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Optimization in Stochastic Hybrid and Switching Systems

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Farshad Ramezan Pour Safaei

Committee in Charge:

Professor João P. Hespanha, Chair

Professor Francesco Bullo

Professor Katie Byl

Professor Andrew Teel

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The dissertation of
Farshad Ramezan Pour Safaei is approved:

Professor Francesco Bullo

Professor Katie Byl

Professor Andrew Teel

Professor João P. Hespanha, Committee Chairperson

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Optimization in Stochastic Hybrid and Switching Systems

To my wife Niloufar.

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Curriculum Vitæ
Farshad Ramezan Pour Safaei

Education

2009 – 2010	MSc in Electrical and Computer Engineering, University of California, Santa Barbara
2004 – 2008	BSc in Electrical and Computer Engineering, University of Tehran, Iran

Experience

2009 – 2013	Graduate Research Assistant, University of California, Santa Barbara.
2012 – 2012	Electrical Engineering Intern, Veeco Instruments, Camarillo, CA
2009 – 2009	Teaching Assistant, University of California, Santa Barbara
2007 – 2007	Electrical Engineering Intern, Iran Khodro Company, Iran

Selected Publications

1. F. R. Pour Safaei, K. Roh, S. Proulx and J. Hespanha, *Quadratic Control of Stochastic Hybrid Systems with Renewal Transitions*, to be submitted to journal publications, 2013.
2. F. R. Pour Safaei, J. Hespanha, G. Stewart, *On Controller Initialization in Multivariable Switching Systems*, Automatica, Dec. 2012.
3. F. R. Pour Safaei, J. Hespanha, S. Proulx, *Infinite Horizon Linear Quadratic Gene Regulation in Fluctuating Environments*, In Proc. of the 51st Conference on Decision and Control, Dec. 2012.
4. K. Roh, F. R. Pour Safaei, J. Hespanha, and S. Proulx, *Evolution of transcription networks in response to temporal fluctuations*, Journal of Evolution, 2012.
5. F. R. Pour Safaei, J. Hespanha, G. Stewart. *Quadratic Optimization for Controller Initialization in MIMO Switching Systems*, In Proc. of the 2010 American Control Conference, June 2010.

Abstract

Optimization in Stochastic Hybrid and Switching Systems

Farshad Ramezan Pour Safaei

This work focuses on optimal quadratic control of a class of hybrid and switching systems. In the first part of this dissertation, we explore the effect of stochastically varying environments on the gene regulation problem. We use a mathematical model that combines stochastic changes in the environments with linear ordinary differential equations describing the concentration of gene products. Motivated by this problem, we study the quadratic control of a class of stochastic hybrid systems for which the lengths of time that the system stays in each mode are independent random variables with given probability distribution functions. We derive a sufficient condition for finding the optimal feedback policy that minimizes a discounted infinite horizon cost. We show that the optimal cost is the solution to a set of differential equations with unknown boundary conditions. Furthermore, we provide a recursive algorithm for computing the optimal cost and the optimal feedback policy. When the time intervals between jumps are exponential random variables, we derive a necessary and sufficient condition for the existence of the optimal controller in terms of a system of linear matrix inequalities.

In the second part of this monograph, we present the problem of optimal controller initialization in multivariable switching systems. We show that by finding optimal values for the initial controller state, one can achieve significantly better transient performance when switching between linear controllers for a not necessarily asymptotically stable MIMO linear process. The initialization is obtained by performing the minimization of a quadratic cost function. By suitable choice of realizations for the controllers, we guarantee input-to-state stability of the closed-loop system when the average number of switches per unit of time is smaller than a specific value. If this is not the case, we show that input-to-state stability can be achieved under a mild constraint in the optimization.

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Introduction

Dynamical systems that integrate continuous and discrete dynamics are usually called *hybrid systems*. Continuous dynamics describe the evolution of continuous (real valued) state, input and output variables, and are typically represented through ordinary differential equations (in continuous time) or difference equations (in discrete time). Whereas the discrete components describe the evolution of discrete (finite or countably valued) state. The defining feature of hybrid systems is the coupling of these two diverse types of dynamics; for example, allowing the flow of the continuous state to depend on the discrete state and the transitions of the discrete state to depend on the continuous state. Traditionally, researchers have focused on either continuous or on discrete behavior. However, a large number of applications (especially in the area of embedded computation and control systems) fall into the class of hybrid systems.

During the past decades, hybrid systems have been the focus of intense research, by control theorists, computer scientists and applied mathematicians [74, 43, 24]. More specifically, many researchers have investigated the applicability of hybrid systems in various applications such as traction control system [8], flight control and management systems [71], chemical reactions and biological applications [34, 2], and TCP/IP networks [27], etc.

Much of the work on hybrid systems have focused on deterministic models that completely characterize the future of the system without allowing any uncertainty. In practice, it is often desirable to introduce some uncertainty in the models, to allow, for example, under-modeling of certain parts of the system. To address this need, researchers in hybrid systems have introduced what are known as non-deterministic models. There are a few intuitive ways to introduce uncertainty in the traditional hybrid system's framework. For instance, one does so in the continuous-time dynamics through the use of stochastic differential equations, rather than the classical ODE's [35]. Another way is to replace the deterministic jumps between discrete states by random jumps governed by some prescribed probabilistic laws [6, 26].

Stability of stochastic differential equations has been studied quite extensively for a number of years, for example, by [7, 36, 45, 80]. By contrast, the problem of designing optimal controllers to stabilize systems of this type has received less attention. From the optimal control perspective, a number of researchers have considered optimization problems on Stochastic Hybrid Systems (SHS). The author of [46] studies the Linear Quadratic Regulator (LQR) problem for Markov Jump Linear (MJL) systems and presents various algorithms to compute the optimal gains. The author of [46] considers both infinite and finite horizon cases and provides a sufficient condition for the existence of solution in the infinite horizon case. Moreover, based on the Stochastic Stabilizability (SS) concept for MJL systems, [36] establishes a necessary and sufficient condition for finite cost in the infinite horizon case. Several researchers have constructed iterative algorithms to solve the system of coupled Riccati equations occurring in jump linear control systems. For instance, [23] proposes the construction of a sequence of Lyapunov algebraic equations whose solutions converge to the solution of the coupled Riccati equations that will appear in this monograph.

In MJL systems, as special class of SHS, the waiting times between consecutive jumps are assumed to be exponentially distributed [46]. Thus, over sufficiently small intervals, the probability of transition to another state is roughly proportional to the length of that interval. The memoryless property of the exponential distribution simplified the analysis of MJL systems, however, in many real world applications, the time intervals between jumps have a probability distribution other than the exponential. As a generalization of MJL systems, one can consider

Stochastic Hybrid System with renewal transitions [6] in which the holding times (times between jumps) are independent random variables with given probability distribution functions, and the embedded jump chain is a Markov chain.

The key challenge in studying SHS with renewal transitions lies in the fact that the Markov property of MJL systems does not hold. This prevents the direct use of approaches based on Dynkin's formula [46]. However, this issue can be overcome by adding a timer to the state of the system that keeps track of the time elapsed since the last transition. Such approach has been introduced in [16].

1.1 Statement of Contribution

Stochastic Gene Regulation in Fluctuating Environments. We start our discussions by formulating the gene regulation problem in stochastically varying environments. This is our main motivation for studying the optimal control of stochastic hybrid systems with renewal transitions. For gene dynamics modelled by a linear system, we derive a mathematical model that represents the total life-time cost of a living organism. Such a cost is the expected square difference between the current protein level and the level that is assumed to be optimal for the current environment plus the cost of protein production/decay, integrated over the life span of the organism. We show that such cost could be represented by a discounted infinite horizon LQR problem with switching equilibria.

This formulation can be used to study how living organisms respond to environmental fluctuations by orchestrating the expression of sets of genes. We illustrate the applicability of our results through a numerical example motivated by the metabolism of sugar by *E. Coli* in the presence and absence of lactose in the environment. Considering linear dynamics for enzymes and mRNA, we compute the optimal gene regulator that minimizes the expected square difference between the current states and the one that would be optimal for the current environment, plus the cost of protein production/decay, integrated over the life span of the bacterium.

Quadratic Control of Markovian Jump Linear Systems. Inspired by the gene regulation problem in stochastically varying environments, we derive an optimal controller that minimizes a discounted infinite horizon LQR problem with switching equilibria in which the holding times (times between jumps) are exponentially distributed. We show that the optimal control is affine in each mode, which turns out to be consistent with the biologically meaningful model for protein degradation considered in [1]. As our contribution, we derive a necessary and sufficient condition for the existence of the optimal control, which can be expressed in terms of a system of Linear Matrix Inequalities (LMIs).

Quadratic Control of Stochastic Hybrid Systems with Renewal Transitions. Following the ideas in [68, 11], we consider quadratic control of SHS with renewal transitions which can be viewed as the generalization of the optimal control of MJL systems. We derive an optimal control policy that minimizes a discounted infinite horizon LQR problem with switching equilibria. We show that the optimal cost is the solution to a set of differential equations (so-called Bellman equations) with unknown boundary conditions. Furthermore, we provide a numerical technique for finding the optimal solution and the corresponding boundary conditions. This is one of our main contributions.

While the proofs of our results are inspired by the extended generator approach for Piecewise Deterministic Markov Processes in [16], we do not require the assumption that the value function belongs to the domain of the extended generator of the closed-loop process. Diverging from [16], we also do not require the vector field of the process be bounded in x uniformly over the control signal, which would not hold even for linear dynamics. We overcome this issue by deriving a Dynkin's-like formula for the "stopped process" [42], which under appropriate assumptions converges to the original process. This is also one of our main contributions.

Controller Initialization in Multivariable Switching Systems. In the Chapters 3 and 4 of this dissertation, we mainly focus on stochastic hybrid systems with renewal transitions and study the optimal LQR problem of such systems. In the final chapter of this monograph, we focus on a slightly different problem. We show how one can achieve significantly better transient performance by taking advantage of an additional degree of freedom that is rarely used by designers. In particular, we investigate optimal controller initialization in multivariable switched systems and show that this results in a significantly smoother transient.

Our main motivation for considering this problem is to ultimately combine the idea of controller initialization with the LQR problem of SHS to achieve the best overall performance. The specific problem formulated in Chapter 5 was first introduced in [33], which provided a method to select controller realizations and initial conditions for the case of an asymptotically stable SISO plant to be controlled. The stability results of [33] were restricted to the case of piecewise constant reference signals. Our contribution is to extend these results to MIMO, possibly unstable processes and show that it is possible to obtain input-to-state stability (ISS) of the closed-loop for arbitrary references. In particular, we show that ISS can be obtained through two alternative mechanisms: when the average number of switches per unit of time is smaller than a specific value, the closed-loop system remains ISS. If this is not the case then the ISS property can still be achieved by adding a mild constraint to the optimization and selecting the initial controller state.

1.2 Organization

In Chapter 2, we start by modelling a simple one-step gene expression process with two discrete environments and then generalize it to a n-step process with an arbitrary number of environments. In Chapter 3, the optimal control strategy for fluctuating environments with exponential holding times is derived. We further establish a necessary and sufficient condition for the existence of solution in terms of linear matrix inequality conditions.

Chapter 4 concerns with a sufficient condition for optimal LQR problem of stochastic hybrid systems with renewal transitions. We derive a set of differential equations (with unknown boundary conditions) to be satisfied by the optimal cost. A numerical algorithm is provided for finding the optimal solution.

As an additional problem, we focus on controller initialization of Multivariable switching systems in Chapter5. We consider a class of switched systems which consists of a linear MIMO and possibly unstable process in feedback interconnection with a multicontroller whose dynamics switch. It is shown how one can achieve significantly better transient performance by selecting the initial condition for every controller when it is inserted into the feedback loop.

1.3 Notation

The following notation will be use throughout this monograph. For a given matrix A, its transpose is denoted by A'. We use $A > 0$ ($A \geq 0$) to denote that a symmetric matrix is positive definite (semi-definite). The identity and zero matrices are denoted by I and 0, respectively. Given a measurable space by $(\Omega, \mathcal{B})$ and probability measure by $P : \mathcal{B} \to [0, 1]$, stochastic process $\mathbf{x} : \Omega \times [0, \infty) \to \mathcal{X} \subset \mathbb{R}^n$ is denoted in **boldface**. We use *wpo* to denote universal quantification with respect to some subset of Ω with probability one. Notation $E_{x_0}\{\mathbf{x}(t)\}$ indicates expectation of the process $\mathbf{x}$ conditioned upon initial condition x_0. I_A denotes an indicator function. We also use the notation $t \wedge s = \min(t, s)$. We denote by $z(t^-)$ and $z(t^+)$ the limits from the left ($\lim_{\tau \uparrow t} z(\tau)$) and the right ($\lim_{\tau \downarrow t} z(\tau)$), respectively.

2

Stochastic Gene Regulation in Fluctuating Environments

Living organisms sense their environmental context and orchestrate the expression of sets of genes to utilize available resources and to survive stressful conditions [57]. Recently, several researchers have considered the effect of stochastically varying environment on gene regulation problems [18, 38, 62]. Following this line of research, we consider a model of gene regulation where the environment switches between discrete states at random time intervals. These states could potentially represent physiological or hormonal states that a cell senses in multicellular organisms. Different environmental conditions have different optimal expression levels, and the performance of the cell improves as the expression level approaches the optimum. For example, a protein that provides a useful function under some environmental conditions may produce deleterious byproducts under other conditions. A recent study of the yeast *Saccharomyces cerevisiae* found that increasing the expression level of a gene leads to slower growth for one fifth of all genes [78]. Therefore, cells need to adjust their expression level to the level which is optimal for the current environment. Our goal is to consider a cost function that represents the expected cost of deviating from the optimal expression level in the current environment plus the cost of protein production/decay over one individual life span of the cell. Motivated by this problem, we study the optimal control of Stochastic Hybrid Systems (SHS) with renewal transitions in Chapters 3 and 4 that can be used to compute the optimal gene regulation strategy in fluctuating environments.

The model that we use to represent the gene regulation problem in fluctuating environments is a special case of Piecewise Deterministic Markov (PDM) processes [16] and Stochastic Hybrid Systems (SHS) [26]. SHSs have been frequently used to model gene regulatory networks. For instance, they can be used to model the uncertainties associated with activation/deactivation of a gene in response to the binding/unbinding of proteins to its promoter. By modeling autoregulatory gene networks as a SHS with two discrete states, [60] analyzes the reduction of intrinsic noise caused by the transition of a promoter between its active and inactive states in a genetic network regulated by negative feedback. In [63], this model is extended to a network of N genes. Moreover, SHS models have been shown to be useful for parameter identification and modeling of subtilin production in *Bacillus subtilis* [14] and nutrient stress response in *E. Coli* [13].

2.1 Systems Dynamics

Inspired by [65], we model the gene regulation problem in stochastically varying environments in a general framework. We consider linear dynamical models in every environmental condition where the parameters depend on the current environment.

2.1.1 Dynamics of a Simple Gene Regulation

Cells living in complex environments can sense a variety of signals. They monitor their environment through such signals and respond to environmental changes by producing appropriate proteins. The rate of protein production is determined by transcription regulatory networks composed of genes that code for special proteins called *transcription factors* [4]. Active transcription factors bind into the promoter region of the DNA and can cause an increase or decrease of the rate at which the target genes are transcribed. The genes are transcribed into mRNA which is then translated into protein, see Figure 2.1. The environmental conditions, mediated through cellular processes, alter the conformation of the

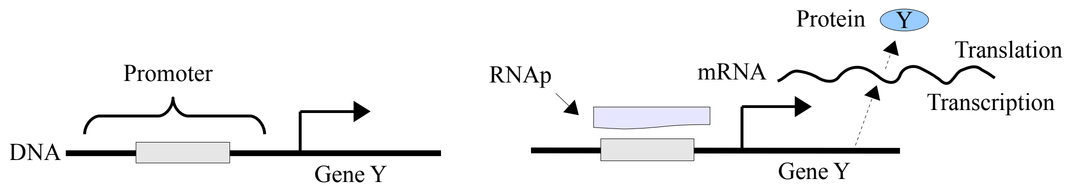


Figure 2.1: Gene expression: process by which information from a gene is used in the synthesis of a protein. Major steps in the gene expression are transcription, RNA splicing, translation, and post-translational modification of a protein.

transcription factors in a way that affects their binding affinities. It is these changes in the transcription factor proteins that regulate the expression of the target gene, creating positive or negative feedback loops.

Gene networks have been described using a variety of modelling approaches. One simplification is to consider ordinary differential equations (ODEs). ODEs can be used to describe the time course of gene product concentrations. We focus on the dynamics of a single gene that is regulated by a single transcription factor. This transcription interaction can be described by Y $\rightarrow$ X which reads "transcription factor Y regulates gene X". Once the transcription factor Y activates the gene X, it begins to be transcribed, the mRNA is translated, and this results in the accumulation of protein X. We assume that the rate of protein production is denoted by u (in units of concentration per unit of time).

The process of protein production is balanced by two additional processes: protein degradation (protein destruction by specialized proteins in the cell) and dilution (due to increase of the cell volume during growth). We denote the total degradation/dilution rate by μ which is the sum of the degradation rate μ_{deg} and the dilution rate μ_{dil},

$$\mu = \mu_{deg} + \mu_{dil}.$$

Thus, the change of concentration of X can be described by the dynamic equation

$$\frac{dx}{dt} = u - \mu x$$

where x describes the protein concentration.

2.1.2 Gene Regulation in Fluctuating Environments

We consider a cell encountering a series of environmental conditions and our goal is to understand what the optimal gene regulation strategy is as the environments fluctuates. Let us start by assuming that the cell encounters two different environmental conditions: environment 0 favors low concentration of protein while environment 1 favors high concentration. These conditions may represent physical parameters such as temperature or osmotic pressure, signaling molecules from other cells, beneficial nutrients, or harmful chemicals. The random environmental shifts can be modeled by exponential waiting times with parameters λ_i (Chapter 3) for which the history does not influence the future states or by more general probability distribution functions (Chapter 4). Given this definition, for the case

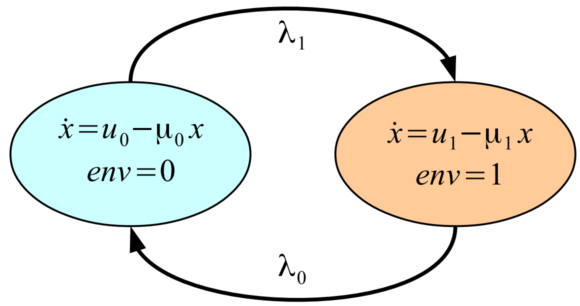


Figure 2.2: Gene regulation in stochastically varying environments.

of exponential waiting time, the expected waiting that the environment stays in state i is $\frac{1}{\lambda_{1-i}}$ for $i \in \{0, 1\}$.

We start by considering a scenario where the optimal concentration of the protein X depends on the current environment, denoted by $\mathbf{env}(t) \in \{0, 1\}$, and the degradation rate is constant. The evolution of protein concentration $\mathbf{x}(t)$ can be modelled by

$$\frac{d\mathbf{x}}{dt} = \mathbf{u}_{\mathbf{env}} - \mu\mathbf{x}(t) \tag{2.1}$$

where $\mathbf{u}_i$ is the rate of transcription in environment $i \in \{0, 1\}$ and μ is the protein degradation/dilution rate. Figure 2.3 shows a sample path of the resulting stochastic system due to changing environments.

Let us consider a simple evolutionary scenario. We assume that the optimal concentration levels of the protein X are 0 and 1 in environments 0 and 1, respectively. At each point in time, we assume that cost of deviation of the protein

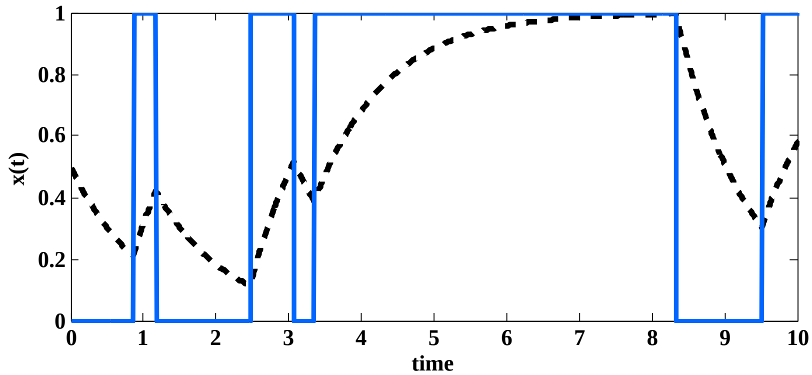


Figure 2.3: A sample path over one individual's life span. The solid line illustrates how the environment changes stochastically while the trajectory of the protein concentration $\mathbf{x}(t)$ over one sample path is depicted by the dashed line.

level from the optimal level in the current environment is a quadratic function of the difference between these values. This cost can be written as $(\mathbf{x}(t) - \mathbf{env}(t))^2$, since we assumed that the optimal protein levels are 0 and 1 in environments 0 and 1, respectively.

We also consider a term in the cost function that reflects energetic costs of producing/decaying mRNA and proteins [75]. This cost may be written as a quadratic function of the current transcription rate $\mathbf{u}(t)$, resulting in a total cost that is given by $(\mathbf{x} - \mathbf{env})^2 + \gamma \mathbf{u}^2$ and defines the penalty in environment $\mathbf{env}$ associated with the protein concentration $\mathbf{x}$ plus the cost of instantaneous protein production/decay. The parameter γ determines the tradeoff between keeping $\mathbf{x}(t)$

close to its ideal value $\mathbf{env}(t)$ and not "wasting" resources in the protein production/decay. One can also consider the case in which γ is environment-dependent.

We assume that organisms die at a rate independent of the strategy they use to regulate gene expression. If the life span ($\mathbf{T}_c$) of a cell is modelled by an exponential random variable with mean $1/\rho$, the probability that an organism is still alive at age t is given by $P(\mathbf{T}_c > t) = e^{-\rho t}$. This assumption is consistent with the experimental data in [69, 64] and [17]. One can show that the total **expected** lifetime cost of an individual is proportional to

$$\int_0^\infty e^{-\rho t}\Big(\big(\mathbf{x}(t) - \mathbf{env}(t)\big)^2 + \gamma \mathbf{u}(t)^2\Big)\ dt. \tag{2.2}$$

Equation (2.2) provides the cost associated with a specific realization of the stochastic process $\mathbf{env}(t)$ that models environmental changes. Since an individual cannot "guess" the future evolution of $\mathbf{env}(t)$, its best bet is to minimize the expected value of such cost, given the current environment and concentration of $\mathbf{x}$

$$J = E_{z_0}\left\{\int_0^\infty e^{-\rho t}\left((\mathbf{x}(t) - \mathbf{env}(t))^2 + \gamma \mathbf{u}(t)^2\right)\ dt\right\} \tag{2.3}$$

conditioned upon the initial condition $z_0 = (\mathbf{x}(0), \mathbf{env}(0))$.

One can also interpret (2.2) by considering a "killed process" $\bar{\mathbf{x}}$ that is equal to $\mathbf{x}$ as long as the cell is alive and $\bar{\mathbf{x}} = \mathbf{env}$ after the organism is dead (which

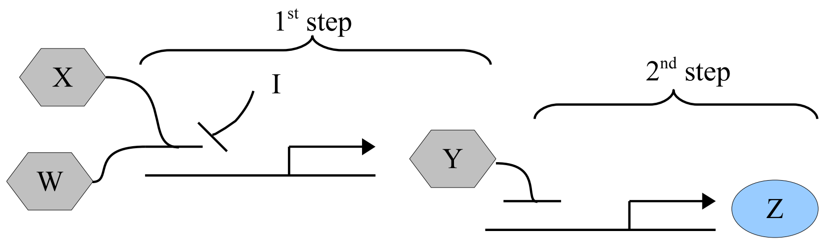


Figure 2.4: An example of multiple-step gene expression process with an arbitrary number of environmental conditions.

generated no further cost with the control $\mathbf{u} = 0$), the total lifetime cost of the killed process is

$$\tilde{J} = E_{z_0}\left\{\int_0^{\infty} (\bar{\mathbf{x}}(t) - \mathbf{env}(t))^2 + \gamma \mathbf{u}(t)^2 \; dt\right\}.$$

One can show that the killed process generates the same cost as (2.3), i.e. $\tilde{J} = J$, see [16, Chapter 3].

2.1.3 Generalization

We now generalize the system described above by considering a multiple-step gene expression process (Figure 2.4) with an arbitrary number of environmental conditions. This can be used to model the multiple-step process in gene production (e.g., the transcription-translation process DNA $\rightarrow$ mRNA $\rightarrow$ protein) and also regulation based on multiple transcription factors.

We model the process of switching between environments by a continuous-time Markov chain $\mathbf{q}(t)$ taking values in the set $\mathcal{S} = \{1, 2, ..., N\}$ with transition rate matrix $\mathbf{P} := \{\lambda_{ij}\}$ where

$$P(\mathbf{q}(t+dt) = j \mid \mathbf{q}(t) = i) = \lambda_{ij} dt + O(dt) \qquad i \neq j. \tag{2.4}$$

Here, $\lambda_{ij} \geq 0$ $(i \neq j)$ is the rate of departing from state i to state j and

$$\lambda_{ii} = -\sum_{j=1,\ j\neq i}^{N} \lambda_{ij}.$$

The different values of $\mathbf{q}(t)$ correspond to distinct linear dynamics according to the following model:

$$\dot{\mathbf{x}} = A_{\mathbf{q}}\mathbf{x} + B_{\mathbf{q}}\mathbf{u} + d_{\mathbf{q}} \tag{2.5}$$

where $\mathbf{x}$ denotes a stochastic process in $\mathbb{R}^n$, $\mathbf{q}$ denotes the current environmental condition, $\mathbf{u} \in \mathbb{R}^m$ an input to be optimized, and $d_{\mathbf{q}}$ is an $\mathbf{q}$-dependent bias term. The affine term $d_{\mathbf{q}}$ in the dynamics is needed for environments that create or consume $\mathbf{x}$ at a fixed rate without control cost. Our goal is to compute the optimal control input $\mathbf{u}$ that minimizes an infinite-horizon discounted criteria of the following form

$$J = E_{z_0}\left\{\int_0^\infty e^{-\rho t}\left((\mathbf{x}-\bar{x}_{\mathbf{q}})' Q_{\mathbf{q}} (\mathbf{x}-\bar{x}_{\mathbf{q}}) + (\mathbf{u}-\bar{u}_{\mathbf{q}})' R_{\mathbf{q}} (\mathbf{u}-\bar{u}_{\mathbf{q}})\right)\ dt\right\} \tag{2.6}$$

by means of a feedback policy that computes $\mathbf{u} = \mu(\mathbf{x}, \mathbf{q})$ where all the Q_i and R_i are positive definite matrices. We will derive the optimal control policy that minimizes the discounted criteria (4.9) in the following chapters.

In the remainder of this chapter, we consider a real gene regulatory network that can be modeled by (2.5). We will return to this example at the end of Chapter 4 where we derive the optimal gene regulator for stochastically varying environments with $\beta-$distributed waiting times.

2.2 Example: Metabolism of Lactose in E. Coli

As discussed before, living organisms respond to changes in their surroundings by sensing the environmental context and by orchestrating the expression of sets of genes to utilize available resources and to survive stressful conditions [48]. As an example, we consider a model for the *lac* operon regulatory network in E. Coli bacterium. E. Coli regulates the expression of many of its genes according to the food sources that are available to it. In the absence of lactose, the Lac repressor in E. Coli binds to the operator region and keeps it from transcribing the *lac* genes. If the bacteria expressed lac genes when lactose was not present, there would likely be an energetic cost of producing an enzyme that was not in use. However, when lactose is available, the lac genes are expressed because allolactose binds to the Lac repressor protein and keeps it from binding to the lac operator. As a result of this change, the repressor can no longer bind to the operator region and falls off. RNA polymerase can then bind to the promoter and transcribe the lac genes. Therefore, depending on the presence or absence of lactose, E. Coli must detect when a specific protein is necessary to produce. This will be studied in more details in Section 4.4.

3

Quadratic Control of Markovian Jump Linear Systems

Hybrid Systems have been the topic of intense research in recent years. Such systems combine continuous dynamics and discrete logic. By introducing randomness in the execution of a hybrid system, one obtains Stochastic Hybrid Systems (SHSs). As surveyed in [54, 44], various models of stochastic hybrid systems have been proposed differing on where randomness comes into play. In most of the models mentioned in these surveys, the solutions are assumed to be unique, however some researchers have recently proposed modelling tools for a class of uncertain hybrid systems with not necessarily unique solutions [70]. Markov Jump Linear (MJL) systems, can be viewed as a special class of stochastic hybrid systems that has been studied in the control community for the past few years. One can trace the applicability of MJL systems to a variety of processes that involve abrupt changes in their structures (e.g. chemical plants, robotic manipulator systems, solar thermal receiver, biological systems, paper mills, etc. [15]).

In Chapter 2, we modelled the gene regulation problem in stochastically varying environments as a Markov Jump system. We considered linear dynamical models in every environmental condition where the parameters depend on the current environment. Motivated by this problem, we derive an optimal controller that minimizes a discounted infinite horizon LQR problem with switching equilibria. We also derive a necessary and sufficient condition for the existence of the optimal control, which can be expressed in terms of a system of Linear Matrix Inequalities (LMIs). The material in this chapter covers the case of exponential waiting times and is based on [48].

When we apply the optimal control results to the computation of optimal gene regulatory responses in variable environments, we conclude that the optimal rate of protein production is affine with respect to the current protein level, which turns out to be consistent with the the biologically meaningful model for protein degradation considered in [1]. Our results also show that the optimal control in a variable environment switches between several (affine) feedback laws, one for each environment. However, the feedback law that corresponds to each environment would typically not be optimal for that specific environment, if the environment was static. The implication of this fact is that an organism that evolved toward optimality in a variable environment will generally not be optimal in a static environment that resembles one of the states of its variable environment. Intuitively, this is because the individual will always be trying to anticipate a change that is never realized. This will be illustrated through a numerical example in Section 3.3.

This chapter is organized as follows. In Section 3.1, we define the Stochastic Stabilizability concept. In Section 3.2, the optimal control strategy for Markov Jump Linear Systems is derived, and we establish a necessary and sufficient condition for the existence of solution in terms of LMIs. Section 3.3 provides a case study and we conclude the chapter in Section 3.4 with some final concluding remarks.

3.1 Stochastic Stabilizability

Our goal is to compute the optimal control input $\mathbf{u}(t)$ that minimizes an infinite-horizon discounted criteria 2.6 by means of a feedback policy that computes

$$\mathbf{u} = \mu(\mathbf{x}, \mathbf{q}) \tag{3.1}$$

where μ is a deterministic state feedback law. Note that (2.6) is conditioned upon the initial condition $z_0 = (x_0, q_0) = \big(\mathbf{x}(0), \mathbf{q}(0)\big)$. Toward this goal, we shall provide a necessary and sufficient condition for the existence of solution, which requires the notion of stochastic stabilizability that we have adapted from [36].

Consider a Markov Jump Linear (MJL) system given by (2.4)-(2.5) and let $\mathbf{x}(t; x_0, q_0)$ denote the trajectory of the process starting from initial condition $z_0 = (\mathbf{x}(t_0), \mathbf{q}(t_0)) = (x_0, q_0)$, and under the feedback control (3.1). The system is *Stochastically Stabilizable* (SS) if there exist a symmetric matrix M and a set of linear gains $\{L_i : i \in \mathcal{S}\}$ such that the solution of (2.4)-(2.5) with $d_i = 0$ and $\mathbf{u}(t) = -L_{\mathbf{q}(t)}\mathbf{x}(t)$ satisfies

$$\lim_{T_f \to \infty} E_{z_0} \left\{ \int_0^{T_f} \mathbf{x}(t)'\mathbf{x}(t)\ dt \right\} \leq x_0' M x_0 \tag{3.2}$$

for all finite $x_0 \in \mathbb{R}^n$ and $q_0 \in \mathcal{S}$. Essentially, stochastic stabilizability of a system is equivalent to the existence of a set of linear feedback gains that make the state mean-square integrable when $d_i = 0, \ \ \forall i \in \mathcal{S}$. The next result from [36, Theorem 1] provides a necessary and sufficient condition for stochastic stabilizability of MJL systems.

Theorem 3.1.1. *The system (2.4)-(2.5) is stochastically stabilizable if and only if there exists a set of matrices $\{L_i : i \in \mathcal{S}\}$ such that for every set of positive definite symmetric matrices $\{N_i : i \in \mathcal{S}\}$, the symmetric solutions $\{M_i : i \in \mathcal{S}\}$*

of the coupled equations

$$(A_i - B_i L_i)' M_i + M_i (A_i - B_i L_i) + \sum_{j=1}^{N} \lambda_{ij} M_j = -N_i \tag{3.3}$$

are positive definite for all $i \in \mathcal{S}$.

3.2 Jump Linear Quadratic Regulator

In the following theorem, we compute the optimal control policy $\mu^*(x, q)$ that minimizes the infinite-horizon discounted criteria (2.6).

