d)$. Consistent with this notation, we can re-write A1 as requiring that $\theta(t) \in \Theta$, $\forall\, t \in \mathbb{Z}_+$ for some compact set Θ. Assume now that the parameter vector $\theta(t)$ takes values inside Θ with bounded variation rate, *i.e.*

$$\theta(t) \in \Theta, \quad |\theta(t+1) - \theta(t)| \le \delta, \qquad \forall\, t \in \mathbb{Z}_+ \tag{29}$$

where $\delta > 0$ defines the variation rate. Further asssume that each controller C_i stabilizes the plant as long as $\theta(t)$ belongs to a certain subset $\Theta_i \subseteq \Theta$ and the variation rate does not exceed a certain threshold δ_{max}, *i.e.* $\delta \le \delta_{max}$. It is well known that, although stability of a time-invariant loop $(P(\theta)/C_i)$, $\theta \in \Theta_i$ need not imply stability of the time-varying loop $(P(\theta(t))/C_i)$, $\theta(t) \in \Theta_i$, such a property holds provided that δ_{max} be small enough (*cf.* (30)). In this respect, we consider the following assumption.

A5 There exists a positive constant β such that, for each $\theta \in \Theta$, there exists at least an index $i \in \underline{N}$ such that $\theta \in \Theta_i$ and

$$\inf_{\theta' \in \Theta \setminus \Theta_i} |\theta - \theta'| \geq \beta$$

•

In words, the sets Θ_i, $i \in \underline{N}$ form a large enough overlapping partition of the parametric uncertainty set Θ.

As shown hereafter, under A5, conclusions similar to those derived in Section VI-A, holds true in case the plant variations satisfy a slow drift conditions. The idea is to derive an inequality of the type given in (25). To this end, consider the following fact. Even if $\{\ell_c\}_{c \in \mathbb{Z}_+}$ need no longer satisfy (29), under the problem feasibility and the additional assumption that $\delta \leq \delta_{max}$, there always exists a controller C_i which stabilizes the plant in the time interval $\{\ell_c, \ldots, \ell_c + \tau\}$ with $\tau := \lfloor \beta/\delta \rfloor$, $\lfloor \alpha \rfloor$ denoting the largest integer less than or equal to $\alpha \in \mathbb{R}_+$. In accordance with (16), this implies that, for every c, there always exists an index s such that

$$\pi_s(t) \leq c_0 + c_1 \,|\, \xi_P \,|\, \lambda^{t+1} + c_2 \| \, w^t \|_\lambda + c_3 \| \, \zeta^{\ell_c - 1} \|_\lambda \lambda^{t - \ell_c + 1}$$
$$\forall \, t \in \{\ell_c, \ldots, \ell_c + \tau\} \tag{30}$$

holds for some finite positive reals c_0, c_1, c_2 and c_3. Notice that these constants will now depend on δ (hence, in general, they will be greater than those in (16) pertaining to the frozen-time analysis). Following the same procedure used to obtain Π^*_{TI} from equation (16), one can introduce a corresponding upper bound for equation (30). In particular, by letting $\hat{\kappa}_0 := c_0$, $\hat{\kappa}_1 := c_1$ and $\hat{\kappa}_2 := (1 - \lambda^2)^{-1/2} \, c_2$, we have that

$$\pi_s(t) + h \;\leq\; \hat{\kappa}_0 + \hat{\kappa}_1 \,|\, \xi_P \,|\, + \hat{\kappa}_2 \,\|\, w \,\|_\infty + h$$
$$=: \; \hat{\Pi}^*_{TI} \tag{31}$$

for every $t \in \{\ell_c, \ldots, \ell_c + \tau\}$.

In essence, $\hat{\Pi}^*_{TI}$ can be viewed as the upper bound on the smallest test functional when the parameter vector $\theta(t)$ remains inside a given subset Θ_i and its variation rate does not exceed the threshold δ_{max}.

Using these properties, the stability analysis becomes similar to that developed in the previous subsection. Indeed, let $\{\hat{\ell}_q\}_{q \in \mathbb{Z}_+}$, with $\hat{\ell}_0 := 0$ by convention, denote the subsequence of plant variations such that $\hat{\ell}_{q+1} := \min\{\ell_c;\ \ell_c - \hat{\ell}_q > \tau\}$. In words, if $\theta(\hat{\ell}_q)$ belongs to Θ_i then $\hat{\ell}_{q+1}$ denotes the first time instant at which a variation in the plant parameter vector can cause θ to leave Θ_i. Thus, (30) can be written in a more convenient form as follows

$$\pi_s(t) \leq c_0 + c_1 \,|\, \xi_P \,|\, \lambda^{t+1} + c_2 \| \, w^t \|_\lambda + c_3 \| \, \zeta^{\hat{\ell}_q - 1} \|_\lambda \lambda^{t-\hat{\ell}_q+1}$$
$$\forall\, t \in \{\hat{\ell}_q, \ldots, \hat{\ell}_{q+1} - 1\}$$

Indeed, by virtue of (30), this inequality holds true on the interval $\{\hat{\ell}_q, \ldots, \hat{\ell}_q + \tau\}$. Moreover, it also holds on the interval $\{\hat{\ell}_q + \tau + 1, \ldots, \hat{\ell}_{q+1} - 1\}$ since, by definition, over that interval no plant variations may occur. On the other hand, from the definition of $\hat{\Pi}^*_{TI}$ it follows immediately that

$$\pi_s(t) + h \;\leq\; \hat{\Pi}^*_{TI} + c_3 \|\zeta^{\hat{\ell}_q - 1}\|_\lambda \lambda^{t-\hat{\ell}_q+1}\,,$$

for every $t \in \{\hat{\ell}_q, \ldots, \hat{\ell}_{q+1} - 1\}$. Notice that this formula parallels the one in (25) obtained for the case of infrequent plant changes.

