# Problem Twenty Eight

## $\mathcal{L}_2$-Induced Gains of Switched Linear Systems

JOÃO P. HESPANHA[1]
Dept. Electrical and Computer Engineering,
University of California, Santa Barbara
USA
hespanha@ece.ucsb.edu

*In the 1999 collection of Open Problems in Mathematical Systems and Control Theory we proposed the problem of computing input-output gains of switched linear systems. Recent developments provided new insights into this problem leading to new questions.*

### 28.1 SWITCHED LINEAR SYSTEMS

A *switched linear system* is defined by a parameterized family of realizations $\{(A_p, B_p, C_p, D_p) : p \in \mathcal{P}\}$, together with a family of piecewise constant *switching signals* $\mathcal{S} := \{\sigma : [0, \infty) \to \mathcal{P}\}$. Here, we consider switched systems for which all the matrices $A_p$, $p \in \mathcal{P}$ are Hurwitz. The corresponding switched system is represented by

$$\dot{x} = A_\sigma x + B_\sigma u, \qquad y = C_\sigma x + D_\sigma u, \qquad \sigma \in \mathcal{S} \tag{28.1}$$

and by a *solution to* (28.1), we mean a pair $(x, \sigma)$ for which $\sigma \in \mathcal{S}$ and $x$ is a solution to the time-varying system

$$\dot{x} = A_{\sigma(t)} x + B_{\sigma(t)} u, \qquad y = C_{\sigma(t)} x + D_{\sigma(t)} u, \qquad t \geq 0. \tag{28.2}$$

Given a set of switching signals $\mathcal{S}$, we define the $\mathcal{L}_2$*-induced gain of* (28.1) by

$$\inf\{\gamma \geq 0 : \|y\|_2 \leq \gamma \|u\|_2,\ \forall u \in \mathcal{L}_2,\ x(0) = 0,\ \sigma \in \mathcal{S}\},$$

where $y$ is computed along solutions to (28.1). The $\mathcal{L}_2$-induced gain of (28.1) can be viewed as a "worst-case" energy amplification gain for the switched system, *over all possible inputs and switching signals* and is an important tool to study the performance of switched systems, as well as the stability of interconnections of switched systems.

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## 28.2 PROBLEM DESCRIPTION

We are interested here in families of switching signals for which consecutive discontinuities are separated by no less than a positive constant called the *dwell-time*. For a given $\tau_D > 0$, we denote by $\mathcal{S}[\tau_D]$ the set of piecewise constant switching signals with interval between consecutive discontinuities no smaller than $\tau_D$. The general problem that we propose is the computation of the function $\mathfrak{g} : [0, \infty) \to [0, \infty]$ that maps each *dwell-time* $\tau_D$ with the $\mathcal{L}_2$-induced gain of (28.1), for the set of dwell-time switching signals $\mathcal{S} := \mathcal{S}[\tau_D]$. Until recently little more was known about $\mathfrak{g}$ other than the following:

i. $\mathfrak{g}$ is monotone decreasing

ii. $\mathfrak{g}$ is bounded below by

$$\mathfrak{g}_{\text{static}} := \sup_{p \in \mathcal{P}} \|C_p(sI - A_p)^{-1}B_p + D_p\|_\infty,$$

where $\|T\|_\infty := \sup_{\Re[s] \geq 0} \|T(s)\|$ denotes the $\mathcal{H}_\infty$-norm of a transfer matrix $T$. We recall that $\|T\|_\infty$ is numerically equal to the $\mathcal{L}_2$-induced gain of any linear time-invariant system with transfer matrix $T$.

Item i is a trivial consequence of the fact that given two dwell-times $\tau_{D_1} \leq \tau_{D_2}$, we have that $\mathcal{S}[\tau_{D_1}] \supset \mathcal{S}[\tau_{D_2}]$. Item ii is a consequence of the fact that every set $\mathcal{S}[\tau_D]$, $\tau_D > 0$ contains all the constant switching signals $\sigma = p$, $p \in \mathcal{P}$. It was shown in [2] that the lower-bound $\mathfrak{g}_{\text{static}}$ is strict and in general there is a gap between $\mathfrak{g}_{\text{static}}$ and

$$\mathfrak{g}_{\text{slow}} := \lim_{\tau_D \to \infty} \mathfrak{g}[\tau_D].$$

This means that even switching arbitrarily seldom, one may not be able to recover the $\mathcal{L}_2$-induced gains of the "unswitched systems." In [2], a procedure was given to compute $\mathfrak{g}_{\text{slow}}$. Opposite to what had been conjectured, $\mathfrak{g}_{\text{slow}}$ is realization dependent and cannot be determined just from the transfer functions of the systems being switched.

The function $\mathfrak{g}$ thus looks roughly like the ones shown in Figure 28.1, where (a) corresponds to a set of realizations that remains stable for arbitrarily fast switching and (b) to a set that can exhibit unstable behavior for sufficiently fast switching [3]. In (b), the scalar $\tau_{\min}$ denotes the smallest dwell-time for which instability can occur for some switching signal in $\mathcal{S}[\tau_{\min}]$.

Several important basic question remain open:

i. Under what conditions is $\mathfrak{g}$ bounded? This is really a stability problem whose general solution has been eluding researchers for a while now (cf., the survey paper [3] and references therein).

ii. In case $\mathfrak{g}$ is unbounded (case (b) in Figure 28.1), how to compute the position of the vertical asymptote? Or equivalently, what is the smallest dwell-time $\tau_{\min}$ for which one can have instability.

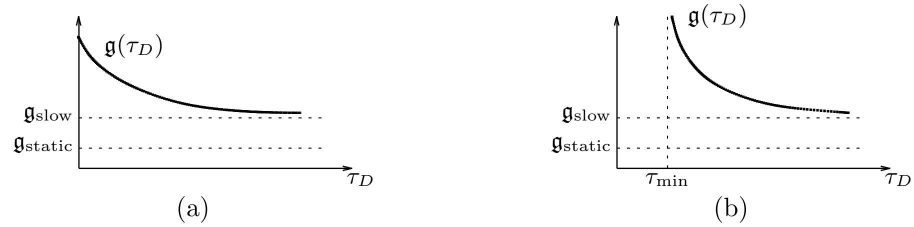


Figure 28.1 $\mathcal{L}_2$-induced gain versus the dwell-time.

iii. Is $\mathfrak{g}$ a convex function? Is it smooth (or even continuous)?

Even if direct computation of $\mathfrak{g}$ proves to be difficult, answers to the previous questions may provide indirect methods to compute tight bounds for it. They also provide a better understanding of the trade-off between switching speed and induced gain. As far as we known, currently only very course upper-bounds for $\mathfrak{g}$ are available. These are obtained by computing a conservative upper-bound $\tau_{\text{upper}}$ for $\tau_{\min}$ and then an upper-bound for $\mathfrak{g}$ that is valid for every dwell-time larger than $\tau_{\text{upper}}$ (cf., e.g., [4, 5]). These bounds do not really address the trade-off mentioned above.

# *Bibliography*

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