Theorem 3.2.1. *Consider the following optimization problem*

$$\min_{\mu} J \qquad \textit{subject to} \qquad \dot{\mathbf{x}} = A_{\mathbf{q}}\mathbf{x} + B_{\mathbf{q}}\mu(\mathbf{x}, \mathbf{q}) + d_{\mathbf{q}} \tag{3.4}$$

with J *given by (2.6). If there exists a solution* $\Lambda_i \in \mathbb{R}^{n\times n}$, $\Gamma_i \in \mathbb{R}^n$, $\Omega_i \in \mathbb{R}$, $i \in \mathcal{S}$ *to the following set of equations*

$$A_i'\Lambda_i + \Lambda_i A_i - \rho\Lambda_i - \Lambda_i B_i R_i^{-1} B_i' \Lambda_i + Q_i + \sum_{j=1}^{N} \lambda_{ij}\Lambda_j = 0 \tag{3.5}$$

$$(A_i' - \Lambda_i B_i R_i^{-1} B_i' - \rho I)\Gamma_i + 2\Lambda_i (B_i \bar{u}_i + d_i) + \sum_{j=1}^{N} \lambda_{ij}\Gamma_j = 2Q_i\bar{x}_i \tag{3.6}$$

$$-\frac{1}{4}\Gamma_i' B_i R_i^{-1} B_i' \Gamma_i + \Gamma_i'(B_i\bar{u}_i + d_i) - \rho\Omega_i + \sum_{j=1}^{N} \lambda_{ij}\Omega_j + \bar{x}_i' Q_i \bar{x}_i = 0, \tag{3.7}$$

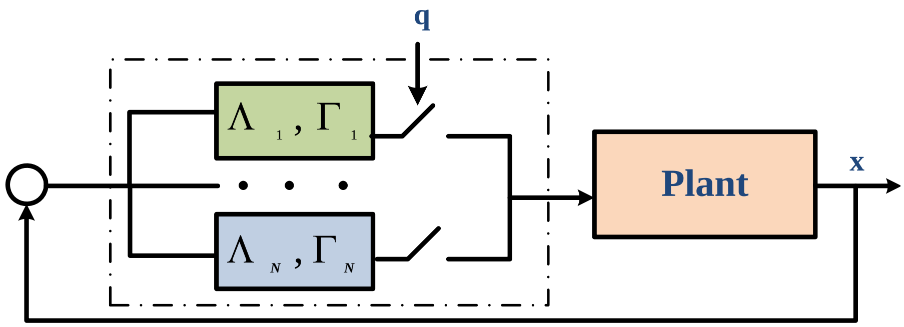


Figure 3.1: Structure of the Jump Linear Quadratic Regulator.

then the minimal cost for $\mathbf{x}(0) = x_0, \mathbf{q}(0) = q_0$ *is given by* $J^* = x_0'\Lambda_{q_0}x_0 + x_0'\Gamma_{q_0} + \Omega_{q_0}$ *and the optimal control is given by*

$$\mu^*(\mathbf{x}, \mathbf{q}) := \bar{u}_{\mathbf{q}} - \frac{1}{2}R_{\mathbf{q}}^{-1}B_{\mathbf{q}}'(2\Lambda_{\mathbf{q}}\mathbf{x} + \Gamma_{\mathbf{q}}). \tag{3.8}$$

Theorem 3.2.1 states that the optimal way of using $\mathbf{x}(t)$ and $\mathbf{q}(t)$ is to feed them back in the control law (3.8) as shown in Figure 3.1.

Proof of Theorem 3.2.1. Let us introduce the value function as $V(x_0, q_0) = \min_{\mu} J$ conditioned on $\mathbf{x}(0) = x_0,\ \mathbf{q}(t_0) = q_0$. From [16], the Hamilton-Jacobi-Bellman (HJB) equation for this problem is given by

$$0 = \min_{u}\{\mathcal{L}V(x,i) - \rho V(x,i) + (x - \bar{x}_i)'Q_i(x - \bar{x}_i) + (u - \bar{u}_i)'R_i(u - \bar{u}_i)\} \tag{3.9}$$

where $\mathcal{L}V$ denotes the extended generator of the Markov pair $\{\mathbf{q}(t), \mathbf{x}(t)\}$, see [26]. The minimization in (3.9) can be done explicitly, leading to the optimal feedback

$$u^* = \bar{u}_i - \frac{1}{2}R_i^{-1}B_i'(\frac{\partial V}{\partial x})',$$

that can be replaced in (3.9). Using (3.5)-(3.7), it is straightforward to verify that $V(x,i) = x'\Lambda_i x + x'\Gamma_i + \Omega_i$ is a piecewise continuous solution to (3.9), since

$$\begin{aligned} 0 =& \frac{\partial V}{\partial x}(A_i x + B_i u^* + d_i) + \sum_{j=1}^{N} \lambda_{ij}(x'\Lambda_j x + x'\Gamma_j + \Omega_j) \\ &- \rho(x'\Lambda_i x + x'\Gamma_i + \Omega_i) + (x - \bar{x}_i)' Q_i (x - \bar{x}_i) \\ &+ (u^* - \bar{u}_i)' R_i (u^* - \bar{u}_i). \end{aligned}$$

Thus, by [16, 42.8], V and $u^* = \mu^*(x, q)$ are optimal which completes the proof.

$\square$

Next, a necessary and sufficient condition for the existence of the optimal regulator will be stated in terms of stochastic stabilizability of the system. We show that under a stochastic stabilizability assumption, the optimal control policy leads to a finite cost for which one can compute a finite upper bound on J. The main result of this section is stated in the following theorem.

Theorem 3.2.2. *Consider the system (2.4)-(2.5) and (2.6) and assume that $\rho > -\lambda_{ii}$ for all $i \in \mathcal{S}$. When the system is stochastically stabilizable, the minimum cost is finite, the equations (3.5)-(3.7) have solutions, and the control policy (3.8) is optimal. Conversely, if for some linear policy the cost (2.6) is bounded then the system is stochastically stabilizable.*

Proof of Theorem 3.2.2. We start by proving the first part of the theorem by showing that Stochastic Stabilizability results in a finite optimal cost. Then, we show that there exists a solution to (3.5)-(3.7) and therefore the optimality of (3.8) follows from Theorem 3.2.1.

Due to the stochastic stabilizability assumption (Theorem 3.1.1), there exists a set of gains $\{L_i\}$ such that for any set of matrices $\{\tilde{N}_i > 0\}$, the corresponding solutions $\{\tilde{M}_i\}$ in (3.3) are positive definite. In what follows, we show that choosing the control $\mathbf{u}(t) = -L_{\mathbf{q}}\mathbf{x}(t)$ (which is not necessarily optimal) results in a finite cost.

We take matrices N_i in (3.3) to be $N_i = Q_i + L_i' R_i L_i > 0$ and M_i to be the corresponding positive definite solutions. Given $\mathbf{x}(0) = x_0$ and $\mathbf{q}(0) = q_0$, one can compute the cost of applying this control policy using

$$\begin{aligned} J &= E_{z_0}\left\{\int_0^\infty e^{-\rho t}\left((\mathbf{x}-\bar{x}_{\mathbf{q}})'Q_{\mathbf{q}}(\mathbf{x}-\bar{x}_{\mathbf{q}}) + (u-\bar{u}_{\mathbf{q}})'R_{\mathbf{q}}(u-\bar{u}_{\mathbf{q}})\right)\, dt\right\} \\ &= E_{z_0}\left\{\int_0^\infty e^{-\rho t}\left(\mathbf{x}'N_{\mathbf{q}}\mathbf{x} - 2\mathbf{x}'(Q_{\mathbf{q}}\bar{x}_{\mathbf{q}} + L_{\mathbf{q}}'R_{\mathbf{q}}\bar{u}_{\mathbf{q}}) + \bar{u}_{\mathbf{q}}'R_{\mathbf{q}}\bar{u}_{\mathbf{q}} + \bar{x}_{\mathbf{q}}'Q_{\mathbf{q}}\bar{x}_{\mathbf{q}}\right)\, dt\right\}. \end{aligned} \tag{3.10}$$

Defining $W(\mathbf{x},\mathbf{q}) = \mathbf{x}'M_{\mathbf{q}}\mathbf{x}$ and applying the extended generator of the stochastic system (2.5)-(2.4), see [26], we obtain $\mathcal{L}W(\mathbf{x},\mathbf{q}) = -\mathbf{x}'N_{\mathbf{q}}\mathbf{x}$. So one can show that

$$\frac{\mathcal{L}W}{W} = -\frac{\mathbf{x}'N_{\mathbf{q}}\mathbf{x}}{\mathbf{x}'M_{\mathbf{q}}\mathbf{x}} \le -\alpha \qquad \alpha := \min_i \frac{\mu_{\min}(N_i)}{\mu_{\max}(M_i)} \qquad \mathbf{q}\in\mathcal{S}$$

where α is positive. So $\mathcal{L}W \le -\alpha W$ and by the Gronwall-Bellman lemma [40]

$$E_{z_0}\{W(\mathbf{x},\mathbf{q})\} \le e^{-\alpha t}W(x_0,q_0).$$

Thus, one can conclude that

$$E_{z_0}\left\{\int_0^{T_f} \mathbf{x}' M_{\mathbf{q}} \mathbf{x}\ dt\right\} \le \left(\int_0^{T_f} e^{-\alpha t}\ dt\right) x_0' M_{q_0} x_0.$$

Lebesgue's Dominated Convergence Theorem in [58] justifies the existence of the limit as $T_f \to \infty$ and we have

$$E_{z_0}\left\{\int_0^{\infty} \mathbf{x}' M_{\mathbf{q}} \mathbf{x}\ dt\right\} = \lim_{T_f\to\infty} E_{z_0}\left\{\int_0^{T_f} \mathbf{x}' M_{\mathbf{q}} \mathbf{x}\ dt\right\} \le \frac{1}{\alpha} x_0' M_{q_0} x_0.$$

We can bound the integral (3.10) which can be written as

$$\begin{aligned} J = {} & E_{z_0}\left\{\int_0^{\infty} e^{-\rho t}(\bar{u}_{\mathbf{q}}' R_{\mathbf{q}} \bar{u}_{\mathbf{q}} + \bar{x}_{\mathbf{q}}' Q_{\mathbf{q}} \bar{x}_{\mathbf{q}})\ dt\right\} + E_{z_0}\left\{\int_0^{\infty} e^{-\rho t}(\mathbf{x}' N_{\mathbf{q}} \mathbf{x})\ dt\right\} \\ & - 2E_{z_0}\left\{\int_0^{\infty} e^{-\rho t}\mathbf{x}'(Q_{\mathbf{q}}\bar{x}_{\mathbf{q}} + L_{\mathbf{r}}' R_{\mathbf{q}} \bar{u}_{\mathbf{q}})\ dt\right\}. \end{aligned} \tag{3.11}$$

Since $\mathcal{S}$ is finite, the first term in (3.11) can be bounded by

$$E_{z_0}\left\{\int_0^{\infty} e^{-\rho t}(\bar{u}_{\mathbf{q}}' R_{\mathbf{q}} \bar{u}_{\mathbf{q}} + \bar{x}_{\mathbf{q}}' Q_{\mathbf{q}} \bar{x}_{\mathbf{q}})\ dt\right\} \le \frac{1}{\rho} \max_{i\in\mathcal{S}}\ (\bar{u}_i' R_i \bar{u}_i + \bar{x}_i' Q_i \bar{x}_i).$$

For the second integral in (3.11), we have

$$\begin{aligned} E_{z_0}\left\{\int_0^{\infty} e^{-\rho t}\mathbf{x}' N_{\mathbf{q}} \mathbf{x}\ dt\right\} & \le \frac{\max_i \mu_{\max}(N_i)}{\min_i \mu_{\min}(M_i)} E_{z_0}\left\{\int_0^{\infty} e^{-\rho t}\mathbf{x}' M_{\mathbf{q}} \mathbf{x}\ dt\right\} \\ & \le \frac{\max_i \mu_{\max}(N_i)}{\min_i \mu_{\min}(M_i)} \cdot \frac{1}{\alpha} x_0' M_{q_0} x_0 \end{aligned}$$

and the third one can be bounded by[1]

$$E_{z_0}\left\{\int_0^\infty e^{-\rho t}\mathbf{x}'(Q_{\mathbf{q}}\bar{x}_{\mathbf{q}} + L'_{\mathbf{q}}R_{\mathbf{q}}\bar{u}_{\mathbf{q}})\ dt\right\}$$
$$\leq \max_i\ |Q_i\bar{x}_i + L'_iR_i\bar{u}_i|E_{z_0}\left\{\int_0^\infty e^{-\rho t}|\mathbf{x}|\ dt\right\}.$$

Defining $\kappa := \max_i |Q_i\bar{x}_i + L'_iR_i\bar{u}_i|$, and using the Cauchy Schwarz inequality for square integrable functions

$$\kappa E_{z_0}\left\{\int_0^\infty e^{-\rho t}|\mathbf{x}|\ dt\right\} \leq \kappa E_{z_0}\left\{\sqrt{\int_0^\infty e^{-2\rho t}\ dt\int_0^\infty |\mathbf{x}|^2\ dt}\right\}$$
$$= \frac{\kappa}{\sqrt{2\rho}}E_{z_0}\left\{\sqrt{\int_0^\infty |\mathbf{x}|^2\ dt}\right\}$$
$$\leq \frac{\kappa}{\sqrt{2\rho\min_i \mu_{\min}(M_i)}}E_{z_0}\left\{\sqrt{\int_0^\infty \mathbf{x}'M_{\mathbf{q}}\mathbf{x}\ dt}\right\}.$$

Note that, by the Cauchy Schwarz inequality, one can show that $E\{\mathbf{y}\} \leq \sqrt{E\{\mathbf{y}^2\}}$, so

$$\frac{\kappa}{\sqrt{2\rho\min_i \mu_{\min}(M_i)}}E_{z_0}\left\{\sqrt{\int_0^\infty \mathbf{x}'M_{\mathbf{q}}\mathbf{x}\ dt}\right\}$$
$$\leq \frac{\kappa}{\sqrt{2\rho\min_i \mu_{\min}(M_i)}}\sqrt{E_{z_0}\left\{\int_0^\infty \mathbf{x}'M_{\mathbf{q}}\mathbf{x}\ dt\right\}}$$
$$\leq \frac{\kappa}{\sqrt{2\rho\min_i \mu_{\min}(M_i)}}\cdot\sqrt{\frac{1}{\alpha}x_0'M_{q_0}x_0},$$

therefore the cost is bounded. This finite quantity (resulting from a not necessarily optimal control) is an upper bound for the optimal cost to go.

[1] We use the Cauchy Schwarz inequality: $|E\{XY\}|^2 \leq E\{X^2\}E\{Y^2\}$ and also $|\int fg\ dx|^2 \leq \int |f|^2\ dx \cdot \int |g|^2\ dx$.

We now show that (3.5)-(3.7) has a solution and therefore the optimality of (3.8) follows from Theorem 3.2.1. Due to the Stochastic Stabilizability assumption, one can guarantee the existence of a set of positive solutions Λ_i to (3.5) [36]. From (3.5), it is straightforward to show that $A_i - B_i R_i^{-1} B_i' \Lambda_i + (\lambda_{ii} - \rho)/2 \; I$ is Hurwitz. Let us define

$$k := \min_{i \in \mathcal{S}} \left| \text{Real} \left\{ \text{eig} \left(A_i - B_i R_i^{-1} B_i' \Lambda_i + \frac{1}{2} (\lambda_{ii} - \rho) \; I \right) \right\} \right|,$$

therefore $(A_i - B_i R_i^{-1} B_i' \Lambda_i + (\lambda_{ii} - \rho + k)/2 \; I)$ is Hurwitz. Since, by assumption $\rho > -\lambda_{ii}$, one can conclude that $(A_i - B_i R_i^{-1} B_i' \Lambda_i + (k/2 - \rho) \; I)$ is a stable matrix. Moreover, knowing Λ_i, (3.6) turns out to be a system of linear equations in Γ_i. Stacking all the entries of the matrix Γ_i in a tall column vector $z \in \mathbb{R}^{n^2}$, we can write (3.6) as $Mz = w$ for an appropriately defined vector $w \in \mathbb{R}^{n^2}$ and with the coefficient matrix M defined as

$$M = \left(\mathbf{P} - \frac{k}{2} I \right) \otimes I_n + \text{diag} \left(A_i' - \Lambda_i B_i R_i^{-1} B_i' + (\frac{k}{2} - \rho) I \right).$$

By the results of [12], the eigenvalues of the transition rate matrix $\mathbf{P}$ are zero or negative therefore $(\mathbf{P} - \frac{k}{2} I) \otimes I_n$ is also Hurwitz. Thus, the system of linear equations (3.6) has a full rank coefficient matrix and has a unique solution. Similarly, knowing the solution of (3.5)-(3.6), (3.7) turns out to be a system of linear equations in Ω_i with the coefficient matrix $\mathbf{P} - \rho I$. Since all the eigenvalues of

$\mathbf{P} - \rho I$ have negative real parts, the coefficient matrix is full rank and (3.7) has a unique solution.

To prove the second part of the Theorem, suppose that the system is not stochastically stabilizable. So there is no linear feedback law that can result in a finite value for (3.2), and this contradicts the existence of a finite cost for a linear policy. □

Theorem 3.2.2 provides a necessary and sufficient condition for the existence of the optimal solution in terms of stochastic stabilizability property. However, for a given set of matrices $\{N_i\}$, the matrix equality (3.3) is bilinear in the unknowns $\{L_i\}$, $\{M_i\}$ and therefore it is not easy to verify if it holds. The following result provides a system of linear matrix inequalities (LMIs) that can be equivalently used to check stochastic stabilizability. Checking feasibility of these LMIs corresponds to a convex optimization problem that can be solved efficiently. There are many software packages that solve LMIs. CVX [25] in particular, is a MATLAB-based package for convex optimization that solves LMIs in a convenient way.

Lemma 3.2.1. *The following statements are equivalent.*

A) The system (2.4)-(2.5) is stochastically stabilizable.

B) There exist sets of matrices $\{L_i\}$ *and* $\{M_i = M_i' > 0\}$ *such that the following Bilinear Matrix Inequality (BMI) holds*

$$(A_i - B_iL_i)'M_i + M_i(A_i - B_iL_i) + \sum_{j=1}^{N} \lambda_{ij}M_j < 0. \tag{3.12}$$

C) There exist sets of matrices $\{P_i\}$ *and* $\{Q_i = Q_i' > 0\}$ *such that the LMI condition*

$$\begin{bmatrix} \begin{matrix}(A_i + \frac{\lambda_{ii}}{2}I)Q_i + Q_i(A_1 + \frac{\lambda_{ii}}{2}I)' \\ -P_i'B_i' - B_iP_i\end{matrix} & Q_i & \dots & Q_i \\ Q_i & -\frac{Q_{j_1}}{\lambda_{ij_1}} & \dots & 0 \\ . & . & . & . \\ . & . & . & . \\ . & . & . & . \\ Q_i & 0 & \dots & -\frac{Q_{j_{N-1}}}{\lambda_{ij_{N-1}}} \end{bmatrix} < 0. \tag{3.13}$$

holds for $\forall i \in \mathcal{S}$ *and* $j_k \in \mathcal{S}\backslash\{i\}$*.*

Moreover, the matrices in (B) and (C) are related by $Q_i = M_i^{-1}$ *and* $P_i = L_iQ_i$*.*

Proof of Lemma 3.2.1. We start by showing that (A) and (B) are equivalent. If the system is stochastically stabilizable, it follows from Theorem 3.1.1 that there exist matrices $\{L_i\}$ such that for any set of positive definite matrices $\{N_i\}$, the solution $\{M_i\}$ to (3.3) are positive definite. By selecting $\{N_i = I\}$ in (3.3), we conclude that (3.12) holds, which proves that stochastic stabilizability is a sufficient condition for (3.12) to hold. To prove necessity, let us assume that the $\{L_i\}$, $\{M_i\}$ are such that for some $\{N_i\}$ we have $(A_i - B_i L_i)' M_i + M_i (A_i - B_i L_i) + \sum_{j=1}^{N} \lambda_{ij} M_j = -N_i < 0$. Our goal is to show that the system is stochastically stabilizable. Let $V(\mathbf{x}, \mathbf{q}) = \mathbf{x}' M_{\mathbf{q}} \mathbf{x}$ be the stochastic Lyapunov function for the system where $\{M_i : i \in \mathcal{S}\}$ satisfy (3.12). Applying the results in [26] to the generator of stochastic hybrid systems, one can compute the time derivative of the expected value of V along the solutions of (2.4)-(2.5). Given any $\mathbf{x}(0) = x_0,\ \mathbf{q}(0) = q_0$,

$$\frac{d}{dt} E_{z_0} \{V(\mathbf{x}, \mathbf{q})\} = E_{z_0} \left\{ \mathbf{x}' (M_{\mathbf{q}} (A_{\mathbf{q}} - B_{\mathbf{r}} L_{\mathbf{q}}) + (A_{\mathbf{q}} - B_{\mathbf{q}} L_{\mathbf{q}})' M_{\mathbf{q}} + \sum_{j=1}^{N} \lambda_{\mathbf{q}j} M_j) \mathbf{x} \right\}$$

Let us define $\alpha := \displaystyle\min_{i \in \mathcal{S}} \frac{\mu_{\min}(N_i)}{\mu_{\max}(M_i)}$ which is a positive number therefore

$$\frac{d}{dt} E_{z_0} \{V(\mathbf{x}, \mathbf{q})\} \leq -\alpha E_{z_0} \{V(\mathbf{x}, \mathbf{q})\}.$$

Using the Gronwall-Bellman lemma [40]

$$E_{z_0} \{V(\mathbf{x}, \mathbf{q})\} \leq e^{-\alpha t} x_0' M_{q_0} x_0.$$

Thus one can conclude

$$E_{z_0} \left\{ \int_0^{T_f} \mathbf{x}(t)' M_{\mathbf{q}} \mathbf{x}(t)\ dt \right\} \leq \left(\int_0^{T_f} e^{-\alpha t}\ dt \right) x_0' M_{q_0} x_0.$$

Lebesgue's Dominated Convergence Theorem in [58] justifies the existence of the limit as $T_f \to \infty$ and we have

$$\lim_{T_f \to \infty} E_{z_0} \left\{ \int_0^{T_f} \mathbf{x}(t)' M_{\mathbf{q}} \mathbf{x}(t) \; dt \right\} \leq x_0' \left(\max_i \frac{M_i}{\alpha ||M_i||} \right) x_0.$$

Therefore, the system is stochastically stabilizable.

We now prove that (B) and (C) are also equivalent. We sketch the proof for $N = 3$ although similar results hold for arbitrarily number of modes. Assume that there exist matrices $\{M_i\}$ and $\{L_i\}$ such that

$$(A_i - B_i L_i)' M_i + M_i (A_i - B_i L_i) + \sum_{j=1}^{N} \lambda_{ij} M_i < 0. \tag{3.14}$$

Define $Q_i := M_i^{-1} > 0$ and $P_i := L_i Q_i$, and multiply both sides of (3.14) by Q_i

$$(A_i + \frac{\lambda_{ii}}{2} I) Q_i + Q_i (A_i + \frac{\lambda_{ii}}{2} I)' - P_i' B_i' - B_i P_i + \lambda_{ij_1} Q_i Q_{j_1}^{-1} Q_i + \lambda_{ij_2} Q_i Q_{j_2}^{-1} Q_1 < 0.$$

Applying the Schur complement [19], one can get

$$\begin{bmatrix} (A_i + \frac{\lambda_{ii}}{2} I) Q_i + Q_i (A_i + \frac{\lambda_{ii}}{2} I)' - P_i' B_i' - B_i P_i & Q_i \\ Q_i & -\lambda_{ij_1}^{-1} Q_{j_1} \end{bmatrix} -$$

$$\begin{bmatrix} Q_i \\ 0 \end{bmatrix} (-\lambda_{ij_2}^{-1} Q_{j_2})^{-1} \begin{bmatrix} Q_i & 0 \end{bmatrix} < 0$$

for $\forall i \in \mathcal{S}, \; j_k \in \mathcal{S} \backslash \{i\}$. By applying the Schur complement again, we get (3.13). Moreover, the proof of necessity follows in a similar fashion. Therefore (B) and (C) are actually equivalent, and this completes the proof. $\square$

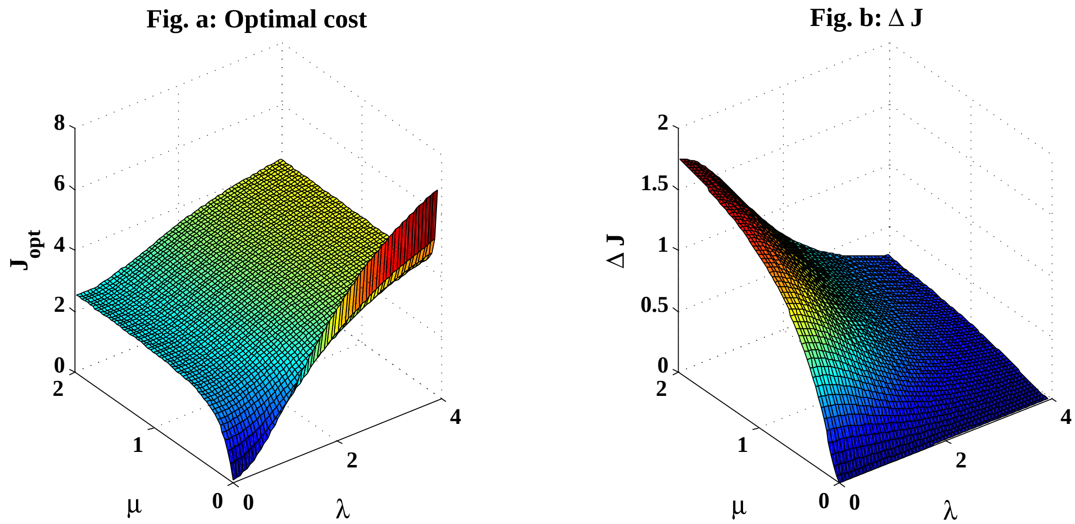


Figure 3.2: Fig. a depicts the cost of using the optimal control (3.8). Fig. b illustrates the additional cost ($\Delta J = J_{\text{nonopt}} - J_{\text{opt}}$) due to the control policy that is obtained by minimizing (2.2) and is optimal for every individual environment when there is no switching. This control results in a larger cost when the environmental switching rate is large, with respect to the protein degradation rate. The system starts from $x_0 = 0.9$ and in environment 1 with $\rho = 0.1$ and $\lambda_0 = \lambda_1 = \lambda$.

3.3 Case Study

We consider the simple gene regulation problem (2.1) with the cost function (2.3). It can be shown that the system (2.1) is stochastically stabilizable for any set of parameters $\{\lambda_0, \lambda_1, \mu\}$ and, using Theorem 3.2.1, one can compute the optimal control (3.8) for this stochastic process.

Let us consider two different scenarios. First, we consider the optimal control policy (3.8) that is obtained for the stochastically varying environment. Second, we compute two policies that are optimal for environments 0 and 1 individu-

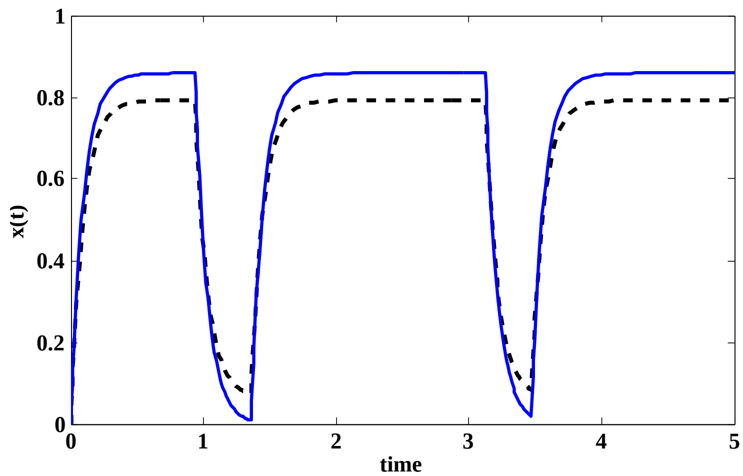


Figure 3.3: Sample paths using the control strategies discussed in Section 3.3. The dashed line corresponds to the optimal controller in fluctuating environment while the solid line is the result of the controller which is optimal in each environment when there is no switching . The system starts from $x_0 = 0$ and in environment 1 with $\rho = 0.1$ and $\lambda_0 = \lambda_1 = 1, \mu = 4$.

ally, assuming that there is no fluctuation in the environment. These policies are obtained by minimizing the cost (2.2) when the probability of changing the environment is zero. If cells were to use these policies when the environment fluctuates, one can show that the cost of applying this control is a quadratic function of the initial protein concentration and depends on the initial environment. Clearly, such cost is always larger than the optimal cost obtained from Theorem 3.2.1.

Figure 3.2 compares the cost of applying the control which is optimal in each environment (if there was no switching) and the optimal control policy (3.8) from Section 2.1.2 that takes into account that the environment changes stochastically.

Figure 3.2.b illustrates that the optimal policy (3.8) results in a much smaller cost when the switching rate of the environment is large, when compared to the degradation rate of the protein. The biological implication of this observation is that an organism that evolved through natural selection in a variable environment is likely to exhibit specialization to the statistics that determine the changes in the environment. Opposite to what one could naively expect, such individual will typically not simply switch between responses that are optimal for the current environment, as if that environment were to remain static forever. Figure 3.3 illustrates sample paths of the system using the two control strategies discussed above. One can see that the controller that is optimal for the changing environment achieves a better cost by being conservative in its response to the environment.

3.3.1 Inference the Environment from Indirect Measurements

In using the result of Section 3.2, we assumed that environmental signal $\mathbf{env}(t)$ is directly and accurately sensed, so the model has direct access to signal $\mathbf{env}(t)$. We now consider an *Indirect Signal* model of the example in Section 3.3. For the indirect signal models, the cell senses the environment through an intermediate process. We adopt a formalism where the signal that the cell receives ($\mathbf{e}_s(t)$) is a continuous variable that becomes closer to the true environmental state as the amount of time in the current environment increases. Mechanistically, this could occur if a signal molecule diffuses into the cell and the external concentration is assumed to be 0 or 1 and the rate of diffusion into the cell is 1. The parameter $\bar{\alpha}$ represents the rate at which the signal molecule is degraded within the cell. The concentration of the environmental signal $\mathbf{e}_s(t)$ follows [57] and is given by

$$\frac{d}{dt}\mathbf{e}_s = -\bar{\alpha}(\mathbf{e}_s(t) - \mathbf{env}(t)). \tag{3.15}$$

Our goal is to compute the probability distribution of the Hidden Markov State $\mathbf{env}(t)$. So, one can replace $\mathbf{env}(t)$ by an estimation of it at the expense of introducing an error.

Given a sequence of observations $O_0, \ldots, O_{t_k}$ of $\mathbf{e}_s$ at discrete times, Forward-Backward algorithm [55] computes the distribution $P(\mathbf{env}(t_k) \mid O_{0:t_k})$ for the hidden Markov state $\mathbf{env}(t_k)$. In the Forward pass, the algorithm computes the probability $P(\mathbf{env}(t_k) \mid O_{0:t_k})$ given the first k observations while the Backward pass computes a set of backward probabilities which provide the probability of observing the remaining observations given any starting point k, i.e. $P(O_{t_{k+1}:t_k} \mid \mathbf{env}(t_k))$. Combining these two steps, one can find the distribution at any time for the given sequence of observations. In our problem, we mainly focus on the Forward pass to find $P(\mathbf{env}(t_k) = 1 \mid \mathbf{e}_s(t_m), m \leq k)$.

