H_∞ Estimation of Continuous-Time Systems with Implicit Outputs from Discrete Noisy Time-Delayed Measurements ⋆

A. Pedro Aguiar [a], João P. Hespanha [b],

[a] *ISR/IST - Institute for Systems and Robotics, Instituto Superior Técnico Torre Norte 8, Av. Rovisco Pais, 1049-001 Lisboa, Portugal*

[b] *Center for Control Engineering and Computation University of California, Santa Barbara, CA 93106-9560*

Abstract

We address the state estimation of a class of continuous-time systems with implicit outputs, whose measurements arrive at discrete-time instants, are time-delayed, noisy, and may not be complete. The estimation problem is formulated in the deterministic H_∞ filtering setting by computing the value of the state that minimizes the induced L_2-gain from disturbances and noise to estimation error, while remaining compatible with the past observations. To avoid weighting the distant past as much as the present, a forgetting factor is also introduced. We show that, under appropriate observability assumptions, the optimal estimate converges globally asymptotically to the true value of the state in the absence of noise and disturbance. In the presence of noise, the estimate converges to a neighborhood of the true value of the state. The estimation of position and attitude of an autonomous robotic vehicle using measurements from an inertial measurement unit (IMU) and a monocular charged-coupled-device (CCD) camera attached to the vehicle illustrates these results. In the context of this application, the robust H_∞ estimator can deal directly with the usual problems associated with vision systems such as noise, latency and intermittency of observations. We present and discuss several simulation results that demonstrate and validate the good performance and applicability of the proposed estimator.

Key words: Observers for nonlinear systems, H_infty estimation, Visual servo control, Autonomous Vehicles.

⋆ Corresponding author A. Pedro Aguiar. Tel. +351 21 841 8056. Fax +351 21 841 8291. This work was supported in part by the FCT-ISR/IST plurianual funding program through the POS_C Program that includes FEDER funds.

Email addresses: pedro@isr.ist.utl.pt (A. Pedro Aguiar),

1 Introduction

Over the past decades there have been an extensive study on the design of observers for nonlinear systems. The extended Kalman filter is a widely used method for estimating the state of a nonlinear system. It is obtained by linearizing the nonlinear dynamics and the observation along the trajectory of the estimate in order to compute the filter gain. However, since it is only a local method, it often fails to converge. The main contribution of this paper is the design of a state estimator algorithm for a particular important class of nonlinear systems with *guaranteed stability and convergence, robustness and performance in the presence of sensor noises and disturbances*.

The class of systems considered are the ones described by

$$\dot{x} = A(x,u) + G(u)w, \tag{1}$$
$$E_j(x,v_j)y_j = C_j(x,u) + v_j, \quad j \in \mathcal{I} := \{1,2,\dots N\} \tag{2}$$

where $x \in \mathbb{R}^n$ denotes the state of the system, $u \in \mathbb{R}^m$ its control input, $y_j \in \mathbb{R}^{q_j}$ its jth measured output, $w \in \mathbb{R}^r$ an input disturbance that cannot be measured, and $v_j \in \mathbb{R}^{p_j}$ measurement noise affecting the jth output. The functions $A(x,u)$, $C_j(x,u)$, and $E_j(x,v_j)$ are affine in x. The initial condition $x(0)$ of (1) and the signals w and v_j are assumed deterministic but unknown. Each measured output y_j is only defined implicitly through (2) and the map $E_j(x,\cdot)$ is such that

$$\operatorname{Image} E_j(x,\cdot) = \Big\{E_{0j}(x) + \sum_{i=1}^{\ell_j} \alpha_{ij}E_{ij} : \alpha_{ij} \in \mathbb{R}\Big\} \tag{3}$$

where $E_{ij} \in \mathbb{R}^{p_j \times q_j}$ and $E_{0j}(x)$ are affine in x. Note that although the implicit representation (2) is affine in x, an explicit representation would generally be nonlinear. We call (1)–(3) a *state-affine system with implicit outputs*, or for short simply a *system with implicit outputs*. These type of systems were introduced in (Aguiar and Hespanha, 2005; Aguiar and Hespanha, 2006*b*) and are motivated by applications in dynamic vision such as the estimation of the motion of a camera from a sequence of images. In particular, in this paper, system (1)–(3) arises when one needs to estimate the relative position and attitude of an autonomous robotic vehicle with respect to a desired coordinate system defined by visual landmarks (see Figure 1). In this setup the measurements are provided by an inertial measurement unit (IMU) and a monocular charged-coupled-device (CCD) camera mounted on-board that observes the apparent motion of stationary points.

System (1)–(3) can also be seen as a generalization of perspective systems introduced by Ghosh et al. (Ghosh *et al.*, 1994). The reader is referred to (Ghosh and Loucks, 1995; Takahashi and Ghosh, 2001) for several other ex-

hespanha@ece.ucsb.edu (João P. Hespanha).

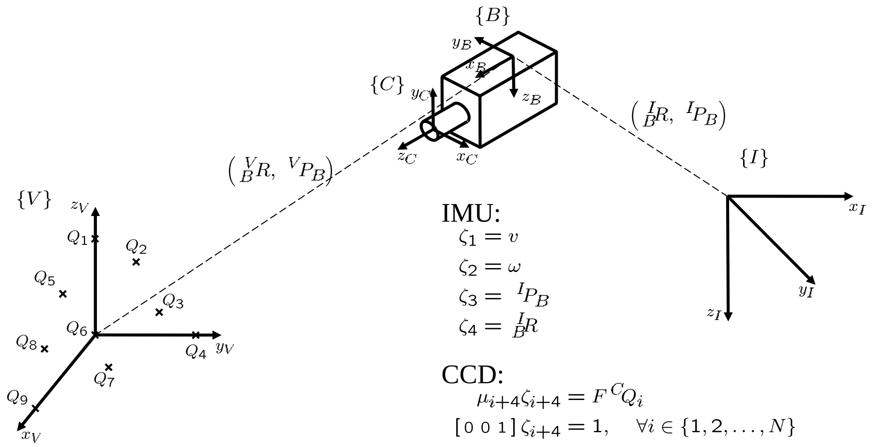

Fig. 1. A schematic of an autonomous vehicle with a monocular CCD camera and an IMU. Body-fixed $\{\mathcal{B}\}$, earth-fixed $\{I\}$, and visual-fixed $\{\mathcal{V}\}$ coordinate frames.

amples of perspective systems in the context of motion and shape estimation. The system with implicitly defined outputs described in (Matveev *et al.*, 2000) and the state-affine systems with multiple perspective outputs considered in (Aguiar and Hespanha, 2006*a*) are also special cases of (1)–(3).

In this paper, we address the state-estimation for the system (1)–(2) supposing that the measurements are acquired only at discrete times t'_i, $i = 0, 1, \ldots, k$, with $t'_0 < t'_1 < \ldots < t'_k$, and that we only have access to them after a time-delay τ_i. Our sequence of measurements from t'_0 to time $t \geq t'_0$ is therefore given by

$$\Big\{t'_i,\ \bar{y}_j(t_i),\ j \in \mathcal{I}_i\Big\}_{i=0,\ldots,k} \tag{4}$$

where k is the number of measurements received from t'_0 to time t (i.e., $t_k \leq t$), $\bar{y}_j(t_i) := y_j(t'_i) = y_j(t_i - \tau_i)$ denotes the time-delay observed variable, and $t_i = t'_i + \tau_i$. In (4), we also assume that the measurements may not be complete, that is, at time t'_i only the outputs $y_j \in \mathbb{R}^{q_j}$ with $j \in \mathcal{I}_i$ are available, where $\mathcal{I}_i \subseteq \mathcal{I}$, and the inclusion may be strict when some measurements are missing.

The problem under consideration is *to design an observer which estimates the continuous-time state vector* $x(t)$ *governed by* (1)*, given the discrete time-delay measurements* $\bar{y}_j(t_i)$ *expressed by the output equation*

$$\begin{aligned} E_j\Big(x(t_i - \tau_i), v(t_i - \tau_i)\Big)\bar{y}_j(t_i) \\ = C_j\Big(x(t_i - \tau_i), u(t_i - \tau_i)\Big) + v_j(t_i - \tau_i), \quad j \in \mathcal{I}_i \end{aligned} \tag{5}$$

Using a H_∞ deterministic approach, we propose an observer that estimates the state vector $x(t)$ given an initial estimate $\hat{x}_0$, the past controls $\{u(\sigma) : 0 \leq \sigma \leq t\}$ and the observations (4), and minimize the induced $\mathcal{L}_2$-gain from disturbances to estimation error. In particular, for the simpler case of no delay

($\tau_i = 0$), given a gain level $\gamma > 0$, the estimate $\hat{x}$ should satisfy

$$\int_0^t \|x(\sigma) - \hat{x}(\sigma)\|^2 \, d\sigma \leq \gamma^2 \Big((x(0) - \hat{x}_0)' P_0 (x(0) - \hat{x}_0) + \int_0^t \|w(\sigma)\|^2 \, d\sigma + \sum_{i=0}^{k} \sum_{j \in \mathcal{I}_i} \|v_j(t_i)\|^2 \Big)$$

where $P_0 > 0$ and $\hat{x}_0$ encode a a-priori information about the state. We also consider the possibility of introducing an exponential forgetting factor that decreases the weight of x, w and v_j from a distant past.

