Robust Event-Triggered Output Feedback Learning Algorithm for Unknown Voltage Source Inverters

Kyriakos G. Vamvoudakis\textsuperscript{1}, Senior Member IEEE, Farshad R. Pour Safaei\textsuperscript{2}, João P. Hespanha\textsuperscript{1}, Fellow IEEE

Abstract—We consider the output feedback event-triggered control of an off-grid voltage source inverter (VSI) with unknown $L - C$ filter dynamics and connected load in the presence of an input disturbance acting at the plant. Due to uncertain dynamics and unmodelled parameters in the $L - C$ filter connected to the VSI, we use an adaptive observer to reconstruct the system’s states. The control mechanism is constructed based on an impulsive actor/critic framework that approximates the cost, the event-triggered controller and the worst case disturbance and generates the desired AC output with the least energy dissipation. We provide rigorous stability proofs and illustrate the applicability of our results through an example.

Index Terms—VSI, output feedback, actor/critic structures, zero-sum game, event-triggered control.

I. INTRODUCTION

Growing concerns about fossil-fuel reserves, energy security, and global warming have drawn a lot of attention to renewable energy resources [6]. The increasing use of such resources [3] with intermittent generations requires “smarter controllers” for balancing power consumption and generation, often through energy exchange. In an electrical utility grid, batteries exchange energy in the form of direct current (DC) electricity. Although there are different applications that can use DC directly, various types of applications require a VSI to convert a DC voltage to an AC voltage. VSIs are widely used for various high performance applications, such as AC motor drives, uninterruptible power supply (UPS) systems, electric vehicles, reactive power compensators and active power filters, AC power supplies and grid connected schemes [14], [20].

Recently, many researchers have focused on high performance control of VSI; see [23] and the references therein. As mentioned in [23], one of the difficulties in designing a desired optimal performance controller for a power system is the lack of a precise model. To overcome this issue, one should design high performance algorithms that are robust to parameter uncertainties and are adaptive to possible changes in the system.

A natural approach to design these controllers is to take advantage of computational intelligent methods and specifically reinforcement learning methods that learn by interacting with the environment, while achieving a desired performance [18]. Such methods have been widely used in optimizing controllers and maintaining the appropriate terminal voltage, increasing real-time responsiveness to varying power loads, component failures, and improving the transient behavior of power systems by using optimal neuro-adaptive controllers [5], [7], [19]. As stated in [25], there is an emergent need for computational intelligent controllers that allow the smart grid to self-heal, resist attacks, allow dynamic optimization of the operation, and improve power quality and efficiency. Following this line of research, [23] proposes an integral reinforcement learning algorithm to achieve the desired voltage magnitude and frequency at the load, without having full information about the dynamics of the systems. Inspired by these ideas, we design a controller for a VSI to produce a desired voltage in the presence of an input disturbance (e.g., transients, interruption, overvoltage (surge), undervoltage (sag), voltage fluctuations, electrical noise, adversary). Unlike predictive controllers that require a good knowledge of the load parameters [21], our technique does not require having any information about the parameters of the system and only depends on measuring the electric voltage at the load, and without requiring any current measurements. We further, consider a bandwidth “effective” implementation of the controller by applying the ideas of event-triggering mechanisms where the controller requests for new current and voltage measurements only when needed. This can be very effective in islanded inverters where battery lifetime is of concern. The event-triggered control algorithms [9], [17] are composed of a feedback controller updated based on sampled state and the event triggering mechanism that determines the transmission time of the output of the controller to a Zero-Order-Hold (ZOH) actuator. The event-triggering mechanisms can potentially have a significant improvement in off-grid inverters in case of computational and communication efficiency. This communication is a limited resource and fast sampling that is required in continuous sampled controllers is impossible in small-battery devices that run 24/7.

Contributions: First, this paper uses reinforcement learning, game theory and adaptive control to design an off-grid VSI to produce the desired AC voltage at the load, while attenuating input disturbances. Since in real scenarios, one can only measure the voltage at the $L - C$ filter, we use an adaptive observer to estimate the states online, while also overcoming the need to know the system dynamics that include the values of the inductors, capacitors of the $L - C$ filter, load and parasitic resistances, which are not known exactly or change with time. Furthermore, the controller that opens and closes the switches is based on an event-triggering mechanism...
that updates its output only when it is about to lose optimality or stability. This is useful especially in off-grid VSIs where periodically monitoring and controlling is expensive. Finally, an actor/critic mechanism based on impulsive dynamics is used to approximate the cost, the event-triggered controller and the worst case input disturbance in real time.