Using (2.4), we consider the following approximation at discrete times $t_k = k\Delta t,\ k = 0, 1, 2, \ldots$

$$\mathbf{env}(t_{k+1}) \approx \begin{cases} 1 \ \text{ w.p. } p_1 & \text{when} \quad \mathbf{env}(t_k) = 0 \\ 0 \ \text{ w.p. } 1 - p_1 & \text{when} \quad \mathbf{env}(t_k) = 0 \\ 0 \ \text{ w.p. } p_0 & \text{when} \quad \mathbf{env}(t_k) = 1 \\ 1 \ \text{ w.p. } 1 - p_0 & \text{when} \quad \mathbf{env}(t_k) = 1. \end{cases} \tag{3.16}$$

By selecting the sampling time Δt sufficiently small, one can choose $p_i = \lambda_i \Delta t$, $i \in \{0, 1\}$. Moreover, from discretizing (3.15), we have

$$\mathbf{e}_s(t_{k+1}) = \alpha \mathbf{e}_s(t_k) + (1 - \alpha)\mathbf{env}(t_k) \tag{3.17}$$

where $\alpha = 1 - \bar{\alpha}\Delta t$. We define

$$\boldsymbol{\beta}(t_k) := \frac{\mathbf{e}_s(t_k) - \alpha \mathbf{e}_s(t_{k-1})}{1 - \alpha}.$$

Let the "transition matrix" T denote the probabilities $P(\mathbf{env}(t_k) \mid \mathbf{env}(t_{k-1}))$. The row index in T represents the starting state while the column index represents the target state. Using (3.16), T can be defined as

$$\mathrm{T} := \begin{bmatrix} 1 - p_1 & p_1 \\ p_0 & 1 - p_0 \end{bmatrix}. \tag{3.18}$$

We also define the "event matrix" O. The elements of O are the probabilities of observing an event $\mathbf{env}(t_k)$ given $\boldsymbol{\beta}(t_k)$, i.e

$$\mathrm{O}_{ij} = P\left(\mathbf{env}(t_k) = |1 - i| \mid \boldsymbol{\beta}(t_k) = |1 - j|\right).$$

Given, $\boldsymbol{\beta}(t_k) = i, i \in \{0, 1\}$, we assume that one can estimate the true value of $\mathbf{env}(t_k)$ with probability $1 - p_e^i$. Therefore, O can be written as

$$\boldsymbol{\beta}(t_k) = 0 \Rightarrow \mathrm{O} = \begin{bmatrix} 1 - p_e^0 & 0 \\ 0 & p_e^1 \end{bmatrix},$$

$$\boldsymbol{\beta}(t_k) = 1 \Rightarrow \mathrm{O} = \begin{bmatrix} p_e^0 & 0 \\ 0 & 1 - p_e^1 \end{bmatrix}.$$

Using Baye's rule, one can show that

$$P\left(\mathbf{env}(t_k) = i \mid \boldsymbol{\beta}(t_s); s \leq k\right) = \frac{P\left(\mathbf{env}(t_k) = i,\ \boldsymbol{\beta}(t_s); s \leq k\right)}{P\left(\boldsymbol{\beta}(t_s); s \leq k\right)} := \hat{f}_{0:t_k}(i).$$

One can show that $\hat{f}_{0:t_k}(i)$ is computed by the following recursion:

$$\hat{f}_{0:t_k} = c_k^{-1}\ \hat{f}_{0:t_{k-1}}\mathrm{TO} \tag{3.19}$$

where c_k is chosen such that it normalizes the probability vector at each step so that entries of $\hat{f}_{0:t_k}$ sum to 1. Using (3.19), one can find the probability $\hat{f}_{0:t_k}(1) = P\left(\mathbf{env}(t_k) = i \mid \boldsymbol{\beta}(t_s); s \leq k\right)$ given all the observations upto and including time t_k. Using (3.19), we can write if $\boldsymbol{\beta}(t_k) = 0$,

$$
\begin{aligned}
&\begin{bmatrix} 1 - \hat{f}_{0:t_k}(1) & \hat{f}_{0:t_k}(1) \end{bmatrix} = \\
&\qquad c_k^{-1} \begin{bmatrix} 1 - \hat{f}_{0:t_{k-1}}(1) & \hat{f}_{0:t_{k-1}}(1) \end{bmatrix} \begin{bmatrix} (1-p_1)(1-p_e^0) & p_1 p_e^1 \\ p_0(1-p_e^0) & (1-p_0)p_e^1 \end{bmatrix} \\
&\qquad = c_k^{-1} \begin{bmatrix} (1-\hat{f}_{0:t_{k-1}}(1))(1-p_1)(1-p_e^0) + \hat{f}_{0:t_{k-1}}(1)p_0(1-p_e^0) \\ (1-\hat{f}_{0:t_{k-1}}(1))p_1 p_e^1 + \hat{f}_{0:t_{k-1}}(1)(1-p_0)p_e^1 \end{bmatrix}'.
\end{aligned}
$$

Hence, c_k and $\hat{f}_{0:t_k}(1)$ are given by

$$
\begin{aligned}
c_k &= (1-p_1)(1-p_e^0) + p_1 p_e^1 + \hat{f}_{0:t_{k-1}}(1)(1-p_0-p_1)(p_e^1 + p_e^0 - 1) \\
\hat{f}_{0:t_k}(1) &= c_k^{-1}\left(p_1 p_e^1 + \hat{f}_{0:t_{k-1}}(1)\left((1-p_0)p_e^1 - p_1 p_e^1\right)\right).
\end{aligned}
$$

Else if $\boldsymbol{\beta}(t_k) = 1$,

$$
\begin{aligned}
&\begin{bmatrix} 1 - \hat{f}_{0:t_k}(1) & \hat{f}_{0:t_k}(1) \end{bmatrix} = \\
&\qquad c_k^{-1} [1 - \hat{f}_{0:t_{k-1}}(1) \quad \hat{f}_{0:t_{k-1}}(1)] \begin{bmatrix} (1-p_1)p_e^0 & p_1(1-p_e^1) \\ p_0 p_e^0 & (1-p_0)(1-p_e^1) \end{bmatrix} \\
&\qquad = c_k^{-1} \begin{bmatrix} (1-\hat{f}_{0:t_{k-1}}(1))(1-p_1)p_e^0 + \hat{f}_{0:t_{k-1}}(1)p_0 p_e^0 \\ (1-\hat{f}_{0:t_{k-1}}(1))p_1(1-p_e^1) + \hat{f}_{0:t_{k-1}}(1)(1-p_0)(1-p_e^1) \end{bmatrix}'.
\end{aligned}
$$

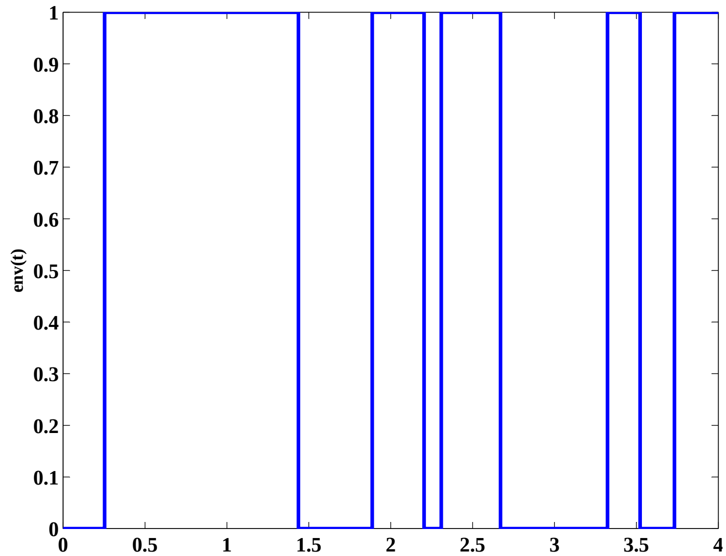
Hence, c_k and $\hat{f}_{0:t_k}(1)$ are given by

$$
\begin{aligned}
c_k &= p_e^0(1-p_1) + p_1(1-p_e^1) + \hat{f}_{0:t_{k-1}}(1)(1-p_0-p_1)(1-p_e^1-p_e^0) \\
\hat{f}_{0:t_k}(1) &= c_k^{-1}(1-p_e^1)\left(p_1 + \hat{f}_{0:t_{k-1}}(1)(1-p_0-p_1)\right).
\end{aligned}
$$

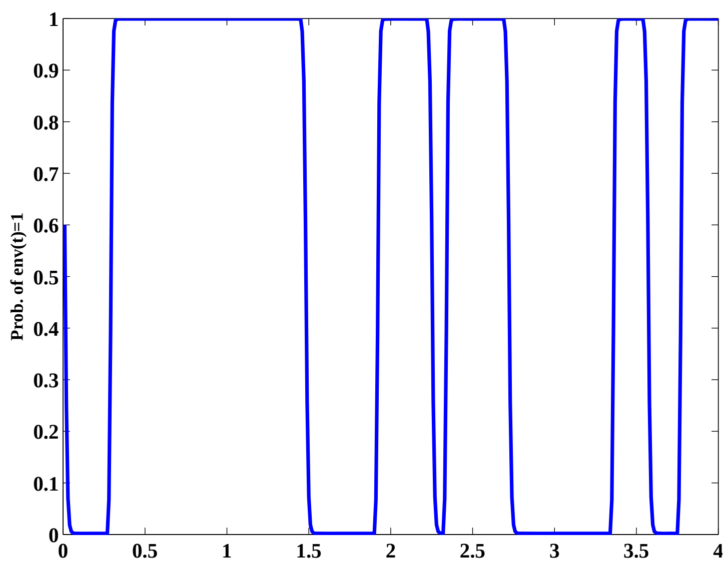
Figure 3.4 (a) illustrates a sample path of the environmental signal for the problem of Section 3.3. The conditional probability of $P(\mathbf{env}(t_k) = 1)$ given all the observations upto time t_k has been shown in Figure 3.4 (b). When the environmental signal is not directly available, one might replace $\mathbf{env}(t_k)$ by $I_{(\hat{f}_{0:t_k}(1)>0.5)}$ at the expense of introducing an error.

3.4 Conclusion

In Chapter 2, we explored the effect of stochastically varying environments on the gene regulation problem. We used a mathematical model that combines stochastic changes in the environments with linear ordinary differential equations describing the concentration of gene product. Based on this model, in Chapter 3, we derived an optimal regulator that minimizes the infinite horizon discounted cost (2.6) with switching equilibria for Markov Jump Linear Systems, and showed that the regulator in each mode $\mathbf{q}$ is an affine function of the continuous state $\mathbf{x}$. We also obtained a necessary and sufficient condition for the existence of an optimal control in terms of a set of LMI conditions. As an extension of the problem in Section 3.2, we will consider scenarios where the waiting times between the environmental changes follow arbitrary probability distribution functions. This is the topic of Chapter 4.



(a) $\mathbf{env}(t)$



(b) $\hat{f}_{0:t}(1)$

Figure 3.4: Figure (a) illustrates a sample path of the environmental signal for the problem of Section 3.3. Figure (b) shows the probability of $\mathbf{env}(t) = 1$ when the environmental signal is not directly available using the method of Section 3.3.1. Such a probability is conditioned upon the observation of an intermediate process described in (3.15). We have chosen $\lambda_0 = 0.5,\ \lambda_1 = 0.7,\ \bar{\alpha} = 0.1$.

4

Quadratic Control of Stochastic Hybrid Systems with Renewal Transitions

In Chapter 3, we investigated the optimal quadratic control of Markov Jump Linear (MJL) systems as a special case of Stochastic Hybrid Systems (SHSs). In MJL systems, the waiting times between consecutive jumps are assumed to be exponentially distributed. Thus, over sufficiently small intervals, the probability of transition to another state is roughly proportional to the length of that interval. The memoryless property of the exponential distribution simplified the analysis of MJL systems, however, in many real world applications, the time intervals between jumps have probability distributions other than the exponential.

In this chapter, we consider a Stochastic Hybrid System with renewal transitions in which the holding times (time between jumps)are independent random variables with given probability distribution functions, and the embedded jump chain is a Markov chain. This can be viewed as a generalization of the Markov Jump Linear systems [46] or as a generalization of the Renewal systems [16] in which there is only one mode. Hence, SHSs with renewal transitions cover a wider range of applications, where the transition rates depend on the length of the time residing in the current mode. This work follows the definition of Piecewise Deterministic Markov Processes in [16], SHS in [26, 11], and in particular, the formulation of SHS with renewal transitions in [6].

The key challenge in studying SHSs with renewal transitions lies in the fact that the Markov property of MJL systems does not hold. This prevents the direct use of approaches based on Dynkin's formula [46]. However, this issue can be overcome by adding a timer to the state of the system that keeps track of the time elapsed since the last transition. Such approach has been introduced in [16].

Inspired by the ideas in [68], we consider the quadratic control of SHSs with renewal transitions. We derive an optimal control policy that minimizes a discounted infinite horizon LQR problem with switching equilibria similar to the one in (2.6). As a generalization of the feedback laws appeared in the previous chapter, we show how the optimal feedback policy depends on the continuous and discrete states as well as the new timer variable. We show that the optimal cost is the solution to a set of differential equations (so-called Bellman equations) with unknown boundary conditions. Furthermore, we provide a numerical technique for finding the optimal solution and the corresponding boundary conditions. These are the main contributions of this chapter. The material in this chapter is based upon the results of [51].

This chapter is organized as follows. Section 4.1 introduces the mathematical model. In Section 4.2, we drive a set of equations to be solved by the value function. In Section 4.3, a sufficient condition for optimal feedback policy is derived. We derive a set of differential equations (with unknown boundary conditions) to be satisfied by the optimal cost. A numerical algorithm is provided for finding the optimal solution. Section 4.4 provides a gene regulation example that has motivated us for solving this problem. We return to *E. Coli* bacterium example that was discussed in Chapter 2. We finally conclude the chapter in Section 4.5 with some final remarks and directions for future research.

4.1 Problem Statement

We consider a class of Stochastic Hybrid Systems (SHSs) with linear dynamics, for which the lengths of the time intervals that the system spends in each mode are independent random variables with given distribution functions. The state space of such system consists of a component $\mathbf{x}$ that takes value in the Euclidean space $\mathbb{R}^n$, and a discrete component $\mathbf{q}$ that takes value in a finite set $\mathcal{S} = \{q_1, .., q_N\}$.

A linear stochastic hybrid system with renewal transitions takes the form of

$$\dot{\mathbf{x}} = A_{\mathbf{q}}\mathbf{x} + B_{\mathbf{q}}\mathbf{u} + d_{\mathbf{q}} \qquad (\mathbf{x}, \mathbf{q}) \in \mathbb{R}^n \times \mathcal{S}, \tag{4.1}$$

where the control input $\mathbf{u}(t) \in \mathbb{R}^m$ may depend on $(\mathbf{x}(s), \mathbf{q}(s))$ for all $s \leq t$ through a causal feedback law and the affine term $d_{\mathbf{q}} \in \mathbb{R}^n$ is a mode-dependent bias term. The causality relation between $\mathbf{u}$ and $(\mathbf{x}, \mathbf{q})$ can be formalized by requiring $\mathbf{u}$ to be adapted to the natural filtration generated by $(\mathbf{x}(s), \mathbf{q}(s))$.

Let $\{\mathbf{t}_k\}$ denote the sequence of jump times. Given $\mathbf{q}(t) = i$, $\forall t \in [\mathbf{t}_k, \mathbf{t}_{k+1})$, the time intervals between consecutive jumps $\mathbf{h}_k := \mathbf{t}_{k+1} - \mathbf{t}_k$ are assumed to be independent random variables with a given cumulative distribution function $F_i(\tau)$ on a finite support $[0, T_i]$ $(0 < T_i < \infty)$

$$0 \leq F_i(\tau) < 1 \qquad \tau \in [0, T_i), \qquad F_i(T_i) = 1. \tag{4.2}$$

This stochastic hybrid system characterizes a stochastic process $\mathbf{x}$ in $\mathbb{R}^n$ called *continuous state*, and a jump process $\mathbf{q}$ called *discrete state* or *mode*. Between jump times, the continuous sate flows according to (4.1) while the discrete state remains constant. When a jump happens, the discrete mode is reset according to a time-homogeneous Markov chain

$$P\left(\mathbf{q}(\mathbf{t}_k) = j | \mathbf{q}(\mathbf{t}_k^-) = i, j \neq i\right) = \mathrm{P}_{ij}, \qquad \sum_{j \neq i} \mathrm{P}_{ij} = 1 \tag{4.3}$$

and $\mathbf{x}$ is reset according to

$$\mathbf{x}(\mathbf{t}_k) = \mathrm{H}_{ij}\mathbf{x}(\mathbf{t}_k^-) \qquad \text{if} \qquad \mathbf{q}(\mathbf{t}_k) = j,\ \mathbf{q}(\mathbf{t}_k^-) = i \tag{4.4}$$

with $\mathrm{H}_{ij} \in \mathbb{R}^{n\times n}$ for all $i, j \in \mathcal{S}$.

We assume that all signals are right continuous, therefore $\mathbf{x}(\mathbf{t}_k^+) = \mathbf{x}(\mathbf{t}_k)$ and $\mathbf{q}(t^+) = \mathbf{q}(t)$ at all times $t \geq 0$ (including the jump times). Even if we were to set $\mathbf{u}(t)$ to be a deterministic function of $(\mathbf{x}(t), \mathbf{q}(t))$, the stochastic process $(\mathbf{x}(t), \mathbf{q}(t))$ might not be a Markov process. The reason is that, at a given time t, the time $\mathbf{t}_{k+1} - t$ until the next jump time $\mathbf{t}_{k+1}$, typically depends on the time $\boldsymbol{\tau} := \ t - \mathbf{t}_k$ elapsed since the last jump $\mathbf{t}_k$, which can be deduced from past values of the state, but not necessarily from the current state. However, given the elapsed time $\boldsymbol{\tau} = t - \mathbf{t}_k$, no other information about the past has any relevance to the process in future. This is due to the assumption that the future intervals $\mathbf{h}_{k+1}, \mathbf{h}_{k+2}, ...$ are independent of the past ones.

Defining a three-component process $(\mathbf{x}(t), \boldsymbol{\tau}(t), \mathbf{q}(t))$ where $\dot{\boldsymbol{\tau}} = 1$ between jumps, and $\boldsymbol{\tau}$ is reset to zero after the jumps, the variable $\boldsymbol{\tau}$ keeps track of the time since the last jump. This has been illustrated in Figure 4.1. It turns out that when the input $\mathbf{u}(t)$ is a deterministic function of $\mathbf{x}(t)$, $\boldsymbol{\tau}(t)$ and $\mathbf{q}(t)$, the process $(\mathbf{x}(t), \boldsymbol{\tau}(t), \mathbf{q}(t))$ is now a Markov process, see [16, Chapter 2].

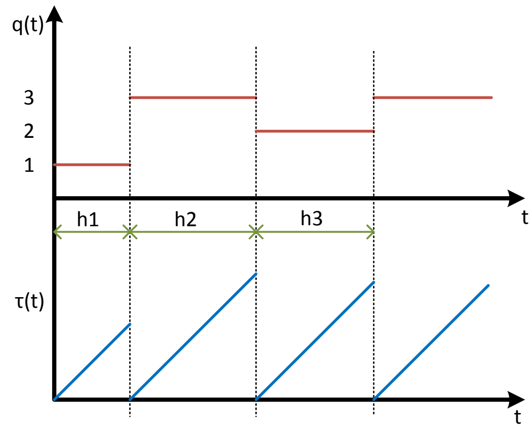


Figure 4.1: Timer $\boldsymbol{\tau}(t)$ keeps track of time between jumps. At every jump time $\mathbf{t}_k$, the time $\boldsymbol{\tau}$ is reset to zero.

We assume that cumulative distribution functions F_i are absolutely continuous and can be written as $F_i(\tau) = \int_0^\tau f_i(s)\, ds$ for some density functions $f_i(\tau) \geq 0$. In this case, one can show that the conditional probability of having a jump in the interval $(t, t+dt]$, given that $\boldsymbol{\tau}(t) = \tau$ is given by

$$P\left(\mathbf{q}(t+dt) = j \;\mid\; \mathbf{q}(t) = i, j \neq i\right) = \mathrm{P}_{ij}\lambda_i(\tau)dt + o(dt)$$

for all $i, j \in \mathcal{S}, \tau \in [0, T_i)$, where

$$\lambda_i(\tau) := \frac{f_i(\tau)}{1 - F_i(\tau)} \qquad \tau \in [0, T_i), \qquad i \in \mathcal{S}$$

is called the *hazard rate* associated with the renewal distribution F_i [5]. The construction of sample paths for this process is similar to that in [26, 11]. For a given initial condition $z := (x, \tau, q)$ with $x \in \mathbb{R}^n, q \in \mathcal{S}, \tau \in [0, T_q]$ construct the processes $(\mathbf{x}(t), \boldsymbol{\tau}(t), \mathbf{q}(t)), t \geq 0$ as follows

(i) If $\tau = T_q$, set $k = 0$ and $\mathbf{t}_0 = 0$.

(ii) If $\tau < T_q$, obtain the jump interval $\mathbf{h}_0$ as a realization of the conditional distribution of $\mathbf{h}_0$ given that $\mathbf{h}_0 > \tau$:

$$F_q(h_0|h_0 > \tau) = \begin{cases} 0 & h_0 < \tau \\ \frac{F_q(h_0)-F_q(\tau)}{1-F_q(\tau)} & \tau \leq h_0 < T_q \\ 1 & h_0 \geq T_q, \end{cases} \tag{4.5}$$

and define $\mathbf{x}(0) = x, \mathbf{q}(0) = q, \boldsymbol{\tau}(0) = \tau$. The continuous state of the SHS in the interval $[0, \mathbf{h}_0 - \tau)$ flows according to (4.1), the timer $\boldsymbol{\tau}$ evolves according to $\dot{\boldsymbol{\tau}} = 1$ and $\mathbf{q}(t)$ remains constant. Set $k = 1$ and $\mathbf{t}_1 = \mathbf{h}_0 - \tau$. One should note that when $\tau < T_q$, the event $\mathbf{t}_1 \leq 0$ ($\mathbf{h}_0 \leq \tau$) happens with zero probability.

(iii) Reset $\boldsymbol{\tau}(\mathbf{t}_k) = 0$, update $\mathbf{q}(\mathbf{t}_k)$ as a realization of a random variable distributed according to (4.3) and reset $\mathbf{x}(\mathbf{t}_k)$ according to (4.4).

(iv) Obtain $\mathbf{h}_k$ as a realization of a random variable distributed according to $F_{\mathbf{q}(\mathbf{t}_k)}$, and set the next jump time $\mathbf{t}_{k+1} = \mathbf{t}_k + \mathbf{h}_k$.

(v) The continuous state of the SHS in the interval $[\mathbf{t}_k, \mathbf{t}_{k+1})$ flows according to (4.1), the timer $\boldsymbol{\tau}$ evolves according to $\dot{\boldsymbol{\tau}} = 1$ and $\mathbf{q}(t)$ remains constant.

(vi) Set $k \to k + 1$, and jump to (iii).

The above algorithm does not guarantee the existence of sample paths on $[0, \infty)$. This construction can fail if either the stochastic process defined by (4.1) has a finite escape time (which could only occur with a non-linear control) or if $\lim_{k\to\infty} \mathbf{t}_k \to L < \infty$. Both cases would lead to a "local-in-time solutions", which we will eventually show that cannot happen for the optimal feedback law.

The following propositions and lemma are direct results of the definition of stochastic hybrid systems with renewal transitions and will be widely used in the rest of this chapter.

Proposition 4.1.1. *For every initial condition $z_0 = (x, \tau, q) \in \mathbb{R}^n \times [0, T_q] \times \mathcal{S}$, we have that $E_{z_0}\{\mathbf{N}(t)\} < \infty, \forall t < \infty$ where $\mathbf{N}(t) := \max\{k : \mathbf{t}_k \leq t\}$ counts the number of jumps up to time t.*

Proof of Proposition 4.1.1. Since the time intervals between consecutive jumps are independent of the initial condition, one can drop the dependency on the initial condition in $E_{z_0}\{\mathbf{N}(t)\}$, an simply consider $E\{\mathbf{N}(t)\}$. By [56, Theorem 3.3.1], all moments of $\mathbf{N}(t)$ are finite, and in particular $E\{\mathbf{N}(t)\} < \infty$ for all $t \geq 0$. $\square$

Proposition 4.1.2. *Let $\{\mathbf{t}_k\}$ denote the sequence of jump times. We have that $\mathbf{t}_k \to \infty$ as $k \to \infty$ with probability one.*

Proof of Proposition 4.1.2. Let $\mathbf{h}_k$ denote the time interval between the jump time $\mathbf{t}_k$ and the subsequent jump $\mathbf{t}_{k+1}$. Suppose $\mathbf{q}(\mathbf{t}_k) = i$ which implies that the system is in mode i during the time interval $[\mathbf{t}_k, \mathbf{t}_{k+1})$. Since $\mathbf{h}_k$ has the probability distribution F_i and $F_i(0) < 1$ for all $i \in \mathcal{S}$, we have $P(\mathbf{h}_k > 0) = 1 - F_i(0) > 0$. Suppose that for all $h > 0$, we have $P(\mathbf{h}_k > h) = 0$. This implies that $F_i(h) = 1$ for all $h \in [0, T_i)$ that contradicts (4.2). Therefore, there exists $h > 0$ such that $P(\mathbf{h}_k > h) > 0$, and by the second Borel-Cantelli Lemma [37], it follows that with probability one, $\mathbf{h}_k > h$ for infinitely many k. Therefore, $\mathbf{t}_{k\to\infty} = \sum_{k=1}^{\infty} \mathbf{h}_k \to \infty$ with probability one. □

Lemma 4.1.1. *Let* $\bar{\mathbf{N}}(t),\ t \geq 0$ *denote the standard Poisson process, then* $\mathbf{N}(t) = \max\{k : \mathbf{t}_k \leq t\}$ *is given by*

$$\mathbf{N}(t) = \bar{\mathbf{N}}\left(\int_0^t \lambda_{\mathbf{q}}(\boldsymbol{\tau}_s)\ ds\right), \qquad \forall t \in [0, \infty). \tag{4.6}$$

Proof of Lemma 4.1.1. Similar to the result of [29], we show how the jumps counter $\mathbf{N}(t) = \max\{k : \mathbf{t}_k \leq t\}$ can be related to the standard Poisson process, $\bar{\mathbf{N}}(t)$ through the following intensity-dependent time scaling

$$\mathbf{N}(t) = \bar{\mathbf{N}}\left(\int_0^t \lambda_{\mathbf{q}}(\boldsymbol{\tau}_s)\ ds\right), \qquad \forall t \in [0, \infty).$$

Denoting by $\mathbf{h}_k := \mathbf{t}_k - \mathbf{t}_{k-1}$, we have $\mathbf{t}_k = \sum_{i=1}^{k} \mathbf{h}_i$. The jump times $\mathbf{t}_k$ are the event times of the standard Poisson process $\bar{\mathbf{N}}(t), t \geq 0$, therefore

$$\begin{aligned}\bar{\mathbf{N}}\left(\int_0^t \lambda_{\mathbf{q}}(\boldsymbol{\tau}_s)\ ds\right) &= \max\left\{k : \mathbf{t}_k \leq \int_0^t \lambda_{\mathbf{q}}(\boldsymbol{\tau}_s)\ ds\right\} \\ &= \max\left\{k : \sum_{i=1}^{k} \mathbf{h}_i \leq \int_0^t \lambda_{\mathbf{q}}(\boldsymbol{\tau}_s)\ ds\right\}.\end{aligned}$$

Our goal is to show that this expression is equal to $\mathbf{N}(t)$. To this effect, take an arbitrary jump time $\mathbf{t}_k$. Since the hazard rate is non-negative, if $\mathbf{t}_k \leq t$, then

$$\int_0^{\mathbf{t}_k} \lambda_{\mathbf{q}}(\boldsymbol{\tau}_s)\ ds \leq \int_0^t \lambda_{\mathbf{q}}(\boldsymbol{\tau}_s)\ ds \Leftrightarrow \qquad \sum_{i=1}^{k} \mathbf{h}_i \leq \int_0^t \lambda_{\mathbf{q}}(\boldsymbol{\tau}_s)\ ds,$$

where we used the fact that

$$\int_0^{\mathbf{t}_k} \lambda_{\mathbf{q}}(\boldsymbol{\tau}_s)\ ds = \sum_{i=1}^{k} \int_{\mathbf{t}_{i-1}}^{\mathbf{t}_i} \lambda_{\mathbf{q}}(\boldsymbol{\tau}_s)\ ds = \sum_{i=1}^{k} \mathbf{h}_i. \tag{4.7}$$

Since $\{k : \mathbf{t}_k \leq t\} \subset \{k : \sum\limits_{i=1}^{k} \mathbf{h}_i \leq \int_0^t \lambda_{\mathbf{q}}(\boldsymbol{\tau}_s)\ ds\}$. we conclude that

$$\begin{aligned}&\mathbf{N}(t) = \max\{k : \mathbf{t}_k \leq t\} \\ &\leq \max\left\{k : \sum_{i=1}^{k} \mathbf{h}_i \leq \int_0^t \lambda_{\mathbf{q}}(\boldsymbol{\tau}_s)\ ds\right\} = \bar{\mathbf{N}}\left(\int_0^t \lambda_{\mathbf{q}}(\boldsymbol{\tau}_s)\ ds\right).\end{aligned} \tag{4.8}$$

To prove that we actually have equality, assume by contradiction that

$$\max\{k : \mathbf{t}_k \leq t\} < \max\left\{k : \sum_{i=1}^{k} \mathbf{h}_i \leq \int_0^t \lambda_{\mathbf{q}}(\boldsymbol{\tau}_s)\ ds\right\}$$

which means that there exists a k^* such that $\mathbf{t}_{k^*-1} \leq t < \mathbf{t}_{k^*}$, but using (4.7), we can show

$$\sum_{i=1}^{k^*} \mathbf{h}_i = \int_0^{\mathbf{t}_{k^*}} \lambda_{\mathbf{q}}(\boldsymbol{\tau}_s)\ ds \leq \int_0^{\mathbf{t}} \lambda_{\mathbf{q}}(\boldsymbol{\tau}_s)\ ds \Leftrightarrow \int_t^{\mathbf{t}_{k^*}} \lambda_{\mathbf{q}}(\boldsymbol{\tau}_s)\ ds \leq 0.$$

However, for $\mathbf{t}_{k^*} > t$ to be a jump time, we must have

$$\begin{aligned}\int_{\mathbf{t}_{k^*-1}}^{\mathbf{t}_{k^*}} \lambda_{\mathbf{q}}(\boldsymbol{\tau}_s)\ ds &= \int_{\mathbf{t}_{k^*-1}}^{t} \lambda_{\mathbf{q}}(\boldsymbol{\tau}_s)\ ds + \int_{t}^{\mathbf{t}_{k^*}} \lambda_{\mathbf{q}}(\boldsymbol{\tau}_s)\ ds \\ &= \mathbf{h}_{k^*} \\ &\Rightarrow \int_{\mathbf{t}_{k^*-1}}^{t} \lambda_{\mathbf{q}}(\boldsymbol{\tau}_s)\ ds \geq \mathbf{h}_{k^*},\end{aligned}$$

which means that $\mathbf{t}_{k^*} \leq t$ and thus contradicts the assumption. Therefore, we actually have equality in (4.8). □

4.1.1 Quadratic Cost Function

The goal is to regulate $\mathbf{x}(t)$ around a nominal point $\bar{x}_{\mathbf{q}}$ that may depend on the current discrete state $\mathbf{q}$, while maintaining the control input $\mathbf{u}$ close to a nominal value $\bar{u}_{\mathbf{q}}$ that may also depend on $\mathbf{q}$. To this effect, we consider an infinite horizon discounted cost function with a quadratic penalty on state and control excursions of the form

$$\int_0^\infty e^{-\rho t}\left((\mathbf{x}-\bar{x}_{\mathbf{q}})' Q_{\mathbf{q}}(\mathbf{x}-\bar{x}_{\mathbf{q}}) + (\mathbf{u}-\bar{u}_{\mathbf{q}})' R_{\mathbf{q}}(\mathbf{u}-\bar{u}_{\mathbf{q}})\right)\ dt. \tag{4.9}$$

As before, the mode-dependent symmetric matrices Q_q and R_q satisfy the conditions $Q_q \geq 0,\ R_q > 0\ ,\forall q \in \mathcal{S}$ and allow us to assign different penalties in different modes. In each mode, the parameters Q_q and R_q determine a trade-off between keeping $\mathbf{x}(t)$ close to its ideal value $\bar{x}_{\mathbf{q}}$ and paying a large penalty for control energy.

Our goal is to find an optimal control $\mathbf{u}$ that minimizes the conditional expected value of the cost in (4.9):

$$J_\mu(x_0, \tau_0, q_0) := E_{z_0}\left\{\int_0^\infty e^{-\rho t}\left((\mathbf{x}-\bar{x}_\mathbf{q})' Q_\mathbf{q}(\mathbf{x}-\bar{x}_\mathbf{q}) + (\mathbf{u}-\bar{u}_\mathbf{q})' R_\mathbf{q}(\mathbf{u}-\bar{u}_\mathbf{q})\right)\, dt\right\} \tag{4.10}$$

given the initial condition $z_0 = (x_0, \tau_0, q_0)$. To minimize (4.10), we consider state feedback laws of the form

$$\mathbf{u}(t) = \mu(\mathbf{x}(t), \boldsymbol{\tau}(t), \mathbf{q}(t)), \tag{4.11}$$

where μ is an appropriately selected deterministic state feedback law. By appropriately choosing the discount factor $\rho > 0$, one can devalue the future cost in (4.10).

4.2 Expectation

In this section, we show that for every feedback policy (4.11), J_μ in (4.10) satisfies a set of differential equations with the associated boundary conditions. For a given feedback policy (4.11) for which the stochastic process (4.1) has a global solution[1], let J_μ be the corresponding cost:

$$J_\mu(x, \tau, q) = \; E_{z_0}\left\{\int_0^\infty e^{-\rho s}\ell_\mu(\mathbf{x}, \boldsymbol{\tau}, \mathbf{q})\, ds\right\} \tag{4.12}$$

[1]The solution of the system has no finite escape time with probability one.

conditioned upon the initial condition $z_0 = (x, \tau, q) \in \mathbb{R}^n \times [0, T_q] \times \mathcal{S}$ with $\ell_\mu(x, \tau, q)$ defined as

$$\ell_\mu(x,\tau,q) := (x - \bar{x}_q)' Q_q (x - \bar{x}_q) + (\mu(x,\tau,q) - \bar{u}_q)' R_q (\mu(x,\tau,q) - \bar{u}_q). \quad (4.13)$$

The following result shows that $J_\mu(x, \tau, q)$ satisfies a boundary condition as τ approaches T_q.