During the last two decades the H_∞ criterion has been applied to filtering problems, cf., e.g., (Başar and Bernhard, 1995; Nagpal and Khargonekar, 1991; Xie *et al.*, 1994; Krener, 1997; Sayed, 2001; Boel *et al.*, 2002; Aguiar and Hespanha, 2005; Aguiar and Hespanha, 2006*b*). In (Aguiar and Hespanha, 2005; Aguiar and Hespanha, 2006*b*), a state-estimator for (1)–(3) was designed using a deterministic H_∞ approach, assuming that measurements were continuously available. Given an initial estimate and the past controls and continuous observations collected up to time t, the optimal state estimate $\hat{x}$ at time t was defined to be the value that minimizes the induced $\mathcal{L}_2$-gain from disturbances to estimation error. Closely related to H_∞ filtering are the minimum-energy estimators, which were first proposed by Mortensen (Mortensen, 1968) and further refined by Hijab (Hijab, 1980). Game theoretical versions of these estimators were proposed by McEneaney (McEneaney, 1998). In (Aguiar and Hespanha, 2006*a*), minimum-energy estimators were derived for systems with perspective outputs and input-to-state stability like properties of the estimation error with respect to disturbances were presented.

In general, both minimum-energy and H_∞ state estimators for nonlinear systems lead to infinite-dimensional observers whose state evolves according to a first-order nonlinear partial differential equation of Hamilton-Jacobi type, driven by the observations. The main contribution of this paper is a closed-form solution that is filtering-like and iterative, which continuously improves the estimates as more measurements become available. More precisely, under appropriate observability assumptions, we prove that the optimal state estimate converges exponentially fast to the true value of the state in the absence of noise and disturbance. In the presence of noise, the estimate converges to a neighborhood of the true value of the state. We can therefore use this state-estimator to design output-feedback controllers by using the estimated state to drive state-feedback controllers. These results extend and complement the ones in (Aguiar and Hespanha, 2005; Aguiar and Hespanha, 2006*b*) in that now the measured outputs are transmitted through a network, that is, the measurements arrive at discrete-time instants, are time-delayed, noisy, and may not be complete.

We apply these results to the estimation of the pose (position and attitude) of an autonomous vehicle using measurements from an IMU and a monocular

CCD camera attached to the vehicle. One of the key contributions is the fact that, contrary to most of the algorithms described in the literature for pose estimation using visual information (cf., e.g., (Matthies *et al.*, 1989; Jankovic and Ghosh, 1995; Soatto *et al.*, 1996; Kaminer *et al.*, 2001; Chiuso *et al.*, 2002; Rehbinder and Ghosh, 2003) and references therein), we propose an observer with guaranteed stability, convergence, and robustness in the presence of disturbances and sensor noises. Furthermore, in the context of this application, the estimator can deal directly with the usual problems associated with vision systems such as noise, latency and intermittency of observations. We present and discuss several simulation results that demonstrate and validate the good performance and applicability of the proposed methodology.

The paper is organized as follows: In Section 2 we formulate the state estimation problem using a H_∞ deterministic approach. Section 3 and Section 4 present the main results of the paper. In Section 3 we derive, using dynamic programming, the equations for the optimal observer. In Section 4 we determine under what conditions the state estimate $\hat{x}$ converges to the true state x. An example in Section 5 illustrates the results. Concluding remarks are given in Section 6.

This paper builds upon and extends previous results by the authors (Aguiar and Hespanha, 2005; Aguiar and Hespanha, 2006*a*; Aguiar and Hespanha, 2006*b*; Aguiar and Hespanha, 2006*c*).

2 Problem Statement

This section formulates the state estimation problem using a H_∞ deterministic approach. Consider the system with implicit outputs (1), (5). From (1), $x(t_i)$ satisfies

$$\begin{aligned} x(t_i) = {} & \Phi(t_i, t_i - \tau_i) x(t_i - \tau_i) \\ & + \int_{t_i-\tau_i}^{t_i} \Phi(t_i,\sigma)\Big[A(0,u(\sigma)) + G(u(\sigma))w(\sigma)\Big]\, d\sigma, \end{aligned} \tag{6}$$

where $\Phi(t, t_0)$ is the transition matrix of system (1) satisfying the linear time-varying differential equation

$$\dot{\Phi} = \nabla A(u)\Phi. \tag{7}$$

In (7), $\nabla A(u)$ denote the gradient of $A(x, u)$ with respect to x. Since $A(x, u)$ is affine in x, it follows that $\nabla A(\cdot)$ only depend on u. From (6) we obtain

$$\begin{aligned} x(t_i - \tau_i) = {} & \Phi^{-1}(t_i, t_i - \tau_i) x(t_i) \\ & - \Phi^{-1}(t_i, t_i - \tau_i) \int_{t_i-\tau_i}^{t_i} \Phi(t_i,\sigma)\Big[A(0,u(\sigma)) + G(u(\sigma))w(\sigma)\Big]\, d\sigma. \end{aligned}$$

$$J(z;t) := \min_{\substack{w:[0,t),\\ \bar{v}_j(t_i)\, i=0,1,\dots k}} \Bigg\{ \gamma^2 e^{-2\lambda t}(x(0)-\hat{x}_0)' P_0 (x(0)-\hat{x}_0) + \gamma^2 \int_0^t e^{-2\lambda(t-\sigma)} \|w(\sigma)\|^2 \, d\sigma + \gamma^2 \sum_{i=0}^{k} \sum_{j\in\mathcal{I}_i} e^{-2\lambda(t_k-t_i)} \|\bar{v}_j(t_i)\|^2$$

$$- \int_0^t e^{-2\lambda(t-\sigma)} \|x(\sigma)-\hat{x}(\sigma)\|^2 \, d\sigma : x(t) = z, \dot{x} = A(x,u) + G(u)w, \bar{E}_{ji}\bar{y}_j(t_i) = \bar{C}_j(x(t_i),u) + \bar{v}_j(t_i) \Bigg\} \tag{10}$$

Substituting this equation in (5) and exploring the fact that $E_j(x,v)$ and $C_j(x,u)$ are affine functions in x,[1] we obtain

$$\bar{E}_{ji}(x(t_i), v_j(t_i - \tau_i))\bar{y}_j(t_i) = \bar{C}_j(x(t_i), u) + \bar{v}_j(t_i), \quad j \in \mathcal{I}_i, \tag{8}$$

where

$$\begin{aligned}
\bar{E}_{ji}(x(t_i), \cdot) &:= E_j\Big(\Phi(t_i - \tau_i, t_i)x(t_i) - \Phi(t_i - \tau_i, t_i) \int_{t_i-\tau_i}^{t_i} \Phi(t_i, \sigma) A(0, u(\sigma))\, d\sigma, \cdot\Big), \\
\bar{C}_j(x(t_i), u) &:= \nabla\bar{C}_j(u) x(t_i) + C_j(0, u(t_i - \tau_i)), \\
\nabla\bar{C}_j(u) &:= \nabla C_j(u(t_i - \tau_i))\Phi(t_i - \tau_i, t_i), \\
\bar{C}_j(0, u) &:= -\nabla\bar{C}_j(u) \int_{t_i-\tau_i}^{t_i} \Phi(t_i, \sigma) A(0, u(\sigma))\, d\sigma + C_j(0, u(t_i - \tau_i)), \\
\bar{v}_j(t_i) &:= \mu_j(t_i) + v_j(t_i - \tau_i), \\
\mu_j(t_i) &:= [-\nabla\bar{C}_j(u) + \mathbf{J}E_j(\bar{y}_j, v_j(t_i - \tau_i))] \int_{t_i-\tau_i}^{t_i} \Phi(t_i, \sigma) G(u(\sigma)) w(\sigma)\, d\sigma.
\end{aligned}$$

The estimation problem can now be stated as follows:

Problem 1 *Consider the continuous-time state equation* (1) *together with the discrete-time implicit output equation* (8)*. For a given gain level* $\gamma > 0$*, initial estimate* $\hat{x}_0$*, input* u *defined on an interval* $[0, t)$*, and measured outputs* $\bar{y}_j(t_i)$*,* $j \in \mathcal{I}_i$ *with* $i = 0, 1, \dots k$*,* $t_0 := 0 \le t_1 \le \cdots \le t_k \le t \le t_{k+1}$*, compute the estimate* $\hat{x}(t)$ *of the state at time* t *defined as*

$$\hat{x}(t) := \arg\min_{z\in\mathbb{R}^n} J(z, t), \tag{9}$$

where $J(z, t)$ *is defined in* (10)*, at the top of the next page,* $P_0 > 0$*, and* $\lambda \ge 0$ *is a forgetting factor.* □

In a broad sense, for a given gain level $\gamma > 0$, the optimal state $\hat{x}$ at time t is defined to be the value for the state that is compatible with the initial estimate $\hat{x}_0$, the past controls and observations collected up to time t, and the dynamics of the system which ensures the prescribed bound γ on the

[1] which implies that $E_j(x,v)y = \mathbf{J}E_j(y,v)x + E_j(0,v)y$ and $C_j(x,u) = \nabla C_j(u)x + C_j(0,u)$, where $\mathbf{J}E_j(y,v)$ is the Jacobian of $E_j(x,v)y$ with respect to x

discounted induced L_2-gain from disturbances and noise to estimation error. The symmetric negative of $J(z,t)$ can also be viewed as the *information state* introduced in (James *et al.*, 1993; James *et al.*, 1994) and interpreted as a measure of the likelihood of state $x = z$ at time t.

3 The observer equations

In this section we present the observer equations, which are derived using dynamic programming. In what follows, given a signal x with a jump at time t, we denote by $x(t^-)$ the limit of $x(\tau)$ as $\tau \uparrow t$ from below, *i.e.*, $x(t^-) := \lim_{\tau\uparrow t} x(\tau)$. Without loss of generality we take all signals to be continuous from above at every point, *i.e.*, $x(t) = \lim_{\tau\downarrow t} x(\tau)$. We propose the following observer and will shortly show that it solves Problem 1.

a) Initial condition

$$t_0 = 0, \quad P(t_0) = P_0, \quad \hat{x}(t_0) = \hat{x}_0 \tag{11}$$

b) Dynamic equations for $t \in [t_i, t_{i+1})$, $i = 0, 1, \ldots, k$

$$\begin{aligned} \dot{P}(t) &= -P(t)(\nabla A(u) + \lambda I) - (\nabla A(u) + \lambda I)' P(t) \\ &\quad - \gamma^{-2}(P(t)G(u)G(u)'P(t) + \gamma^2 I), \end{aligned} \tag{12}$$

$$\dot{\hat{x}}(t) = A(\hat{x}, u), \tag{13}$$

with $P(t_i) = P_i$, and $\hat{x}(t_i) = \hat{x}_i$.

c) Impulse equations at $t = t_{i+1}$, $i = 0, 1, \ldots, k-1$

$$P(t_{i+1}) = P(t_{i+1}^-) + \gamma^2 \sum_{j\in\mathcal{I}_{i+1}} \Psi_j(t_{i+1}), \tag{14}$$

$$\hat{x}(t_{i+1}) = \hat{x}(t_{i+1}^-) - P(t_{i+1})^{-1}\gamma^2 \sum_{j\in\mathcal{I}_{i+1}} \Big[\Psi_j(t_{i+1})\hat{x}(t_{i+1}^-) + \psi_j(t_{i+1})\Big] \tag{15}$$

where

$$\begin{aligned} \Psi_j(t_i) &:= \Big(\mathbf{J}E_{0j}(\bar{y}_j(t_i))\Phi(t_i - \tau_i, t_i) - \nabla\bar{C}_j\Big)' \Big(I - Y_{ji}Y_{ji}^{\perp}\Big)' \\ &\quad \times \Big(I - Y_{ji}Y_{ji}^{\perp}\Big)\Big(\mathbf{J}E_{0j}(\bar{y}_j(t_i))\Phi(t_i - \tau_i, t_i) - \nabla\bar{C}_j\Big), \\ \psi_j(t_i) &:= \Big(\mathbf{J}E_{0j}(\bar{y}_j(t_i))\Phi(t_i - \tau_i, t_i) - \nabla\bar{C}_j\Big)' \Big(I - Y_{ji}Y_{ji}^{\perp}\Big) \\ &\quad \times \Big(I - Y_{ji}Y_{ji}^{\perp}\Big)\Big(E_{0j}(0)\bar{y}_j(t_i) - \mathbf{J}E_{0j}(\bar{y}_j(t_i)) \\ &\quad \times \Phi(t_i - \tau_i, t_i)\int_{t_i-\tau_i}^{t_i} \Phi(t_i, \sigma)A(0, u(\sigma))\, d\sigma - \bar{C}_j(0, u)\Big), \\ Y_{ji} &:= \Big[E_{1j}\bar{y}_j(t_i)|E_{2j}\bar{y}_j(t_i)|\cdots|E_{\ell_j j}\bar{y}_j(t_i)\Big], \end{aligned} \tag{16}$$

$Y_{ji}^{\perp}$ denotes the pseudo-inverse of Y_{ji}, $\nabla A(u)$ and $\mathbf{J}E_{0j}(y)$ are respectively the gradient of $A(x,u)$ and the Jacobian of $E_{0j}(x)y$ both with respect to x. The following result solves Problem 1.

Theorem 2 *The H_∞ state estimate $\hat{x}(t)$ defined by* (9)–(10) *can be obtained from the* impulse system (11)–(15). *Furthermore, the cost function $J(z;t)$ defined in* (10) *is quadratic and can be written as*

$$J(z;t) = \Big(z - \hat{x}(t)\Big)' P(t) \Big(z - \hat{x}(t)\Big) + c(t), \tag{17}$$

where $c(t)$ satisfies an appropriate impulse equation (cf. (22), (27) *below).*

Proof. The function $J(z;t)$, $z \in \mathbb{R}^n$, $t \geq 0$ defined in (10) can be viewed as a cost-to-go and computed using dynamic programming. Take some $t \in (t_i, t_{i+1})$. To derive the dynamic programming operator we can consider an elementary time interval dt and write

$$\begin{aligned} J(z;t) = \min_w \Big\{ & \gamma^2 e^{-2\lambda dt} \Big(\|w\|^2 - \frac{1}{\gamma^2} \|z - \hat{x}\|^2 \Big) dt \\ & + e^{-2\lambda dt} J(z - (A(z,u) + G(u)w)dt; t - dt) \Big\} \end{aligned}$$

Subtracting $J(z; t - dt)$ from both sides of the above equation, dividing by dt, and taking the limit as $dt \to 0$, leads to

$$\begin{aligned} J_t(z;t) &= \min_w \Big\{ \gamma^2 \Big(\|w\|^2 - \frac{1}{\gamma^2} \|z - \hat{x}\|^2 \Big) \\ &\qquad - J_z(z;t)(A(z,u) + G(u)w) - 2\lambda J(z;t) \Big\} \\ &= -\frac{1}{4\gamma^2} \|G'(u) J_z'(z;t)\|^2 - \|z - \hat{x}\|^2 \\ &\qquad - J_z(z;t) A(z,u) - 2\lambda J(z;t), \end{aligned} \tag{18}$$

where J_t and J_z denote the partial derivatives of J with respect to t and z, respectively. For $i = 0$, the value of $J(z;t)$, $\forall t \in [t_0, t_1)$, is determined from the linear partial differential equation (18), with initial condition

$$J(z;0) = (z - \hat{x}_0)' P_0 (z - \hat{x}_0), \qquad z \in \mathbb{R}^n. \tag{19}$$

and can be written as (17) for appropriately defined signals $\hat{x}(t)$ and $c(t)$. The signal $\hat{x}$ thus minimize $J(z;t)$ and is therefore the estimate for the state x of the implicit system (1), (8). Moreover, matching (19) with (17) we conclude that

$$P(0) = P_0, \quad \hat{x}(0) = \hat{x}_0, \quad c(0) = 0.$$

To verify that the solution to (18)–(19) can indeed be written as (17), we substitute (17) into (18) and obtain

$$-2(z-\hat{x})'P\dot{\hat{x}}+(z-\hat{x})'\dot{P}(z-\hat{x})+\dot{c}=-\frac{1}{\gamma^2}\|G'P(z-\hat{x})\|^2-\|z-\hat{x}\|^2-2(z-\hat{x})'PA(z,u)$$
$$-2\lambda(z-\hat{x})'P(z-\hat{x})-2\lambda c.$$