Notation: $\mathbb{N}^+$ denotes the natural numbers set without zero, $\mathbb{R}^+$ denotes the set $\{x \in \mathbb{R} : x > 0\}$. Moreover we write $\Delta(M)$ for the minimum eigenvalue of matrix $M$.

II. Problem Formulation

Our goal is to design a controller that attenuates an input disturbance while converting DC voltage to AC voltage by appropriately opening and closing switches, namely a single phase off-grid VSI, with guaranteed performance, robustness, limited bandwidth and without any knowledge of the system’s dynamics and load. Figure 1 shows the VSI with an $L - C$ filter that is used to reduce the switching harmonics entering the distribution network.

![Fig. 1. VSI filtered by an $L - C$ filter connected to a linear load.](image)

Before we proceed to the design of the controller, let us consider a VSI with an $L - C$ filter as shown in Figure 1 and a linear resistive load. Later, we shall see that our proposed algorithm is independent of the load, which could be modeled by any linear or nonlinear function.

A. VSI State Space Description

The state-space representation of the system shown in Figure 1 is given by,

$$
\dot{x} = Ax(t) + Bu(t) + Kd(t)
$$

$$
y = C^T x \equiv \begin{bmatrix} 0 & 1 \\ -\frac{1}{C_C} & \frac{1}{C_C} \end{bmatrix} x + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u + Kd,
$$

where $x_1 = i$, $x_2 = V_C$ are the states of the system, $u \in \mathbb{R}$ denotes the control input denoted as $V_{dc}$ in Figure 1, $K \in \mathbb{R}^{2 \times 1}$ is the disturbance matrix, and $d \in \mathbb{R}$ is the unknown input disturbance (e.g., transients, interruption, overvoltage (surge), undervoltage (sag), voltage fluctuations, electrical noise) with a known upper bound $d_{\text{M}}$. The output of this system is the voltage across the load. In order to achieve the desired frequency $\omega_0$ and RMS voltage $V$, we use the following exosystem initialized at the right amplitude:

$$
\dot{Z} = A_Z Z \equiv \begin{bmatrix} 0 & \omega_0 \\ -\omega_0 & 0 \end{bmatrix} Z, \quad Z(0) = \begin{bmatrix} \sqrt{2} V \\ 0 \end{bmatrix}.
$$

where $Z \in \mathbb{R}^2$ denotes the state of the exosystem and $y_z \in \mathbb{R}$ its output. To save resources, the controller works with a version of the state that is sampled at a monotonically increasing sequence of sampling instants $\{r_j\}_{j=0}^\infty$, where $r_j$ is the $j$-th sampling instant with $r_{j+1} > r_j, j \in \mathbb{N}$. The output of the sampled-data component is a sequence of sampled states $\bar{x}_j, \ j \in \mathbb{N}$, where $\bar{x}_j = x(r_j)$ for all $t \in (r_j, r_{j+1}), \ j \in \mathbb{N}$. The controller maps the sampled states onto a control vector $\bar{u}_j, \ j \in \mathbb{N}$, which after using a zero-order hold (ZOH) becomes a piecewise continuous input signal.

B. Performance Design

We shall focus on designing a performance index that tracks the output of the exosystem (2), by using the least amount of energy, attenuating the disturbance with the input voltage inside the interval $[-V_{dc}, V_{dc}]$. We shall see later that this interval will be finally mapped to the discrete set $\{-V_{dc}, V_{dc}\}$. Hence, our goal is to determine values for $u$ and $d$ to minimize and maximize, respectively an infinite horizon cost functional of the following form:

$$
J(x(0), (y(0) - y_e(0)); u, d) = \int_0^\infty \left( x^T Q x + R_u(u) 
- \gamma^2 \|d\|^2 + (y - y_e)^T Q_r(y - y_e) \right) dt
$$

where $Q, Q_r$ are user defined non-negative matrices of appropriate dimensions, $\gamma > \gamma^* \geq 0$, where $\gamma^*$ is the smallest $\gamma$ for which the criterion (3) can be made finite [24]. In order to force bounded inputs $|u| \leq V_{dc}$ one should use $R_u(u) = 2 \int_0^\infty (\theta - 1(v))v^T dv := 2 \int_0^\infty (V_{dc} \text{tanh}^{-1}(v/V_{dc})) v^T dv, \ \forall u$ used to map $u$ onto the interval $(-V_{dc}, V_{dc})$. The term $x^T Q x$ penalizes the current and the voltage to encourage a smooth response, whereas the term $(y - y_e)^T Q_r(y - y_e)$ favors good tracking. Inspired by the work on implicit model following in [1], we shall use an augmented system $x_{\text{aug}} := \begin{bmatrix} x \\ Z \end{bmatrix} \in \mathbb{R}^8$.