Accordingly, let

$$\hat{\Pi}^*_{TV} := \hat{\Pi}^*_{TI} + c\, Z(\hat{\Pi}^*_{TI} + \nu)$$

where $c := \max\{c_3,\, 2\mu^{-1}\}$ and further assume that

$$\tau \;\geq\; \left\lceil \log_\lambda \frac{\nu}{c_3\, Z(\hat{\Pi}^*_{TI} + \nu)} \right\rceil + 2\Delta(\hat{\Pi}^*_{TV}) =: \Xi(\hat{\Pi}^*_{TV}) \tag{32}$$

Since under (32) we have that $\hat{\ell}_{q+1} - \hat{\ell}_q > \Xi(\hat{\Pi}^*_{TV})$ for every $q \in \mathbb{Z}_+$, next result follows along the same lines as that of Theorem 3.

Theorem 4: Let the HSL-R switched system Σ_σ be based on the test functionals (7) and the resetting rule (18). Let A1$-$A5 hold. Then, Σ_σ is stable provided that

$$\delta \;\leq\; \min\left\{ \delta_{max},\, \frac{\beta}{\Xi(\hat{\Pi}^*_{TV})} \right\} \tag{33}$$

■

In (33), we used the fact that $\Xi(\hat{\Pi}^*_{TV}) \leq \beta/\delta$ implies $\Xi(\hat{\Pi}^*_{TV}) \leq \lfloor \beta/\delta \rfloor = \tau$, since $\Xi(\hat{\Pi}^*_{TV})$ is a positive integer.

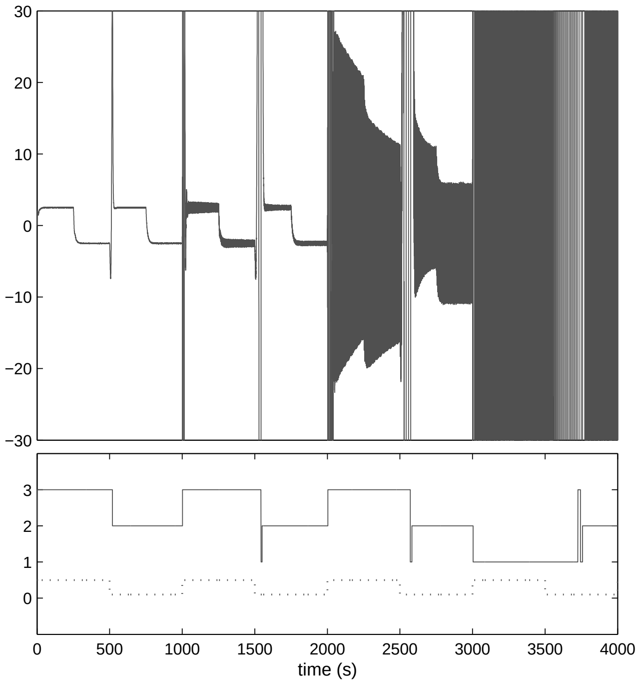

Fig. 6. Supervision based on HSL-∞ in the presence of infrequent plant variations. Top: Plant Output; Bottom: Switching sequence (solid line), and time variations of K (dotted line).

C. *Example 2*

Consider the continuous-time LTI plant with transfer function

$$P(s) = \frac{K}{s+\tau}\,(1+\Delta_m(s))$$

controlled by feeding its input via a zero-order holder and sampling its output every $0.2s$. Furthermore, $K \in [0.1, 1]$, $\tau \in [-0.5, .0.5]$ while Δ_m represents the plant unmodeled dynamics. Three PI controllers have been designed so as to provide good performance over the uncertainty set specified by K and τ. In particular, C_1 and C_2 are the same as in Example 1, while $C_3(s) = (1.7424-1.6325\,d)/(1-d)$. We first compare HSL-$\infty$ and HSL-R under the set-up used Example 1 for the case where the plant variations are subject to infrequent large jumps. In particular, we consider an unstable plant with $\tau = -0.4$ and gain K that switches between 0.5 and 0.1 every $500\,s$. Notice that for $K = 0.1$ only C_2 is stabilizing while for $K = 0.5$ only C_1 and C_3 are stabilizing. Fig. 6-7 show the plant output response for $\Delta_m(s) = 1/(s^2+100)$ and disturbances modeled by taking n_u and n_y to be uniformly distributed in $[-0.1, 0.1]$. One sees that, while the

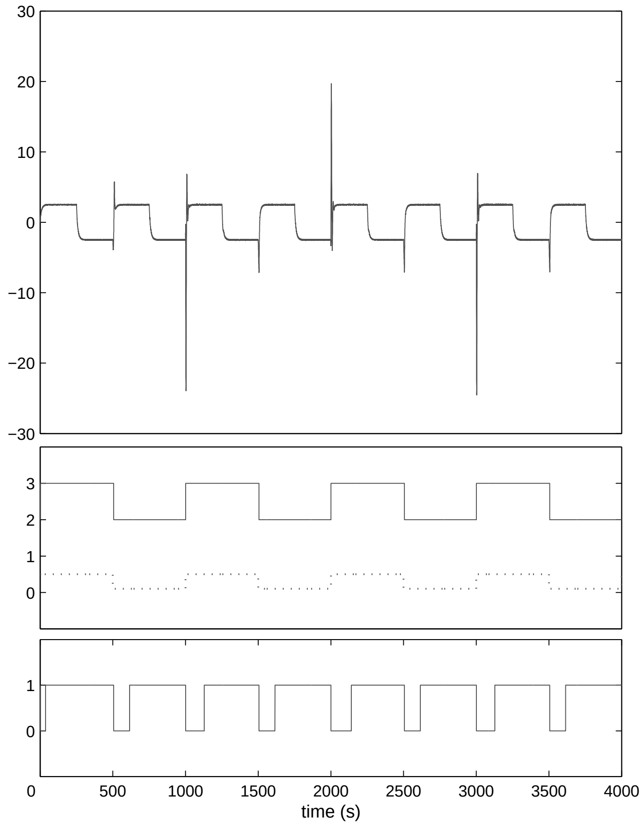

Fig. 7. Supervision based on HSL-R in the presence of infrequent plant variations. From top to bottom: Plant Output; Switching sequence (solid line) and time variations of K (dotted line); Resetting sequence (1 stands for resetting)