Lemma 4.2.1. *Suppose that the solutions of the stochastic process (4.1) with a feedback policy* (4.11) *exist globally, and the corresponding cost J_μ in (4.12) is finite for all $(x, \tau, q) \in \mathbb{R}^n \times [0, T_q] \times \mathcal{S}$. Then J_μ satisfies*

$$J_\mu(x, T_q, q) = \sum_{j \neq q} \mathrm{P}_{qj}\ J_\mu(\mathrm{H}_{qj} x, 0, j). \quad (4.14)$$

Proof of Lemma 4.2.1. Take $T_q - \epsilon \in [0, T_q)$ for some $\epsilon > 0$. For every $(x, q) \in \mathbb{R}^n \times \mathcal{S}$, we have

$$J_\mu(x, T_q - \epsilon, q) = E_{z_0} \left\{ \int_0^\infty e^{-\rho s} \ell_\mu(\mathbf{x}, \boldsymbol{\tau}, \mathbf{q})\ ds \right\}, \qquad z_0 = (x, T_q - \epsilon, q).$$

Let $\mathbf{t}_1$ denote the first jump of the process. Given that $\boldsymbol{\tau}(0) = T_q - \epsilon$, we have that $\mathbf{t}_1$ belongs to $[0, \epsilon)$ with probability one. One can write

$$\begin{aligned} J_\mu(x, T_q - \epsilon, q) &= E_{z_0} \left\{ \int_0^{\mathbf{t}_1} e^{-\rho s} \ell_\mu(x_s, \tau_s, q_s)\ ds + \int_{\mathbf{t}_1}^\infty e^{-\rho s} \ell_\mu(\mathbf{x}_s, \boldsymbol{\tau}_s, \mathbf{q}_s)\ ds \right\} \\ &= E_{z_0} \left\{ \int_0^{\mathbf{t}_1} e^{-\rho s} \ell_\mu(x_s, \tau_s, q_s)\ ds + E_{z_{\mathbf{t}_1}} \left\{ \int_{\mathbf{t}_1}^\infty e^{-\rho s} \ell_\mu(\mathbf{x}_s, \boldsymbol{\tau}_s, \mathbf{q}_s)\ ds \right\} \right\} \\ &= E_{z_0} \left\{ \int_0^{\mathbf{t}_1} e^{-\rho s} \ell_\mu(x_s, \tau_s, q_s)\ ds \right\} + \sum_{j \neq q} \mathrm{P}_{qj} E_{z_0} \left\{ e^{-\rho \mathbf{t}_1} J_\mu(\mathrm{H}_{qj} \mathbf{x}_{\mathbf{t}_1^-}, 0, j) \right\}, \end{aligned}$$

where $(\mathbf{x}_t, \boldsymbol{\tau}_t, \mathbf{q}_t)$ denotes the solution of (4.1) at time $t \geq 0$, starting from initial point $z_0 = (x, T_q - \epsilon, q)$. Here, we used the conditional expectation property $E\{E\{X|\mathfrak{F}\}|\mathfrak{G}\} = E\{X|\mathfrak{G}\}$ if $\mathfrak{G} \subset \mathfrak{F}$. As $\epsilon \downarrow 0$[2], the first integral on the right tends to zero and the second integral tends to $\sum_{j \neq q} \mathrm{P}_{qj} J_\mu(\mathrm{H}_{qj} x, 0, j)$. Therefore, the limit is given by (4.14). □

We further define the following notations that will be used in the rest of this chapter. For every function $g(x, \tau, q)$ such that for fixed $q \in \mathcal{S}$, g is continuously differentiable with respect to its first and second arguments on $\mathbb{R}^n \times [0, T_q)$, we define the operators $\mathcal{L}^f_\nu$ and $\mathcal{L}_\nu$ as

$$\mathcal{L}^f_\nu g(x, \tau, q) := \frac{\partial g(x, \tau, q)}{\partial x}(A_q x + B_q \nu) + \frac{\partial g(x, \tau, q)}{\partial \tau} \tag{4.15}$$

$$\mathcal{L}_\nu g(x, \tau, q) := \mathcal{L}^f_\nu g(x, \tau, q) + \lambda_q(\tau) \sum_{j \neq q} \mathrm{P}_{qj} \big(g(\mathrm{H}_{qj} x, 0, j) - g(x, \tau, q) \big) \tag{4.16}$$

$\forall q \in \mathcal{S}$ and all $x \in \mathbb{R}^n, \tau \in [0, T_q),\ \nu \in \mathbb{R}^m$. When the vector ν is set equal to a function of (x, τ, q), e.g. $\nu = \mu(x, \tau, q)$, we use the notation $\mathcal{L}_\mu$ instead of $\mathcal{L}_{\mu(x,\tau,q)}$. The definition in (4.15) is inspired by the extended generator in the Markov process literature [16, 26], but here we further extend this operator to every continuously differentiable function g (regardless of what the domain of the extended generator may be).

[2] We have $\tau \to T$ and $\mathbf{t}_1 \to 0$ almost surely.

The next theorem, provides a set of differential equations to be solved by a finite J_μ in (4.12) with a state feedback law μ.

Theorem 4.2.1. *Let μ be a state feedback policy such that the solutions of* (4.1) *with* (4.11) *exist globally. Suppose that for every initial condition $z_0 = (x, \tau, q)$, the cost function J_μ in* (4.12) *is differentiable with respect to its first and second arguments and is finite. Then J_μ satisfies*

$$\mathcal{L}_\mu J_\mu(x, \tau, q) - \rho J_\mu(x, \tau, q) + \ell_\mu(x, \tau, q) = 0 \tag{4.17}$$

with $\ell_\mu(x, \tau, q)$ defined in (4.13).

Proof of Theorem 4.2.1. Similar to the calculation of [16, Theorem 32.2], take an arbitrary $t \geq 0$, and suppose $\mathbf{t}_1$ be the first jump time of the process.[3] $J_\mu(x, \tau, q)$ can be written as

$$\begin{aligned} J_\mu(x, \tau, q) &= E_{z_0}\left\{\int_0^\infty e^{-\rho s}\ell_\mu(\mathbf{x}_s, \boldsymbol{\tau}_s, \mathbf{q}_s)\ ds\right\} \\ &= E_{z_0}\left\{\int_0^{t\wedge\mathbf{t}_1} e^{-\rho s}\ell_\mu(\mathbf{x}_s, \boldsymbol{\tau}_s, \mathbf{q}_s)\ ds\right\} + E_{z_0}\left\{\int_{t\wedge\mathbf{t}_1}^\infty e^{-\rho s}\ell_\mu(\mathbf{x}_s, \boldsymbol{\tau}_s, \mathbf{q}_s)\ ds\right\}. \end{aligned}$$

[3] $t \wedge \mathbf{t}_1$ is a stopping time.

Using the conditional expectation property $E\{E\{X|\mathfrak{F}\}|\mathfrak{G}\} = E\{X|\mathfrak{G}\}$ for sigma algebras $\mathfrak{G} \subset \mathfrak{F}$, one can write

$$\begin{aligned}
J_\mu(x,\tau,q) &= E_{z_0}\left\{\int_0^{t\wedge\mathbf{t}_1} e^{-\rho s}\ell_\mu(x_s,\tau_s,q_s)\ ds\right\} \\
&\quad + E_{z_0}\left\{E\left\{\int_{t\wedge\mathbf{t}_1}^{\infty} e^{-\rho s}\ell_\mu(\mathbf{x}_s,\boldsymbol{\tau}_s,\mathbf{q}_s)\ ds \mid \mathfrak{F}_{t\wedge\mathbf{t}_1}\right\}\right\} \\
&= E_{z_0}\ \left\{\int_0^{t\wedge\mathbf{t}_1} e^{-\rho s}\ell_\mu(x_s,\tau_s,q_s)\ ds\right\} \\
&\quad + E_{z_0}\left\{e^{-\rho(t\wedge\mathbf{t}_1)}J_\mu\left(\mathbf{x}(t\wedge\mathbf{t}_1),\boldsymbol{\tau}(t\wedge\mathbf{t}_1),\mathbf{q}(t\wedge\mathbf{t}_1)\right)\ \right\}
\end{aligned}$$

where $\mathfrak{F}_t$ is the natural filtration of the process $(\mathbf{x},\boldsymbol{\tau},\mathbf{q})$. Therefore,

$$\begin{aligned}
J_\mu(x,\tau,q) = E_{z_0}\ &\left\{\int_0^t e^{-\rho s}\ell_\mu(x_s,\tau_s,q_s)\ ds\ I_{(t<\mathbf{t}_1)}\right. \\
&+ \int_0^{\mathbf{t}_1} e^{-\rho s}\ell_\mu(x_s,\tau_s,q_s)\ ds\ I_{(t\geq\mathbf{t}_1)} + e^{-\rho t}J_\mu(x(t),\tau(t),q(t))\ I_{(t<\mathbf{t}_1)} \\
&\left. + e^{-\rho\mathbf{t}_1}\sum_{j\neq q}\mathrm{P}_{qj}J_\mu(\mathrm{H}_{qj}x(\mathbf{t}_1^-),0,j)\ I_{(t\geq\mathbf{t}_1)}\right\}.
\end{aligned} \tag{4.18}$$

The density function of the first jump time is $\lambda_q(\tau(s))e^{-\theta_q(s)}$ where $\theta_q(s) = \int_0^s \lambda_q(\tau(u))\ du$, hence one can re-write the expectation in (4.18)[4]

$$\begin{aligned}
J_\mu(x,\tau,q) &= e^{-\theta_q(t)}\int_0^t e^{-\rho s}\ell_\mu(x_s,\tau_s,q_s)\ ds \\
&+ \int_0^t \lambda_q(\tau(s))e^{-\theta_q(s)}\int_0^s e^{-\rho w}\ell_\mu(x_w,\tau_w,q_w)\ dw\ ds + e^{-\theta_q(t)-\rho t}J_\mu(x_t,\tau_t,q_t) \\
&+ \int_0^t \lambda_q(\tau(s))e^{-\theta_q(s)-\rho s}\sum_{j\neq q}\mathrm{P}_{qj}J_\mu(\mathrm{H}_{qj}x_s,0,j)\ ds.
\end{aligned} \tag{4.19}$$

[4] $E\{I_{(t<\mathbf{t}_1)}\} = P(t<\mathbf{t}_1) = e^{-\theta_q(t)}$ and $E\{f(\mathbf{t}_1)I_{(t\geq\mathbf{t}_1)}\} = \int_0^t f(s)\lambda_q(\tau_s)e^{-\theta_q(s)}\ ds$.

The first two integrals on the right-hand side of (4.19) can be written as

$$
\begin{aligned}
&e^{-\theta_q(t)} \int_0^t e^{-\rho s} \ell_\mu(x_s, \tau_s, q_s)\, ds + \int_0^t \lambda_q(\tau_s) e^{-\theta_q(s)} \int_0^s e^{-\rho w} \ell_\mu(x_w, \tau_w, q_w)\, dwds \\
&= e^{-\theta_q(t)} \int_0^t e^{-\rho s} \ell_\mu(x_s, \tau_s, q_s)\, ds + \int_0^t e^{-\rho w} \ell_\mu(x_w, \tau_w, q_w) \int_w^t \lambda_q(\tau_s) e^{-\theta_q(s)}\, dsdw \\
&= \int_0^t e^{-\rho s - \theta_q(s)} \ell_\mu(x_s, \tau_s, q_s)\, ds.
\end{aligned}
$$

Hence (4.19) can be written as

$$
\begin{aligned}
J_\mu(x, \tau, q) &= \int_0^t e^{-\rho s - \theta_q(s)} \ell_\mu(x_s, \tau_s, q_s)\, ds + e^{-\rho t - \theta_q(t)} J_\mu(x(t), \tau(t), q(t)) \\
&\quad + \int_0^t \lambda_q(\tau(s)) e^{-\rho s - \theta_q(s)} \sum_{j \neq q} \mathrm{P}_{qj} J_\mu(\mathrm{H}_{qj} x_s, 0, j)\, ds \\
&= e^{-\rho t - \theta_q(t)} J_\mu(x(t), \tau(t), q(t)) \\
&\quad + \int_0^t e^{-\rho s - \theta_q(s)} \left(\ell_\mu(x_s, \tau_s, q_s) + \lambda_q(\tau_s) \sum_{j \neq q} \mathrm{P}_{qj} J_\mu(\mathrm{H}_{qj} x_s, 0, j) \right) ds
\end{aligned}
$$

which implies

$$
\begin{aligned}
&J_\mu(x(t), \tau(t), q(t)) = \exp\left(\int_0^t \rho + \lambda_q(\tau(w))\, dw \right) J_\mu(x, \tau, q) \\
&- \int_0^t e^{\left(\int_s^t \rho + \lambda_q(\tau(w))\, dw\right)} \left(\ell_\mu(x_s, \tau_s, q_s) + \lambda_q(\tau_s) \sum_{j \neq q} \mathrm{P}_{qj} J_\mu(\mathrm{H}_{qj} x_s, 0, j) \right) ds.
\end{aligned}
$$

By an elementary calculation, one can show

$$
\begin{aligned}
e^{-\rho t} J_\mu(x(t), \tau(t), q(t)) - J_\mu(x, \tau, q) &= \int_0^t e^{-\rho s} \lambda_q(\tau_s) J_\mu(x_s, \tau_s, q_s)\, ds \\
&\quad - \int_0^t e^{-\rho s} \left(\ell_\mu(x_s, \tau_s, q_s) + \lambda_q(\tau_s) \sum_{j \neq q} \mathrm{P}_{qj} J_\mu(\mathrm{H}_{qj} x_s, 0, j) \right) ds.
\end{aligned}
$$

This implies that J_μ is absolutely continuous in t. Taking the derivative of J_μ, one gets

$$\frac{d}{dt}J_\mu(x(t),\tau(t),q(t)) = -\ell_\mu(x_t,\tau_t,q_t) + \rho J_\mu\left(x(t),\tau(t),q\right) \\ -\lambda_q(\tau(t))\sum_{j\neq q}\mathrm{P}_{qj}\big(J_\mu(\mathrm{H}_{qj}x(t),0,j) - J_\mu(x(t),\tau(t),q(t))\big). \tag{4.20}$$

$J_\mu(x,q,\tau)$ is continuously differentiable with respect to (x,τ). For the absolutely continuous function J_μ, we showed that $t \to J_\mu(x(t),\tau(t),q(t))$ is differentiable almost everywhere except at jump times $\mathbf{t}_k$, therefore the value of its derivative should be equal to

$$\frac{\partial J_\mu}{\partial x}(A_q x + B_q\mu(x,\tau,q)) + \frac{\partial J_\mu}{\partial \tau}$$

at points $(x',\tau',q') = (x(t),\tau(t),q(t))$. Equation (4.20) implies that this derivative is equal to $-\ell_\mu(x',\tau',q') + \rho J_\mu(x',\tau',q') + \lambda_{q'}(\tau')\sum_{j\neq q'}\mathrm{P}_{q'j}(J_\mu(\mathrm{H}_{q'j}x',0,j) - J_\mu(x',\tau',q'))$. Thus, by the same argument as in [16, Theorem 32.2], $J_\mu(x,\tau,q)$ satisfies (4.17) for all $(x,\tau,q)\in\mathbb{R}^n\times[0,T_q)\times\mathcal{S}$. □

4.3 Optimal Control

In this section, we compute the optimal feedback policy and provide a recursive algorithm for finding the optimal regulator and the optimal cost. The following theorem provides a sufficient condition for the existence of an optimal feedback policy μ^* that minimizes the infinite horizon discounted criteria (4.10).

Theorem 4.3.1. *Consider the following optimization problem*

$$J^* = \min_{\mu} J_\mu \qquad \textit{subject to} \qquad \dot{\mathbf{x}} = A_{\mathbf{q}}\mathbf{x} + B_{\mathbf{q}}\ \mu(\mathbf{x}, \boldsymbol{\tau}, \mathbf{q}) + d_{\mathbf{q}},$$

where J_μ given in (4.10). If there exist C^1 and bounded solutions $\Lambda_i : [0, T_i] \to \mathbb{R}^{n\times n}$, $\Gamma_i : [0, T_i] \to \mathbb{R}^n$ and $\Omega_i : [0, T_i] \to \mathbb{R}$, $i \in \mathcal{S}$ to the following differential equations with the associated boundary conditions,

$$\begin{cases} -\frac{\partial \Lambda_i}{\partial \tau} = A_i'\Lambda_i + \Lambda_i A_i - \rho\Lambda_i - \Lambda_i B_i R_i^{-1} B_i' \Lambda_i + Q_i \\ \qquad +\lambda_i(\tau) \sum\limits_{j\neq i} \mathrm{P}_{ij} \left(\mathrm{H}_{ij}'\Lambda_j(0)\mathrm{H}_{ij} - \Lambda_i(\tau)\right) \\ \Lambda_i(T_i) = \sum\limits_{j\neq i} \mathrm{P}_{ij}\mathrm{H}_{ij}'\Lambda_j(0)\mathrm{H}_{ij} \end{cases} \tag{4.21}$$

$$\begin{cases} -\frac{\partial \Gamma_i}{\partial \tau} = (A_i' - \Lambda_i B_i R_i^{-1} B_i' - \rho I)\Gamma_i - 2Q_i\bar{x}_i \\ \qquad +\lambda_i(\tau) \sum\limits_{j\neq i} \mathrm{P}_{ij} \left(\mathrm{H}_{ij}'\Gamma_j(0) - \Gamma_i(\tau)\right) + 2\Lambda_i(B_i\bar{u}_i + d_i) \\ \Gamma_i(T_i) = \sum\limits_{j\neq i} \mathrm{P}_{ij}\mathrm{H}_{ij}'\Gamma_j(0) \end{cases} \tag{4.22}$$

$$\begin{cases} -\frac{\partial \Omega_i}{\partial \tau} = -\frac{1}{4}\Gamma_i' B_i R_i^{-1} B_i' \Gamma_i + \Gamma_i'(B_i\bar{u}_i + d_i) - \rho\Omega_i \\ \qquad +\lambda_i(\tau) \sum\limits_{j\neq i} \mathrm{P}_{ij} \left(\Omega_j(0) - \Omega_i(\tau)\right) + \bar{x}_i' Q_i \bar{x}_i \\ \Omega_i(T_i) = \sum\limits_{j\neq i} \mathrm{P}_{ij}\Omega_j(0) \end{cases} \tag{4.23}$$

such that for every $q \in \mathcal{S}$, $V^*(x, \tau, q) := x'\Lambda_q(\tau)x + x'\Gamma_q(\tau) + \Omega_q(\tau)$ is a non-negative function, then the feedback policy

$$\mu^*(x, \tau, q) := \bar{u}_q - \frac{1}{2}R_q^{-1}B_q'\left(2\Lambda_q(\tau)x + \Gamma_q(\tau)\right) \tag{4.24}$$

is optimal over all feedback policies for which we have $E_{z_0}\{||\mathbf{x}(t)||^2\} \leq k_1 e^{-k_2 t}||x_0||$ *for some* $k_1 > 0, k_2 > -\rho$*. Moreover, the minimal cost is given by* $J^*(x_0, \tau_0, q_0) = V^*(x_0, \tau_0, q_0)$*.*

It is worth noting that (4.21) is a system of ordinary differential Riccati-like equations that should satisfy the given boundary conditions. If we were given *initial conditions* for the ODEs in (4.21)-(4.23), we could solve the ODE in (4.21) using the methods developed in [39, 59]. With the solution to (4.21), we could then proceed to solve (4.22) and (4.23). However, in general we do not know a priori the initial conditions for the ODEs in (4.21)-(4.23). This difficulty shall be overcome in Section 4.3.1, where we provide a numerical algorithm that computes the $\Lambda_i, \Gamma_i, \Omega_i$ based on a recursive system of differential equations.

The following result shows that the feedback policy (4.24) results in a global solution on $[0, \infty)$ with probability one.

Lemma 4.3.1. *Suppose* $\Lambda_i(\tau)$ *and* $\Gamma_i(\tau)$ *are bounded* C^1 *functions* $\forall i \in \mathcal{S}$*. Then the solution of stochastic process (4.1) with the feedback policy* μ^* *in (4.24) exists globally with probability one.*

Proof of Lemma 4.3.1. Between jumps, the right hand side of (4.1) with $\mathbf{u} = \mu^*(\mathbf{x}, \boldsymbol{\tau}, \mathbf{q})$ can be written as $\dot{\mathbf{x}} = \bar{A}(t)\mathbf{x} + \bar{g}(t)$, where $\bar{A}(t)$ and $\bar{g}(t)$ are continuous functions on t and the elements of $\bar{A}(t)$ are bounded. Thus, by [40, Theorem 3.2], the solution exists between jumps. When a jump happens, $\mathbf{x}(\mathbf{t}_k)$ is reset according to the linear reset map in (4.4). Therefore, the sample paths of system (4.1) do not have a finite escape time. Moreover, by Proposition 4.1.2, $\mathbf{t}_k \overset{\text{wpo}}{\to} \infty$. Thus, the sample paths of the system exist globally with probability one. □

The proof of Theorem 4.3.1 depends on a Dynkin's-like formula for a stopped process constructed based on $(\mathbf{x}, \boldsymbol{\tau}, \mathbf{q})$ [29]. This proof is inspired by the "Poisson driven" process in [42] and the Piecewise Deterministic Markov Processes in [16] but avoids some strong assumptions in these references. We first define the stopped process as in [29, Section 5] and then derive a Dynkin's-like formula for the stochastic process (4.1) under a feedback policy (4.11).

We denote $\mathbf{X} := (\mathbf{x}, \boldsymbol{\tau}, \mathbf{q})$. For every $q \in \mathcal{S}$, let Q_τ^q and Q_x be two compact subsets of $[0, T_q)$ and $\mathbb{R}^n$, respectively. Define $Q^q := Q_x \times Q_\tau^q$ which is also compact in $\mathbb{R}^n \times [0, T_q)$, see [76, Theorem 17.8] and $Q := \cup_{q \in \mathcal{S}} Q^q$. The Q-stopped process from the process $\mathbf{X}$ will be defined as following. We define the Q-stopping time

$$\mathbf{T}_Q := \sup\left(\{\bar{t} \in [0, \infty) : \forall t \in [0, \bar{t}), \mathbf{q}(t) = q \Rightarrow (\mathbf{x}(t), \boldsymbol{\tau}(t)) \in \operatorname{Int}(\mathbb{Q}^q)\} \cup \{0\}\right)$$

where $\mathrm{Int}(\mathbb{Q}^q)$ denotes the interior of $\mathbb{Q}^q$.[5] Based on $\mathbf{T}_Q$, we define the Q- stopped process as

$$\mathbf{X}_Q(t) := \begin{cases} \mathbf{X}(t) & t \in [0, \mathbf{T}_Q) \\ \mathbf{X}(\mathbf{T}_Q) & t \in [\mathbf{T}_Q, \infty) \end{cases} \tag{4.25}$$

and the Q-stopped jump counter as

$$\mathbf{N}_Q(t) := \begin{cases} \mathbf{N}(t) & t \in [0, \mathbf{T}_Q) \\ \mathbf{N}(\mathbf{T}_Q) & t \in [\mathbf{T}_Q, \infty) \end{cases} \tag{4.26}$$

where $\mathbf{N}(t) := \max\{k : \mathbf{t}_k \leq t\}$ counts the number of jumps up to time t. These stochastic processes are well-defined for all times since the system has no finite escape time and by Proposition 4.1.2, $\mathbf{t}_k \to \infty$ with probability one. The following result will be used in deriving the Dynkin's formula.

Lemma 4.3.2. *For any compact sets $Q_x \subseteq \mathbb{R}^n$ and $Q_\tau^q \subseteq [0, T_q)$, $\forall q \in \mathcal{S}$, the expected value of $\mathbf{N}_Q(t)$ is finite on any finite time interval. In particular,*

$$E\left\{\mathbf{N}_Q(s_2) - \mathbf{N}_Q(s_1)\right\} \leq \lambda_{\max}(s_2 - s_1),$$

for all $s_2 \geq s_1 > 0$ and $\lambda_{\max} := \sup_{q\in\mathcal{S}} \sup_{\tau \in Q_\tau^q} \lambda_i(\tau)$.

Proof of Lemma 4.3.2. Since the hazard rate $\lambda_q(\tau)$ is continuous on $[0, T_q)$, $\lambda_q(\tau)$ is locally bounded on $(0, T_q)$ for all $q \in \mathcal{S}$, and $\lambda_{\max}$ always exists over compact

[5] $\mathbf{T}_Q$ is a stopping time for the natural filtration of the process $\mathbf{X}$.

sets Q^q_τ. Since there are no more jumps after $\mathbf{T}_Q$,

$$\begin{aligned}E\left\{\mathbf{N}_Q(s_2)-\mathbf{N}_Q(s_1)\right\} &= E\left\{\mathbf{N}_Q(s_2\wedge \mathbf{T}_Q)-\mathbf{N}_Q(s_1\wedge \mathbf{T}_Q)\right\}\\ &= E\left\{\mathbf{N}(s_2\wedge \mathbf{T}_Q)-\mathbf{N}(s_1\wedge \mathbf{T}_Q)\right\}.\end{aligned}$$

Moreover, one can show that $\mathbf{N}(t)=\bar{\mathbf{N}}(\int_0^t \lambda_{\mathbf{q}}(\boldsymbol{\tau}_s)\,ds)$ where $\bar{\mathbf{N}}(t)$ is the standard Poisson process, see Lemma 4.1.1. Therefore, we have

$$\begin{aligned}&E\left\{\mathbf{N}(s_2\wedge \mathbf{T}_Q)-\mathbf{N}(s_1\wedge \mathbf{T}_Q)\right\}\\ &= E\left\{\bar{\mathbf{N}}\left(\int_0^{s_2\wedge \mathbf{T}_Q}\lambda_{\mathbf{q}}(\boldsymbol{\tau}_s)\,ds\right)-\bar{\mathbf{N}}\left(\int_0^{s_1\wedge \mathbf{T}_Q}\lambda_{\mathbf{q}}(\boldsymbol{\tau}_s)\,ds\right)\right\}\\ &= E\left\{\bar{\mathbf{N}}\left(\int_0^{s_1\wedge \mathbf{T}_Q}\lambda_{\mathbf{q}}(\boldsymbol{\tau}_s)\,ds+\int_{s_1\wedge \mathbf{T}_Q}^{s_2\wedge \mathbf{T}_Q}\lambda_{\mathbf{q}}(\boldsymbol{\tau}_s)\,ds\right)-\bar{\mathbf{N}}\left(\int_0^{s_1\wedge \mathbf{T}_Q}\lambda_{\mathbf{q}}(\boldsymbol{\tau}_s)\,ds\right)\right\}\\ &\le E\left\{\bar{\mathbf{N}}\left(\int_0^{s_1\wedge \mathbf{T}_Q}\lambda_{\mathbf{q}}(\boldsymbol{\tau}_s)\,ds+\lambda_{\max}(s_2-s_1)\right)-\bar{\mathbf{N}}\left(\int_0^{s_1\wedge \mathbf{T}_Q}\lambda_{\mathbf{q}}(\boldsymbol{\tau}_s)\,ds\right)\right\}\\ &= \lambda_{\max}(s_2-s_1).\end{aligned}$$

Here, we used the fact that for a Poisson process $\bar{\mathbf{N}}$, we have $E\left\{\bar{\mathbf{N}}(b)-\bar{\mathbf{N}}(a)\right\}=b-a$ for all $b\ge a\ge 0$ [37]. □

Recall $\mathcal{L}_\nu\, g(x,\tau,q)$ from (4.15). Given compact sets $\mathbb{Q}^q\subset\mathbb{R}^n\times[0,T_q)$ $q\in\mathcal{S}$, we define

$$\mathcal{L}^Q_\nu g(x,\tau,q) := \begin{cases}\mathcal{L}_\nu g(x,\tau,q) & (x,\tau)\in \mathrm{Int}(\mathbb{Q}^q)\\ 0 & \text{otherwise}\end{cases}$$

$\forall q\in\mathcal{S}$ and all $x\in\mathbb{R}^n, \tau\in[0,T_q),\ \nu\in\mathbb{R}^m$. Next, we use [29, Theorem 6.2], and derive a Dynkin's-like formula for the stopped process.

Lemma 4.3.3. *Suppose that $Q_x \subset \mathbb{R}^n$ and $Q_\tau^q \subset [0, T_q), \forall q \in \mathcal{S}$ are all compact sets and define $Q := \cup_{q\in\mathcal{S}} Q_x \times Q_\tau^q$. Let μ in* (4.11) *be a feedback policy such that the stochastic process defined by (4.1) exist globally with probability one. For every function $V(X) = V(x, \tau, q)$ that is continuously differentiable with respect to its first and second arguments, every initial mode $q \in \mathcal{S}$, every inital condition $(x, \tau) \in \mathbb{R}^n \times [0, T_q]$ and every $0 \leq t < \infty$, we have that*

$$\begin{aligned} e^{-\rho t} E_z \left\{ V\left(\mathbf{X}_Q(t)\right)\right\} &= V\left(x, \tau, q\right) \\ &+ I_{(\tau=T_q)} \big(\sum_{j\neq q} \mathrm{P}_{qj} V\left(\mathrm{H}_{qj} x, 0, j\right) - V\left(x, \tau, q\right) \big) \\ &+ E_z \left\{ \int_0^t e^{-\rho s} \left(\mathcal{L}_\mu^Q V\left(\mathbf{X}_Q(s)\right) - \rho V\left(\mathbf{X}_Q(s)\right)\right) \, ds \right\}, \end{aligned} \tag{4.27}$$

where $\mathbf{X}_Q$ is the stopped process defined by (4.25) *with the process $\mathbf{X}$ constructed according to procedure described in Section 4.1, initialized with $z = (x, \tau, q)$.*

Proof of Lemma 4.3.3. Let $\mathbf{N}_Q(t)$ denote the Q-stopped jump timer in (4.26). One should note that $\mathbf{N}_Q(t)$ does not count a jump that may happen at 0 due to $\tau = T_q$. Between jump times, $\mathbf{X} = (\mathbf{x}, \boldsymbol{\tau}, \mathbf{q})$ is absolutely continuous and evolves according to the vector field of the process, and at a jump time $\mathbf{t}_k,\ k \leq \mathbf{N}_Q(t)$, we have an instantaneous jump from $\mathbf{X}(\mathbf{t}_k^-)$ to $\mathbf{X}(\mathbf{t}_k) = (\mathbf{x}(\mathbf{t}_k), 0, \mathbf{q}(\mathbf{t}_k))$. Therefore, one can conclude that

$$\begin{aligned}
e^{-\rho(t\wedge \mathbf{T}_Q)}V(\mathbf{X}(t\wedge \mathbf{T}_Q)) \overset{\text{wpo}}{=} & V(x,\tau,q) \\
& + I_{(\tau=T_q)}\big[V\left(\mathrm{H}_{q\mathbf{q}(0)}\ x,0,\mathbf{q}(0)\right) - V\left(x,\tau,q\right)\big] \\
& + \int_0^{t\wedge \mathbf{T}_Q} e^{-\rho s}\left(\mathcal{L}_\mu^f V(\mathbf{X}_s) - \rho V\left(\mathbf{X}_s\right)\right)\, ds \\
& + \sum_{k=1}^{\infty} e^{-\rho \mathbf{t}_k}\mathbf{I}_{(k\leq \mathbf{N}_Q(t))}\left(V(\mathbf{X}(\mathbf{t}_k)) - V(\mathbf{X}(\mathbf{t}_k^-))\right)
\end{aligned}$$

with $\mathbf{I}_{(k\leq \mathbf{N}_Q(t))}$ denoting the indicator function of the event $k \leq \mathbf{N}_Q(t)$ and $I_{(\tau=T_q)}$ indicates if a jump happens at time 0. We have $\mathbf{X}_Q(s) = \mathbf{X}(s)$ for all $s\in[0, t\wedge \mathbf{T}_Q]$ and these processes are also equal at any jump time in $\mathbf{t}_k$, $k\leq \mathbf{N}_Q(t)$. Moreover, $\mathbf{X}_Q$ remains constant on $[t\wedge \mathbf{T}_Q, t]$. Thus, we conclude that

$$\begin{aligned}
e^{-\rho t}V(\mathbf{X}_Q(t)) \overset{\text{wpo}}{=} & V(x,\tau,q) + I_{(\tau=T_q)}\big[V\left(\mathrm{H}_{q\mathbf{q}(0)}\ x,0,\mathbf{q}(0)\right) - V\left(x,\tau,q\right)\big] \\
& + \int_0^{t} e^{-\rho s}\left(\mathcal{L}_\mu^{f,Q} V(\mathbf{X}_Q(s)) - \rho V(\mathbf{X}_Q(s))\right)\, ds \\
& + \sum_{k=1}^{\infty} e^{-\rho \mathbf{t}_k}\mathbf{I}_{(k\leq \mathbf{N}_Q(t))}\left(V(\mathbf{X}_Q(\mathbf{t}_k)) - V(\mathbf{X}_Q(\mathbf{t}_k^-))\right),
\end{aligned} \tag{4.28}$$

where for every $q\in\mathcal{S}$,

$$\mathcal{L}_\mu^{f,Q} g(x,\tau,q) := \begin{cases} \mathcal{L}_\mu^f g(x,\tau,q) & (x,\tau)\in \mathrm{Int}(Q^q) \\ 0 & \text{otherwise.} \end{cases}$$

and $Q^q = Q_x \times Q_\tau^q$. We take the expected value of the both sides of (4.28), conditioned upon the initial condition $z=(x,\tau,q)$. Starting with the expectation of the second term on the right-hand side, we get

$$E_z \left\{ I_{(\tau=T_q)} \left[V\left(\mathrm{H}_{q\mathbf{q}(0)}\ x, 0, \mathbf{q}(0) \right) - V\left(x,\tau,q\right) \right] \right\}$$

$$= I_{(\tau=T_q)} (\sum_{j\neq q} \mathrm{P}_{qj} V(\mathrm{H}_{qj}x, 0, j) - V(x,\tau,q)) \tag{4.29}$$

This is due to steps (i) and (iii) in the construction of the sample paths. We also define

$$h(x,\tau,q) := \sum_{j\neq q} \mathrm{P}_{qj} V(\mathrm{H}_{qj}x, 0, j) - V(x,\tau,q)).$$