The dynamics of the augmented states are given by,

$$
x_{\text{aug}} = \begin{bmatrix} A & 0_{2 \times 2} \\ 0_{2 \times 2} & A_Z \end{bmatrix} x_{\text{aug}} + \begin{bmatrix} \frac{1}{C_C} \\ 0 \\ \frac{K}{C_C} \end{bmatrix} u, d.
$$

Following [1], we minimize the conflict between the need to minimize the tracking error $(y - y_e)^T Q_r(y - y_e)$ and to keep $x^T Q x$ small, by adjusting (3) as follows:

$$
J(x_{\text{aug}}(0); u, d) = \int_0^\infty \left( (R_u(u) - \gamma^2 \|d\|^2 + x_{\text{aug}}^T \hat{Q} x_{\text{aug}}) \right) dt
$$

$$
:= \int_0^\infty c(x_{\text{aug}}, u, d) dt,
$$

where,

$$
\hat{Q} := \begin{bmatrix} \hat{Q}_{11} & \hat{Q}_{12} \\ \hat{Q}_{21} & \hat{Q}_{22} \end{bmatrix}
$$

with $\hat{Q}_{11} := C^T Q C_r + C Q_C C^T, \quad C_r := [I - C(C^T C)^{-1} C], \quad \hat{Q}_{12} := -(C^T Q_C)^{-1} C^T, \quad \hat{Q}_{21} := -C^T Q C_r C(C^T C)^{-1} C_r^T, \quad \hat{Q}_{22} := C^T C C^{-1} C_r^T C(Q_C C^T)^{-1} C_r^T. \quad \text{We}
are interested in finding a Nash equilibrium policy (saddle point) \( u^*, d^* \) [2] that satisfies,
\[
J(:;u^*, d^*) \leq J(:;u^*, d^*) \leq J(:u, d^*), \quad \forall u, d
\]
where for simplicity, we have omitted the dependence on initial conditions.

III. ADAPTIVE OBSERVER

Since the parasitic quantities in the \( L - C \) filter can vary or are not known exactly, we cannot construct the controller directly. To overcome this difficulty, a state observer based on the work of [12], [15] is employed that overcomes the need to know the system dynamics by only measuring the output voltage. Since in our observer design, we consider unknown dynamics, we rewrite (1) in a more general form as,
\[
\dot{x} = A_0x + F(x) + g(x)u + Kd, \\
y = C^T x,
\]
(8)
where \( A_0 \in \mathbb{R}^{2 \times 2} \) is a known matrix such that the pair \((A_0, C)\) is observable, but the functions \( F(x) \) and \( g(x) \) are unknown. By using universal approximation properties [11], we know that, for a given compact set \( \Omega \subseteq \mathbb{R}^2 \), there exists a sufficiently large number of basis functions \( \mathcal{N} \) such that,
\[
F(x) = W_1^T \Phi(x) + \epsilon_1(x), \quad \forall x,
\]
(9)
where \( W_1 \in \mathbb{R}^{N \times 2} \) are the ideal weights bounded by a constant over the compact set \( \Omega \) as \( \|W_1\| \leq W_{1\text{max}} \). \( \Phi(x) \) is the basis function bounded by a constant \( \Phi_M \) and \( \epsilon_1(x) \in \mathbb{R}^2 \) is the reconstruction error bounded above in the compact set \( \Omega \) by \( \epsilon_{1\text{max}} \). We can similarly for the unknown function \( g(x) \) to write,
\[
g(x) = W_2^T \Phi(x) + \epsilon_2(x), \quad \forall x,
\]
(10)
where \( W_2 \in \mathbb{R}^{N \times 2} \) are the ideal weights bounded by a constant over the compact set \( \Omega \) as \( \|W_2\| \leq W_{2\text{max}} \) and \( \epsilon_2(x) \in \mathbb{R}^2 \) is the reconstruction error bounded above in the compact set \( \Omega \) by \( \epsilon_{2\text{max}} \).