control self-reconfiguration capability progressively degrades under HSL-∞, this issue does not arise in the HSL-R because of resetting. Fig. 8 finally depicts simulation results for HSL-R when, in addition to the abrupt changes of K, τ varies slowly according to $\tau(t) = 0.4 \sin(0.02\,\pi t)$. Notice that in this example Δ_m was chosen to be a marginally stable transfer function. This is consistent with the fact that the proposed method allows large plant modeling uncertainties, including norm-unbounded unmodeled dynamics.

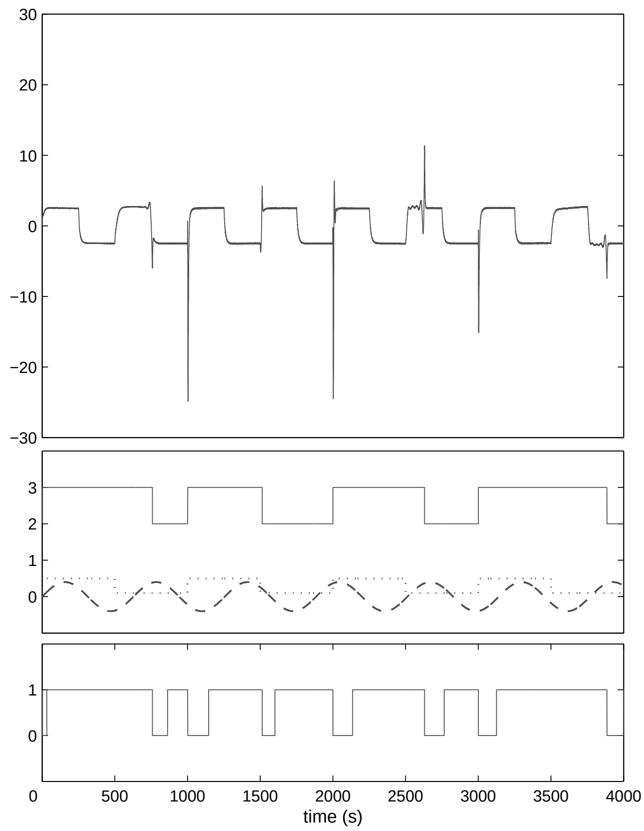

Fig. 8. Supervision based on HSL-R in the presence of slow parameter drift. From top to bottom: Plant Output; Switching sequence (solid line), time variations of K (dotted line) and time variations of τ (dashed line); Resetting sequence (1 stands for resetting)

VII. CONCLUSIONS

Consideration has been given to the control of uncertain time-varying plants by means of adaptive switching control techniques. We have introduced a novel class of algorithms based on hysteresis switching, which, when combined with appropriate test functionals, makes it possible to achieve stability for time-varying systems under large plant modeling errors, unmodeled dynamics and persistent disturbances.

The unique feature of this type of supervisory scheme is that the supervisor orchestrates the switching by means of a specially devised mechanism which, from time to time, determinates whether past data are still relevant to achieve closed-loop stability. In particular, this mechanism consists of a logic according to which the test functionals are reset to zero whenever the measured data indicates that the information contained in the test functionals is no longer needed to achieve the desired stability properties. Although the major emphasis has been on the stabilization of time-varying systems, simulation results indicate that this supervisory scheme compares favorably with HSL-∞ even when applied to time-invariant systems, since it does not rely on a finite switching stopping-time.

These results lend themselves to be extended in various directions. First, the idea of selecting online the memory of the test functionals can be extended to rules more elaborated than a simple resetting logic. As a second point, notice that the motivation for considering supervisory schemes based on unfalsified control was mainly dictated by the goal of achieving robustness against large modeling uncertainties. Nonetheless, the approach here introduced could be used within supervisory schemes alternative to model-free switching control. In this respect, it is known that the adoption of model-based test functionals can significantly improve the transients performance [7], [19]. Hence, a natural question, whose answer is yet unavailable, is how a resetting logic can be adapted to supervisory schemes based on multiple models.

APPENDIX

Throughout the appendix, we shall make use of the following properties.

i) $\| \, x|_{\alpha}^{t} \, \| \leq \| \, x|_{\alpha}^{t} \, \|_{\lambda} \, \lambda^{-(t-\alpha)}$

ii) $\| \, x|_{\alpha}^{t} \, \|_{\lambda} \leq \| \, x|_{h+1}^{t} \, \|_{\lambda} + \| \, x|_{\alpha}^{h} \, \|_{\lambda} \, \lambda^{t-h}$

iii) $\| \, x|_{\alpha}^{t-h} \, \|_{\lambda} \leq \| \, x|_{\alpha}^{t-\ell} \, \|_{\lambda} \, \lambda^{-(h-\ell)}, \quad \ell \leq h$

To prove Lemma 2 we use the following result.