Using the fact that $A(z,u)=\nabla A(u)z+A(0,u)$, we obtain

$$\begin{aligned}
&z'\Big[\dot{P}+\frac{1}{\gamma^2}PGG'P+P\nabla A+\nabla A'P+2\lambda P+I\Big]z\\
&\quad+2z'\Big[-P\dot{\hat{x}}-\dot{P}\hat{x}-\frac{1}{\gamma^2}PGG'P\hat{x}-\nabla A'P\hat{x}+PA(0,u)-2\lambda P\hat{x}-\hat{x}\Big]\\
&\quad\quad+\dot{c}+2\hat{x}'P\dot{\hat{x}}+\hat{x}'\dot{P}\hat{x}+\frac{1}{\gamma^2}\hat{x}'PGG'P\hat{x}-2\hat{x}'PA(0,u)+2\lambda\hat{x}'P\hat{x}\\
&\qquad\qquad+2\lambda c+\|\hat{x}\|^2=0
\end{aligned}$$

Since this equation must hold for all $z\in\mathbb{R}^n$ we conclude that

$$\dot{P}+\frac{1}{\gamma^2}PGG'P+P\nabla A+\nabla A'P+2\lambda P+I=0 \quad (20)$$

$$-P\dot{\hat{x}}-\dot{P}\hat{x}-\frac{1}{\gamma^2}PGG'P\hat{x}-\nabla A'P\hat{x}+PA(0,u)-2\lambda P\hat{x}-\hat{x}=0 \quad (21)$$

$$\dot{c}+2\hat{x}'P\dot{\hat{x}}+\hat{x}'\dot{P}\hat{x}+\frac{1}{\gamma^2}\hat{x}'PGG'P\hat{x}-2\hat{x}'PA(0,u)$$
$$+2\lambda\hat{x}'P\hat{x}+2\lambda c+\|\hat{x}\|^2=0 \quad (22)$$

Substituting (20) in (21), we conclude (12), (13), and (17) for $t\in[0,t_1)$.

Consider now the case $t=t_k$, $k>0$. From (10), we notice that $J(z;t_k)$ can be written as shown in (23), at the top of the next page. From the definition of $\bar{E}_{ji}$ and since $E_j(z,\bar{v}_j(t_k)+\mu_j(t_k))$ satisfies (3), the minimization in (10) over $\bar{v}_j(t_k)$ can be transformed into a minimization over $\alpha_{jk}:=(\alpha_{1j},\alpha_{2j},\dots,\alpha_{\ell j}j)'$. Thus,

$$J(z;t_k)=J(z;t_k^-)+\gamma^2\min_{\alpha_{jk}}\sum_{j\in\mathcal{I}_k}\|\bar{E}_{0j}(z)\bar{y}_j(t_k)+Y_{jk}\alpha_{jk}-\bar{C}_j(z,u)\|^2 \quad (24)$$

where Y_{jk} is defined in (16) and

$$\begin{aligned}
\bar{E}_{0j}(z)\bar{y}_j(t_k)&=\Big[E_{0j}\Big(\Phi(t_i-\tau_i,t_i)z-\Phi(t_i-\tau_i,t_i)\int_{t_i-\tau_i}^{t_i}\Phi(t_i,\sigma)A(0,u(\sigma))\,d\sigma\Big)\Big]\bar{y}_j(t_k)\\
&=\mathbf{J}E_{0j}(\bar{y}_j(t_k))\Phi(t_i-\tau_i,t_i)z\\
&\quad-\mathbf{J}E_{0j}(\bar{y}_j(t_k))\Phi(t_i-\tau_i,t_i)\int_{t_i-\tau_i}^{t_i}\Phi(t_i,\sigma)A(0,u(\sigma))\,d\sigma+E_{0j}(0)\bar{y}_j(t_k)
\end{aligned}$$

For $k=1$ we already saw that $J(z,t_1^-)$ is given by (17). Assuming that it has the same form at time t_1, substituting it in the left and right-hand-side of

$$
\begin{aligned}
J(z;t_k) = \min_{\bar v_j(t_k)} \Bigg\{ &\min_{\substack{w:[0,t_k),\bar v_j(t_i)\\ i=0,1,\dots k-1}} \Big\{ \gamma^2 e^{-2\lambda t_k}(x(0)-\hat x_0)'P_0(x(0)-\hat x_0) + \gamma^2\int_0^{t_k} e^{-2\lambda(t_k-\sigma)}\|w(\sigma)\|^2\,d\sigma \\
&+ \gamma^2 \sum_{j\in\mathcal{I}_k} \|\bar E_{jk}(x(t_k),v_j(t_k-\tau_k))\bar y_j(t_k) - \bar C_j(x(t_k),u)\|^2 \\
&+ \gamma^2 \sum_{i=0}^{k-1}\sum_{j\in\mathcal{I}_i} e^{-2\lambda(t_k-t_i)}\|\bar E_{ji}(x(t_i),v_j(t_i-\tau_i))\bar y_j(t_i) - \bar C_j(x(t_i),u)\|^2 \\
&- \int_0^{t_k} e^{-2\lambda(t_k-\sigma)}\|x(\sigma)-\hat x(\sigma)\|^2\,d\sigma : x(t_k^-) = x(t_k) = z, \dot x = A(x,u)+G(u)w \Big\} \Bigg\} \\
= \min_{\bar v_j(t_k)} &\Big\{ J(z;t_k^-) + \gamma^2\sum_{j\in\mathcal{I}_k} \|\bar E_{jk}(z,\bar v_j(t_k)-\mu_j(t_k))\bar y_j(t_k) - \bar C_j(z,u)\|^2 \Big\} \qquad (23)
\end{aligned}
$$

(24), we obtain

$$
\begin{aligned}
&z'\Big[P(t_k) - P(t_k^-) - \gamma^2\sum_{j\in\mathcal{I}_k}\Psi_j(t_k)\Big]z \\
&\quad + 2z'\Big[-P(t_k)\hat x(t_k) + P(t_k^-)\hat x(t_k^-) - \gamma^2\sum_{j\in\mathcal{I}_k}\psi(t_k)\Big] \\
&\quad + c(t_k) + \hat x(t_k)'P(t_k)x(t_k) - \hat x(t_k^-)'P(t_k^-)x(t_k^-) - c(t_k^-) \\
&\quad - \gamma^2\sum_{j\in\mathcal{I}_k}\Big(\bar E_{0j}(0)\bar y_j(t_k) - \bar C_j(0,u)\Big)'\Big(I - Y_{jk}Y_{jk}^{\perp}\Big)'\times \\
&\qquad \Big(I - Y_{jk}Y_{jk}^{\perp}\Big)\Big(\bar E_{0j}(0)\bar y_j(t_k) - \bar C_j(0,u)\Big) = 0.
\end{aligned}
$$

This equation holds for $k=1$, provided that

$$
P(t_k) - P(t_k^-) - \gamma^2\sum_{j\in\mathcal{I}_k}\Psi_j(t_k) = 0, \qquad (25)
$$

$$
-P(t_k)\hat x(t_k) + P(t_k^-)\hat x(t_k^-) - \gamma^2\sum_{j\in\mathcal{I}_k}\psi(t_k) = 0, \qquad (26)
$$

$$
c(t_k) + \hat x(t_k)'P(t_k)x(t_k) - \cdots = 0. \qquad (27)
$$

Thus, substituting (25) in (26) we conclude that (14) and (15) hold.

Notice that $P_1 := P(t_1) = P(t_1^-) + \gamma^2\sum_{j\in\mathcal{I}_1}\Psi_j(t_1)$ is positive definite because $P(t_1^-) > 0$ as it was proved above, and $\Psi_j(t_i) \geq 0$, $i = 1,\dots,k$. Therefore, substituting the initial condition (19) by

$$
J(z;t_1) = (z-\hat x_1)'P_1(z-\hat x_1), \quad z\in\mathbb{R}^n
$$

with $\hat x_1 = \hat x(t_1)$, and solving the linear partial differential equation (18), it follows that (12)–(15) hold for $t\in[0,t_2)$. Applying this line of reasoning successively until $i=k$ we conclude that (17) holds and that $\hat x(t)$ given by (12)–(15) is indeed the solution to Problem 1. □

To guarantee that the H_∞ state estimate has a global solution ($T = \infty$), the value of γ should be sufficiently large. In particular, a sufficient condition for this is given by the following observability condition.