Next, the conditional expectation of the summation in (4.28) can be computed as

$$E_z \left\{ \sum_{k=1}^{\infty} e^{-\rho \mathbf{t}_k} \mathbf{I}_{(k\leq \mathbf{N}_Q(t))} \left(V\left(\mathbf{X}_Q(\mathbf{t}_k)\right) - V\left(\mathbf{X}_Q(\mathbf{t}_k^-)\right) \right) \right\}$$
$$= E_z \left\{ \sum_{k=1}^{\infty} e^{-\rho \mathbf{t}_k} \mathbf{I}_{(k\leq \mathbf{N}_Q(t))} h\left(\mathbf{X}_Q(\mathbf{t}_k^-)\right) \right\},$$

due to step (iii) in the construction of the sample paths (regardless of whether $\tau = T_q$ or $\tau < T_q$). We use the Dominated Convergence Theorem [41] to move the series out of the expectation: To verify that the theorem is applicable, note that for $k \leq \mathbf{N}_Q(t)$ and $\mathbf{q}(\mathbf{t}_k^-) = q$, $(\mathbf{x}_Q(\mathbf{t}_k^-), \boldsymbol{\tau}_Q(\mathbf{t}_k^-)) \in Q^q$ with probability one, and we have

$$\left| e^{-\rho \mathbf{t}_k} \mathbf{I}_{(k\leq \mathbf{N}_Q(t))} h\left(\mathbf{X}_Q(\mathbf{t}_k^-)\right) \right| \overset{\text{wpo}}{\leq} h_Q$$

$$h_Q := \sup_{q\in\mathcal{S}} \sup_{(x,\tau)\in Q^q} |h(X)| < \infty.$$

Here, we used the fact that $V(x,\tau,q)$ is C^1 and therefore $h(x,\tau,q)$ is locally bounded for any $q \in \mathcal{S}$. Moreover, since the number of jump times $\mathbf{t}_k$ with

$k \leq \mathbf{N}_Q(t)$ is upper bounded by $\mathbf{N}_Q(t)$, with probability one, we conclude that for every finite integer M

$$\left|\sum_{k=1}^{M} e^{-\rho \mathbf{t}_k} \mathbf{I}_{(k \leq \mathbf{N}_Q(t))} h(\mathbf{X}_Q(\mathbf{t}_k^-))\right| \overset{\text{wpo}}{\leq} \sum_{k=1}^{\infty} \mathbf{I}_{(k \leq \mathbf{N}_Q(t))} \left|h(\mathbf{X}_Q(\mathbf{t}_k^-))\right| \leq h_Q \mathbf{N}_Q(t)$$

and by Lemma 4.3.2, we know that $E_z\left\{h_Q \mathbf{N}_Q(t)\right\} < h_Q \lambda_{\max} t < \infty$ for every $t < \infty$. We therefore conclude from the Dominated Convergence Theorem [41] that

$$E_z\left\{\sum_{k=1}^{\infty} e^{-\rho \mathbf{t}_k} \mathbf{I}_{(k \leq \mathbf{N}_Q(t))} h\left(\mathbf{X}_Q(t_k^-)\right)\right\} = \sum_{k=1}^{\infty} E_z\left\{e^{-\rho \mathbf{t}_k} \mathbf{I}_{(k \leq \mathbf{N}_Q(t))} h\left(\mathbf{X}_Q(t_k^-)\right)\right\}.$$

Moreover, using the density functions of the inter-jump intervals and defining $\mathbf{t}_0 := 0$, one can show that (Appendix C)

$$\sum_{k=1}^{\infty} E_z\left\{e^{-\rho \mathbf{t}_k} \mathbf{I}_{(k \leq \mathbf{N}_Q(t))} h\left(\mathbf{X}_Q(t_k^-)\right)\right\} = \sum_{k=1}^{\infty} E_z\left\{\int_{t \wedge \mathbf{t}_{k-1}}^{t \wedge \mathbf{t}_k} e^{-\rho s} (h\lambda)_Q(\mathbf{X}_Q(s))\, ds\right\} \tag{4.30}$$

with $(h\lambda)_Q(x, \tau, q) := h(x, \tau, q)\lambda_q(\tau)$ inside Q^q and zero otherwise. We are able to use the result of Appendix C, because V is continuous, the solutions exist globally, and therefore $t \to h(X(t))$ is integrable on any finite time interval $[0, t]$. We now use the Dominated Convergence Theorem to move the series back inside the expectation in (4.30): To verify that this is allowed, note that by defining $\lambda_{\max} := \sup_{q \in \mathcal{S}} \sup_{\tau \in Q_\tau^q} \lambda_q(\tau)$ as before, any finite summation is dominated by

$$\left|\sum_{k=1}^{M} \int_{t \wedge \mathbf{t}_{k-1}}^{t \wedge \mathbf{t}_k} e^{-\rho s} (h\lambda)_Q(\mathbf{X}_Q(s))\, ds\right| = \left|\int_0^{t \wedge \mathbf{t}_M} e^{-\rho s} (h\lambda)_Q(\mathbf{X}_Q(s))\, ds\right| \leq h_Q \lambda_{\max} t$$

for all $M \geq 1$ and $t < \infty$. By Proposition 4.1.2, $\mathbf{t}_M$ becomes larger than any $t < \infty$ with probability one, as M tends to infinity, therefore

$$\sum_{k=1}^{\infty} \int_{t\wedge \mathbf{t}_{k-1}}^{t\wedge \mathbf{t}_k} e^{-\rho s}(h\lambda)_Q(\mathbf{X}_Q(s))\ ds = \lim_{M\to\infty} \int_0^{t\wedge \mathbf{t}_M} e^{-\rho s}(h\lambda)_Q(\mathbf{X}_Q(s))\ ds$$
$$\overset{\text{wpo}}{=} \int_0^t e^{-\rho s}(h\lambda)_Q(\mathbf{X}_Q(s))\ ds.$$

We therefore conclude from the Dominated Convergence Theorem that, one can move the series inside the expectation in the left-hand side of (4.30) and obtain

$$E_z\left\{\sum_{k=1}^{\infty} e^{-\rho \mathbf{t}_k} \mathbf{I}_{(k\leq \mathbf{N}_Q(t))} \left(V\left(\mathbf{X}_Q(\mathbf{t}_k)\right) - V\left(\mathbf{X}_Q(\mathbf{t}_k^-)\right)\right)\right\}$$
$$= E_z\left\{\int_0^t e^{-\rho s}(h\lambda)_Q(\mathbf{X}(s))\ ds\right\}.$$

Using this equality and (4.29), when we take the expectation of both sides of (4.28) conditioned upon initial condition, we obtain

$$\begin{aligned} e^{-\rho t} E_z\left\{V\left(\mathbf{X}_Q(t)\right)\right\} &= V\left(x,\tau,q\right) + I_{(\tau=T_q)}\Big(\sum_{j\neq q} \mathrm{P}_{qj} V\left(\mathrm{H}_{qj}x, 0, j\right) - V\left(x,\tau,q\right)\Big) \\ &+ E_z\left\{\int_0^t e^{-\rho s}\left(\mathcal{L}_\mu^{f,Q} V\left(\mathbf{X}_Q(s)\right) - \rho V\left(\mathbf{X}_Q(s)\right)\right)\ ds\right\} \\ &+ E_z\left\{\int_0^t e^{-\rho s}(h\lambda)_Q(\mathbf{X}_Q(s))\ ds\right\}. \end{aligned}$$

Therefore,

$$\begin{aligned} e^{-\rho t} E_z\left\{V\left(\mathbf{X}_Q(t)\right)\right\} &= V\left(x,\tau,q\right) + I_{(\tau=T_q)}\Big(\sum_{j\neq q} \mathrm{P}_{qj} V\left(\mathrm{H}_{qj}x, 0, j\right) - V\left(x,\tau,q\right)\Big) \\ &+ E_z\left\{\int_0^t e^{-\rho s}\left(\mathcal{L}_\mu^{Q} V\left(\mathbf{X}_Q(s)\right) - \rho V\left(\mathbf{X}_Q(s)\right)\right)\ ds\right\} \end{aligned}$$

which concludes the proof. □

We are now ready to prove Theorem 4.3.1.

Proof of Theorem 4.3.1. Inspired by the result of Theorem 4.2.1, we consider the following minimization problem. For given $q \in \mathcal{S}$, $x \in \mathbb{R}^n$ and $\tau \in [0, T_q)$, consider the following equation:

$$\begin{aligned}\min_{\nu \in \mathbb{R}^m} \{&\mathcal{L}_\nu V(x,\tau,q) - \rho V(x,\tau,q) \\ &+(x-\bar{x}_q)' Q_q (x-\bar{x}_q) + (\nu - \bar{u}_q)' R_q (\nu - \bar{u}_q)\} = 0. \end{aligned} \tag{4.31}$$

It is straightforward to show that $\nu^* := \mu^*(x,\tau,q)$ defined in (4.24) achieves the minimum in (4.31). Since $\Lambda_q(\tau), \Gamma_q(\tau), \Omega_q(\tau)$ satisfy (4.21)-(4.23), one can show that defining $V^*(x,\tau,q) = x'\Lambda_q(\tau)x + x'\Gamma_q(\tau) + \Omega_q(\tau)$, this function is a solution to (4.31) and satisfies the boundary condition

$$V^*(x, T_q, q) = \sum_{j \neq q} \mathrm{P}_{qj} V^*(\mathrm{H}_{qj} x, 0, j). \tag{4.32}$$

We first show that $V^*(x,\tau,q)$ is the cost due to the feedback policy μ^* and then that μ^* is indeed the optimal policy. One should note that by Lemma 4.3.1, the stochastic process defined by (4.1) with feedback policy μ^* exists globally with probability one.

Denoting $X := (x,\tau,q)$, the function $V^*(X) = V^*(x,\tau,q)$ is non-negative, C^1 with respect to (x,τ), and uniformly bounded in τ over $[0, T_q]$ for a fixed value of x. For $m \geq 0$, define $Q_x(m) := \{x \in \mathbb{R}^n : \sup_{q\in\mathcal{S},\tau\in[0,T_q]} V(x,\tau,q) \leq m\}$, $Q_\tau^q(m) := [0,(1-e^{-m})T_q]$ and

$$Q^m := \cup_{q\in\mathcal{S}} Q_x(m) \times Q_\tau^q(m).$$

As noted above, V^* is a solution to (4.31), so we have that

$$\mathcal{L}_{\mu^*} V^* - \rho V^* = -\ell_{\mu^*} \leq 0 \tag{4.33}$$

with μ^* given in (4.24). Since $\mathcal{L}_{\mu^*} V^*, V^*$ and $\ell_{\mu^*}(x,\tau,q)$ are quadratic functions in x and $\Lambda_i(\tau), \Gamma_i(\tau), \Omega_i(\tau)$ are bounded in τ for every $i \in \mathcal{S}$, there exists a positive constant c such that $\mathcal{L}_{\mu^*} V^* - \rho V^* \leq -cV^*$. Our goal is to use (4.27) but with $\mathbf{X}_{Q^m}$ replaced by $\mathbf{X}$, which we will achieve by making $m \to \infty$. From the definition of the stopped process, we have $\mathbf{X}(t \wedge \mathbf{T}_{Q^m}) = \mathbf{X}_{Q^m}(t), \forall t \geq 0$, and using (4.27), we get

$$\begin{aligned} e^{-\rho t} E_z \left\{ V^* \left(\mathbf{X}_{Q^m}(t) \right) \right\} &= V^* \left(x, \tau, q \right) \\ &+ I_{(\tau=T_q)} \big(\sum_{j\neq q} \mathrm{P}_{qj} V^* \left(\mathrm{H}_{qj} x, 0, j \right) - V^* \left(x, \tau, q \right) \big) \\ &+ E_z \left\{ \int_0^t e^{-\rho s} \mathcal{L}_{\mu^*} V^* \left(\mathbf{X}_{Q^m}(s) \right) - \rho V^* \left(\mathbf{X}_{Q^m}(s) \right) \right\} \, ds \end{aligned} \tag{4.34}$$

for all $t < \infty$. We show next that, for the function V^* considered here, the second term on the right in (4.34) is always zero. For $\tau < T_q$, we have $I_{(\tau=T_q)} = 0$, and for $\tau = T_q$, one can use (4.32) to conclude that

$$I_{(\tau=T_q)} \left(\sum_{j\neq q} \mathrm{P}_{qj} V^* \left(\mathrm{H}_{qj} x, 0, q \right) - V^* \left(x, \tau, q \right) \right) = 0,$$

for all $q \in \mathcal{S}$ and $(x, \tau) \in \mathbb{R}^n \times [0, T_q]$. Using Fubini's Theorem [41], one can interchange the expectation and the integration in (4.34), and obtain

$$\begin{aligned} e^{-\rho t} E_z \left\{ V^* \left(\mathbf{X}_{Q^m}(t) \right) \right\} &= V^*(x, \tau, q) \\ &+ \int_0^t e^{-\rho s} E_z \left\{ \mathcal{L}_{\mu^*} V^* \left(\mathbf{X}_{Q^m}(s) \right) - \rho V^* \left(\mathbf{X}_{Q^m}(s) \right) \right\} \, ds \end{aligned} \tag{4.35}$$

for all $t < \infty$. Hence $t \to E_z \left\{ e^{-\rho t} V^*(\mathbf{X}_{Q^m}(t)) \right\}$ is absolutely continuous and therefore differentiable almost everywhere with

$$\frac{d}{dt} E_z \left\{ e^{-\rho t} V^*(\mathbf{X}_{Q^m}(t)) \right\} = e^{-\rho t} E_z \left\{ \mathcal{L}_{\mu^*} V^*(\mathbf{X}_{Q^m}) - \rho V^*(\mathbf{X}_{Q^m}) \right\}.$$

We therefore have

$$\begin{aligned} \frac{d}{dt} E_z \left\{ e^{(c-\rho)t} V^*(\mathbf{X}_{Q^m}(t)) \right\} &= e^{(c-\rho)t} E_z \left\{ \mathcal{L}_{\mu^*} V^*(\mathbf{X}_{Q^m}(t)) \right\} \\ &+ (c - \rho) e^{(c-\rho)t} E_z \left\{ V^*(\mathbf{X}_{Q^m}(t)) \right\} \leq 0. \end{aligned}$$

From this, we conclude that

$$0 \leq E_z \left\{ e^{-\rho t} V^*(\mathbf{X}_{Q^m}(t)) \right\} \leq e^{-ct} V^*(x, \tau, q). \tag{4.36}$$

Since $\lim_{t \to \infty} \lim_{m \to \infty} e^{-ct} V^*(x, \tau, q) = 0$, we conclude from (4.36) that

$$\lim_{t \to \infty} \lim_{m \to \infty} E_z \left\{ e^{-\rho t} V^*(\mathbf{X}_{Q^m}(t)) \right\} = 0. \tag{4.37}$$

Moreover, for the feedback policy μ^*, the integral term on the right-hand side of (4.34) can be bounded by

$$\begin{aligned}
&\left|\int_0^t e^{-\rho s}\left(\mathcal{L}_{\mu^*}^{Q^m} V^*\left(\mathbf{X}_{Q^m}(s)\right) - \rho V^*\left(\mathbf{X}_{Q^m}(s)\right)\right)\, ds\right| \\
&\quad = \left|\int_0^{t\wedge \mathbf{T}_{Q^m}} e^{-\rho s}\left(\mathcal{L}_{\mu^*} V^*\left(\mathbf{X}(s)\right) - \rho V^*\left(\mathbf{X}(s)\right)\right)\, ds\right| \\
&\quad \leq \int_0^t e^{-\rho s}\left|\mathcal{L}_{\mu^*} V^*\left(\mathbf{X}(s)\right) - \rho V^*\left(\mathbf{X}(s)\right)\right|\, ds,
\end{aligned}$$

and by the Dominated Convergence Theorem, we get

$$\begin{aligned}
&\lim_{m\to\infty} E_z\Big\{\int_0^{t\wedge T_{Q^m}} e^{-\rho s}\left(\mathcal{L}_{\mu^*} V^*\left(\mathbf{X}_s\right) - \rho V^*\left(\mathbf{X}_s\right)\right) ds\Big\} \\
&\quad = E_z\Big\{\int_0^t e^{-\rho s}\left(\mathcal{L}_{\mu^*} V^*\left(\mathbf{X}_s\right) - \rho V^*\left(\mathbf{X}_s\right)\right)\, ds\Big\}.
\end{aligned} \tag{4.38}$$

Now, we take the limit of (4.34) as $m \to \infty$ and $t \to \infty$. By (4.37), the limit on the left of (4.34) is zero and from (4.38) and (4.33), we conclude

$$V^*(x,\tau,q) = E_z\left\{\int_0^\infty e^{-\rho s}\ell_{\mu^*}(\mathbf{x}_s,\boldsymbol{\tau}_s,\mathbf{q}_s)\, ds\right\}$$

for all $q \in \mathcal{S}$ and $x \in \mathbb{R}^n, \tau \in [0, T_q]$. This implies that $V^*(x,\tau,q)$ is the cost due to the feedback policy μ^*, conditioned upon the initial condition $z = (x,\tau,q)$.

Now let μ be an arbitrary feedback control for which the process $\mathbf{X}(t)$ exists globally and we have $E_z\left\{||\mathbf{x}(t)||^2\right\} \leq k_1 e^{-k_2 t}||x_0||$ for some $k_1 > 0, k_2 > -\rho$. For such a process, we still have

$$\lim_{t\to\infty}\lim_{m\to\infty} e^{-\rho t} E_z\left\{V^*(\mathbf{X}_{Q^m}(t)\right\} = 0,$$

but since μ typically does not minimize (4.31), we have $\mathcal{L}_\mu V^*(x,\tau,q) - \rho V^* \geq -\ell_\mu(x,\tau,q)$ instead of (4.33). In this case, the argument above applies but with the appropriate equalities replaced by inequalities, leading to

$$V^*(x,\tau,q) \leq \lim_{t\to\infty}\lim_{m\to\infty} e^{-\rho t} E_z\left\{V^*(\mathbf{X}_{\mathbf{Q}^\mathbf{m}}(t)\right\} + E_z\left\{\int_0^\infty e^{-\rho s}\ell_\mu(\mathbf{x}_s,\boldsymbol{\tau}_s,\mathbf{q}_s)\ ds\right\}$$
$$= E_z\left\{\int_0^\infty e^{-\rho s}\ell_\mu(\mathbf{x}_s,\boldsymbol{\tau}_s,\mathbf{q}_s)\ ds\right\}. \tag{4.39}$$

Therefore, the cost V^* associated with μ^* is always no larger than the cost J_μ associated with another policy μ, which proves the optimality of μ^*. □

4.3.1 Recursive computations

In this subsection, we examine a recursive algorithm that provides a numerical technique for computing the optimal discounted cost in Theorem 4.3.1. The main challenge in solving (4.21)-(4.23) in Theorem 4.3.1 is due to the unknown boundary conditions, but the following algorithm can be used to compute the optimal cost and the optimal policy.

Algorithm 4.3.1.

(a) Set $k=0$, $\Lambda_q^{(0)}(\tau) = 0_{n\times n}$, $\Gamma_q^{(0)}(\tau) = 0_{n\times 1}$, and $\Omega_q^{(0)}(\tau) = c_q \in [0,\infty)$, $\forall \tau \in [0,T_q]$ and $\forall q \in \mathcal{S}$.

(b) Compute solutions that are C^1 in the interval $[0, T_q]$ for the following ODEs and

$$\begin{cases} -\frac{\partial \Lambda_q^{(k+1)}}{\partial \tau} = A_q' \Lambda_q^{(k+1)} + \Lambda_q^{(k+1)} A_q - \rho \Lambda_q^{(k+1)} \\ \quad -\Lambda_q^{(k+1)} B_q R_q^{-1} B_q' \Lambda_q^{(k+1)} + Q_q \\ \quad +\lambda_q(\tau) \sum_{j \neq q} \mathrm{P}_{qj} (\mathrm{H}_{qj}' \Lambda_j^{(k)}(0) \mathrm{H}_{qj} - \Lambda_q^{(k+1)}(\tau)) \\ \Lambda_q^{(k+1)}(T_q) = \sum_{j \neq q} \mathrm{P}_{qj} \mathrm{H}_{qj}' \Lambda_j^{(k)}(0) \mathrm{H}_{qj} \end{cases} \tag{4.40}$$

$$\begin{cases} -\frac{\partial \Gamma_q^{(k+1)}}{\partial \tau} = (A_q' - \Lambda_q^{(k+1)} B_q R_q^{-1} B_q' - \rho I) \Gamma_q^{(k+1)} \\ \quad -2 Q_q \bar{x}_q + 2 \Lambda_q^{(k+1)} (B_q \bar{u}_q + d_q) \\ \quad +\lambda_q(\tau) \sum_{j \neq q} \mathrm{P}_{qj} (\mathrm{H}_{qj}' \Gamma_j^{(k)}(0) - \Gamma_q^{(k+1)}(\tau)) \\ \Gamma_q^{(k+1)}(T_q) = \sum_{j \neq q} \mathrm{P}_{qj} \mathrm{H}_{qj}' \Gamma_j^{(k)}(0) \end{cases} \tag{4.41}$$

$$\begin{cases} -\frac{\partial \Omega_q^{(k+1)}}{\partial \tau} = -\frac{1}{4} \Gamma_q^{(k+1)'} B_q R_q^{-1} B_q' \Gamma_q^{(k+1)} + \bar{x}_q' Q_q \bar{x}_q \\ \quad +\Gamma_q^{(k+1)'} (B_q \bar{u}_q + d_q) - \rho \Omega_q^{(k+1)} \\ \quad +\lambda_q(\tau) \sum_{j \neq q} \mathrm{P}_{qj} (\Omega_j^{(k)}(0) - \Omega_q^{(k+1)}(\tau)) \\ \Omega_q^{(k+1)}(T_q) = \sum_{j \neq q} \mathrm{P}_{qj} \Omega_j^{(k)}(0) \end{cases} \tag{4.42}$$

(c) Set $k \to k+1$ and go back to step (b).

The following Theorem implies that as k increases, one can achieve a cost that can be made arbitrarily close to the optimal cost by selecting large k.

Theorem 4.3.1. *Suppose that for every $k > 0$ in Algorithm 4.3.1, the functions $\Lambda_q^{(k)} : [0, T_q] \to \mathbb{R}^{n \times n}$, $\Gamma_q^{(k)} : [0, T_q] \to \mathbb{R}^n$, $\Omega_q^{(k)} : [0, T_q] \to \mathbb{R}$, $q \in \mathcal{S}$ to* (4.40)-(4.42) *have the property that the function* $G^k(x,\tau,q) := x'\Lambda_q^{(k)}(\tau)x + x'\Gamma_q^{(k)}(\tau) + \Omega_q^{(k)}(\tau)$ *is non-negative. Then, for every initial mode* $q \in \mathcal{S}$ *and every initial condition* $(x,\tau) \in \mathbb{R}^n \times [0,T_q]$,

$$G^k(x,\tau,q) = \min_{\mu} E_z\left\{\int_0^{\mathbf{t}_k} e^{-\rho s}\ell_\mu(\mathbf{x}_s,\boldsymbol{\tau}_s,\mathbf{q}_s)\ ds + e^{-\rho\mathbf{t}_k}c_{\mathbf{q}(\mathbf{t}_k)}\right\} \tag{4.43}$$

Moreover, for every $\epsilon > 0$, *there exists a sufficiently large* k *so that the feedback law*

$$\mu_k^*(x,\tau,q) := \bar{u}_q - \frac{1}{2}R_q^{-1}B_q'\big(2\Lambda_q^{(k)}(\tau)x + \Gamma_q^{(k)}(\tau)\big) \tag{4.44}$$

leads to a cost that is above the optimal cost V^* *associated with the optimal policy* μ^* *(4.24) by less than* ϵ.

Theorem 4.3.1 implies that the minimum cost in (4.10) is given by $\lim_{k\to\infty} G^k(x,\tau,q)$ and such a cost is achieved by $\mu_k^*(x,\tau,q)$.

The following result will be used in the Proof of Theorem 4.3.1.

Lemma 4.3.1. *For every function* $\psi(.,.,q) : \mathbb{R}^n \times [0,T_q] \to \mathbb{R}$ *that is continuous with respect to its first and second arguments, we have*

$$\begin{aligned} E_z\left\{e^{-\rho\mathbf{t}_1}\psi(\mathbf{x}_{\mathbf{t}_1^-},\boldsymbol{\tau}_{\mathbf{t}_1^-},\mathbf{q}_{\mathbf{t}_1^-})\right\} &= I_{(\tau=T_q)}E_z\{\psi(\mathbf{x}_{\mathbf{t}_0},\boldsymbol{\tau}_{\mathbf{t}_0},\mathbf{q}_{\mathbf{t}_0})\} \\ &+ E_z\left\{\int_0^{\mathbf{t}_1} e^{-\rho s}\lambda_{\mathbf{q}}(\boldsymbol{\tau})\psi(\mathbf{x},\boldsymbol{\tau},\mathbf{q})\ ds\right\} \end{aligned} \tag{4.45}$$

for every initial mode $q \in \mathcal{S}$ *and every initial condition* $(x,\tau) \in \mathbb{R}^n \times [0,T_q]$ *with* $z = (x,\tau,q)$.

Proof of Lemma 4.3.1. We start by computing the expectation of the last term on the right-hand side of (4.45) when $\tau < T_q$. Until the first jump, we have $\boldsymbol{\tau}(t) = t + \tau$, so by using (4.5), we get

$$\begin{aligned} &E_z\left\{\int_0^{\mathbf{t}_1} e^{-\rho s}\lambda_{\mathbf{q}}(\boldsymbol{\tau}_s)\psi(\mathbf{x}_s, \boldsymbol{\tau}_s, \mathbf{q}_s)\ ds\right\} \\ &= \int_\tau^{T_q}\int_0^{h_0-\tau} e^{-\rho s}\lambda_q(\tau+s)\psi(x_s, \tau_s, q_s)\ \tfrac{f_q(h_0)}{1-F_q(\tau)}\ ds\ dh_0 \\ &= \int_0^{T_q-\tau}\int_0^{w} e^{-\rho s}\lambda_q(\tau+s)\psi(x_s, \tau_s, q_s)\ \tfrac{f_q(w+\tau)}{1-F_q(\tau)}\ ds\ dw \end{aligned}$$

where we made the change of integration variable $w = h_0 - \tau$ with $f_q(w)$ denoting the probability density function of the jump interval in mode q. By changing the order of integration, we have that

$$\begin{aligned} &E_z\left\{\int_0^{\mathbf{t}_1} e^{-\rho s}\lambda_{\mathbf{q}}(\boldsymbol{\tau}_s)\psi(\mathbf{x}_s, \boldsymbol{\tau}_s, \mathbf{q}_s)\ ds\right\} \\ &= \int_0^{T_q-\tau} e^{-\rho s}\lambda_q(\tau+s)\psi(x_s, \tau_s, q_s)\int_{s+\tau}^{T_q}\tfrac{f_q(w)}{1-F_q(\tau)}\ dw\ ds \\ &= \int_0^{T_q-\tau} e^{-\rho s}\lambda_q(\tau+s)\psi(x_s, \tau_s, q_s)\tfrac{(1-F_q(s+\tau))}{1-F_q(\tau)}\ ds \\ &= \int_0^{T_q-\tau}\tfrac{f_q(\tau+s)}{1-F_q(\tau)}e^{-\rho s}\psi(x_s, \tau_s, q_s)\ ds \\ &= E_z\left\{e^{-\rho\mathbf{t}_1}\psi(\mathbf{x}_{\mathbf{t}_1^-}, \boldsymbol{\tau}_{\mathbf{t}_1^-}, \mathbf{q}_{\mathbf{t}_1^-})\right\}. \end{aligned} \tag{4.46}$$

The proof of the equation (4.46) follows along the lines of the proof of equation (4.54). Moreover, for $\tau = T_q$, we have[6]

$$E_z\{e^{-\rho\mathbf{t}_1}\psi(\mathbf{x}_{\mathbf{t}_1^-}, \boldsymbol{\tau}_{\mathbf{t}_1^-}, \mathbf{q}_{\mathbf{t}_1^-})\} = E_z\left\{\psi(x_{\mathbf{t}_0}, \boldsymbol{\tau}_{\mathbf{t}_0}, \mathbf{q}_{\mathbf{t}_0}) + E_{z_{\mathbf{t}_0}}\{e^{-\rho\mathbf{t}_1}\psi(\mathbf{x}_{\mathbf{t}_1^-}, \boldsymbol{\tau}_{\mathbf{t}_1^-}, \mathbf{q}_{\mathbf{t}_1^-})\}\right\}$$

[6] We used the conditional expectation property $E\{E\{X|\mathfrak{F}\}|\mathfrak{G}\} = E\{X|\mathfrak{G}\}$ for σ-algebras $\mathfrak{G} \subset \mathfrak{F}$.

with $z_{\mathbf{t}_0} = (\mathbf{x}_{\mathbf{t}_0}, \boldsymbol{\tau}_{\mathbf{t}_0}, \mathbf{q}_{\mathbf{t}_0})$. Similar to (4.46), one can show that

$$\begin{aligned} & E_z\left\{E_{z_{\mathbf{t}_0}}\{\textstyle\int_0^{\mathbf{t}_1} e^{-\rho s}\lambda_{\mathbf{q}}(\boldsymbol{\tau}_s)\psi(\mathbf{x}_s, \boldsymbol{\tau}_s, \mathbf{q}_s)\ ds\}\right\} \\ &= E_z\left\{E_{z_{\mathbf{t}_0}}\{e^{-\rho \mathbf{t}_1}\psi(\mathbf{x}_{\mathbf{t}_1^-}, \boldsymbol{\tau}_{\mathbf{t}_1^-}, \mathbf{q}_{\mathbf{t}_1^-})\}\right\} \\ &= E_z\left\{e^{-\rho \mathbf{t}_1}\psi(\mathbf{x}_{\mathbf{t}_1^-}, \boldsymbol{\tau}_{\mathbf{t}_1^-}, \mathbf{q}_{\mathbf{t}_1^-})\right\}. \end{aligned} \tag{4.47}$$

Hence, from (4.46)-(4.47), we get

$$\begin{aligned} E_z\left\{e^{-\rho \mathbf{t}_1}\psi(\mathbf{x}_{\mathbf{t}_1^-}, \boldsymbol{\tau}_{\mathbf{t}_1^-}, \mathbf{q}_{\mathbf{t}_1^-})\right\} &= I_{(\tau=T_q)}E_z\left\{\psi(\mathbf{x}_{\mathbf{t}_0}, \boldsymbol{\tau}_{\mathbf{t}_0}, \mathbf{q}_{\mathbf{t}_0})\right\} \\ &+E_z\left\{\textstyle\int_0^{\mathbf{t}_1} e^{-\rho s}\lambda_{\mathbf{q}}(\boldsymbol{\tau}_s)\psi(\mathbf{x}_s, \boldsymbol{\tau}_s, \mathbf{q}_s)\ ds\right\}. \end{aligned}$$

□

We are now ready to prove Theorem 4.3.1.

Proof of Theorem 4.3.1. Similar to the proof of Theorem 4.3.1, for given $k > 0$, $q \in \mathcal{S}$, $x \in \mathbb{R}^n$ and $\tau \in \times[0, T_q)$, consider the following equation

$$\begin{aligned} \min_{\nu\in\mathbb{R}^m} & \left\{ \tfrac{\partial G^k}{\partial x}(A_q x + B_q \nu) + \tfrac{\partial G^k}{\partial \tau} - \rho G^k(x, \tau, q) \right. \\ & +\lambda_q(\tau)\sum_{j\neq q} \mathrm{P}_{qj}\left(G^{k-1}(\mathrm{H}_{qj}x, 0, j) - G^k(x, \tau, q)\right) \\ & \left. +(x - \bar{x}_q)'Q_q(x - \bar{x}_q) + (\nu - \bar{u}_q)'R_q(\nu - \bar{u}_q)\right\} = 0. \end{aligned} \tag{4.48}$$

It is straightforward to show that $\nu^* := \mu_k^*(x, q, \tau)$ defined in (4.44) achieves the minimum in (4.48). Moreover, since $\Lambda_q^{(k)}, \Gamma_q^{(k)}, \Omega_q^{(k)}$ satisfy (4.40)-(4.42), one can show that $G^k(x, \tau, q) = x'\Lambda_q^{(k)}(\tau)x + x'\Gamma_q^{(k)}(\tau) + \Omega_q^{(k)}(\tau)$ is a solution to (4.48). Moreover, for every $q \in \mathcal{S}$, G^k satisfies the boundary condition

$$G^k(x, T_q, q) = \sum_{j\neq q} \mathrm{P}_{qj}G^{k-1}(\mathrm{H}_{qj}x, 0, j). \tag{4.49}$$

Since $\Lambda_i^{(k)}(\tau)$ and $\Gamma_i^{(k)}(\tau)$ are bounded C^1 functions of τ, by Lemma 4.3.1, one can conclude that the stochastic process (4.1) with the feedback policy $\mu_k^*(x,\tau,q)$ exists globally with probability one.