Proposition 1: Under assumption A1, there exist finite nonnegative constants g_0, g_1 and g_2 such that

$$\|\zeta^t\|_{\lambda} \leq \, g_0 \, | \, \xi_P \, | \, \lambda^{t+1} + g_1 \| \, w^t \, \|_{\lambda} + g_2 \|\zeta^{t-1}\|_{\lambda} \qquad \forall \, t \in \mathbb{Z}_+ \tag{34}$$

Proof of Proposition 1. For any LTI feedback loop (P_*/C_i) we have (dropping the arguments d and t)

$$\begin{bmatrix} R_i & -S_i \\ B_* & A_* \end{bmatrix} \zeta = \begin{bmatrix} 0 & 0 & 0 \\ A_* & -B_* & -A_* \end{bmatrix} w \tag{35}$$

By A1, we also get

$$|\zeta(t)| \le c_0 \| \, \zeta|_{t-q}^{t-1} \, \| + c_1 \| \, w|_{t-q}^{t} \, \|, \qquad \forall\, t \in \mathbb{Z}_+$$

for some positive constants c_0 and c_1, with $q = \max\{n^*, m\}$. According to i), by letting $c_2 := c_0\, \lambda^{-(q-1)}$ and $c_3 := c_1\, \lambda^{-q}$ we get

$$\begin{aligned} \| \, \zeta^t \, \|_\lambda &\le |\zeta(t)| + \| \, \zeta^{t-1} \, \|_\lambda \\ &\le (c_2 + \lambda) \| \, \zeta^{t-1} \, \|_\lambda + c_3 \| \, w|_{t-q}^{t} \, \|_\lambda, \quad \forall\, t \in \mathbb{Z}_+ \end{aligned}$$

where the second inequality follows recalling that $\zeta(k) = 0$, $k = -1, -2, \ldots$. Finally,

$$\begin{aligned} \| \, w|_{t-q}^{t} \, \|_\lambda &\le \| \, w|_{-q}^{t} \, \|_\lambda \le \| \, w^t \, \|_\lambda + \| \, w|_{-q}^{-1} \, \|_\lambda \lambda^{t+1} \\ &\le \| \, w^t \, \|_\lambda + | \, \xi_P \, | \, \lambda^{t+1}, \quad \forall\, t \in \mathbb{Z}_+ \end{aligned} \tag{36}$$

where the second inequality follows by ii) with $\alpha = -q$ and $h = -1$, and the third since $n_u(k) = u_P(k)$, $n_y(k) = y_P(k)$ $k = -1, \ldots, -n^*$, and $n_u(k) = n_y(k) = 0$, $k = -n^* - 1, -n^* - 2, \ldots$. This completes the proof. ■

Proof of Lemma 2. Consider an arbitrary $k \in \mathbb{Z}_+$ and let $\mathbb{T}_{k[j]} := \big\{ t_{k[j]}, \cdots, t_{k[j+1]} - 1 \big\}$, $t_{k[0]} := t_k$, represent the j-th subinterval of $\mathbb{T}_k$ over which the switching signal is constant.

For clarity, the proof is divided into three steps.

— *Basic recursive equation.*

First, notice that

$$\begin{aligned} \| \, \zeta^t \, \|_\lambda &\le \| \, \zeta^t_{\sigma(t)} \, \|_\lambda + \| \, (v_{\sigma(t)} - r)^t \, \|_\lambda \\ &\le \pi_{\sigma(t)}(t) \, \big(\mu + \| \, r^t \, \|_\lambda \big) \\ &+ \big(1 + \pi_{\sigma(t)}(t) \big) \| \, (v_{\sigma(t)} - r)^t \, \|_\lambda, \; \forall\, t \in \mathbb{Z}_+ \end{aligned}$$

Consider now an arbitrary $\mathbb{T}_{k[j]}$. Accordingly, for some $i \in \underline{N}$, one has $\sigma(t) = i$, $\forall\, t \in \mathbb{T}_{k[j]}$. Let $\delta_i := v_i - r$. By exploiting ii) with respect to δ_i, with $\alpha = 0$ and $h = t_{k[j]} - 1$, we get

$$\begin{aligned}
\|\, \delta_i^t \,\|_\lambda &\leq \|\, \delta_i|_{t_{k[j]}}^{t} \,\|_\lambda + \|\, \delta_i^{t_{k[j]}-1} \,\|_\lambda \lambda^{t-t_{k[j]}+1} \\
&\leq g_i \,\|\, \delta_i|_{t_{k[j]}-q}^{t_{k[j]}-1} \,\|\lambda^{t-t_{k[j]}} + \|\, \delta_i^{t_{k[j]}-1} \,\|_\lambda \lambda^{t-t_{k[j]}+1} \\
&\leq \tilde{g}_i \,\|\, \delta_i|_{t_{k[j]}-q}^{t_{k[j]}-1} \,\|_\lambda \lambda^{t-t_{k[j]}+1} + \|\, \delta_i^{t_{k[j]}-1} \,\|_\lambda \lambda^{t-t_{k[j]}+1} \\
&\leq \hat{g}_i \,\|\, \delta_i^{t_{k[j]}-1} \,\|_\lambda \lambda^{t-t_{k[j]}+1}, \quad \forall\, t \in \mathbb{T}_{k[j]}
\end{aligned} \tag{37}$$

In (37), we made use of the following facts: The second inequality follows, for some finite positive constant g_i, because $S_i(d)\delta_i(t) = 0$ for every $t \in \mathbb{T}_{k[j]}$ and $S_i(d)$ is a λ-Hurwitz polynomial; the third inequality is obtained using i) and letting $\tilde{g}_i := g_i \lambda^{-q}$; the last inequality follows with $\hat{g}_i := 1 + \tilde{g}_i$ since, for all candidate indices, (6) is initialized at time zero from zero initial conditions, *viz.* $\delta_i(k) = 0$, $k = -1, -2, \ldots$.

By A3 one sees that the map from ζ to δ_i,

$$S_i(d)\,\delta_i(t) = S_i(d)\,(v_i(t) - r(t)) = [R_i(d) \;\; - S_i(d)]\,\zeta(t)$$

has bounded $\lambda-$weighted ℓ_2-norm. In particular, since (6) is initialized at time zero from zero initial conditions, it readily follows that $\|\delta_i^t\|_\lambda \;\leq\; \overline{g}_i \,\|\zeta^t\|_\lambda$, $\forall\, t \in \mathbb{Z}_+$,$\forall\, i \in \underline{N}$, and some finite nonnegative constant $\overline{g}_i$.