Lemma 3 *The H_∞ estimator* (11)–(15) *has a global solution and*

$$P(t) \geq \delta I > 0, \qquad \forall t \geq 0, \tag{28}$$

for some $\delta > 0$, if there exists a sufficiently large $\gamma > 0$ such that the following condition

$$\gamma^2 W_0(t) \geq \int_0^t \bar{\Phi}(\tau, t)' \bar{\Phi}(\tau, t) d\tau + \delta I \quad \forall t \geq 0 \tag{29}$$

holds, where

$$W_0(t) := \sum_{i=1}^{k} \sum_{j \in \mathcal{I}_i} \bar{\Phi}(t_i, t)' \Psi_j(t_i) \bar{\Phi}(t_i, t), \tag{30}$$

$$\bar{\Phi}(\tau, \sigma) := \begin{cases} \bar{\Phi}_i(\tau,\sigma), & i=j \\ \bar{\Phi}_i(\tau,t_{i+1})\bar{\Phi}_{i+1}(t_{i+1},t_{i+2})\cdots\bar{\Phi}_j(t_j,\sigma), & i<j \end{cases}$$

$\forall \tau \in [t_i, t_{i+1})$ and $\forall \sigma \in [t_j, t_{j+1})$, and $\bar{\Phi}_i(t, \tau)$ denotes the state transition matrix of $\dot{z} = (\nabla A + \gamma^{-2} G G' P + \lambda I) z$, $\forall \tau \in [t_i, t_{i+1})$.

Proof. See the Appendix. □

4 Estimator convergence

We are now interesting in determining under what conditions the state estimate $\hat{x}$ converges to the true state x. As in (Aguiar and Hespanha, 2006*a*), the following technical assumption is needed:

Assumption 4 *Let* $\mathrm{Num}(t, \sigma)$, $0 \leq \sigma < t$ *denote the number of time instants at which measurement arrive in the open interval* (σ, t). *There exist finite positive constants τ_D and N_0, for which the following condition holds:*

$$\mathrm{Num}(t, \sigma) \leq N_0 + \frac{t - \sigma}{\tau_D}.$$

The constant τ_D is called the average dwell-time *and N_0 the* chatter bound. □

This assumption roughly speaking guarantees that the average interval between consecutive arrival of measurements is no less than τ_D. In this way, the summation in (10) will not grow unbounded due to "too frequent" measurements.

The following result establishes the convergence of the state estimate.

Theorem 5 *Assuming that the solutions to the system with implicit outputs* (1), (5) *exists on* $[0, T)$, $T \in [0, \infty]$, *the solution to the impulse state estimator*

(11)–(15) *also exists on* $[0, T)$. *Moreover, if* $P(t) \geq \delta I$, $\forall t \in [0, T)$, $\delta > 0$, *then there exist positive constants* $c > 0, r < 1, \gamma_w, \gamma_1, \ldots, \gamma_N$ *such that*

$$\|\tilde{x}(t_k)\| \leq c\, r^k \|\tilde{x}(0)\| + \gamma_w \sup_{\sigma \in (0, t_k)} \|w(\sigma)\| + \sum_{j=1}^{N} \gamma_j \sup_{\sigma \in (0, t_k)} \|\bar{v}_j(\sigma)\|, \quad t_k > 0 \tag{31}$$

where $\tilde{x}(t) := \hat{x}(t) - x(t)$ *denotes the state estimation error.*

Proof. From (1) and (13) we conclude that for all $t \in [t_i, t_{i+1})$, the state estimation error evolves according to

$$\dot{\tilde{x}} = \nabla A(u)\tilde{x} - G(u)w.$$

Defining, $V := \tilde{x}P\tilde{x}$, and computing its time derivative, it follows that for all $t \in [t_i, t_{i+1})$

$$\begin{aligned} \dot{V} &= \tilde{x}'(\dot{P} + P\nabla A + \nabla A' P)\tilde{x} - 2\tilde{x}' P G w \\ &= -\tilde{x}' \Big[\frac{1}{\gamma^2} P G G' P + 2\lambda P + I \Big] \tilde{x} - 2\tilde{x}' P G w. \end{aligned}$$

By completing the squares, we further conclude that

$$\begin{aligned} \dot{V} &= -\tilde{x}'(2\lambda P + I + \frac{1}{2\gamma^2}(PGG'P))\tilde{x} \\ &\quad - \frac{1}{2\gamma^2} \|G' P \tilde{x} + 2\gamma^2 w\|^2 + 2\gamma^2 \|w\|^2 \\ &\leq -\bar{\lambda} V + 2\gamma^2 \|w\|^2 \end{aligned}$$

where $\bar{\lambda} := 2\lambda + 1/\lambda_{max}(P) + 1/(2\gamma^2)\delta_G \lambda_{min}(P)$, and $\delta_G \geq 0$ satisfies $\delta_G I \leq G(u)G(u)'$. Therefore, for all $t \in [t_i, t_{i+1})$,

$$V(t) \leq V(t_i) e^{-\bar{\lambda}(t - t_i)} + \frac{2\gamma^2}{\bar{\lambda}} \sup_{\sigma \in [t_i, t)} \|w(\sigma)\|^2. \tag{32}$$

Consider now $t = t_{i+1}$. From (14)–(15), the estimation error $\tilde{x}$ at time $t = t_{i+1}$ can be written as

$$\begin{aligned} \tilde{x}(t_{i+1}) &= \hat{x}(t_{i+1}^-) - P(t_{i+1})^{-1}\gamma^2 \sum_{j \in \mathcal{I}_{i+1}} \Big[\Psi_j(t_{i+1})\hat{x}(t_{i+1}^-) + \psi_j(t_{i+1}) \Big] - x(t_{i+1}) \\ &= \Big[I - P(t_{i+1})^{-1}\gamma^2 \sum_{j \in \mathcal{I}_{i+1}} \Psi(t_{i+1}) \Big] \tilde{x}(t_{i+1}^-) - P(t_{i+1})^{-1}\eta, \end{aligned} \tag{33}$$

where

$$\eta := \gamma^2 \sum_{j\in\mathcal{I}_{i+1}} \Big(\mathbf{J}E_{0j}(\bar{y}_j(t_{i+1}))\Phi(t_{i+1}-\tau_{i+1}, t_{i+1}) - \nabla\bar{C}_j\Big)' \\ \times \Big(I - Y_{j,i+1}Y_{j,i+1}^{\perp}\Big)'\Big(I - Y_{j,i+1}Y_{j,i+1}^{\perp}\Big)\bar{v}_j(t_{i+1}).$$

Thus,

$$\begin{aligned} V(t_{i+1}) &= \tilde{x}(t_{i+1}^-)'P(t_{i+1})\tilde{x}(t_{i+1}) \\ &= \tilde{x}(t_{i+1}^-)'\Big[P(t_{i+1}) - 2\gamma^2\sum_{j\in\mathcal{I}_{i+1}}\Psi(t_{i+1}) \\ &\quad + \gamma^2\sum_{j\in\mathcal{I}_{i+1}}\Psi(t_{i+1})P(t_{i+1})^{-1}\gamma^2\sum_{j\in\mathcal{I}_{i+1}}\Psi(t_{i+1})\Big]\tilde{x}(t_{i+1}^-) \\ &\quad - 2\tilde{x}(t_{i+1}^-)'\Big[I - \gamma^2\sum_{j\in\mathcal{I}_{i+1}}\Psi(t_{i+1})P(t_{i+1})^{-1}\Big]\eta \\ &\quad + \eta'P(t_{i+1})^{-1}\eta \end{aligned} \tag{34}$$

Simplifying the notation by dropping the time dependence, using (14) and resorting to the matrix inversion lemma[2], the first terms in brackets on the right-hand-side of (34) can be written as

$$\begin{aligned} &P(t_{i+1}) - 2W + WP(t_{i+1})^{-1}W \\ &\quad = P - W + W[P+W]^{-1}W \\ &\quad = P - W^{\frac{1}{2}}\Big[I - W^{\frac{1}{2}}\Big[P + W^{\frac{1}{2}}IW^{\frac{1}{2}}\Big]^{-1}W^{\frac{1}{2}}\Big]W^{\frac{1}{2}} \\ &\quad = P - W^{\frac{1}{2}}FW^{\frac{1}{2}} \end{aligned}$$

where $F := \Big[I + W^{\frac{1}{2}}P^{-1}W^{\frac{1}{2}}\Big]^{-1}$, $W := \gamma^2\sum_{j\in\mathcal{I}_{i+1}}\Psi_j(t_{i+1})$, and $P = P(t_{i+1}^-)$. In this setting, given a positive semidefinite matrix M, we denote by $M^{\frac{1}{2}}$ any matrix such that $(M^{\frac{1}{2}})'M^{\frac{1}{2}} = M$. The others terms in (34) can be written as

$$\begin{aligned} P(t_{i+1})^{-1} &= (P+W)^{-1} \\ &= P^{-1} - P^{-1}W^{\frac{1}{2}}FW^{\frac{1}{2}}P^{-1}, \\ I - W(t_{i+1})P(t_{i+1})^{-1} &= I - W^{\frac{1}{2}}IW^{\frac{1}{2}}\Big[P + W^{\frac{1}{2}}IW^{\frac{1}{2}}\Big]^{-1} \\ &= I - W^{\frac{1}{2}}FW^{\frac{1}{2}}P^{-1}. \end{aligned}$$