Let \( \hat{W}_1, \hat{W}_2 \) denote the estimates of the ideal weights \( W_1 \) and \( W_2 \), respectively. Defining the weight estimation errors as, \( \hat{W}_i := W_i - \hat{W}_i, \quad i = 1, 2 \) we can write,
\[
F(x) - F(\hat{x}) = W_1^T (\Phi(\hat{x}) - \Phi(x)) + \epsilon_1(x) \quad \text{and} \quad g(x) - g(\hat{x}) = W_2^T (\Phi(\hat{x}) - \Phi(x)) + \epsilon_2(x)
\]
and \( \hat{\Phi}(x) = W_1^T \Phi(x) + \epsilon_1(x) \) and \( \tilde{\Phi}(x) = W_2^T \Phi(x) + \epsilon_2(x) \) where \( \hat{x} \in \mathbb{R}^2 \) is the observed state. Since the basis functions are bounded, which means that \( (\Phi(x) - \tilde{\Phi}(x)) \) is bounded above by \( \Phi_M \). Using the approximations of (9) and (10), a state estimator for (8) can be expressed as,
\[
\dot{\tilde{x}} = A_0 \tilde{x} + P^{-1} C \left( W_1^T \phi(x) + W_2^T \phi(\hat{x}) \right) + L_0 (y - C^T \tilde{x}), \quad \tilde{y} = C^T \tilde{x},
\]
where \( \tilde{y} \) denotes the measured output, the observer gain \( L_0 \in \mathbb{R}^2 \) is chosen such that \( A_{OC} : = A_0 - L_0 C \) is Hurwitz and \( P \) is a positive definite matrix such that, \( A_{OC}^T P + P A_{OC} = -Q I_2 \) with \( Q \in \mathbb{R}^{+} \) being a design parameter. We will define the state and output errors as, \( \tilde{x} := x - \tilde{x} \) and \( \tilde{y} := y - \tilde{y} \), respectively. We now can write the error state dynamics as,
\[
\dot{\tilde{x}} = A_{OC} \tilde{x} + P^{-1} C \left( W_1^T \Phi(x) + W_2^T \Phi(\hat{x}) + \epsilon_1(x) + \left( W_1^T \phi(x) + W_2^T \phi(\hat{x}) \right) u \right), \quad \tilde{y} = C^T \tilde{x}.
\]

The following theorem shows how to tune the weights \( \hat{W}_1 \) and \( \hat{W}_2 \) to drive the estimation errors to zero.

**Theorem 1:** Suppose that the control input \( u(t) \) is upper bounded by \( u_M \) and that the state remains inside the compact set \( \Omega \). Consider the observer in (8) with, \( \hat{W}_1 = -\alpha_1 B_1 \hat{W}_1 + (y - \tilde{y}) B_1 \Phi(\hat{x}) \) and \( \hat{W}_2 = -\alpha_2 B_2 \hat{W}_2 + (y - \tilde{y}) B_2 \Phi(\hat{x}) \) where \( \alpha_1, \alpha_2 \in \mathbb{R}^+ \) and positive definite symmetric matrices \( B_1, B_2 \). Then the state error \( \tilde{x} \) and the weight estimation errors \( \hat{W}_1, \hat{W}_2 \) are uniformly ultimately bounded (UBB).

**Proof.** The proof is similar to [12], [15].

Now the augmented system dynamics from (4) are given by,
\[
\dot{\hat{x}}_{\text{aug}} = \tilde{f}(\hat{x}_{\text{aug}}) + \tilde{g}(\hat{x}_{\text{aug}})u + \begin{bmatrix} K \\ 0_2 \end{bmatrix} d,
\]
(11)
where \( \hat{f}(\hat{x}_{\text{aug}}) := \begin{bmatrix} A_0 \hat{x} + P^{-1} C W_1^T \Phi(\hat{x}) + L_0 (y - C^T \hat{x}) \\ A_2 \end{bmatrix} \) and \( \hat{g}(\hat{x}_{\text{aug}}) := \begin{bmatrix} P^{-1} C W_2^T \Phi(\hat{x}) \\ 0_2 \end{bmatrix} \).

IV. EVENT-TRIGGERED ZERO-SUM GAME DESIGN

Our next step is to design an event-based control mechanism based on the state of the observer defined in the previous section, which is now fully available for feedback.

The ultimate goal is to compute the cost function \( V^* \) defined by,
\[
V^*(\hat{x}_{\text{aug}}(t)) := \min_{u \in \mathcal{U}} \max_{d \in \mathcal{D}} \int_0^T c(\hat{x}_{\text{aug}}, u, d) \, dt, \quad \forall t \geq 0,
\]
(12)
such that the system dynamics constraint (4) and given bounded inputs inside the interval \([-V_{oc}, V_{oc}]\).