Hence, letting $\hat{g} := \max_{i\in\underline{N}}\{\hat{g}_i\}$ and $\overline{g} := \max_{i\in\underline{N}}\{\overline{g}_i\}$, we finally get

$$\begin{aligned}
\|\, \zeta^t \,\|_\lambda \;&\leq\; \pi_{\sigma(t)}(t)\, \left(\mu + \|\, r^t \,\|_\lambda\right) \\
&+\; g\, \left(1 + \pi_{\sigma(t)}(t)\right) \|\, \zeta^{t_{k[j]}-1} \,\|_\lambda \lambda^{t-t_{k[j]}+1}, \\
&\qquad \forall\, t \in \mathbb{T}_{k[j]}, \quad j = 0, \cdots, \mathfrak{N}_k
\end{aligned} \tag{38}$$

where $g := \hat{g}\,\overline{g}$.

— *Upper-bound on data between switching.*

From Lemma 1, we have $\pi_{\sigma(t+1)}(t) \leq \Pi^k$ for every $t \in \mathbb{T}_k$. Then,

$$\begin{aligned}
\|\, \zeta^t \,\|_\lambda &\leq \Pi^k\, \left(\mu + \|\, r^t \,\|_\lambda\right) \\
&\quad + g\, \left(1 + \Pi^k\right) \|\, \zeta^{t_{k[j]}-1} \,\|_\lambda \lambda^{t-t_{k[j]}+1}, \\
&\forall\, t \in \mathbb{T}_{k[j]} \setminus \left\{t_{k[j+1]} - 1\right\},\; j = 0, \cdots, \mathfrak{N}_k
\end{aligned} \tag{39}$$

In words, the performance signal related to the switched-on controller cannot exceed Π^k, apart from the time instants right before switching. Notice that by Lemma 1 this may happen at most $\mathfrak{N}_k$-times, since, after $\mathfrak{N}_k$ switching, no more switching occurs over $\mathbb{T}_k$.

— *Proof of (11).*

In order to prove (11) we use (39) and exploit the results of Proposition 1 at the time instants $\left\{t_{k[j+1]}-1\right\}$. Without loss of generality, let $g_2, g \geq 1$, g_2 and g as in (34) and, respectively, (39). It is also convenient to let $g_3 := \max\{g_0,\, g_1\}$ with g_0 and g_1 as in (34), and write in more compact form $L\,(t, \xi_P, \mu, w) := \mu + |\,\xi_P\,|\,\lambda^{t+1} + \|\,w^t\,\|_\lambda$. Then, combining (34) and (39), we get

$$\begin{aligned}
\|\,\zeta^t\,\|_\lambda \;&\leq\; g_2\|\,\zeta^{t-1}\,\|_\lambda + g_3\,L\,(t, \xi_P, \mu, w)\\
&\leq\; (g_2\Pi^k + g_3)\,L\,(t,\, \xi_P,\, \mu,\, w)\\
&+ g_2\,g\,\left(1+\Pi^k\right)\,\|\,\zeta^{t_{k[j]}-1}\,\|_\lambda \lambda^{t-t_{k[j]}+1}\\
&\leq G_k\,\left(\|\,\zeta^{t_{k[j]}-1}\,\|_\lambda \lambda^{t-t_{k[j]}+1} + L\,(t, \xi_P, \mu, w)\right),\\
&\qquad\qquad \forall\, t \in \mathbb{T}_{k[j]}, \quad j = 0, \cdots, \mathfrak{N}_k
\end{aligned} \tag{40}$$

In the second inequality we used the fact that $\|r^t\|_\lambda \leq \|w^t\|_\lambda$, while $G_k := g_3 + g_2\,g(1+\Pi^k)$.

By induction, it is easy to show that

$$\begin{aligned}
\|\,\zeta^t\,\|_\lambda \;\;&\leq\;\; \left(\sum_{m=1}^{j+1} G_k^m\right)\,\left(\|\,\zeta^{t_k-1}\,\|_\lambda \lambda^{t-t_k+1} + L\,(t, \xi_P, \mu, w)\right),\\
&\qquad \forall\, t \in \bigcup_{m=0}^{j} \mathbb{T}_{k[m]}, \quad j = 0, \cdots, \mathfrak{N}_k
\end{aligned} \tag{41}$$

Indeed, consider the following facts. By (40), equation (41) holds for $j = 0$ since $t_{k[0]} = t_k$. Assume next that (41) holds up to a certain $j < \mathfrak{N}_k$. Then, by (40),

$$\begin{aligned}
\|\,\zeta^t\,\|_\lambda &\leq G_k\,\left(\left\|\zeta^{t_{k[j+1]}-1}\right\|_\lambda \lambda^{t-t_{k[j+1]}+1} + L\,(t, \xi_P, \mu, w)\right)\\
&\leq G_k\left\{\left(\sum_{m=1}^{j+1} G_k^m\right)\left(\|\,\zeta^{t_k-1}\,\|_\lambda \lambda^{t_{k[j+1]}-t_k}\right.\right.\\
&\left.\left.+ L\left(t_{k[j+1]}-1, \xi_P, \mu, w\right)\right)\lambda^{t-t_{k[j+1]}+1} + L\,(t, \xi_P, \mu, w)\right\}\\
&\leq \left(\sum_{m=1}^{j+2} G_k^m\right)\,\left(\|\,\zeta^{t_k-1}\,\|_\lambda \lambda^{t-t_k+1}\right.\\
&\qquad\qquad \left.+ L\,(t, \xi_P, \mu, w)\right), \quad \forall\, t \in \mathbb{T}_{k[j+1]}
\end{aligned} \tag{42}$$