Thus,

$$\begin{aligned} V(t_{i+1}) &= \tilde{x}'P\tilde{x} - \tilde{x}'W^{\frac{1}{2}}FW^{\frac{1}{2}}\tilde{x} + \eta'(P^{-1} - P^{-1}W^{\frac{1}{2}}FW^{\frac{1}{2}}P^{-1})\eta \\ &\quad + 2\tilde{x}'(I - W^{\frac{1}{2}}FW^{\frac{1}{2}}P^{-1})\eta. \end{aligned}$$

---

[2] Let A, C, and $A^{-1}+B'C^{-1}B$ be non-singular matrices, then $(A^{-1}+B'C^{-1}B)^{-1} = A - AB'(BAB'+C)^{-1}BA$. Another useful matrix identity is the following $(A^{-1} + B'C^{-1}B)^{-1}B'C^{-1} = AB'(BAB'+C)^{-1}$.

By completing the squares, we further conclude that

$$V(t_{i+1}) \leq (1+\epsilon)V(t_{i+1}^-) + \Big(1 + \frac{1}{\epsilon}\Big)\eta' P^{-1}\eta,$$

where ϵ is an arbitrary small positive constant. Therefore, resorting to (32), $V(t_{i+1})$ satisfies

$$V(t_{i+1}) \leq (1+\epsilon)V(t_i)e^{-\bar{\lambda}(t_{i+1}-t_i)} + \frac{1}{\epsilon}a_{i+1} + b_{i+1},$$

where

$$a_{i+1} := \lambda_{max}(P^{-1})\|\eta\|^2, \qquad b_{i+1} := (1+\epsilon)\frac{2\gamma^2}{\bar{\lambda}} \sup_{\sigma\in[t_i,t_{i+1})} \|w(\sigma)\|^2 + a_{i+1}.$$

Furthermore, solving this inequality recursively, we get

$$V(t_k) \leq (1+\epsilon)^k e^{-\bar{\lambda}(t_k-t_0)}V(t_0) + \sum_{j=0}^{k-1}(1+\epsilon)^j e^{-\bar{\lambda}(t_k-t_{k-j})}\Big(\frac{1}{\epsilon}a_{k-j} + b_{k-j}\Big).$$

Applying Assumption 4, we first notice that

$$t_k - t_{k-j} \geq [j - N_0]\tau_D,\ j = 0, 1, \ldots, k-1.$$

Consequently

$$\begin{aligned} V(t_k) \leq & \Big[(1+\epsilon)e^{-\bar{\lambda}\tau_D}\Big]^k e^{\bar{\lambda}N_0\tau_D}V(t_0) \\ & + \sum_{j=0}^{k-1}\Big[(1+\epsilon)e^{-\bar{\lambda}\tau_D}\Big]^j \Big(\frac{1}{\epsilon}a_{k-j} + b_{k-j}\Big)e^{\bar{\lambda}N_0\tau_D}. \end{aligned} \tag{35}$$

From this inequality, we further conclude that by picking $\epsilon > 0$ such that

$$r := (1+\epsilon)e^{-\bar{\lambda}\max_j \tau_D} < 1,$$

it follows that V is bounded and $V(t_k) \to \frac{1}{1-r}(\frac{1}{\epsilon}\max_j a_j + \max_j b_j)e^{\bar{\lambda}\max_j\{N_0\tau_D\}}$ as $k \to \infty$. Since for every finite time P is positive definite, V must be finite on any finite interval and therefore so must be $\tilde{x}$ and $\hat{x}$. Global existence of solution follows. It is also straightforward to conclude from (35) that the ISS-like bound (31) holds. □

Combining Theorem 5 and Lemma 3 we obtain the following:

Corollary 6 *When Assumption 4 and the persistence of excitation condition* (29) *hold, the state-estimate* $\hat{x}$ *converges exponentially fast to the state* x *in the absence of disturbance input and measurement noise. When the disturbance and noise are bounded but nonzero,* $\hat{x}$ *converges to a neighborhood of the true state* x. □

5 Illustrative Example

To illustrate the results of the paper, we consider the problem of estimate the position and attitude of an autonomous vehicle with respect to an inertial coordinate frame defined by visual landmarks. The measurements are provided by an inertial measurement unit (IMU) and a monocular charged-coupled-device (CCD) camera mounted on-board. More precisely, consider the coordinates frames represented in Figure 1 where $\{\mathcal{V}\}$ denotes an inertial coordinate frame defined by visual landmarks, $\{\mathcal{B}\}$ a body-fixed coordinate frame whose origin is located e.g., at the center of mass of the vehicle, and $\{\mathcal{I}\}$ another inertial coordinate frame. In this setup, the IMU provides the vehicle's linear velocity $v \in \mathbb{R}^3$, angular velocity $\omega \in \mathbb{R}^3$, and pose (position and attitude) $(p, R) \in \mathrm{SE}(3)$ of $\{\mathcal{B}\}$ with respect to $\{\mathcal{I}\}$. The camera attached to the vehicle sees N points $Q_i \in \mathbb{R}^3$, $i \in \{1, 2, \ldots, N\}$ whose coordinates expressed in $\{\mathcal{V}\}$ are assumed to be known. The objective is to determine the position ${}^{\mathcal{V}}P_{\mathcal{B}} \in \mathbb{R}^3$ and orientation ${}^{\mathcal{V}}_{\mathcal{B}}R \in \mathrm{SO}(3)$ of the vehicle with respect to the visual coordinate system $\{\mathcal{V}\}$. It is assumed that the position and orientation of $\{\mathcal{I}\}$ with respect to $\{\mathcal{V}\}$ are unknown.

In (Aguiar and Hespanha, 2005; Aguiar and Hespanha, 2006*b*), we have formulated this problem in the framework of state estimation of a system with implicit outputs of the form (1)–(3). Denoting the measurements by ζ_i, $i \in \{1, \ldots, 4+N\}$, where the first four are obtained from the IMU, that is,

$$\zeta_1 := v, \qquad \zeta_2 := \omega, \qquad \zeta_3 := p, \qquad \zeta_4 := R,$$

and the rest of them are obtained from the CCD camera and satisfy

$$\mu_{i+4}\zeta_{i+4} = F\,{}^{\mathcal{C}}Q_i, \tag{36}$$

$$\begin{bmatrix} 0 & 0 & 1 \end{bmatrix} \zeta_{i+4} = 1, \quad \forall i \in \{1, 2, \ldots, N\} \tag{37}$$

where ${}^{\mathcal{C}}Q_i$ is the position of Q_i expressed in the camera's frame $\{\mathcal{C}\}$, $\mu_{i+4} \in \mathbb{R}$ captures the depth of the point ${}^{\mathcal{C}}Q_i$ (which is unknown), and F is an upper triangular matrix with the camera's intrinsic parameters, the system with implicit outputs is given by

$$\begin{aligned}
\overset{\cdot}{\overbrace{{}^{\mathcal{V}}_{\mathcal{B}}R'\;{}^{\mathcal{V}}P_{\mathcal{I}}}} &= -S(\omega)\,{}^{\mathcal{V}}_{\mathcal{B}}R'\;{}^{\mathcal{V}}P_{\mathcal{I}}, \\
\mathrm{stack}({}^{V}_{I}\dot{R}') &= 0_{9\times 1} \\
{}^{B}\dot{Q}_1 &= -S(\omega)\,{}^{B}Q_1 - v + \left({}^{V}\dot{Q}_1' \otimes I_{3\times 3}\right)\mathrm{stack}({}^{\mathcal{V}}_{\mathcal{B}}R'), \\
\mathrm{stack}({}^{V}_{B}\dot{R}') &= \left(I_{3\times 3} \otimes -S(\omega)\right)\mathrm{stack}({}^{\mathcal{V}}_{\mathcal{B}}R'), \\
y_1 &= \left({}^{V}Q_1' \otimes I_{3\times 3}\right)\mathrm{stack}({}^{\mathcal{V}}_{\mathcal{B}}R') - {}^{B}Q_1 - P_I, \\
\left({}^{\mathcal{V}}_{I}R \otimes I_{3\times 3}\right)y_2 &= \mathrm{stack}({}^{\mathcal{V}}_{\mathcal{B}}R'), \\
\mu_{i+4}y_{2+i} &= F\,{}^{\mathcal{C}}P_{\mathcal{B}} + \left[\left({}^{V}Q_i - {}^{V}Q_1\right)' \otimes F\,{}^{\mathcal{C}}_{\mathcal{B}}R\right]\mathrm{stack}({}^{\mathcal{V}}_{\mathcal{B}}R') + F\,{}^{\mathcal{C}}_{\mathcal{B}}R\,{}^{B}Q_1
\end{aligned}$$

(see (Aguiar and Hespanha, 2005; Aguiar and Hespanha, 2006*b*) for details).