Let $(\mathbf{x}_t, \boldsymbol{\tau}_t, \mathbf{q}_t)$ denote the stochastic process defined by (4.1) at time $t \geq 0$, starting from the initial mode $q \in \mathcal{S}$ and the initial condition $(x,\tau) \in \mathbb{R}^n \times [0, T_q]$ with the feedback control $\mu_k^*(x,\tau,q)$. For every $t \in [0, \mathbf{t}_1)$, the value of the derivative $\frac{d}{dt}G^k(\mathbf{x}(t), \boldsymbol{\tau}(t), \mathbf{q}(t))$ along the solution of the system is given by

$$\begin{aligned}
\tfrac{d}{dt}G^k(\mathbf{x}(t), \boldsymbol{\tau}(t), \mathbf{q}(t)) &\overset{\text{wpo}}{=} \tfrac{\partial G^k}{\partial x}(A_{\mathbf{q}}\mathbf{x} + B_{\mathbf{q}}\mu_k^*(\mathbf{x}, \boldsymbol{\tau}, \mathbf{q})) + \tfrac{\partial G^k}{\partial \tau} \\
&= -\ell_{\mu_k^*}(\mathbf{x}, \boldsymbol{\tau}, \mathbf{q}) + \rho G^k(\mathbf{x}, \boldsymbol{\tau}, \mathbf{q}) \\
&\quad -\lambda_{\mathbf{q}}(\boldsymbol{\tau}) \sum_{j \neq \mathbf{q}} \mathrm{P}_{\mathbf{q}j} \left(G^{k-1}(\mathrm{H}_{\mathbf{q}j}\mathbf{x}, 0, j) - G^k(\mathbf{x}, \boldsymbol{\tau}, \mathbf{q}) \right)
\end{aligned}$$

where the second equality results from the fact that G^k is a solution to (4.48). Therefore

$$\begin{aligned}
e^{-\rho \mathbf{t}_1} G^k(\mathbf{x}_{\mathbf{t}_1^-}, \boldsymbol{\tau}_{\mathbf{t}_1^-}, \mathbf{q}_{\mathbf{t}_1^-}) - G^k(x,\tau,q) &\overset{\text{wpo}}{=} I_{(\tau = T_q)} \left[G^k\left(\mathbf{x}_{\mathbf{t}_0}, \boldsymbol{\tau}_{\mathbf{t}_0}, \mathbf{q}_{\mathbf{t}_0}\right) - G^k(x,\tau,q) \right] \\
&- \int_0^{\mathbf{t}_1} e^{-\rho s} \ell_{\mu_k^*}(\mathbf{x}_s, \boldsymbol{\tau}_s, \mathbf{q}_s)\, ds \\
&- \int_0^{\mathbf{t}_1} \sum_{j \neq \mathbf{q}} \mathrm{P}_{\mathbf{q}j} e^{-\rho s} \lambda_{\mathbf{q}}(\boldsymbol{\tau}_s) \left(G^{k-1}(\mathrm{H}_{\mathbf{q}j}\mathbf{x}_s, 0, j) - G^k(\mathbf{x}_s, \boldsymbol{\tau}_s, \mathbf{q}_s) \right) ds.
\end{aligned} \tag{4.50}$$

By the result of Lemma 4.3.1 with $\psi = G^k$, we have that

$$\begin{aligned}
E_z \left\{ e^{-\rho \mathbf{t}_1} G^k(\mathbf{x}_{\mathbf{t}_1^-}, \boldsymbol{\tau}_{\mathbf{t}_1^-}, \mathbf{q}_{\mathbf{t}_1^-}) \right\} &= I_{(\tau = T_q)} E_z \left\{ G^k\left(\mathbf{x}_{\mathbf{t}_0}, \boldsymbol{\tau}_{\mathbf{t}_0}, \mathbf{q}_{\mathbf{t}_0}\right) \right\} \\
&+ E_z \left\{ \int_0^{\mathbf{t}_1} e^{-\rho s} \lambda_{\mathbf{q}}(\boldsymbol{\tau}_s) G^k(\mathbf{x}_s, \boldsymbol{\tau}_s, \mathbf{q}_s)\, ds \right\}.
\end{aligned} \tag{4.51}$$

By taking the expectation of both sides of (4.50), conditioned to the initial condition and using (4.51), we obtain

$$\begin{aligned} -G^k(x,\tau,q) \ &= -I_{(\tau=T_q)}G^k(x,T_q,q) - E_z\Big\{ \int_0^{\mathbf{t}_1} e^{-\rho s}\ell_{\mu_k^*}(\mathbf{x}_s,\boldsymbol{\tau}_s,\mathbf{q}_s)\ ds\Big\} \\ -E_z\Big\{ \sum_{j\neq \mathbf{q}} \mathrm{P}_{\mathbf{q}j} \int_0^{\mathbf{t}_1} e^{-\rho s}\lambda_{\mathbf{q}}(\boldsymbol{\tau}_s)G^{k-1}(\mathrm{H}_{\mathbf{q}j}\mathbf{x}_s,0,j)\ ds\Big\} \end{aligned} \tag{4.52}$$

Due to step (iii) in the construction of the sample paths, for any $\tau \in [0, T_q]$, we get

$$E_z\left\{e^{-\rho\mathbf{t}_1}G^{k-1}(\mathbf{x}_{\mathbf{t}_1},\boldsymbol{\tau}_{\mathbf{t}_1},\mathbf{q}_{\mathbf{t}_1})\right\} = E_z\left\{\sum_{j\neq\mathbf{q}} \mathrm{P}_{\mathbf{q}j}e^{-\rho\mathbf{t}_1}G^{k-1}(\mathrm{H}_{\mathbf{q}j}\mathbf{x}_{\mathbf{t}_1^-},0,j)\right\}. \tag{4.53}$$

To compute the right-hand side of (4.53), we consider two different cases when $\tau < T_q$ and $\tau = T_q$. Similar to (4.46), for $\tau < T_q$, one can obtain

$$\begin{aligned} &E_z\left\{\int_0^{\mathbf{t}_1} e^{-\rho s}\lambda_q(\tau_s)G^{k-1}(H_{qj}x_s,0,j)\right\} \\ &= \int_0^{T_q-\tau}\int_0^{w} e^{-\rho s}\lambda_q(\tau_s)G^{k-1}(\mathrm{H}_{qj}x_s,0,j)\ \tfrac{f_q(w+\tau)}{1-F_q(\tau)}\ ds\ dw \\ &= \int_0^{T_q-\tau} e^{-\rho s}\lambda_q(\tau_s)G^{k-1}(\mathrm{H}_{qj}x_s,0,j)\tfrac{(1-F_q(s+\tau))}{1-F_q(\tau)}\ ds \\ &= \int_0^{T_q-\tau} \tfrac{f_q(\tau+s)}{1-F_q(\tau)}e^{-\rho s}G^{k-1}(\mathrm{H}_{qj}x_s,0,j)\ ds \\ &= E_z\left\{e^{-\rho\mathbf{t}_1}G^{k-1}(\mathrm{H}_{qj}\mathbf{x}_{\mathbf{t}_1^-},0,j)\right\}. \end{aligned} \tag{4.54}$$

Moreover, for $\tau = T_q$, following the lines of the proof of (4.47), we get

$$\begin{aligned} E_z\left\{e^{-\rho\mathbf{t}_1}G^{k-1}(\mathrm{H}_{qj}\mathbf{x}_{\mathbf{t}_1^-},0,j)\right\} &= E_z\left\{\sum_{j\neq\mathbf{q}} \mathrm{P}_{\mathbf{q}j}\int_0^{\mathbf{t}_1} e^{-\rho s}\lambda_{\mathbf{q}}G^{k-1}(\mathrm{H}_{\mathbf{q}j}\mathbf{x}_s,0,j)\right\} \\ &+\sum_{j\neq q}\mathrm{P}_{qj}G^{k-1}(\mathrm{H}_{qj}x,0,j). \end{aligned} \tag{4.55}$$

Combining (4.54) and(4.55), we get

$$E_z\left\{e^{-\rho\mathbf{t}_1}G^{k-1}(\mathbf{x}_{\mathbf{t}_1},\boldsymbol{\tau}_{\mathbf{t}_1},\mathbf{q}_{\mathbf{t}_1})\right\} = E_z\left\{\sum_{j\neq\mathbf{q}}\mathrm{P}_{\mathbf{q}j}\int_0^{\mathbf{t}_1}e^{-\rho s}\lambda_{\mathbf{q}}G^{k-1}(\mathrm{H}_{\mathbf{q}j}\mathbf{x}_s,0,j)\right\}$$
$$+I_{(\tau=T_q)}\sum_{j\neq q}\mathrm{P}_{qj}G^{k-1}(\mathrm{H}_{qj}x,0,j). \tag{4.56}$$

Thus, we conclude from (4.49), (4.52) and (4.56) that

$$G^k(x,\tau,q) = E_z\left\{e^{-\rho\mathbf{t}_1}G^{k-1}(\mathbf{x}_{\mathbf{t}_1},\boldsymbol{\tau}_{\mathbf{t}_1},\mathbf{q}_{\mathbf{t}_1})\right\} + E_z\left\{\int_0^{\mathbf{t}_1}e^{-\rho s}\ell_{\mu_k^*}(x_s,\tau_s,q_s)\,ds\right\}$$

Now let μ be an arbitrary feedback policy for which the process $(\mathbf{x}(t),\boldsymbol{\tau}(t),\mathbf{q}(t)$ exists globally. Since μ typically does not minimize (4.48), we have

$$\frac{\partial G^k}{\partial x}(A_qx+B_q\mu(x,\tau,q))+\frac{\partial G^k}{\partial \tau}$$
$$\geq -\ell_\mu(x,\tau,q)+\rho G^k(x,\tau,q)-\lambda_q\sum_{j\neq q}\mathrm{P}_{qj}\left(G^{k-1}(\mathrm{H}_{qj}x,0,j)-G^k(x,\tau,q)\right)$$

instead of (4.48). In this case, the argument above applies but with the appropriate equalities replaced by inequalities, leading to

$$G^k(x,\tau,q) \quad \leq E_z\left\{\int_0^{\mathbf{t}_1}e^{-\rho s}\ell_\mu(x_s,\tau_s,q_s)\,ds\right\} + E_z\left\{e^{-\rho\mathbf{t}_1}G^{k-1}(\mathbf{x}_{\mathbf{t}_1},\boldsymbol{\tau}_{\mathbf{t}_1},\mathbf{q}_{\mathbf{t}_1})\right\}.$$

Therefore, the cost G^k associated with μ_k^* is always no larger than the cost associated with another policy μ, which proves the optimality of μ_k^*:

$$G^k(x,\tau,q) \quad = \min_\mu E_z\left\{\int_0^{\mathbf{t}_1}e^{-\rho s}\ell_\mu(x_s,\tau_s,q_s)\,ds + e^{-\rho\mathbf{t}_1}G^{k-1}(\mathbf{x}_{\mathbf{t}_1},\boldsymbol{\tau}_{\mathbf{t}_1},\mathbf{q}_{\mathbf{t}_1})\right\}. \tag{4.57}$$

We now prove (4.43) for all $k \geq 1$ by induction on k. The base of induction $k = 1$ follows directly from (4.57). Assuming now that (4.43) holds for some $k > 1$, we shall show that it holds for $k+1$

$$
\begin{aligned}
&G^k(x,\tau,q) = \min_{\mu} E_z \Big\{ \textstyle\int_0^{\mathbf{t}_1} e^{-\rho s} \ell_\mu(x_s,\tau_s,q_s)\ ds \\
&+ e^{-\rho \mathbf{t}_1} \min_{\bar\mu} E_{z_{\mathbf{t}_1}} \Big\{ \textstyle\int_0^{\mathbf{t}_2-\mathbf{t}_1} e^{-\rho w} \ell_{\bar\mu}(x_w,\tau_w,q_w)\ dw + e^{-\rho(\mathbf{t}_2-\mathbf{t}_1)} G^{k-2}(\mathbf{x}_{\mathbf{t}_2},\boldsymbol{\tau}_{\mathbf{t}_2},\mathbf{q}_{\mathbf{t}_2}) \Big\} \Big\} \\
&= \min_{\mu} E_z \Big\{ \textstyle\int_0^{\mathbf{t}_1} e^{-\rho s} \ell_\mu(x_s,\tau_s,q_s)\ ds \\
&+ e^{-\rho \mathbf{t}_1} \min_{\bar\mu} E_{z_{\mathbf{t}_1}} \Big\{ \textstyle\int_{\mathbf{t}_1}^{\mathbf{t}_2} e^{-\rho(s-\mathbf{t}_1)} \ell_{\bar\mu}(x_s,\tau_s,q_s)\ ds + e^{-\rho(\mathbf{t}_2-\mathbf{t}_1)} G^{k-2}(\mathbf{x}_{\mathbf{t}_2},\boldsymbol{\tau}_{\mathbf{t}_2},\mathbf{q}_{\mathbf{t}_2}) \Big\} \Big\} \\
&= \min_{\mu} E_z \Big\{ \textstyle\int_0^{\mathbf{t}_1} e^{-\rho s} \ell_\mu(x_s,\tau_s,q_s)\ ds \\
&+ E_{z_{\mathbf{t}_1}} \Big\{ \textstyle\int_{\mathbf{t}_1}^{\mathbf{t}_2} e^{-\rho s} \ell_\mu(x_s,\tau_s,q_s)\ ds + e^{-\rho \mathbf{t}_2} G^{k-2}(\mathbf{x}_{\mathbf{t}_2},\boldsymbol{\tau}_{\mathbf{t}_2},\mathbf{q}_{\mathbf{t}_2}) \Big\} \Big\} \\
&= \min_{\mu} E_z \Big\{ \textstyle\int_0^{\mathbf{t}_1} e^{-\rho s} \ell_\mu(x_s,\tau_s,q_s)\ ds \\
&+ \textstyle\int_{\mathbf{t}_1}^{\mathbf{t}_2} e^{-\rho s} \ell_\mu(\mathbf{x}_s,\boldsymbol{\tau}_s,\mathbf{q}_s)\ ds + e^{-\rho \mathbf{t}_2} G^{k-2}(\mathbf{x}_{\mathbf{t}_2},\boldsymbol{\tau}_{\mathbf{t}_2},\mathbf{q}_{\mathbf{t}_2}) \Big\} \\
&= \min_{\mu} E_z \Big\{ \textstyle\int_0^{\mathbf{t}_2} e^{-\rho s} \ell_\mu(\mathbf{x}_s,\boldsymbol{\tau}_s,\mathbf{q}_s)\ ds + e^{-\rho \mathbf{t}_2} G^{k-2}(\mathbf{x}_{\mathbf{t}_2},\boldsymbol{\tau}_{\mathbf{t}_2},\mathbf{q}_{\mathbf{t}_2}) \Big\}
\end{aligned}
$$

which completes the proof of (4.43). In the above calculations, we used the fact that the feedback law μ is mode-dependent.

From Proposition 4.1.2, we know that $\mathbf{t}_k \overset{\text{wpo}}{\to} \infty$ as $k \to \infty$. Therefore, $e^{-\rho \mathbf{t}_k} G^0(\mathbf{x}_{\mathbf{t}_k},\boldsymbol{\tau}_{\mathbf{t}_k},\mathbf{q}_{\mathbf{t}_k}) \to 0$ with probability one, and by Bounded Convergence Theorem $E_z\{e^{-\rho \mathbf{t}_k} G^0(\mathbf{x}_{\mathbf{t}_k},\boldsymbol{\tau}_{\mathbf{t}_k},\mathbf{q}_{\mathbf{t}_k})\} \to 0$. Since the integral on the right of (4.43) is an increasing function of k, by the Monotone Convergence Theorem [58], we get

$$\lim_{k\to\infty} G^k(x,\tau,q) = \min_{\mu} E_z \left\{ \int_0^\infty e^{-\rho s} \ell_\mu(\mathbf{x}_s, \boldsymbol{\tau}_s, \mathbf{q}_s)\ ds \right\}.$$

Thus, for every initial condition $z = (x, \tau, q)$, as k increases the policy μ_k^* achieves a cost G^k that can be made arbitrarily close to the optimal cost by selecting large k. □

4.4 Metabolism of Lactose in E. Coli (re-visited)

We now return to example of Section 2.2, and study how E.Coli responds to the availability of lactose in the environment. Inspired by the model of *lac* operon in [22], we consider the case of two specific enzymes in E. Coli: the *lactose permease* and *β-galactosidase*. The first enzyme allows the bacterium to allow external lactose to enter the cytoplasm of the cell while the latter one is used for degrading lactose into glucose which is its main source of energy. We denote the concentration of *β-galactosidase* and *lactose permease* by $\mathbf{x}(t)$ and $\mathbf{y}(t)$, respectively; and we denote the concentration of the RNA molecules issued from transcription of *lac* genes by $\mathbf{z}(t)$. We consider a simple evolutionary scenario. Assume that the cell experiences two different environmental conditions $\{0, 1\}$ where 0 denotes the absences of lactose and 1 corresponds to the presence of lactose and absence of glucose. At each point in time, we assume that cost of deviation of the states

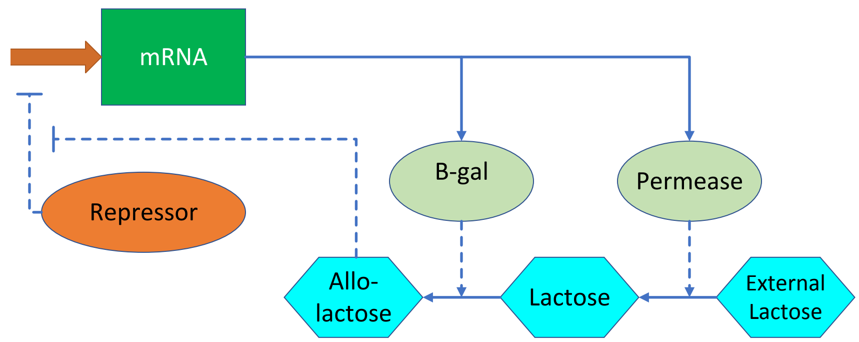


Figure 4.2: Schematic diagram of lactose metabolism in E. Coli.

from the optimal levels in the current environment is a quadratic function of the difference between these values. We also consider a term in the cost function that reflects the energetic costs of producing/decaying mRNA and proteins [75]. If the life span of the cell is modeled by an exponential random variable with mean $1/\rho$, one can model the total expected life-time cost of the cell similar to (4.10), see [48] and references in.

Concerning the concentration of the enzymes and mRNA, we use the linear model given in [72]

$$\begin{cases} \dot{\mathbf{z}} = \mathbf{u_q} - 0.716\mathbf{z}, \\ \dot{\mathbf{x}} = 9.4\mathbf{z} - 0.033\mathbf{x}, \\ \dot{\mathbf{y}} = 18.8\mathbf{z} - 0.033\mathbf{y} \end{cases} \tag{4.58}$$

where the transcription rate of $\mathbf{z}$ depends on the environmental condition, and states of the systems. From the data given in [72], one can compute the opti-

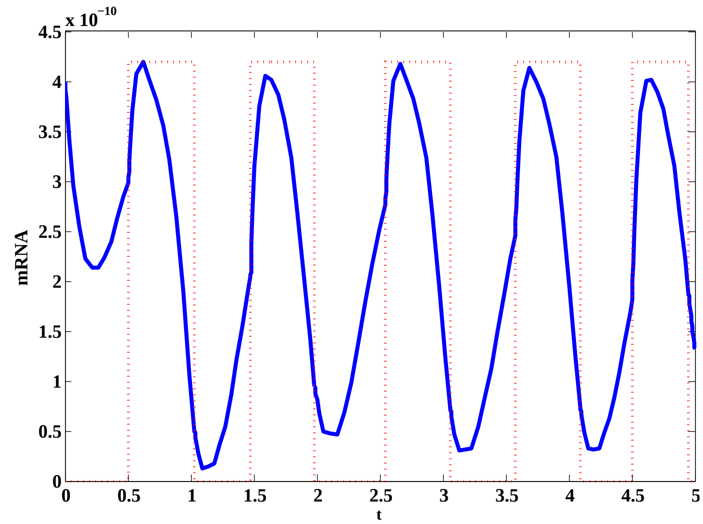


Figure 4.3: A sample path of the process (4.58) with the optimal feedback policy (4.24). The time interval between environmental jumps are identically independent random variables with Beta(40,40) distribution. One can see that the optimal control law is conservative in its response to the environment by anticipating the change of environment. The biological implication of this observation is that an organism that evolved through natural selection in a variable environment is likely to exhibit specialization to the statistics that determine the changes in the environment.

mal level of concentrations of $\mathbf{z}, \mathbf{x}$, and $\mathbf{y}$ in every environment which are [6.98e-13, 1.99e-10, 3.98e-10] and [4.20e-10, 1.19e-7, 2.39e-07].

Suppose that the process of switching between environments is a continuous time random variable that has a finite support on $[0, T]$. For instance, one can consider time intervals between environmental jumps that are independent random variables with Beta distribution $\beta(a, b)$ on $[0, 1]$. By appropriately choosing the parameters a, b one can obtain a wide family of density functions. For example

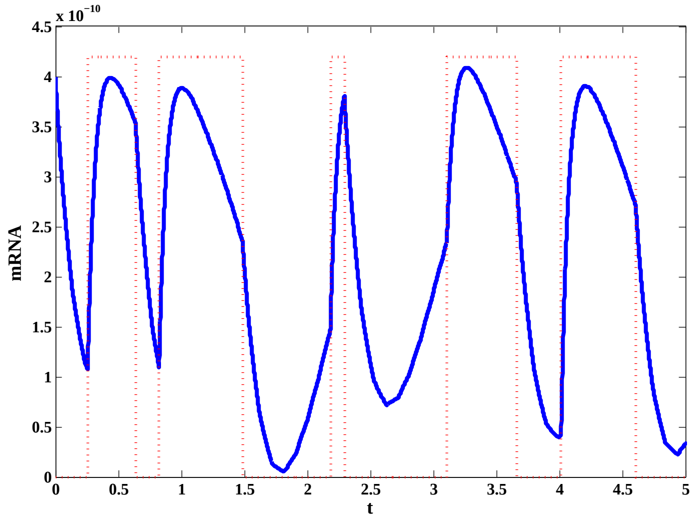


Figure 4.4: A sample path of the process (4.58) with the optimal feedback policy (4.24). The time interval between environmental jumps are uniformly distributed on $[0, 1]$ and are independent.

$a = b = 1$ results in uniform distribution on $[0, 1]$. When both a and b take large values, the sample paths of the environmental fluctuations are almost periodic.

Figure 4.3 illustrates a sample path of this stochastic process using the optimal feedback policy (4.24) when $a = b = 40$, i.e. the environmental change is almost periodic with period 0.5 unit of time. In our simulations, we have chosen $\rho = 0.1$,

$$Q_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 10^{-3} & 0 \\ 0 & 0 & 10^{-3} \end{bmatrix}, \quad Q_2 = \begin{bmatrix} 5 & 0 & 0 \\ 0 & 10^{-3} & 0 \\ 0 & 0 & 10^{-3} \end{bmatrix},$$

$R_1 = 0.1$, $R_2 = 0.1/5$ and $\mathrm{T}_{ij} = \mathrm{I}$ with the initial condition $[\mathbf{z}(0), \mathbf{x}(0), \mathbf{y}(0)]$=[4e-10, 6e-8, 12e-8] and $\mathbf{q}(0) = 0$. One can see that the controller that is optimal for the changing environment tries to anticipate the change of the environment instead of simply reacting to changes in $\mathbf{q}(t)$ and that the the optimal control law depends on the value of the timer $\boldsymbol{\tau}(t)$ that keeps track of the time elapsed since the last environmental change.

A similar result has been illustrated in Figure 4.4 when the holding times (time between jumps) are modelled by a uniform random variable on $[0, 1]$. In this case, the optimal feedback policy anticipates the change of the environment too. The required cell machinery to "implement" such a control law is a topic for future research.

4.5 Conclusion

We studied quadratic control of stochastic hybrid systems with renewal transitions for which the lengths of time intervals that the system spends in each mode are independent random variables with given probability distribution functions on finite supports. We derived a Bellman-like equation for this problem and showed that if the solution V^* to this equation satisfies a boundary condition on τ, then V^* is the minimum value of (4.10). A recursive algorithm was provided for computing the optimal cost since the boundary conditions are unknown. The applicability of our result was illustrated through a numerical example, motivated by stochastic gene regulation in biology.

A topic for future research is to obtain a condition that guarantees existence of the solution to (4.21)-(4.23). Furthermore, we plan to consider H_∞ control of SHS with renewal transitions. In this problem, one would like to characterise a feedback controller for disturbance attenuation in terms of a set of differential matrix inequalities.

5

On Controller Initialization in Multivariable Switching Systems

The classical approach to satisfy multiple conflicting requirements in a linear control system has some inherent limitations that have been well recognized in the literature [9]. A solution to such problems is to design several controllers with distinct transfer functions $\{K_q(s) : q \in \mathcal{P}\}$, and switch between them. Two examples of this approach are the fault tolerant control, [20], and the gain scheduling problem, [66]. To achieve the best overall performance, it is the job of the designer to choose which controller to use at each time instant, based on the current operating conditions (e.g. [3]).

In designing a multicontroller in a linear switching system, there are several degrees of freedom available to designers. Two of them, selecting different controller transfer functions and deciding which controller to use at each instant of time, are widely used. In addition, one can take advantage of another degree of freedom which is rarely used by designers: selecting suitable initial conditions for the controllers when they are inserted into the feedback loop. In this chapter, we mainly focus on exploring this degree of freedom to increase the performance of the overall system. Moreover, since stability of switching systems depends on controller realization, by choosing appropriate state-space realizations for controller transfer functions, we guarantee closed-loop stability under very mild assumptions on the switching mechanism. We assume that the set of controller transfer functions is given and that we have no control over the mechanism by which it is decided when the different controllers are switched in and out of the feedback loop.

Switching between controllers has the potential to cause adverse transient responses and, consequently, the mitigation of such transients has been addressed in the literature. [73] proposed to address this by minimizing the difference between the control signal that is being fed to the process and the control signals produced by out-of-loop controllers. This minimization was formulated as a linear quadratic (LQ) optimization and resulted in a static feedback law for the out-of-loop controllers. Based on this work, [82] and [81] have analyzed the steady-state gain of the infinite horizon LQ bumpless transfer topologies and steady state bumpless control transfer under uncertainties. This type of approach is motivated by the intuitive appeal of "bumpless transfer:" by switching into the feedback loop a controller whose output is close to the current control signal, the transient should be small. Here, we consider a more direct approach in that we will directly address the problem of improving transient performance of the system, without making assumption on whether or not bumpless transfer will lead to a better transient. We will show that transients due to switching can be mostly suppressed by carefully initializing the state of a new controller to be inserted into the feedback loop, so as to minimize a quadratic cost that combines the control signal, error signal and controlled output.

The authors of [73] assumed that both on- and off-line controllers are stabilizing for the plant in question however nothing can be concluded about stability of the overall system under arbitrary switching, [43]. The authors of [32] considered the problem of selecting controller realizations and choosing initial conditions to stabilize a switched system, but no improvement in the transient response of the switching controller was considered. The optimal control problem for switched systems was considered in [77] where they assume that a prespecified sequence of active subsystems is given and optimal switching times are driven. Also [21] derives a switching law to improve the performance under a stabilizing controller. Moreover, [67] presents a technique to improve the performance of each sub-controller in discrete-time systems which is further pursued in [33] for controller initialization.

The specific problem formulated here was first introduced in [33], which provided a method to select controller realizations and initial conditions for the case of an asymptotically stable SISO plant to be controlled. We consider a class of switched systems which consists of a linear MIMO and possibly unstable process in feedback interconnection with a multicontroller whose dynamics switch. It is shown how one can achieve significantly better transient performance by selecting the initial condition for every controller when it is inserted into the feedback loop. This initialization is obtained by performing the minimization of a quadratic cost function of the tracking error, controlled output, and control signal. We guarantee input-to-state stability of the closed-loop system when the average number of switches per unit of time is smaller than a specific value. If this is not the case then stability can still be achieved by adding a mild constraint to the optimization.

The remainder of this chapter is organized as follows. In section 5.1, the mathematical problem statement is introduced. Section 5.2 shows how to choose the controller reset map to minimize a quadratic criteria. Section 5.3 proves the stability of the closed loop under a mild assumption on the switching signal. Simulation results that compare the performance of our switching controller with those in [32] for a MIMO unstable process are provided in sections 5.4. Finally, section 5.5 provides conclusions and directions for future work. The material in this chapter is based upon the works [50, 49].

5.1 Problem Statement

5.1.1 Controller Architecture

The switched systems considered here arises from the feedback interconnection of a Linear Time-Invariant (LTI) plant Σ to be controlled with a *multicontroller* $\mathbf{C}(\sigma)$ whose inputs are the usual tracking error $e_T(t)$ as well as a piecewise constant *switching signal* $\sigma : [0, +\infty) \to \mathcal{P}$ that determines which controller transfer function to use at each time instant.

More specifically, when $\sigma(t)$ is set constant equal to some $q \in \mathcal{P}$, the multicontroller should behave as an LTI system with transfer function $K_q(s)$. In particular, given n-dimensional state space realizations E_q, F_q, V_q, W_q for each $K_q(s)$, $q \in \mathcal{P}$, the state $x_{\text{mult}}(t)$ of the multicontroller $\mathbf{C}(\sigma)$ evolves according to

$$\begin{cases} \dot{x}_{\text{mult}}(t) = E_{\sigma(t)} x_{\text{mult}}(t) + F_{\sigma(t)} e_T(t) \\ u(t) = V_{\sigma(t)} x_{\text{mult}}(t) + W_{\sigma(t)} e_T(t) \end{cases} \tag{5.1}$$

on any time interval on which the switching signal $\sigma(t)$ remains constant, and according to

$$x_{\text{mult}}(t) = \psi(x_{\text{mult}}(t^-), \sigma(t^-), \sigma(t), r(t)), \tag{5.2}$$

at every time t, called a *switching time*, at which $\sigma(t)$ is discontinuous. The function $\psi(.)$ is called the *reset map* and it determines the initial state of the controllers at the switching times. We let such initial state depend on the state $x_{\mathrm{mult}}(t^-)$ of the controller just before the switching time, the index p of the previous controller ($\sigma(t^-) = p$), the index q of the new controller ($\sigma(t) = q$), and also on the current reference value $r(t)$. Our goal is to select the reset map so as to achieve a smooth transient at the controller switching times. In particular, we select this function so as to minimize a quadratic cost function that penalizes the tracking error, the controlled output, and the control signal, all integrated over an interval that starts at the switching time.

The construction of the multicontroller follows [28] and is inspired by the Q-parameterization of all the stabilizing controllers, [79]. In this parameterization, one can define the set of all possible stabilizing controllers as a function of the so-called parameter Q which is a stable transfer matrix. This is a useful tool in problems where one wishes to optimize a performance objective with a stability constraint.

Consider a multivariable LTI process Σ with transfer function $\mathrm{G}_{\mathrm{pl}}(s)$ from the input $u(t)$ to the output $y(t)$, and assume given a finite family of stabilizing controller transfer functions $\{K_q(s) : q \in \mathcal{P}\}$ from the tracking error $e_T(t)$ to the control input $u(t)$, where $r(t)$ denotes a reference signal.

Let (A, B, C) denote a stabilizable and detectable n_{pl}-dimensional realization for the process transfer function $\mathrm{G}_{\mathrm{pl}}(s)$:

$$\begin{cases} \dot{x}_{\mathrm{pl}} = Ax_{\mathrm{pl}} + Bu, \\ y = Cx_{\mathrm{pl}}, \qquad\qquad x_{\mathrm{pl}} \in \mathbb{R}^{n_{\mathrm{pl}}}, u \in \mathbb{R}^k, y \in \mathbb{R}^m, z \in \mathbb{R}^l \\ z = Gx_{\mathrm{pl}} + Hu, \end{cases} \tag{5.3}$$

where x_{pl} denotes the process state, u the control signal, y the measured output and z the controlled output. The measured output $y(t)$ corresponds to the signal that is available for control whereas $z(t)$ corresponds to the signal that one would like to make as small as possible in the shortest possible time. It should be noted that the matrix H represents a direct feedthrough term between u and controlled output z. Such matrix is needed if we want to penalize the control signal. One could also consider a similar feedthrough term between u and the measured output y. For simplicity, we are ignoring such term. If such term exists, we can get similar results by applying the parameterization techniques introduced in [19, Chapter 5].

As stated in the following theorem, one can select an LQG/LQR Q-augmented realization for each stabilizing controller of the form shown in Figure 5.1.

Theorem 5.1.1. *Assume that $A - LC$ and $A - BK$ are Hurwitz matrices. For every controller transfer matrix $K_q(s)$ that asymptotically stabilizes* (5.3)*, there exists a BIBO (Bounded-Input-Bounded-Output) stable transfer matrix $Q_q(s)$ such*

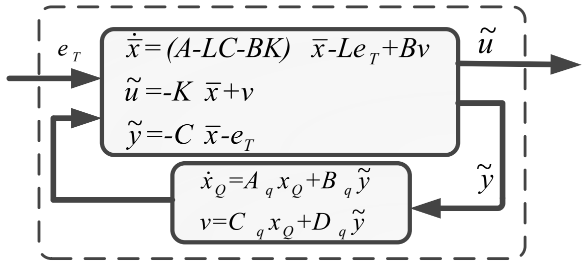


Figure 5.1: Controller $K_q(s)$

that for every stabilizable and detectable realization (A_q, B_q, C_q, D_q) of $Q_q(s)$, the controller transfer matrix $K_q(s)$ admits a stabilizable and detectable realization (E_q, F_q, V_q, W_q) with

$$
\begin{aligned}
E_q &:= \begin{bmatrix} A - LC - BK - BD_qC & BC_q \\ -B_qC & A_q \end{bmatrix}, \\
F_q &:= \begin{bmatrix} -L - BD_q \\ -B_q \end{bmatrix}, \\
V_q &:= [\ -K - D_qC \quad C_q\ ], \qquad W_q := \ -D_q
\end{aligned}
\tag{5.4}
$$

where A_q matrices are selected so that $A_q + A_q' < 0$.