In (42), the second inequality follows since, by assumption, (41) holds true up to j; the last inequality follows since

$$\begin{aligned} & L\left(t_{k[j+1]}-1, \xi_P, \mu, w\right) \lambda^{t-t_{k[j+1]}+1} \\ & \quad = \left(\mu + |\, \xi_P \,| \, \lambda^{t_{k[j+1]}} + \| \, w^{t_{k[j+1]}-1} \, \|_\lambda\right) \lambda^{t-t_{k[j+1]}+1} \\ & \quad \leq \mu + |\, \xi_P \,| \, \lambda^{t+1} + \| \, w^t \, \|_\lambda = L\left(t, \xi_P, \mu, w\right) \end{aligned}$$

where the inequality is obtained using iii) with respect to $\left\| w^{t_{k[j+1]}-1} \right\|_\lambda$, with $\alpha = 0$, $\ell = 0$ and $h = t - t_{k[j+1]} + 1$. Hence, (42) holds true for $j+1$, and the proof follows by letting $\mathfrak{g}\left(\Pi^k\right) := \sum_{m=1}^{\mathfrak{N}_k+1} G_k^m$. ∎

Proof of Lemma 3. Let $\theta(t) \in \mathbb{R}^{2n^*}$, $t \in \mathbb{Z}_+$, denote the vector of time-varying parameters composed by the coefficients of A_t and B_t. Consistent with this notation, we can re-write A1 as requiring that $\theta(t) \in \Theta$, $\forall\, t \in \mathbb{Z}_+$ for some compact set Θ. By assumption, $\theta(t) \equiv \theta$ for all $t \in \{\tau, \tau+1, \ldots, T\}$ and some $\theta \in \Theta$, i.e. $P = P(\theta) = B(\theta)/A(\theta)$ for all $t \in \{\tau, \tau+1, \ldots, T\}$. Consider now that, under assumption A3, the virtual references in (6) are well-defined. Thus, combining (6) and (1) we can write $\Phi_{\theta/i}\, \zeta_i = \Psi_\theta\, w_i$, where

$$\Phi_{\theta/i} := \begin{bmatrix} R_i & -S_i \\ B(\theta) & A(\theta) \end{bmatrix}, \quad \Psi_\theta := \begin{bmatrix} 0 & 0 & 0 \\ A(\theta) & -B(\theta) & -A(\theta) \end{bmatrix} \tag{43}$$

Consider next that, for any candidate controller C_i such that $(P(\theta)/C_i)$ is λ-*stable* there exists a polynomial matrix $I_{\theta/i}(d)$ such that

$$\zeta_i(t) = \Phi_{\theta/i}^{-1}(d)\, I_{\theta/i}(d) \,|\, \xi_i|_{\tau-q}^{\tau-1} \,| + \Phi_{\theta/i}^{-1}(d)\, \Psi_\theta(d)\, w_i(t) \tag{44}$$

where $\xi_i := [\, \zeta_i' \;\; w_i' \,]'$, $q = \max\{n^*, m\}$, and $\Phi_{\theta/i}$ is such that $\det \Phi_{\theta/i}$ is a λ-*Hurwitz* polynomial. Consider now that, by assumption A2, for any $\theta_j \in \Theta$ there exist a candidate controller C and an open ball Θ_j around θ_j such that $(P(\theta)/C)$ is λ-*stable* for all $\theta \in \Theta_j$. Thus, an infinite open covering for Θ exits. In turn, in view of assumptions A1 and A2, this implies the existence of a finite (closed) covering $\bar{\Theta}_1, \ldots, \bar{\Theta}_{\bar{N}}$ for Θ such that, for each j, all the plants $P(\theta)$, $\theta \in \bar{\Theta}_j$, are stabilized by a common controller $C_{\rho(j)}$, $\rho(j) \in \underline{N}$. As a consequence, from (44), it is therefore immediate to conclude that there exist positive reals m_{0j} and m_{1j} such that, for any $\theta \in \bar{\Theta}_j$,

$$\| \, \zeta_{\rho(j)}|_\tau^t \, \|_\lambda \leq m_{0j} \| \, \xi_{\rho(j)}|_{\tau-q}^{\tau-1} \, \| \lambda^{t-\tau+1} + m_{1j} \| \, w_{\rho(j)}|_\tau^t \, \|_\lambda, \quad \forall\, t \in \{\tau, \tau+1, \ldots, T\} \tag{45}$$

holds true where $m_{0j} := \sup_{\theta\in\bar{\Theta}_j} \|\Phi^{-1}_{\theta/\rho(j)}\, I_{\theta/\rho(j)}\|_{\infty,\lambda}$, $m_{1j} := \sup_{\theta\in\Theta_j} \|\Phi^{-1}_{\theta/\rho(j)}\, \Psi_\theta\|_{\infty,\lambda}$ and $\|H\|_{\infty,\lambda}$ denotes the λ-weighted $\mathcal{H}_\infty$ norm of H. Hence, eq. (45) holds true over Θ with finite positive constants $m_0 := \max_{j\in\bar{N}}\, m_{0j}$ and $m_1 := \max_{j\in\bar{N}}\, m_{1j}$.

Now, consider that using ii) with $\alpha = 0$ and $h = \tau - 1$ we have $\|\zeta_s^t\|_\lambda \le \|\, \zeta_s|_\tau^t \,\|_\lambda + \|\zeta_s^{\tau-1}\|_\lambda \lambda^{t-\tau+1}$. Moreover, using i) we also have

$$\begin{aligned} \|\, \xi_s|_{\tau-q}^{\tau-1} \,\| &\le \|\, \xi_s|_{\tau-q}^{\tau-1} \,\|_\lambda \lambda^{-(q-1)} \\ &\le \left(\|\, \zeta_s|_{\tau-q}^{\tau-1} \,\|_\lambda + \|\, w_s|_{\tau-q}^{\tau-1} \,\|_\lambda \right) \lambda^{-(q-1)} \end{aligned} \tag{46}$$

Consider next that $\|\, \zeta_s|_{\tau-q}^{\tau-1} \,\|_\lambda = \|\zeta_s^{\tau-1}\|_\lambda$ because of the zero initial condition constraint, while $\|\, w_s|_{\tau-q}^{\tau-1} \,\|_\lambda \le \|w_s^{\tau-1}\|_\lambda + \|\, w_s|_{\tau-q}^{-1} \,\|_\lambda \lambda^\tau \le \|w_s^{\tau-1}\|_\lambda + |\xi_P|\lambda^\tau$, the first inequality being obtained from ii) with $\alpha = \tau - q$ and $h = -1$.