In the above equations, the symbol $\otimes$ denotes the Kronecker product and $\text{stack}(\cdot)$ is the stack operator that stacks the columns of the argument one on top of each other, with the first column on top. Given two frames $\{A\}$, $\{B\}$, the symbol ${}^{B}_{A}R$ is the rotation matrix from $\{A\}$ to $\{B\}$, ${}^{B}Q$ is the position of the vector Q expressed in $\{B\}$, and ${}^{B}P_A$ the position of the origin of frame $\{A\}$ expressed in $\{B\}$.

Defining the vectors $x \in \mathbb{R}^{24}$, $y \in \mathbb{R}^{12+N}$, and $u \in \mathbb{R}^6$ as

$$x := \begin{bmatrix} {}^{\mathcal{V}}_{\mathcal{B}}R' \; {}^{\mathcal{V}}P_{\mathcal{I}} \\ \text{stack}({}^{\mathcal{V}}_{I}R') \\ {}^{B}Q_1 \\ \text{stack}({}^{\mathcal{V}}_{\mathcal{B}}R') \end{bmatrix}, \quad \begin{aligned} y_1 &:= \zeta_4'\zeta_3 \\ y_2 &:= \text{stack}(\zeta_4) \\ y_{2+i} &:= \zeta_{4+i},\ i = 1, \dots, N \end{aligned} \qquad u := \begin{bmatrix} \zeta_1 \\ \zeta_2 \end{bmatrix}$$

we obtain (1)–(3) with

$$\begin{aligned} A(x,u) &:= \begin{bmatrix} -S(\omega) & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -S(\omega) & {}^{V}\dot{Q}_1' \otimes I_{3\times 3} \\ 0 & 0 & 0 & I_{3\times 3} \otimes -S(\omega) \end{bmatrix} x + \begin{bmatrix} 0 \\ 0 \\ -v \\ 0 \end{bmatrix}, \\ C_1(x,u) &:= \begin{bmatrix} -I & 0 & -I & {}^{V}Q_1' \otimes I_{3\times 3} \end{bmatrix} x, \\ C_2(x,u) &:= \begin{bmatrix} 0 & 0 & 0 & I \end{bmatrix} x \\ C_{2+i}(x,u) &:= \begin{bmatrix} 0 & 0 & F\,{}^{\mathcal{C}}_{\mathcal{B}}R & ({}^{V}Q_i - {}^{V}Q_1)' \otimes F\,{}^{\mathcal{C}}_{\mathcal{B}}R \end{bmatrix} x + F\,{}^{\mathcal{C}}P_{\mathcal{B}}, \end{aligned}$$

$\forall i \in \{1, \dots, N\}$. By introducing additive noise to (36) we conclude that

$$\begin{aligned} E_1(x, v_1) &:= I \\ E_2(x, v_2) &:= {}^{\mathcal{V}}_{I}R \otimes I_{3\times 3} \\ E_{2+i}(x, v_{2+i}) &:= \begin{bmatrix} 0 & 0 & 1 \end{bmatrix} F \Big[{}^{\mathcal{C}}P_{\mathcal{B}} + {}^{\mathcal{C}}_{\mathcal{B}}R\, {}^{\mathcal{V}}_{\mathcal{B}}R' \Big({}^{V}Q_i - {}^{V}Q_1 \Big) \\ &\quad + {}^{\mathcal{C}}_{\mathcal{B}}R\, {}^{B}Q_1 \Big] + v_{2+i}, \quad \forall i \in \{1, 2, \dots, N\} \end{aligned}$$

(see (Aguiar and Hespanha, 2005; Aguiar and Hespanha, 2006*b*) for details). The image of $E_j(x, v_j)$ satisfies (3) with

$$\begin{aligned} E_{01}(x) &:= I, & \ell_1 &= 0, & & \\ E_{02}(x) &:= {}^{\mathcal{V}}_{I}R \otimes I_{3\times 3}, & \ell_2 &= 0, & & \\ E_{0,2+i}(x) &:= 0, & \ell_{2+i} &= 1, & E_{1,2+i} &:= 1 \end{aligned}$$

$\forall i \in \{1, \dots, N\}$. We can now use the results given in the previous sections to compute $\hat{x}$. From ${}^{\mathcal{B}}\hat{Q}_1$ and ${}^{\mathcal{V}}_{\mathcal{B}}\hat{R}'$, the position ${}^{\mathcal{V}}P_{\mathcal{B}}$ can also be estimated using

$${}^{\mathcal{V}}\hat{P}_{\mathcal{B}} = {}^{\mathcal{V}}Q_1 - {}^{\mathcal{V}}_{\mathcal{B}}\hat{R}\, {}^{\mathcal{B}}\hat{Q}_1.$$

We now illustrate the performance of the proposed estimator through computer simulation. The autonomous vehicle starts at the origin ${}^{\mathcal{V}}P_{\mathcal{B}} = 0$ with orientation ${}^{\mathcal{V}}_{\mathcal{B}}R = I$ and follows a circular path with a camera looking up

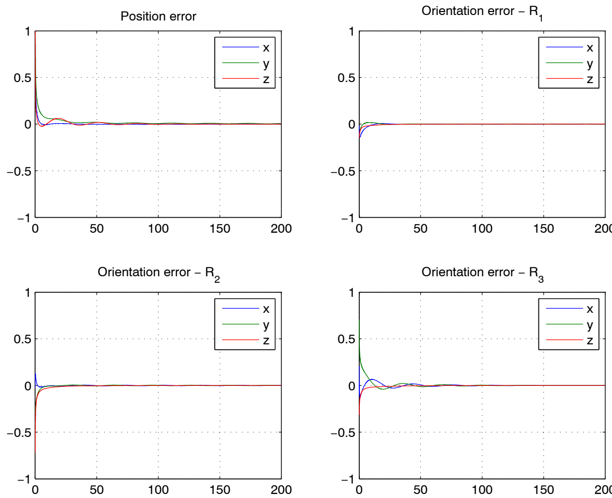

Fig. 2. Time evolution of the estimation errors in position and orientation when there are no measurement noises and disturbances.

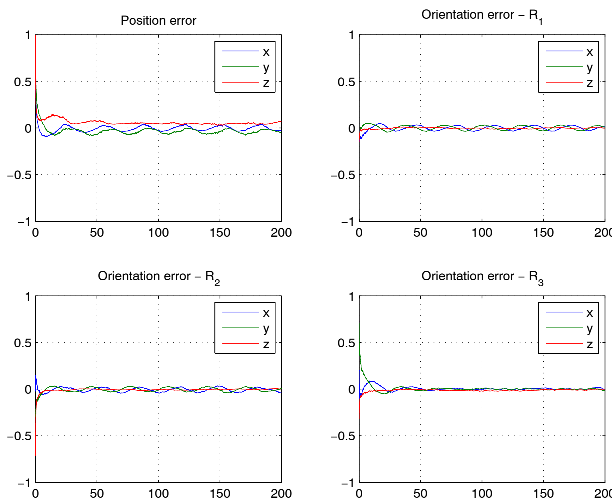

Fig. 3. Time evolution of the estimation errors in position and orientation in the presence of measurement noises.

at four non-coplanar points. The linear velocity is ${}^{\mathcal{B}}v = [0.3, 0, 0]'\, m/s$ and the angular velocity is ${}^{\mathcal{B}}\omega = [0, 0, 0.2]'\, rad/s$. The vision sampling interval is $T_{CCD} = 0.4\, s$ and the time-delay is $\tau_{CCD} = 0.05\, s$. The IMU sampling interval is $T_{IMU} = 0.1\, s$ and there is no time-delay. The estimator was initialized with ${}^{\mathcal{V}}\hat{P}_{\mathcal{B}} = [1, 1, 1]'\, m/s$ and ${}^{\mathcal{V}}_{\mathcal{B}}\hat{R} = \begin{bmatrix} 0.9814 & -0.0179 & 0.1913 \\ -0.1246 & 0.6983 & 0.7049 \\ -0.1462 & -0.7156 & 0.6831 \end{bmatrix}$.