The controller realization, introduced in Theorem 5.1.1, corresponds to the diagram in Figure 5.1.

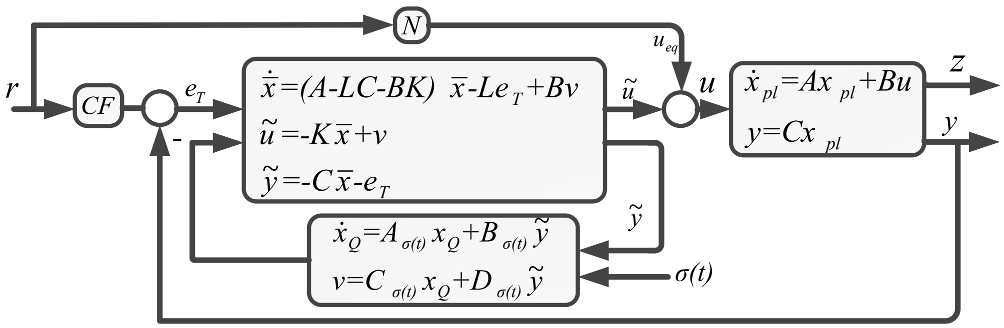


Figure 5.2: Closed-loop Architecture

Proof of Theorem 5.1.1. The existence of Q system and realizations (5.4) are proved in [28]. We further show that one can always find A_q matrices that satisfy

$$A_q + A_q' < 0, \quad \forall q \in \mathcal{P}. \tag{5.5}$$

Consider a finite family of Hurwitz matrices $\mathcal{S} = \{\tilde{\mathcal{A}}_q : q \in \mathcal{P}\}$. By using a similarity transformation, we show how to compute A_q matrices for every $\tilde{\mathcal{A}}_q$ such that it satisfies (5.5). Since $\tilde{\mathcal{A}}_q$ matrices are asymptotically stable, the family of Lyapunov equations

$$\mathcal{Q}_q\tilde{\mathcal{A}}_q + \tilde{\mathcal{A}}_q'\mathcal{Q}_q = -I \qquad q \in \mathcal{P} \tag{5.6}$$

always have positive definite symmetric solutions $\mathcal{Q}_q$ which can be written as $\mathcal{Q}_q = Y_q'Y_q$. Let $A_q = Y_q\tilde{\mathcal{A}}_qY_q^{-1}$. From (5.6), we can conclude that

$$Y_q'(A_q + A_q')Y_q = -I$$

which yields to $A_q + A_q' = -(Y_q')^{-1}(Y_q)^{-1} < 0$. □

We now propose to use a multicontroller $C(\sigma)$ with state $x_{\text{mult}} = [\bar{x}', \ x'_Q]'$ that evolves according to (5.1), with realization (5.4), on any time interval on which the switching signal $\sigma(t)$ remains constant and is reset according to the equation (5.2) at every switching time t. In (5.2), only the x_Q component of the state of the multi-controller is reset according to a map $\psi_Q(.)$ to be defined in section 5.2.

5.1.2 Closed-loop Configuration

The optimal set point problem can be reduced to that of optimal regulation by considering an auxiliary system with state $\tilde{x} := \ x - x_{eq}$, [28]. This motivates us to consider the closed-loop configuration as shown in Figure 5.2 with appropriately defined matrices N and F. If one wants the controlled output z to converge to a given nonzero constant set-point value $r \in \mathbb{R}^l$, there should exist an equilibrium point (x_{eq}, u_{eq}) that satisfies the linear equation

$$\begin{bmatrix} -A & B \\ -G & H \end{bmatrix} \begin{bmatrix} -x_{eq} \\ u_{eq} \end{bmatrix} = \begin{bmatrix} 0 \\ r \end{bmatrix} \qquad x_{eq} \in \mathbb{R}^{n_{\text{pl}}}, \ u_{eq} \in \mathbb{R}^k. \tag{5.7}$$

Depending on the number of inputs k and the number of controlled outputs l, three different cases should be considered: (i) $k < l$ (*under-actuated system*) for which (5.7) generally does not have a solution (ii) $k > l$ (*over-actuated system*) for which (5.7) generally has more than one solution (iii) $k = l$ for which (5.7) always has a solution as long as the coefficient matrix in (5.7) is nonsingular. If there exists a solution to (5.7), it can be written in the form $x_{eq} = Fr$, $u_{eq} = Nr$ for appropriately defined matrices F and N, which corresponds to the control architecture in Figure 5.2. The feed-forward term Nr is absent when the state matrix A has an eigen-value at the origin and this mode is observable through z. Note that in this case the process has an integrator. Also, when $z = y$, we have $G = C, H = 0$, and in this case $Cx_{eq} = r$. This corresponds to $CF = 1$.

We are now ready to find the closed-loop realization of the system shown in Figure 5.2. Connecting the plant Σ with realization (A, B, C) to the multicontroller $\mathbf{C}(\sigma)$, through the negative feedback interconnection, results in a switched system with a state $x(t) = \left[x_{\text{pl}}(t)', \bar{x}(t)', x_Q(t)'\right]'$ that evolves according to

$$\begin{cases} \dot{x}(t) = \hat{A}_{\sigma(t)}x(t) + \hat{B}_{\sigma(t)}r(t), \\ y(t) = \hat{C}_{\sigma(t)}x(t), \\ z(t) = \hat{G}_{\sigma(t)}x(t) + \hat{H}_{\sigma(t)}r(t) \end{cases} \tag{5.8}$$

on any time interval on which $\sigma(t)$ remains constant and

$$x(t) = \begin{bmatrix} x_{\text{pl}}(t) \\ \bar{x}(t) \\ x_Q(t) \end{bmatrix} = \begin{bmatrix} x_{\text{pl}}(t^-) \\ \bar{x}(t^-) \\ \psi_Q(x(t^-), \sigma(t^-), \sigma(t), r(t)) \end{bmatrix} \tag{5.9}$$

at every switching time t. The matrices in (5.8) are defined by

$$\hat{A}_q := \begin{bmatrix} A + BD_qC & -B(K + D_qC) & BC_q \\ (L + BD_q)C & A - LC - BK - BD_qC & BC_q \\ B_qC & -B_qC & A_q \end{bmatrix},$$

$$\hat{B}_q := \begin{bmatrix} BN - BD_qCF \\ -(L + BD_q)CF \\ -B_qCF \end{bmatrix},$$

$$\hat{C}_q := [\ C \quad 0 \quad 0\ ],$$

$$\hat{G}_q := [\ G + HD_qC \quad H(-K - D_qC) \quad C_q\ ],$$

$$\hat{H}_q := H(N - D_qCF),$$

for $\forall q \in \mathcal{P}$. Since each controller transfer function $K_q(s)$ asymptotically stabilizes the process, all $\hat{A}_q$ matrices are Hurwitz.

In the remainder of this chapter, we focus on the goal of selecting an appropriate reset map $\psi_Q(\cdot)$ in (5.9) that achieves an optimal transient performance at switching times, while maintaining the stability of the closed-loop system. The idea of optimizing transient performance is the subject of the next section.

5.2 Optimization of Transient Performance

5.2.1 Quadratic Cost Function

As discussed in Section 5.1.2, only $x_Q(t)$ component of (5.9) jumps at switching times, and the other two components evolve continuously. Suppose that there is a switching at $t = t_0$, which results in a jump in the switching signal from $\sigma(t_0^-) = p$ to $\sigma(t_0) = q$. In what follows, we try to select an appropriate post-switching state defined by the reset map

$$x_Q(t_0) = \psi_Q(x(t_0^-), p, q, r(t_0)) \tag{5.10}$$

such that it optimizes the resulting transient performance, as measured by a quadratic cost of the following form

$$J = \int_{t_0}^{t_1} \Big(e_T(t)' R e_T(t) + z(t)' W z(t) + u(t)' K u(t) \Big) dt + \big(x(t_1) - x_\infty\big)' T \big(x(t_1) - x_\infty\big) \tag{5.11}$$

where e_T, z and u are shown in Figure 5.2. In the above equation, t_1 is a time larger than t_0 and possibly equal to $+\infty$; R, W, K, T are appropriately selected positive semi-definite symmetric matrices, and $x_\infty := \; -\hat{A}_q^{-1} \hat{B}_q \, r(t_0)$. Note that the vector x_∞ that appears in the terminal term in (5.11) is the steady-state value to which $x(t)$ would converge as $t \to \infty$ if both $\sigma(t)$ and $r(t)$ were to remain constant. Moreover, it is worth noting that all the results in this section hold for an infinite horizon, as we make $t_1 \to \infty$ in (5.11), in which case the terminal cost term in (5.11) disappears.

One can define a transient performance by appropriately defining R, W, K, T matrices in (5.11) which correspond to penalizing the tracking error, controlled output, control effort, and final state magnitude, respectively. In this optimization, we assume that the switching signal $\sigma(t)$ and the reference $r(t)$ remain constant and equal to q and $r(t_0)$ over the optimization horizon $[t_0, t_1]$. If $\sigma(t)$ turns out to switch again before t_1, the value to which x_Q was reset at time t_0 will generally not be optimal, but in Section 5.3 we will make sure that stability is preserved even if unexpected switches occur.

5.2.2 Optimal Reset Map

To find the value of $x_Q(t_0)$ in (5.10) that minimizes (5.11) we need to introduce the following notation: Let Q_q denote the symmetric solution to the following Lyapunov equation

$$Q_q \hat{A}_q + \hat{A}_q' Q_q = -P_q,$$

where

$$P_q := \hat{C}_q' R \hat{C}_q + \hat{G}_q' W \hat{G}_q + \tilde{C}_q' K \tilde{C}_q \geq 0$$

and

$$\tilde{C}_q := \begin{bmatrix} D_q C & -(K + D_q C) & C_q \end{bmatrix}.$$

Such Q_q always exists and is positive semi-definite because $\hat{A}_q$ is a Hurwitz matrix. We also define the positive semi-definite matrix $M_q := Q_q + e^{\hat{A}_q' \Delta}(T - Q_q)e^{\hat{A}_q \Delta}$, the length of optimization interval $\Delta := t_1 - t_0$, and the vector

$$\begin{aligned} g_q' := \; & 2r(t_0)'\Big(\big(-F'C'R\hat{C}_q + \hat{H}_q' W \hat{G}_q + \\ & (N - D_q CF)'K\tilde{C}_q + \hat{B}_q' Q_q\big)\hat{A}_q^{-1}\big(I - e^{\hat{A}_q \Delta}\big) \\ & + \hat{B}_q'\big(e^{\hat{A}_q' \Delta}(\hat{A}_q')^{-1}(Q_q - T) - (\hat{A}_q')^{-1} Q_q\big)e^{\hat{A}_q \Delta}\Big). \end{aligned}$$

We further need to block-partition the symmetric matrix M_q and the vector g_q according to the partition in (5.9) of the state vector:

$$M_q = \begin{bmatrix} M_{11}^q & M_{12}^q & M_{13}^q \\ M_{21}^q & M_{22}^q & M_{23}^q \\ M_{31}^q & M_{32}^q & M_{33}^q \end{bmatrix}, \qquad g_q = \begin{bmatrix} g_1^q \\ g_2^q \\ g_3^q \end{bmatrix}$$

and perform a singular value decomposition of

$$M_{33}^q = \begin{bmatrix} U_1^q & U_2^q \end{bmatrix} \begin{bmatrix} \Lambda_q & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} (V_1^q)' \\ (V_2^q)' \end{bmatrix}$$

(with Λ_q nonsingular). We are now ready to provide the solution to the above quadratic optimization problem.

Theorem 5.2.1. *Assume that there is a single switching at $t = t_0$, where $\sigma(t_0^-) = p$, $\sigma(t_0) = q$ and $r(t) = r(t_0)$ $\forall t \in [t_0, t_1]$. The reset map (with the smallest Euclidean norm)*

$$x_Q^*(t_0) = V_1^q \Lambda_q^{-1} (U_1^q)' \left(\frac{1}{2} g_3^q - (M_{13}^q)' x_{\mathrm{pl}}(t_0^-) - M_{32}^q \bar{x}(t_0^-) \right). \tag{5.12}$$

provides a global minimum to (5.11).

Proof of Theorem 5.2.1. We first compute the quadratic cost function J along a solution to the closed loop dynamics in the optimization interval:

$$\begin{cases} x(t) = e^{\hat{A}_q(t-t_0)} x(t_0) + \int_{t_0}^{t} e^{\hat{A}_q(t-\tau)} \hat{B}_q \cdot r(t_0)\, d\tau, \\ u(t) = \tilde{C}_q x(t) + (N - D_q C F) r(t) \end{cases} \qquad \forall t \in [t_0, t_1].$$

Straightforward algebra shows that the terminal term in (5.11) is given by

$$\begin{aligned} \big((x(t_1) - x_\infty\big)' T \big(x(t_1) - x_\infty\big) = & \left[e^{\hat{A}_q(t_1-t_0)} x(t_0) + \int_{t_0}^{t_1} e^{\hat{A}_q(t_1-\tau)} \hat{B}_q \cdot r(t_0)\, d\tau + \right. \\ & \left. + \hat{A}_q^{-1} \hat{B}_q r(t_0) \right]' T \left[e^{\hat{A}_q(t_1-t_0)} x(t_0) + \int_{t_0}^{t_1} e^{\hat{A}_q(t_1-\tau)} \hat{B}_q \cdot r(t_0)\, d\tau + \hat{A}_q^{-1} \hat{B}_q r(t_0) \right] \\ & = x(t_0)' e^{\hat{A}_q' \Delta} T e^{\hat{A}_q \Delta} x(t_0) + 2 r(t_0)' \hat{B}_q' e^{\hat{A}_q' \Delta} (\hat{A}_q^{-1})' T e^{\hat{A}_q \Delta} x(t_0) + r(t_0)'. \\ & \hat{B}_q{}' (e^{\hat{A}_q' \Delta} - I)(\hat{A}_q^{-1})' T \hat{A}_q^{-1} (e^{\hat{A}_q \Delta} - I) \hat{B}_q r(t_0) \\ & + r(t_0)' \hat{B}_q' (\hat{A}_q^{-1})' T \hat{A}_q^{-1} (e^{\hat{A}_q \Delta} - \frac{1}{2} I) \hat{B}_q r(t_0). \end{aligned} \tag{5.13}$$

Since

$$e_T(t)'Re_T(t) = r(t_0)'F'C'RCFr(t_0) - 2r(t_0)'F'C'R\hat{C}_q x(t) + x(t)'\hat{C}_q'R\hat{C}_q x(t),$$

$$\begin{aligned}
u(t)'Ku(t) &= \big(\tilde{C}_q x(t) + (N - D_qCF)r(t_0)\big)'K\big(\tilde{C}_q x(t) + (N - D_qCF)r(t_0)\big) \\
&= \big(x(t)'\tilde{C}_q' - r(t_0)'D_q'\big)K\big(\tilde{C}_q x(t) - D_q r(t_0)\big) \\
&= x(t)'\tilde{C}_q'K\tilde{C}_q x(t) + 2r(t_0)'(N' - F'C'D_q')K\tilde{C}_q x(t) \\
&+ r(t_0)(N' - F'C'D_q')K(N - D_qCF)r(t_0),
\end{aligned}$$

$$\begin{aligned}
z(t)'Wz(t) &= (\hat{G}_q x(t) + \hat{H}_q r(t_0))'W(\hat{G}_q x(t) + \hat{H}_q r(t_0)) \\
&= x(t)'\hat{G}_q'W\hat{G}_q x(t) + 2r(t_0)'\hat{H}_q'W\hat{G}_q x(t) + r(t_0)'\hat{H}_q'W\hat{H}_q r(t_0)),
\end{aligned}$$

the integral terms in (5.11) are given by

$$\begin{aligned}
&\int_{t_0}^{t_1} \Big(e_T(t)'Re_T(t) + z(t)'Wz(t) + u(t)'Ku(t)\Big)dt \\
&\quad = \int_{t_0}^{t_1} \Big(r(t_0)'\big(F'C'RCF + \hat{H}_q'W\hat{H}_q + (N' - F'C'D_q')K(N - D_qCF)\big)r(t_0) \\
&\quad + x(t)'P_q x(t) + c_q'x(t)\Big)dt, \qquad (5.14)
\end{aligned}$$

where c_q' is defined by

$$c_q' := \; 2r(t_0)'(-F'C'R\hat{C}_q + \hat{H}_q'W\hat{G}_q + (N - CFD_q)'K\tilde{C}_q).$$

Those terms in (5.14) that depend on $x(t)$ are given by

$$\int_{t_0}^{t_1}\Big(x(t)'P_q x(t)+c_q'x(t)\Big)dt=\int_{t_0}^{t_1}\Big(2\,r(t_0)'\hat{B}_q'Q_q x(t)-\frac{d}{dt}(x'Q_q x)+c_q'x(t)\Big)dt$$

$$\begin{aligned}&=\big(2r(t_0)'\hat{B}_q'Q_q+c_q'\big)\hat{A}_q^{-1}\big(e^{\hat{A}_q\Delta}-I\big)x(t_0)\\&+\big(2r(t_0)'\hat{B}_q'Q_q+c_q'\big)\hat{A}_q^{-1}\big(\hat{A}_q^{-1}e^{\hat{A}_q\Delta}-\hat{A}_q^{-1}-\Delta I\big)\hat{B}_q r(t_0)\\&-x(t_0)'e^{\hat{A}_q'\Delta}Q_q e^{\hat{A}_q\Delta}x(t_0)-2r(t_0)'\hat{B}_q'(e^{\hat{A}_q'\Delta}-I)(\hat{A}_q^{-1})'Q_q e^{\hat{A}_q\Delta}x(t_0)\\&+x(t_0)'Q_q x(t_0)+r(t_0)'\hat{B}_q'(e^{\hat{A}_q'\Delta}-I)(\hat{A}_q^{-1})'Q_q\hat{A}_q^{-1}(e^{\hat{A}_q\Delta}-I)\hat{B}_q r(t_0).\end{aligned}$$

Combing the latest result and (5.13), we conclude that

$$J=x(t_0)'M_q x(t_0)-x(t_0)'g_q+*$$

where $*$ stands for additive terms that do not depend on the $x_Q(t_0)$. Such terms will thus not affect the optimal value for $x_Q(t_0)$. Since our optimization is only performed on the component $x_Q(t_0)$ of $x(t_0)$, we further re-write

$$J=x_Q(t_0)'M_{33}^q x_Q(t_0)+x_Q(t_0)'\big(2((M_{13}^q)'x_{\mathrm{pl}}(t_0)+M_{32}^q\bar{x}(t_0)-g_3^q\big)+*. \tag{5.15}$$

Also M_{33}^q is positive semi-definite, (5.15) is convex on $x_Q(t_0)$ and any vector $x_Q^*(t_0)$ satisfying the first order optimality condition

$$M_{33}^q x_Q^*(t_0)=\frac{1}{2}g_3^q-(M_{13}^q)'x_{\mathrm{pl}}(t_0)-M_{32}^q\bar{x}(t_0) \tag{5.16}$$

provides a global minimum to J, see [10]. In general, (5.16) may not be solvable, but in our specific problem it can be proved that a solution always exists. The proof is based on a contradiction argument is similar too the one in [33]. Let w be the right hand side of (5.16)

$$w := \frac{1}{2}g_3^q - (M_{13}^q)'x_{\mathrm{pl}}(t_0) - M_{32}^q\bar{x}(t_0).$$

Assume that there is no solution to $M_{33}^q x_Q^*(t_0) = w$. This implies that $w \notin (\mathrm{Ker}M_{33}^q)^\perp$ where $(\mathrm{Ker}M_{33}^q)^\perp$ is the orthogonal complement of the null space of M_{33}^q. Therefore, there exists a vector $z \in (\mathrm{Ker}M_{33}^q)$ such that $w'z \neq 0$. If one sets $x_Q^*(t_0) = \alpha z$ with a sufficiently large α, then the cost (5.15) becomes negative. This is a contradiction since the cost is always non-negative.

Moreover, one can conclude that the minimum Euclidean norm solution to (5.16) is given by (5.12), see [53, Chapter 5]. $\square$

Remark 5.2.1. Theorem 5.2.1 provides an explicit formula for the optimal reset map. In practice, we cannot implement (5.12) since the left-hand side depends on the state of the process that may not be directly available. It turns out that the signal $\hat{x} := \bar{x} + x_{eq}$ that can be directly obtained from the state of $\mathbf{C}(\sigma)$ converges exponentially to the process state $x_{\mathrm{pl}}(t)$ regardless of the control signal $u(t)$. To verify that this is so, we define the state estimation error $e = x_{\mathrm{pl}} - \hat{x}$ that, because of (5.3) and (5.1), with the matrices defined in (5.4), evolves according to

$$\dot{e} = \dot{x}_{\mathrm{pl}} - \dot{\hat{x}} = \dot{x}_{\mathrm{pl}} - \dot{\bar{x}} = (A - LC)e$$

where $A - LC$ is a Hurwitz matrix. So, we can replace x_{pl} in (5.12) by $\hat{x} = \bar{x} + x_{eq}$ at the expense of introducing an error that vanishes to zero exponentially fast for every control input. This motivates the following reset map for (5.2):

$$\psi_Q(x(t_0^-), p, q, r(t_0)) := \; V_1^q \Lambda_q^{-1} (U_1^q)' \Big(\frac{1}{2} \hat{g}_3^q - (M_{32}^q + (M_{13}^q)') \bar{x}(t_0^-) \Big) \tag{5.17}$$

where $\hat{g}_3^q = g_3^q - 2(M_{13}^q)' F r(t_0)$. If one were to include exogenous disturbances and/or measurement noise in the process model (5.3), then the error $e(t)$ would not converge to zero and (5.17) would not be optimal. However, in this case one can select the matrix L to minimize the steady-state mean-square error between x_{pl} and $\hat{x}$. This would correspond to a standard LQG estimation problem. As we shall see, our results do not depend on a specific choice for the matrix L, other that is must make $A - LC$ Hurwitz, so selecting L based on a LQG optimization is a very reasonable option to minimize the loss of optimality due to discrepancies between x_{pl} and $\hat{x}$. Another option to minimize the influence of the state estimation error is to use a finite-time observer in parallel with the main observer used in the controller. In this case, one can use the result of finite-time observer in linear time-varying systems found in [52].

Remark 5.2.2. *Robust optimization under parameter uncertainties:* The optimal solution (5.12) involves M_{33}^q matrices that are associated with the plant. In the presence of parameter uncertainty, one can use results from robust quadratic programming to compute the optimal solution, cf. [10]: Let us consider a variation of (5.15) that includes uncertainty in the matrix M_{33}^q. We assume that this matrix belongs to a set $\mathcal{E}$ which is specified by a nominal value $M_{33}^{q^0}$ plus a bound on the eigenvalues of the deviation

$$\mathcal{E} = \{M_{33}^q | -\gamma I \leq M_{33}^q - M_{33}^{q^0} \leq \gamma I\}.$$

We define the robust quadratic program as

$$\begin{aligned} \min_{x_Q(t_0)} \sup_{M_{33}^q \in \mathcal{E}} & \; x_Q(t_0)' M_{33}^q x_Q(t_0) + x_Q(t_0)' \\ & \big(2((M_{13}^q)' x_{\mathrm{pl}}(t_0) + M_{32}^q \bar{x}(t_0) - g_3^q\big) + *. \end{aligned} \tag{5.18}$$

For a given x, the supremum of $x'(M_{33}^q - M_{33}^{q^0})x$ over $\mathcal{E}$ is given by $\gamma x'x$. Therefore, we can express the robust quadratic programming as

$$\min_{x_Q(t_0)} \; x_Q(t_0)'(M_{33}^{q^0} + \gamma I)x_Q(t_0) + x_Q(t_0)'\big(2((M_{13}^q)' x_{\mathrm{pl}}(t_0) + M_{32}^q \bar{x}(t_0) - g_3^q\big) + *$$

leading to a reset map that minimizes (5.18).

5.3 Input-to-State Stability

In Section 5.2, we assumed that no further switching would occur in the interval $[t_0, t_1]$ but if there are further switchings, then the reset map (5.17) may destabilize the switched system even though all individual subsystems are asymptotically stable, e.g., [43]. In this section, we show that we can achieve Input-to-State-Stability (ISS) of the impulsive system (5.8)-(5.9) in the sense of [30].

We first recall the following standard definitions from [40]: A continuous function $\alpha : [0, \infty) \to [0, \infty)$ is said to belong to class $\mathcal{K}$, when it is strictly increasing, and $\alpha(0) = 0$. If α is also unbounded, then we say it is of class $\mathcal{K}_\infty$. A continuous function $\beta : [0, \infty) \times [0, \infty) \to [0, \infty)$ is of class $\mathcal{KL}$ when $\beta(., t)$ is of class $\mathcal{K}$ for each fixed $t \geq 0$, and $\beta(r, t)$ decreases to 0 as $t \to \infty$ for each fixed $r \geq 0$.

It is often of interest to characterize input-to-state stability over classes of switching sequences. Suppose that $\mathcal{S}$ is the set of admissible switching signals. We say that the switching system (5.8)-(5.9) is *uniformly input-to-state stable* (ISS) over $\mathcal{S}$ if there exist functions $\beta \in \mathcal{KL}$ and $\gamma \in \mathcal{K}_\infty$ such that for every initial condition, every reference signal $r(t)$, and every $\sigma \in \mathcal{S}$ the corresponding solution to (5.8)-(5.9) exists globally and satisfies

$$|x(t)| \leq \beta\left(|x(t_0)|, t - t_0\right) + \gamma\left(||r||_{[t_0,t]}\right) \qquad \forall t \geq t_0$$

where $||.||_{\mathrm{I}}$ denotes the supremum norm on an interval I. Moreover, let $\mathcal{S}_{avg}[\tau^*, N_0]$, with $\tau^* > 0$ and $N_0 > 0$, denote the set of Average Dwell Time (ADT) [31] switching sequences that satisfy

$$N(t, s) \leq \frac{t - s}{\tau^*} + N_0 \qquad \forall t \geq s \geq t_0 \tag{5.19}$$

where $N(t,s)$ is the number of discontinuities of switching signal σ, in the open interval (s,t). The constant τ^* is called *average dwell time* and N_0 the *chatter bound*. If $N_0 = 1$ then (5.19) implies that σ cannot switch twice on any interval of length smaller that τ^*. Switching signals with this property are the switching signals with *dwell time* τ^*. Also note that $N_0 = 0$ corresponds to the case of no switching, since σ cannot switch on any interval of length smaller than τ^*. In general, if the first N_0 switches are discarded, then the average time between consecutive switches is at least τ^*.

We further need the following concepts introduced in [30]. Consider the following system:

$$\begin{cases} \dot{x}(t) = f_{\sigma(t)}\left(x(t), r(t)\right), & t \neq t_k \\ x(t) = h_{\sigma(t)}\left(x(t^-), r(t^-)\right), & t = t_k. \end{cases} \tag{5.20}$$

We say that a function $V : \mathbb{R}^n \to \mathbb{R}$ is a *candidate exponential ISS-Lyapunov function* for (5.20) with rate coefficient $c, d \in \mathbb{R}$ if V is locally Lipschitz, positive definite, radially unbounded, and satisfies

$$\nabla V(x) f_\sigma(x,r) \leq -cV(x) + \mathcal{X}_1(|r|) \qquad \forall x \text{ a.e.}, \forall r \tag{5.21}$$

$$V(h_\sigma(x^-, r^-)) \leq e^{-d} V(x) + \mathcal{X}_2(|r|) \qquad \forall x, r \tag{5.22}$$

for some functions $\mathcal{X}_i \in \mathcal{K}_\infty$ and all admissible switching signals $\sigma \in \mathcal{S}$. In (5.21), "$\forall x$ a.e." should be interpreted as "for every $x \in \mathbb{R}^n$ except, possibly, on a set of zero Lebesgue-measure in $\mathbb{R}^n$".

Theorem 5.3.1. *Let $\bar{\tau} > 0$ be given by*

$$\bar{\tau} = \inf_{\substack{c>0,\\ Q_1,Q_2,Q_3>0}} \tau \qquad \textit{subject to}$$

$$\begin{bmatrix} (e^{\tau c/2}-1)(Q_1+Q_2) & (e^{\tau c/2}-1)(-Q_2) \\ (e^{\tau c/2}-1)(-Q_2) & (e^{\tau c/2}-1)Q_2 - \phi_q' Q_3 \phi_q \end{bmatrix} \geq 0 \tag{5.23}$$

$$\begin{bmatrix} \begin{array}{c} -(A-BK)'Q_1 + c\ Q_1 \\ -Q_1(A-BK) \end{array} & -Q_1 B C_q & Q_1 B (K + D_q C) \\ -C_q' B' Q_1 & \begin{array}{c} -A_q' Q_3 - Q_3 A_q \\ +c\ Q_3 \end{array} & Q_3 B_q C \\ (K+D_qC)'B'Q_1 & C'B_q'Q_3 & \begin{array}{c} -(A-LC)'Q_2 + c\ Q_2 \\ -Q_2(A-LC) \end{array} \end{bmatrix} \geq 0 \tag{5.24}$$

for $\forall q \in \mathcal{P}$ and

$$\phi_q := \; - V_1^q \Lambda_q^{-1} (U_1^q)' \left((M_{13}^q)' + M_{32}^q\right).$$

Then, for every average dwell-time switching sequence $\mathcal{S}_{avg}[\tau^*, N_0]$ with $\tau^* > \bar{\tau}$, the switched system (5.8)-(5.9) *with reset map (5.17) is ISS.*

For fixed values of τ and c, (5.23)-(5.24) turn out to be linear matrix inequality conditions. One could solve the optimization problem of Theorem 5.3.1 by griding on c, τ (with an appropriate resolution) and checking the feasibility of (5.23)-(5.24). Thus, the solution would be the smallest τ for which (5.23)-(5.24) are feasible. One can also check if the closed-loop system remains ISS for a given τ by performing a line search on c and checking the feasibility of (5.23)-(5.24) at each step.

Proof of Theorem 5.3.1. We define $V(x(t)) := x(t)'Qx(t)$ as the Lyapunov function for the closed-loop system

$$Q := \Gamma' \begin{bmatrix} Q_1 & 0 & 0 \\ 0 & Q_3 & 0 \\ 0 & 0 & Q_2 \end{bmatrix} \Gamma \qquad \Gamma := \begin{bmatrix} I & 0 & 0 \\ 0 & 0 & I \\ -I & I & 0 \end{bmatrix}.$$

Straightforward algebra and (5.24) show that

$$Q\hat{A}_q + \hat{A}_q' Q \leq -c\, Q < 0.$$

This implies that

$$\nabla V(x) \left(\hat{A}_{\sigma(t)} x(t) + \hat{B}_{\sigma(t)} r(t) \right) \leq -cV(x) + b|x||\hat{B}_{\sigma(t)} r(t)| \qquad (5.25)$$

from which (5.21) follows by appropriate choice of $\mathcal{X}_1(|r|)$. We will further need to show the inequality (5.22) is satisfied for any $\tau^* > \bar{\tau}$, from which $V(x)$ turns out to be a candidate exponential Lyapunov function for the closed-loop system. Let us consider the function

$$K(s) := \min\left\{\frac{s}{2}, \frac{|\bar{\tau}c|}{2} \inf_{\substack{|z|\geq s/2, \\ \nabla V(z)\neq 0}} \frac{V(z)}{|\nabla V(z)|}\right\}.$$

By construction K is positive for $s \neq 0$, monotone nondecreasing, and radially unbounded. This construction does not guarantee that $K \in \mathcal{K}_\infty$. However, for simplicity, we assume that $K \in \mathcal{K}_\infty$, because if this is not the case, we can replace it by a smaller function in $\mathcal{K}_\infty$. By a slight abuse of notation, we define $g(x,r) := \psi_Q(x,p,q,r)$ where the function g depends on p and q. For every $p, q \in \mathcal{P}$, pick arbitrary (x,r) for which $v := g(x,r) - g(x,0)$ is small in the sense that

$$|v| = |g(x,r) - g(x,0)| < K(|g(x,0)|).$$

Following the same procedure as in the proof of [30, Theorem 3], one can show that

$$V\left(g(x,r)\right) \leq e^{\bar{\tau}c/2} V(g(x,0)). \tag{5.26}$$

Moreover, straightforward algebra shows that the constraint (5.23) of the minimization problem makes sure that

$$V(g(x,0)) \leq e^{\bar{\tau}c/2} V(x). \tag{5.27}$$

So one can conclude that (5.26) and (5.27) lead to

$$V(g(x,r)) \leq e^{\bar{\tau}c} V(x). \tag{5.28}$$

Suppose now that we pick (x, r) for which v is large in the sense that it satisfies

$$|g(x, 0)| \leq K^{-1}(|v|).$$

We now have

$$\begin{aligned} V(g(x, r)) = V(v + g(x, 0)) &\leq \alpha(|v| + |g(x, 0)|) \\ &\leq \alpha \circ (id + K^{-1})(|v|) \leq \mathcal{X}_2(|r|) \end{aligned} \tag{5.29}$$

where α is a class $\mathcal{K}_\infty$ function with the property that $V(x) \leq \alpha(x)$ and id denotes the identity function. We also define $\mathcal{X}_2 = \alpha \circ (id + K^{-1}) \circ \gamma$ with function γ as the growth estimate of $g(x, r) - g(x, 0)$. The existence of $\mathcal{X}_2$ is guaranteed in [30]. Combining (5.28) and (5.29), we conclude that (5.22) is satisfied for any $\tau^* > \bar{\tau} > 0$. So $V(x)$ turns out to be a candidate exponential Lyapunov function for the closed-loop system, and according to [30, Theorem 1] the closed-loop system is ISS over $\mathcal{S}_{avg}[\tau^*, N_0]$. □

Theorem 5.3.1 guarantees input-to-state stability of the closed-loop system when the average number of switches per unit of time is smaller than a specific value. If this is not the case then stability can still be achieved by adding a mild constraint to the optimization (5.15). This is the result of the next theorem.