Overall, we get

$$\begin{aligned} \|\, \zeta_s^t \,\|_\lambda \le\; & m_1 \,\|\, w_s^t \,\|_\lambda + m_2 \,\|\, \zeta_s^{\tau-1} \,\|_\lambda \lambda^{t-\tau+1} \\ & + m_3 \,|\, \xi_P \,|\, \lambda^{t+1} + m_3 \,\|\, w_s^{\tau-1} \,\|_\lambda \lambda^{t-\tau+1} \end{aligned} \tag{47}$$

where $m_2 := 1 + m_0\lambda^{-(q-1)}$ and $m_3 := m_0\lambda^{-(q-1)}$. Finally, by noting that $\|\zeta_s^{\tau-1}\|_\lambda \le \|\zeta^{\tau-1}\|_\lambda + \|v_s^{\tau-1}\|_\lambda + \|w^{\tau-1}\|_\lambda$ and $\|w_s^{\tau-1}\|_\lambda \le \|v_s^{\tau-1}\|_\lambda + \|w^{\tau-1}\|_\lambda$ and exploiting iii) with respect to v_s and w, with $\alpha = 0$, $h = t - \tau + 1$ and $\ell = 0$, we get

$$\begin{aligned} \|\, \zeta_s^t \,\|_\lambda \le\; & m_4 \,\|\, v_s^t \,\|_\lambda + m_2 \,\|\, \zeta^{\tau-1} \,\|_\lambda \lambda^{t-\tau+1} \\ & + m_3 \,|\xi_P|\, \lambda^{t+1} + m_4 \,\|\, w^t \,\|_\lambda \end{aligned} \tag{48}$$

where $m_4 := m_1 + m_2 + m_3$. Hence, (16) immediately follows from the definition of the π_i's. ∎

Proof of Lemma 4. Consider the same notation as in the proof of Lemma 2. Since by assumption $\Pi^k \le \Pi^*$ for every $k \in \mathbb{Z}_+$, the number of switching on every interval $\mathbb{T}_k$ is upper bounded by $\mathfrak{N}_*$. Now, using (38) and (39) we obtain

$$\begin{aligned} \pi_*(t) \le \pi_{\sigma(t+1)}(t) + \epsilon_{k[j]}(t)\, \frac{1}{\mu + \|\, r^t \,\|_\lambda}, \\ \forall\, t \in \mathbb{T}_{k[j]} \setminus \left\{ t_{k[j+1]} - 1 \right\} \end{aligned} \tag{49}$$

for every $j \leq \mathfrak{N}_k$, where

$$\begin{aligned}\epsilon_{k[j]}(t) &:= g\ \left(1+\pi_{\sigma(t+1)}(t)\right) \|\, \zeta^{t_{k[j]}-1}\,\|_\lambda \lambda^{t-t_{k[j]}+1} \\ &< G_k \,\|\, \zeta^{t_{k[j]}-1}\,\|_\lambda \lambda^{t-t_{k[j]}+1}, \quad t \in \mathbb{T}_{k[j]} \setminus \left\{t_{k[j+1]}-1\right\} .\end{aligned}$$

We first derive an upper bound for $\epsilon_{k[j]}$. By (41),

$$\begin{aligned}\|\, \zeta^{t_{k[j]}-1}\,\|_\lambda \leq \left(\sum_{m=1}^{\mathfrak{N}_k} G_k^m\right) \ &\left(\|\, \zeta^{t_k-1}\,\|_\lambda \lambda^{t_{k[j]}-t_k}\right. \\ &\left.+L\left(t_{k[j]}-1, \xi_P, \mu, w\right)\right)\end{aligned}$$

and, hence,

$$\begin{aligned}\epsilon_{k[j]}(t) < \mathfrak{g}\left(\Pi^k\right)\ &\left(\|\, \zeta^{t_k-1}\,\|_\lambda \lambda^{t_{k[j]}-t_k}\right. \\ &\left.+L\left(t_{k[j]}-1, \xi_P, \mu, w\right)\right) \lambda^{t-t_{k[j]}+1} \\ \leq& \mathfrak{g}\left(\Pi^k\right)\ \left\{\left(\Pi^{k-1}+\epsilon\right)\left(\mu+\|\, r^{t_k-1}\,\|_\lambda\right)\lambda^{t_{k[j]}-t_k}\right. \\ &\left.+L\left(t_{k[j]}-1, \xi_P, \mu, w\right)\right\} \lambda^{t-t_{k[j]}+1}\end{aligned}$$

where the second inequality follows from the definition of admissible resetting times (see equation (14)). Consider further that $\|r^{t_k-1}\|_\lambda \leq (1-\lambda^2)^{-1/2}\|w\|_\infty$, and that $L\left(t, \xi_P, \mu, w\right) \leq \mu + |\xi_P| + (1-\lambda^2)^{-1/2}\|w\|_\infty$ for every t, from which we finally obtain

$$\begin{aligned}\epsilon_{k[j]}(t) < &\left\{\mathfrak{g}\left(\Pi^*\right)\ |\xi_P|\right. \\ &\left.+(1-\lambda^2)^{-1/2}\mathfrak{g}\left(\Pi^*\right)\left(1+\Pi^*+\epsilon\right)\left(\mu+\|\, w\,\|_\infty\right)\right\}\ \lambda^{t-t_{k[j]}+1} = \ Z\left(\Pi^*\right)\lambda^{t-t_{k[j]}+1}\end{aligned} \tag{50}$$

where $Z(\cdot)$ is as in (19).