Figures 2 and 3 display the time evolution of the estimation errors without and with measurements corrupted with additive Gaussian noise, respectively. The orientation errors labeled in the figures by R_1, R_2, and R_3 correspond to the estimation errors for the first, second, and third columns of ${}^{\mathcal{V}}_{\mathcal{B}}R$, respectively. The Gaussian noise has standard deviation equal to roughly 5% of the measurements. It can be seen that the estimated pose without noise converges to zero and in the presence of noise tend to a small neighborhood

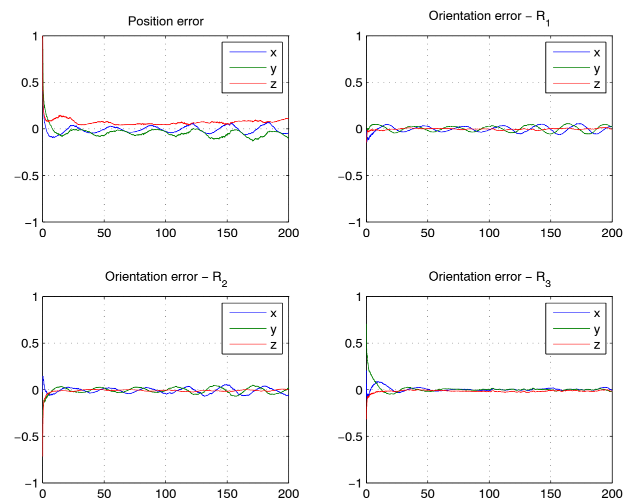

Fig. 4. Time evolution of the estimation errors in position and orientation in the presence of noise. At $t = 100\,s$ the standard deviation of the measurement noise increases twice its previous value.

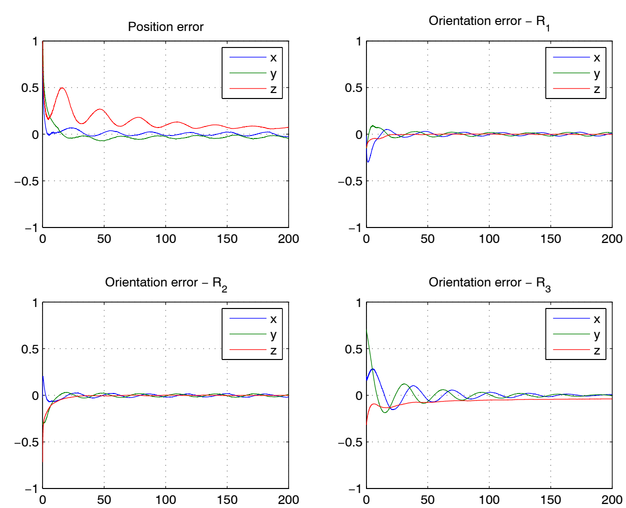

Fig. 5. Time evolution of the estimation errors in position and orientation in the presence of noise and only measurements from the CCD camera.

of the true value. Figure 4 shows the evolution of the estimation errors for the same situation as the one reported in the previous figure, but at $t = 100\,s$ the standard deviation of the measurement noise increases twice its previous value.

To illustrate the benefit of having measurements from the IMU, Figure 5 shows the time evolution of the estimation errors when there is only measurements from the CCD camera. As expected, although the errors converge to a small neighborhood of the origin, the transients are worst than the ones displayed in Figure 3. In Figure 6 we can also see what happens when the observer does not receive measurements at all from $t = 20\,s$ to $t = 100\,s$.

As a final example, we test the behavior of the observer when the config-

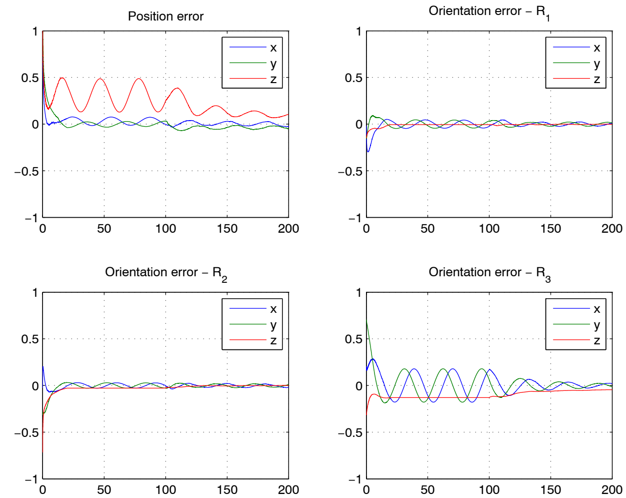

Fig. 6. Time evolution of the estimation errors in position and orientation in the presence of noise, only measurements from the CCD camera, and no measurements from $t \in (20, 100)s$.

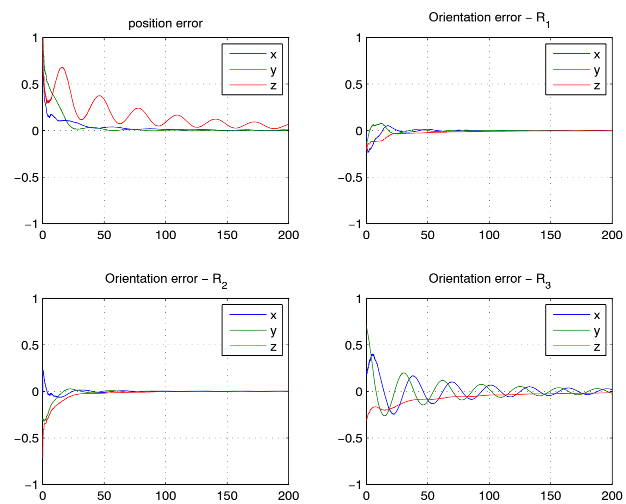

Fig. 7. Time evolution of the estimation errors in position and orientation in the presence of noise, no IMU measurements, and random measurements from the CCD camera.

uration of the feature points randomly change. To this effect, we inserted at the output of the measurements points four random binary signals (one for each point) using a Bernoulii distribution with parameter $\lambda = (\lambda_1, \lambda_2, \lambda_3, \lambda_4)$ to emulate the presence or not of the feature points. Figure 7 shows a particular simulation result with $\lambda = (0.2, 0.4, 0.6, 0.8)$. In this case, it was also considered that there were no measurements from the IMU (which is a worst situation). It can be seen that the estimate error still converges to a small neighborhood of zero and the performance is not significantly affected (compare with Figure 5).

6 Conclusions

We considered the problem of estimating the state of a system with implicit outputs, whose measurements arrive at discrete-time instants, are time-delayed, noisy, and may not be complete. We designed a estimator using a deterministic H_∞ approach that is globally convergent under appropriate observability assumptions and can therefore, be used to design output-feedback controllers. These results were applied to the estimation of position and attitude of an autonomous vehicle using measurements from an inertial measurement unit and a monocular charged-coupled-device camera attached to the vehicle. Future work will address the derivation of conditions to test observability that are easier to test than the ones provided by Lemma 3.

A Appendix

Proof of Lemma 3 From (12) we note that the evolution of $P(t)$ for $t \in [t_i, t_{i+1})$ satisfies

$$\begin{aligned}\dot{P} = &-P(\nabla A + \gamma^{-2}GG'P + \lambda I) - (\nabla A + \gamma^{-2}GG'P + \lambda I)'P \\ &+ \gamma^{-2}PGG'P - I.\end{aligned} \tag{A.1}$$

Thus, from (A.1) and (14), we conclude that

$$\begin{aligned}P(t) = &\bar{\Phi}(0,t)'P_0\bar{\Phi}(0,t) \\ &+ \int_0^t \bar{\Phi}(\tau,t)'\Big(\gamma^{-2}PGG'P - I\Big)\bar{\Phi}(\tau,t)d\tau + \gamma^2 W_0(t).\end{aligned} \tag{A.2}$$

Now, since $\bar{\Phi}(t,0)P_0\bar{\Phi}(t,0)' > 0$ and $\gamma^{-2}PGG'P \geq 0$, from (A.2) and (29)–(30) we conclude that $P(t) \geq \Phi(0,t)'P_0\Phi(0,t) + \delta I$ for all $t \geq 0$. Therefore, the smallest singular value of P remains strictly positive for every finite time t, which implies that $P^{-1}(t)$ remains bounded for every finite t. Global existence of solutions to (11)–(15) follows. □

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