Theorem 5.3.2. *Let c and positive definite matrices Q_1, Q_2, Q_3 satisfy the matrix inequality (5.24), and $\mathcal{S}_{avg}[\tau^*, N_0]$ denote average dwell time switching sequences* *with* $\tau^* > 0$ *and* $N_0 > 0$*. The switched system* (5.8)-(5.9) *is ISS for a reset map* (5.10) *that is obtained by solving the following minimization problem*

$$\min_{x_Q(t_0)} x_Q(t_0)' M_{33}^q x_Q(t_0) + x_Q(t_0)' \big(2((M_{13}^q)' x_{\mathrm{pl}}(t_0^-) + M_{32}^q \bar{x}(t_0^-) - g_3^q)\big) \tag{5.30}$$

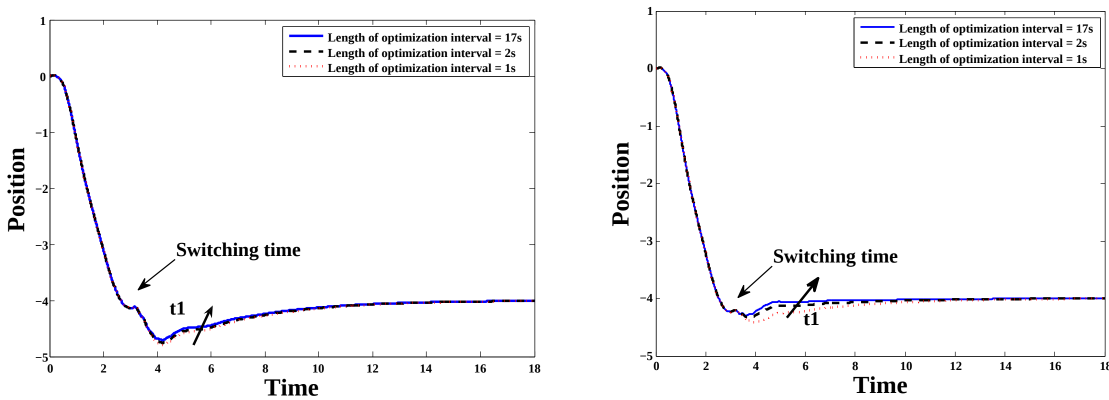
subject to

$$x_Q(t_0)' Q_3 x_Q(t_0) \le e^{\tau^* c} x_Q(t_0^-)' Q_3 x_Q(t_0^-) + \mathcal{X}_2(|r(t_0)|^2) \tag{5.31}$$
$$+ (e^{\tau^* c} - 1) \begin{bmatrix} x_{\mathrm{pl}}(t_0^-)' & \bar{x}(t_0^-)' \end{bmatrix} \begin{bmatrix} Q_1 + Q_2 & -Q_2 \\ -Q_2 & Q_2 \end{bmatrix} \begin{bmatrix} x_{\mathrm{pl}}(t_0^-) \\ \bar{x}(t_0^-) \end{bmatrix}$$

In the minimization problem of Theorem 5.3.2, the cost function (5.30) is the same as (5.15). The additional constraint (5.31) may lead to some increase in the value of the criterion (5.15), but it makes sure that the closed-loop system is ISS under the given ADT.

A natural choice for c and Q_1, Q_2, Q_3 in Theorem 5.3.2 would be the constants that result from the optimization in the Theorem 5.3.1, since they minimize the allowable average dwell time.

Note that M_q and consequently M_q^{33} are positive semi-definite matrices, and the constraint is convex, so the minimization problem in Theorem 5.3.2 turns out to be convex. The convexity of the problem makes it computationally easy, however (5.30)-(5.31) should be solved at every switching time. Although this may



(a) $R = I$, $W = I$, $K, T = 0$ (b) $R = I$, $W, K, T = 0$

Figure 5.3: Transient responses for different weighting matrices and different lengths of the optimization interval $[t_0, t_1]$ ($t_1 = 4, 5, 20$). In both plots there is a single controller switching at time $t_0 = 3$ sec. By comparing the above plots, one can see how the penalty coefficient matrix W for the output rate of change affects the transient responses. Details on the process and controllers being switched can be found in Section 5.4.

require more online computational resources, the convex optimization problem can be solved very efficiently using the results of [47].

Proof of Theorem 5.3.2. Using the same Lyapunov function $V(x) = x(t)'Qx(t)$ as before, we have already shown that (5.21) holds for $c > 0$. Moreover, for any given Average Dwell Time switching sequences $\mathcal{S}_{avg}[\tau^*, N_0]$, we can always find $d < 0$ such that $|d| \leq \tau^* c$; therefore the constraint (5.31) makes sure that for the given switching sequences we have

$$V(x(t_0)) \leq e^{-d} V(x(t_0^-)) + \mathcal{X}_2(|r(t_0)|)$$

at every switching time t_0. So $V(x)$ turns out to be a candidate exponential Lyapunov function for the closed-loop system, and according to [30, Corollary 1] the closed-loop system is ISS over $\mathcal{S}_{avg}[\tau^*, N_0]$. $\square$

Remark 5.3.1. It should be noted that choosing a large $\mathcal{X}_2 \in \mathcal{K}_\infty$ in (5.31) leads to large ISS gain of the closed-loop system. Since the ISS gain of the system is larger than max $\{\mathcal{X}_1, \mathcal{X}_2\}$ in (5.21)-(5.22), one may choose $\mathcal{X}_2$ in (5.31) no larger than $\mathcal{X}_1$ in (5.25).

Remark 5.3.2. When the process model (5.3) includes exogenous inputs such as measurement model and/or input disturbances, the results of Theorems 5.3.1 and 5.3.2 are still true for an input that includes the reference signal as well as these additional exogenous inputs. In this case, we would have to select $\mathcal{X}_i$ to make sure that (5.21)-(5.22) hold for a more general input.

5.4 Simulation Results

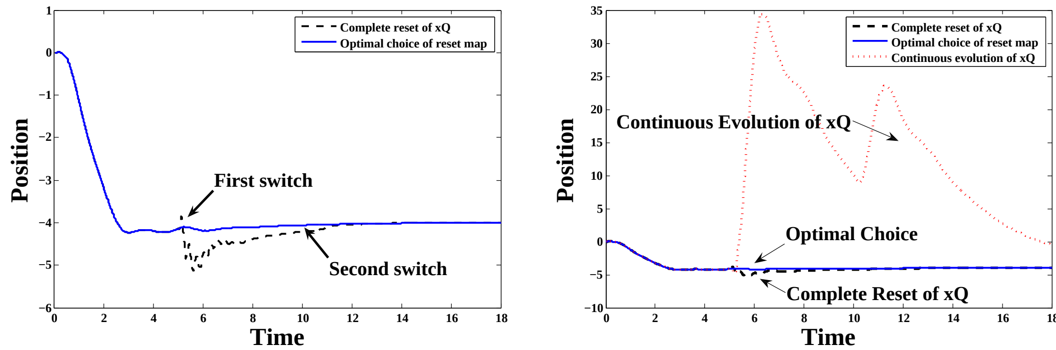
In this section, we show simulation results for a closed−loop system that switches between two controllers for a MIMO flexible beam. The following equations approximately model the flexible beam described in [61]:

$$A = \begin{bmatrix} 0 & 0.31 & 0.79 & 1.07 & 1 & 0 & 0 & 0 \\ 0 & -0.25 & -0.58 & -0.8 & 0 & 1 & 0 & 0 \\ 0 & -0.01 & -0.5 & -0.11 & 0 & 0 & 1 & 0 \\ 0 & 0 & -0.01 & -1.25 & 0 & 0 & 0 & 1 \\ 0 & 224.65 & 1457.28 & 4169.8 & 0 & 0.31 & 0.79 & 1.07 \\ 0 & -181.05 & -1101.27 & -3151.13 & 0 & -0.25 & -0.58 & -0.8 \\ 0 & -26.49 & -636.68 & -491.65 & 0 & -0.01 & -0.50 & -0.11 \\ 0 & -9.59 & -62.23 & -3941.2 & 0 & 0 & -0.01 & -1.25 \end{bmatrix}$$

$$B = \begin{bmatrix} 0 & 0 & 0 & 0 & 43.48 & -32.86 & -5.13 & -1.86 \end{bmatrix}'$$

$$C = \begin{bmatrix} 1.13 & 1.66 & -1.35 & 1.13 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1.94 & -17.9 & -31.02 & 0 & 0 & 0 & 0 \end{bmatrix} \qquad D = \begin{bmatrix} 0 & 0 & 0 \end{bmatrix}'$$

where the input of the system is the torque applied at the base, the first output denotes the tip position of the beam, and the other outputs are base angular velocity and strain gauge measurement, respectively. This system is unstable and contains several lightly damped modes that can be easily excited by controller switching.



(a) Switching times: 5 and 10 sec (b) Switching times: 5 and 10 sec

Figure 5.4: Transient responses for the multicontroller proposed here and for the two alternative multicontrollers proposed in [32] (for $R = I$ and $W, K, T = 0$). The plots show the transients due to two control switchings at times 5 and 10 sec. The optimization intervals are [5,10] and [10,30]. In Fig. (a) The "Optimal Choice" and "Complete Reset of x_Q ($\psi_Q = 0$)" are shown, while in Fig. (b) the "Optimal Choice" and "Complete Reset of x_Q" are compared to the "Continuous evolution of x_Q ($\psi_Q = I$)".

The model used for the flexible beam has a stabilizable and detectable state space realization with two poles at the origin. The two controllers to be used in the feedback loop are given by

$$K = \begin{bmatrix} 0.01 & 5.00 & 31.89 & 95.68 & 0.05 & -0.05 & -0.25 & -0.16 \end{bmatrix}$$

$$L = \begin{bmatrix} 0.96 & 1.00 & -0.16 \\ 0.33 & -0.75 & 0.49 \\ -0.01 & -0.12 & -0.08 \\ 0.00 & -0.04 & -0.01 \\ -0.02 & 43477.88 & 1.96 \\ 0.95 & -32856.38 & -1.64 \\ -0.07 & -5126.38 & -0.46 \\ -0.00 & -1856.68 & -0.06 \end{bmatrix}$$

$$A_1 = \begin{bmatrix} -93.54 & 80.17 & -6.60 & -78.19 & 58.41 \\ -9.91 & -77.52 & -93.03 & 28.72 & -77.20 \\ 2.40 & 3.33 & -46.58 & 17.18 & -15.50 \\ 14.70 & 11.36 & -11.71 & -34.01 & 35.16 \\ 9.99 & 7.72 & -7.96 & 14.27 & -31.10 \end{bmatrix}$$

$$A_2 = \begin{bmatrix} -20.67 & -22.03 & -57.02 & -44.77 & -8.23 \\ -30.67 & -99.46 & -84.03 & -62.60 & 61.91 \\ 12.05 & 15.41 & -53.98 & -90.30 & 58.64 \\ 20.99 & 26.67 & -2.70 & -57.13 & 4.79 \\ 32.96 & 41.88 & -4.16 & 2.91 & -51.50 \end{bmatrix}$$

$$B_1 = \begin{bmatrix} 3.11 & -5.66 & -4.11 \\ -9.68 & 0.86 & -4.03 \\ -8.79 & -7.09 & -1.55 \\ 5.09 & -5.41 & -5.67 \\ -7.62 & 4.99 & 0.52 \end{bmatrix} \quad B_2 = \begin{bmatrix} 1.87 & -1.88 & -1.24 \\ 5.64 & 2.62 & -1.12 \\ 8.93 & -1.58 & 0.39 \\ 10.26 & 4.18 & -0.59 \\ -7.18 & 5.57 & -0.66 \end{bmatrix}$$

$$C_1 = \begin{bmatrix} -1026.96 & -8164.16 & -10168.89 & 8722.19 & -6697.03 \end{bmatrix}$$

$$C_2 = \begin{bmatrix} 45.60 & 903.46 & 609.30 & 2830.85 & -3604.13 \end{bmatrix}$$

$$D_1 = \begin{bmatrix} -1446.62 & -467.20 & 278.67 \end{bmatrix} \quad D_2 = \begin{bmatrix} -635.44 & -244.67 & -30.62 \end{bmatrix}.$$

These controllers are chosen to decrease the settling time with a reasonable amount of overshoot and to limit the torque in step response. The first controller requires smaller torque while the second one is faster but needs a larger torque.

Figure 5.3 depicts the result of numerical simulations illustrating how varying the length of the optimization interval may influence the system's behavior. We can see that the transient response improves as we increase the length of optimization interval. Figure 5.4 (a) illustrates that we can achieve significantly better performance with the multicontroller proposed here than with the multicontroller proposed in [32], which simply sets the left-hand side of (5.12) to zero. In Figure

5.4 (b), the results shown in Figure 5.4 (a) are compared to the case that there is no reset in the controller state.

Figure 5.5 shows the impact of using the reset map (5.17) instead of (5.12) in the presence of persistent noise and disturbance. As suggested in Remark 3.1, we selected the matrix L to minimize the steady-state mean-square error between x_{pl} and $\hat{x}$.

Using the results of Theorem 5.3.1, we could show that the closed-loop system with reset map (5.17) remains ISS for average dwell time switching sequences $\mathcal{S}_{avg}[\tau, N_0]$ with τ as low as 10^{-16} . Since matrices associated with (5.17) are computed offline, one can initialize the controllers by performing matrix multiplications at switching times.

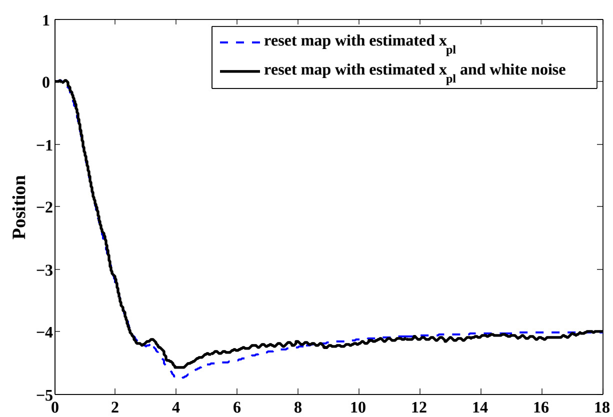


Figure 5.5: This plot illustrates the impact of using the reset map 5.17), instead of the optimal map (5.12), in the presence of persistent measurement noise. The solid line shows the output trajectory when the output measurement is excited by a white noise with power 0.01. There is a switching at $t_0 = 3$.

5.5 Conclusion

In this chapter, we showed that by finding optimal values for the initial controller state, one can achieve significantly better transients when switching between linear controllers for a not necessarily asymptotically stable MIMO linear process. The initialization was obtained by performing the minimization of a quadratic cost function of the tracking error, controlled output, and control signal.

By suitable choice of realizations for the controllers, we guaranteed input-to-state stability of the closed-loop system when the average number of switches per unit of time is smaller than a specific value. If this is not the case, we showed that ISS property can be achieved under a mild constraint in optimization. A direction for future research is to consider the order of controller realization as a degree of freedom to guarantee closed-loop stability and to improve transient performance. Moreover, we assumed that the switching signal $\sigma(t)$ was provided externally however, in future, one may add a criterion for selecting appropriate switching times in the minimization of J in (5.11). In many applications of switching control, destabilizing controllers are sometimes (temporarily) placed in the feedback loop. Such scenarios, which are not allowed here, are also an important direction for future research.

Moreover, as an important direction for future research, we would like to combine the idea of controller initialization (in this chapter) and the LQR problem of stochastic hybrid systems (Chapter 4) to get the best overall performance.

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Appendices

Appendix A

Concepts in Probability Theory

In this section, we summarize some concepts related to probability theory that we use throughout this dissertation. More details can be found in [56, 16].

Definition A.0.1 (Convergence with probability one). Let X_n be a sequence of random variables on a probability space $(\Omega, \mathfrak{F}, P)$. The sequence X_n converges *almost surely* or *almost everywhere* or with *probability one* or *strongly* towards X if

$$P\left(\lim_{n\to\infty} X_n = X\right) = 1.$$

This means that the events that X_n does not converge to X have probability zero. For generic random elements X_n on a metric space (S, d), almost surely convergence is defined similarly:

$$P\left(\omega \in \Omega \ : \ d\left(X_n(\omega), X(\omega)\right) \stackrel{n\to\infty}{\longrightarrow} 0\right) = 1.$$

Theorem A.0.1 (Monotone Convergence)**.** *If X_n and X are non-negative random variables on a probability space $(\Omega, \mathfrak{F}, P)$ such that $X_n \uparrow X$ with $E\{X\} < \infty$ then for any sub σ-field $\mathfrak{G}$ in $\mathfrak{F}$,*

$$E\{X_n \mid \mathfrak{G}\} \uparrow E\{X \mid \mathfrak{G}\}.$$

Theorem A.0.2 (Dominated Convergence)**.** *If X_n, X and Y are random variables on a probability space $(\Omega, \mathfrak{F}, P)$ such that Y is integrable ($E\{Y\} < \infty$), $|X_n| \leq Y$ for all n and $X_n \to X$ almost surely then for any sub σ-field $\mathfrak{G}$ in $\mathfrak{F}$, X is integrable and*

$$\lim_{n\to\infty} E\{X_n \mid \mathfrak{G}\} = E\{X \mid \mathfrak{G}\}.$$

A special case is bounded convergence *in which Y is a constant.*

Theorem A.0.3 (Properties of conditional expectations)**.** *In the following statements, whenever the equality of random variables is used it is understood in the almost sure sense. Consider a probability space $(\Omega, \mathfrak{F}, P)$ and suppose $\mathfrak{G}$ and $\mathfrak{H}$ are sub σ-fields of $\mathfrak{F}$:*

1. *If X is $\mathfrak{G}$-measurable, then*

$$E\{XY \mid \mathfrak{G}\} = XE\{Y \mid \mathfrak{G}\}.$$

2. *If $\mathfrak{G} \subset \mathfrak{H}$, then*

$$E\left\{E\left\{X \mid \mathfrak{H}\right\} \mid \mathfrak{G}\right\} = E\left\{X|\mathfrak{G}\right\}.$$

3. *If $\sigma(X)$ and $\mathfrak{G}$ are independent, then*

$$E\left\{X|\mathfrak{G}\right\} = E\left\{X\right\}.$$

Lemma A.0.1 (Borel-Cantelli)**.** *For events $\{A_n\}$ such that $\sum_n P\left(A_n\right) < \infty$, we have*

$$P\left(\{\omega \ : \ \omega \in A_n \text{ for infinitely many } n\}\right) = 0.$$

A related result, sometimes called the second Borel-Cantelli Lemma, is a partial converse of the first Borel-Cantelli lemma. The lemma states: If the events $\{A_n\}$ are independent and the sum of the probabilities of the $\{A_n\}$ diverges to infinity, then the probability that infinitely many of them occur is one.

Appendix B

Piecewise-Deterministic Markov Processes

The model that we used for stochastic hybrid systems with renewal transitions, whose formal definitions can be found in Section 4.1 was introduced in [6], and is heavily inspired by the Piecewise-Deterministic Markov Process (PDPs) in [16]. In a PDP, state trajectories are right-continuous with finitely many jumps on a finite interval. The continuous evolution of the process is described by a deterministic flow whereas the jumps can occur at random times. In this section, we provide a simplified version of the PDPs defined in [16], since we do not allow the process to have deterministic jumps.

We consider state variable x and q to be in $\mathbb{R}^n$ and $\mathcal{S}$, respectively where $\mathcal{S}$ is a finite set. During flows, the continuous state $\mathbf{x}(t)$ evolves according to the vector field $f(x, q)$, whereas the discrete state $\mathbf{q}(t)$ remains constant and changes only with jumps. For a fixed $q \in \mathcal{S}$, we denote by $\phi(t; x, q)$ the continuous flow at time t defined by the vector field $f(.,q)$ and starting at x at time 0. The conditional probability that at least one jump occurs between the time instants t and s, $0 < s < t$, given $\mathbf{x}(s)$ and $\mathbf{q}(s)$, is

$$1 - \exp\left(-\int_s^t \lambda\left(\phi(\tau - s; \mathbf{x}(s), \mathbf{q}(s)), \mathbf{q}(s)\right)\ d\tau\right),$$

where the non-negative function $\lambda(x,q)$ is called *jump rate* at $(x,q) \in \mathbb{R}^n \times \mathcal{S}$. At each jump, the state $\mathbf{X} := (\mathbf{x}, \mathbf{q})$ assumes a new value distributed according to the *jump kernel* Q. Namely, if $\{\mathbf{t}_k\}$ denote the sequence of jump times, then

$$P\left(\mathbf{X}(\mathbf{t}_k) \in A \mid \mathbf{X}(\mathbf{t}_k^-) = X\right) = Q(X, A)$$

for some $A \in \mathfrak{B}(\Omega)$. Under the standard Assumption B.0.1, it is proven in [16] that the defined PDP is a strong Markov process.

Assumption B.0.1.

1. $x \to f(x,q)$ is locally Lipschitz for each $q \in \mathcal{S}$ and the solution admits no finite escape time.
2. $\lambda : \mathbb{R}^\times\mathcal{S} \to \mathbb{R}^+$ is a measurable function such that map $t \to \lambda(\phi(t;x,q),q)$ is locally integrable.
3. The probability kernel Q is such that $Q(X, \{X\}) = 0$ for all $X \in \mathbb{R}^n \times \mathcal{S}$.
4. $E_X[\sum_{k=0}^{\infty} I_{(\mathbf{t}_k \le t)}] < \infty$ for all $t > 0$ and $X \in \mathbb{R}^n \times \mathcal{S}$.

The *extended generator* of the PDP was first characterized in [16]. For path-differentiable function h, define the operator (extended generator)

$$\mathcal{D}_\phi h(x,q) := \frac{d}{dt} h\left(\phi(t;x,q),q\right)|_{t=0}\,.$$

The characterization of the extended generator is given in [16, 26.14] which is stated in the following Theorem.

Theorem B.0.4. *The domain of the extended generator of the PDP* $\mathbf{X}$ *consists of all measurable functions* h *that satisfy*

1. *The map* $t \to h(\phi(t;x,q),q)$ *is absolutely continuous for all* $(x,q) \in \mathbb{R}^n \times \mathcal{S}$

2. *The process*

$$\sum_{\mathbf{t}_k < t} h(\mathbf{X}(\mathbf{t}_k)) - h(\mathbf{X}(\mathbf{t}_k^-))$$

   is locally integrable.

The extended generator is given by

$$\mathcal{L}h := \mathcal{D}_\phi h(x,q) + \lambda Q h - \lambda h$$

for a function h *in the domain of the extended generator and*

$$Qh(y) := \int h(s) Q(y, ds).$$

Appendix C

We use the (conditional) probability density functions of the inter-jump interval to show that

$$\sum_{k=1}^{\infty} E_z \left\{ e^{-\rho \mathbf{t}_k} \mathbf{I}_{(k \leq \mathbf{N}_Q(t))} h\left(\mathbf{X}_Q(t_k^-)\right)\right\} = E_z \left\{ \int_0^{t\wedge \mathbf{t}_1} e^{-\rho t} (h\lambda)_Q \left(\mathbf{X}_Q(t)\right)\ dt \right\}$$
$$+ \sum_{k=2}^{\infty} E_z \left\{ \int_{t\wedge \mathbf{t}_{k-1}}^{t\wedge \mathbf{t}_k} e^{-\rho t} (h\lambda)_Q \left(\mathbf{X}_Q(t)\right)\ dt \right\}. \tag{C.1}$$

From the construction of the SHS, $\forall t \in (\mathbf{t}_{k-1}, \infty)$ and $k > 1$, the probability density functions of the event $(\mathbf{t}_k < t)$ conditioned to the natural filtration $\mathfrak{F}_{\mathbf{t}_{k-1}}$ is given by

$$\begin{aligned}
f_k(t|\mathfrak{F}_{\mathbf{t}_{k-1}}) &= \frac{d}{dt} P(\mathbf{t}_k \leq t | \mathfrak{F}_{\mathbf{t}_{k-1}}) \\
&= \frac{d}{dt} P(\mathbf{h}_{k-1} \leq t - \mathbf{t}_{k-1} | \mathfrak{F}_{\mathbf{t}_{k-1}}) \\
&= \frac{dF_{\mathbf{q}(\mathbf{t}_{k-1})}(\boldsymbol{\tau}(t))}{dt} \\
&= \lambda_{\mathbf{q}(\mathbf{t}_{k-1})}(\boldsymbol{\tau})(1 - F_{\mathbf{q}(\mathbf{t}_{k-1})}(\boldsymbol{\tau})) \\
&= \lambda_{\mathbf{q}(\mathbf{t}_{k-1})}(\boldsymbol{\tau}) E\{\mathbf{I}_{\mathbf{t}_k > t} | \mathfrak{F}_{\mathbf{t}_{k-1}}\}
\end{aligned}$$

from which we conclude

$$
\begin{aligned}
&E\left\{\mathbf{I}_{(k\le \mathbf{N}_Q(t))} e^{-\rho \mathbf{t}_k} h\left(\mathbf{X}_Q(t_k^-)\right)|\mathfrak{F}_{\mathbf{t}_{k-1}}\right\} \\
&\quad = \mathbf{I}_{\left(\mathbf{X}_Q(\mathbf{t}_{k-1})\in \mathrm{Int}(Q)\right)} \int_{t\wedge \mathbf{t}_{k-1}}^{t\wedge T_Q(\mathbf{t}_{k-1})} e^{-\rho t} h\left(\mathbf{X}(t)\right) f_k(t|\mathfrak{F}_{\mathbf{t}_{k-1}})\ dt \\
&\quad = \mathbf{I}_{\left(\mathbf{X}_Q(\mathbf{t}_{k-1})\in \mathrm{Int}(Q)\right)} \int_{t\wedge \mathbf{t}_{k-1}}^{t\wedge T_Q(\mathbf{t}_{k-1})} e^{-\rho t} (h\lambda)\left(\mathbf{X}(t)\right) E\{\mathbf{I}_{\mathbf{t}_k > t}|\mathfrak{F}_{\mathbf{t}_{k-1}}\}\ dt \qquad \text{(C.2)} \\
&\quad = E\left\{\mathbf{I}_{\left(\mathbf{X}_Q(\mathbf{t}_{k-1})\in \mathrm{Int}(Q)\right)} \int_{t\wedge \mathbf{t}_{k-1}}^{t\wedge T_Q(\mathbf{t}_{k-1})} e^{-\rho t} (h\lambda)\left(\mathbf{X}(t)\right) \mathbf{I}_{\mathbf{t}_k > t}\ dt|\mathfrak{F}_{\mathbf{t}_{k-1}}\right\} \\
&\quad = E\left\{\mathbf{I}_{\left(\mathbf{X}_Q(\mathbf{t}_{k-1})\in \mathrm{Int}(Q)\right)} \int_{t\wedge \mathbf{t}_{k-1}}^{t\wedge \mathbf{t}_k \wedge T_Q(\mathbf{t}_{k-1})} e^{-\rho t} (h\lambda)\left(\mathbf{X}(t)\right)\ dt|\mathfrak{F}_{\mathbf{t}_{k-1}}\right\}
\end{aligned}
$$

with $(h\lambda)(x,\tau,q) := h(x,\tau,q)\lambda_q(\tau)$. Here $T_Q(\mathbf{t}_{k-1})$ denotes the first time that the solution of the system leaves the interior of Q after the jump $\mathbf{t}_{k-1}$. Moreover, since $\forall t \in [\mathbf{t}_{k-1}, \mathbf{t}_k \wedge T_Q(\mathbf{t}_{k-1})]$,

$$
\mathbf{X}_Q(t) = \begin{cases} \mathbf{X}(t) & \mathbf{X}_Q(\mathbf{t}_{k-1}) \in \mathrm{Int}(Q) \\ \mathbf{X}_Q(\mathbf{t}_{k-1}) & \mathbf{X}_Q(\mathbf{t}_{k-1}) \notin \mathrm{Int}(Q), \end{cases}
$$

we can re-write the right-hand side of (C.2) compactly as

$$
\begin{aligned}
&E\left\{\mathbf{I}_{(k\le \mathbf{N}_Q(t))} e^{-\rho \mathbf{t}_k} h\left(\mathbf{X}_Q(t_k^-)\right)|\mathfrak{F}_{\mathbf{t}_{k-1}}\right\} = \\
&\qquad E\left\{\int_{t\wedge \mathbf{t}_{k-1}}^{t\wedge \mathbf{t}_k} e^{-\rho t} (h\lambda)_Q\left(\mathbf{X}_Q(t)\right)\ dt|\mathfrak{F}_{\mathbf{t}_{k-1}}\right\} \qquad \text{(C.3)}
\end{aligned}
$$

with $(h\lambda)_Q(x,\tau,q) := h(x,\tau,q)\lambda_q(\tau)$ inside Q^q and zero otherwise. Similarly for $\forall t \in (0,\infty)$ and $k = 1$, conditional probability density functions of the event $(\mathbf{t}_1 < t\ )$ given the initial condition $z = (x,\tau,q) \in \mathbb{R}^n \times [0, T_q) \times \mathcal{S}$ can be computed as

$$\begin{aligned} f_1(t|z) &= \frac{d}{dt}P(\mathbf{t}_1 \le t|z) \\ &= \frac{d}{dt}P(\mathbf{h}_0 \le t|0) = \frac{dF_q(\tau(t))}{dt} \\ &= \lambda_q(\tau(t))(1 - F_q(\tau(t))) = \lambda_q(\tau(t))E\{\mathbf{I}_{\mathbf{t}_1>t}|z\} \end{aligned}$$

from which we obtain

$$\begin{aligned} &E_z\left\{\mathbf{I}_{(1\le\mathbf{N}_Q(t))}e^{-\rho\mathbf{t}_1}h\left(\mathbf{X}_Q(t_1^-)\right)\right\} \\ &\quad= \mathbf{I}_{\left(z\in\mathrm{Int}(Q)\right)}\int_0^{t\wedge T_Q} e^{-\rho t}h\left(\mathbf{X}(t)\right)f_1(t|z)\ dt \\ &\quad= \mathbf{I}_{\left(z\in\mathrm{Int}(Q)\right)}\int_0^{t\wedge T_Q} e^{-\rho t}(h\lambda)\left(\mathbf{X}(t)\right)E\{\mathbf{I}_{\mathbf{t}_1>t}|z\}\ dt \\ &\quad= E_z\left\{\mathbf{I}_{\left(z\in\mathrm{Int}(Q)\right)}\int_0^{t\wedge T_Q} e^{-\rho t}(h\lambda)\left(\mathbf{X}(t)\right)\mathbf{I}_{\mathbf{t}_1>t}\ dt\right\} \qquad \text{(C.4)} \\ &\quad= E_z\left\{\mathbf{I}_{\left(z\in\mathrm{Int}(Q)\right)}\int_0^{t\wedge\mathbf{t}_1\wedge T_Q} e^{-\rho t}(h\lambda)\left(\mathbf{X}(t)\right)\ dt\right\} \\ &\quad= E_z\left\{\int_0^{t\wedge\mathbf{t}_1} e^{-\rho t}(h\lambda)_Q\left(\mathbf{X}_Q(t)\right)\ dt\right\} \end{aligned}$$

Combining (C.3) and (C.4), the left-hand-side of (C.1) can be written as

$$\begin{aligned} &\sum_{k=1}^{\infty}E_z\left\{e^{-\rho\mathbf{t}_k}\mathbf{I}_{(k\le\mathbf{N}_Q(t))}h\left(\mathbf{X}_Q(t_k^-)\right)\right\} = E_z\{e^{-\rho\mathbf{t}_1}\mathbf{I}_{(1\le\mathbf{N}_Q(t))}h\left(\mathbf{X}_Q(t_1^-)\right\}\} \\ &+\sum_{k=2}^{\infty}E_z\left\{E\left\{e^{-\rho\mathbf{t}_k}\mathbf{I}_{(k\le\mathbf{N}_Q(t))}h\left(\mathbf{X}_Q(t_k^-)\right)|\mathfrak{F}_{\mathbf{t}_{k-1}}\right\}\right\} \\ &= E_z\left\{\int_0^{t\wedge\mathbf{t}_1} e^{-\rho t}(h\lambda)_Q\mathbf{X}_Q\ dt\right\} + \sum_{k=2}^{\infty}E_z\left\{E\left\{\int_{t\wedge\mathbf{t}_{k-1}}^{t\wedge\mathbf{t}_k} e^{-\rho t}(h\lambda)_Q\mathbf{X}_Q\ dt|\mathfrak{F}_{\mathbf{t}_{k-1}}\right\}\right\} \\ &= E_z\left\{\int_0^{t\wedge\mathbf{t}_1} e^{-\rho t}(h\lambda)_Q\left(\mathbf{X}_Q(t)\right)\ dt\right\} + \sum_{k=2}^{\infty}E_z\left\{\int_{t\wedge\mathbf{t}_{k-1}}^{t\wedge\mathbf{t}_k} e^{-\rho t}(h\lambda)_Q\left(\mathbf{X}_Q(t)\right)\ dt\right\}. \end{aligned}$$