Consider now that, because of (49), we have $\epsilon_{k[j]}(t) > \epsilon\mu$ for every $t \in \mathbb{T}_{k[j]} \setminus \left\{t_{k[j+1]}-1\right\}$, and, hence,

$$t_{k[j+1]} \leq t_{k[j]} + \left\lceil \log_\lambda \frac{\epsilon\mu}{Z\left(\Pi^*\right)}\right\rceil . \tag{51}$$

Indeed, if the latter condition were not satisfied we would then have a reset before $t_{k[j+1]}$, contradicting the fact that $t_{k[j+1]} \leq t_{k[\mathfrak{N}_k+1]} = t_{k+1}$. Applying (51) recursively, we get $t_{k[j]} \leq t_k + j\ \left\lceil \log_\lambda \epsilon\mu/Z\left(\Pi^*\right)\right\rceil$. Then, the claim follows by recalling that the number of switching is upper bounded by $\mathfrak{N}_*$. ∎

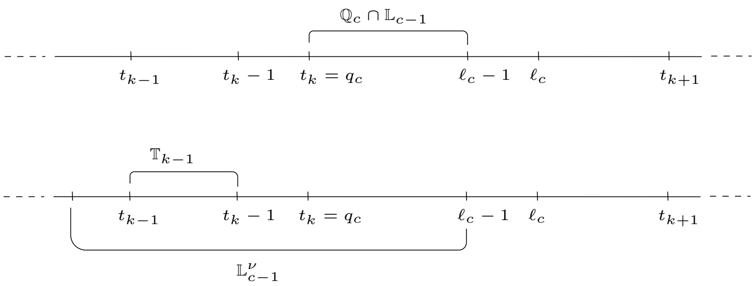

Fig. 9. Segment of time axis for dwell-time computation.

Proof of Lemma 5. Consider the subsequence $\{q_c\}_{c\in\mathbb{Z}_+}$ of resetting instants defined by $q_c := \max_{k\in\mathbb{Z}_+}\{t_k;\ t_k \le \ell_c\}$. Notice that, under (27), $\{q_c\}_{c\in\mathbb{Z}_+}$ exists well-defined since $q_0 = 0$ and $q_i < q_j$ if $i < j$. Let $\mathbb{Q}_c := \{q_c, \dots, q_{c+1} - 1\}$ and note that $\cup_{c\in\mathbb{Z}_+}\mathbb{Q}_c = \mathbb{Z}_+$. Thus, in order to prove that the switched system is stable, it is sufficient to prove that the upper bound Π^*_{TV} holds over each interval $\mathbb{Q}_c$ since this implies that the same upper bound will hold over each $\mathbb{T}_k$.

Decompose $\mathbb{Q}_c$ as $\mathbb{Q}_c = (\mathbb{Q}_c \cap \mathbb{L}_{c-1}) \cup (\mathbb{Q}_c \cap \mathbb{L}_c)$ and first assume that the interval $\mathbb{Q}_c \cap \mathbb{L}_{c-1}$ is non-empty, as depicted at the top Figure 9. By definition q_c necessarily belongs to $\mathbb{L}^{\nu}_{c-1}$, Then, from equation (25) it follows that on this interval one has the upper bound $\Pi^*_{TI} + \nu$. Thus, (11) implies

$$\|\,\zeta^t\,\|_\lambda \;\le\; \mathfrak{g}\,(\Pi^*_{TI} + \nu)\,(\,\mu + |\,\xi_P\,|\,\lambda^{t+1} + \|\,w^t\,\|_\lambda + \|\,\zeta^{q_c-1}\,\|_\lambda \lambda^{t-q_c+1}), \quad \forall\, t \in \mathbb{Q}_c \cap \mathbb{L}_{c-1} \tag{52}$$

Indeed, equation (52) follows immediately by extending the conclusions of Lemma 2 to any truncation of a resetting interval ($\mathbb{Q}_c$ in this case). In addition, by (27) there exists a resetting interval, say $\mathbb{T}_{k-1}$, right before $\mathbb{Q}_c$, which is contained into $\mathbb{L}^{\nu}_{c-1}$ (bottom of Fig. 9) and therefore such that $\Pi^{k-1} \le \Pi^*_{TI} + \nu$. Since by virtue of (18) the sequence of reset times is admissible, the sequence $\{\zeta^{q_c-1}\}_{c\in\mathbb{Z}_+}$ is such that $\|\zeta^{q_c-1}\|_\lambda \le (\Pi^*_{TI} + \nu + \epsilon)(\mu + \|r^{q_c-1}\|_\lambda)$. Combining this inequality with (52), it follows immediately that $\|\zeta^t\|_\lambda \le Z\,(\Pi^*_{TI} + \nu)$ for every $t \in \mathbb{Q}_c \cap \mathbb{L}_{c-1}$. Thus, from the definition of $Z(\cdot)$ and taking (24) into account we get $\pi_i(t) \le 2\mu^{-1} Z\,(\Pi^*_{TI} + \nu)$ for every $t \in \mathbb{Q}_c \cap \mathbb{L}_{c-1}$. Finally, substituting (26) into (25) it follows that for some index s we have that $\pi_s(t) + h \le \Pi^*_{TI} + g_3\, Z\,(\Pi^*_{TI} + \nu)$ holds on the interval $\mathbb{Q}_c \cap \mathbb{L}_c$ from which the upper bound (28) follows.

If instead $\mathbb{Q}_c \cap \mathbb{L}_{c-1} = \emptyset$, then $\mathbb{Q}_c \subseteq \mathbb{L}_c$ and $q_c = \ell_c$. In such a case (28) follows immediately by (25) and (26). ∎

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