The controller works with a sampled version of the state and is updated only when an event is triggered. In other words, we have
\[
\hat{x}_{\text{aug}}(t) = \begin{cases} \hat{x}_{\text{aug}}(r_j), & t = r_j \\ \hat{x}_{\text{aug}}(t), & \forall t \in (r_j, r_{j+1}] \end{cases}
\]
(13)

We define the gap between the sampled version of the state and the actual state as, \( e(t) := \hat{x}_{\text{aug}}(t) - \hat{x}_{\text{aug}}(t) \). We shall see that when a condition is violated, or about to be violated a new event will be triggered to force the gap to zero, i.e.
\[
e(t) = \begin{cases} 0, & t = r_j \\ \hat{x}_{\text{aug}}(t) - \hat{x}_{\text{aug}}(t), & \forall t \in (r_j, r_{j+1}] \end{cases}
\]

We can derive the Hamiltonian associated with the system (11) as follows,
\[
H(\hat{x}_{\text{aug}}, u, d, \frac{\partial V^*(\hat{x}_{\text{aug}})}{\partial \hat{x}_{\text{aug}}}) = \\
\frac{\partial V^*(\hat{x}_{\text{aug}})}{\partial \hat{x}_{\text{aug}}}^T \left( \hat{f}(\hat{x}_{\text{aug}}) + \hat{g}(\hat{x}_{\text{aug}})u + \begin{bmatrix} K \\ 0_2 \end{bmatrix} d \right) \\
+ c(\hat{x}_{\text{aug}}, u, d), \forall \hat{x}_{\text{aug}}, u, d.
\]
(13)

We need to find the control input and the disturbance such that the performance (12) is minimized with respect to the control
and maximized with respect to the disturbance. For the optimal control input we have,
\[
u^*\left(\tilde{x}_{aug}\right) = -\theta \left( \frac{1}{2} \tilde{g}(\tilde{x}_{aug})^T \frac{\partial V^*\left(\tilde{x}_{aug}\right)}{\partial \tilde{x}_{aug}} \right)
\]  
(14)
and since we have event-triggering controllers, we shall write (14) as
\[
u^*\left(\tilde{x}_{aug}\right) = -\theta \left( \frac{1}{2} \tilde{g}(\tilde{x}_{aug})^T \frac{\partial V^*\left(\tilde{x}_{aug}\right)}{\partial \tilde{x}_{aug}} \right),
\quad \text{for } t \in \{r_j, r_{j+1}\} \text{ and } j \in \mathbb{N},
\]  
(15)
and for the disturbance (where the controller holds a copy of it) we get
\[
d^*\left(\tilde{x}_{aug}\right) = \frac{1}{2\gamma^2} \left[ \begin{array}{c} K \\ 0_2 \end{array} \right]^T \frac{\partial V^*\left(\tilde{x}_{aug}\right)}{\partial \tilde{x}_{aug}}, \forall t \geq 0.
\]  
(16)
Substituting (14) and (16) into (13), we get a Hamilton-Jacobi-Isaacs (HJI) equation with continuous control updates, \(V_{\tilde{x}_{aug}}\) of the following form,
\[
H(\tilde{x}_{aug}, u^*(\tilde{x}_{aug}), d^*(\tilde{x}_{aug}), \frac{\partial V^*(\tilde{x}_{aug})}{\partial \tilde{x}_{aug}}) = 0,
\]  
(17)
and after substituting (15) and (16) into (13) we get a Hamilton-Jacobi-Isaacs (HJI) equation with event-triggered control updates, \(V_{\tilde{x}_{aug}}\) of the following form,
\[
H(\tilde{x}_{aug}, u^*(\tilde{x}_{aug}), d^*(\tilde{x}_{aug}), \frac{\partial V^*(\tilde{x}_{aug})}{\partial \tilde{x}_{aug}}) = \frac{\partial V^*(\tilde{x}_{aug})}{\partial \tilde{x}_{aug}} \left( \tilde{f}(\tilde{x}_{aug}) + \frac{\partial V^*(\tilde{x}_{aug})}{\partial \tilde{x}_{aug}} u^*(\tilde{x}_{aug}) \right) + \left[ K \right] d^*(\tilde{x}_{aug}) + c(\tilde{x}_{aug}, u^*(\tilde{x}_{aug}), d^*(\tilde{x}_{aug})),
\]  
(18)
which are eventually the equations we would like to solve.

The following assumptions are classical in event-triggered control [22] and are needed before one can proceed to the design of the triggering condition, the existence of solution and stability theorem for the event-triggered scheme.

**Assumption 1:** The controller is globally Lipschitz continuous \(V_{\tilde{x}_{aug}}\), \(\tilde{x}_{aug}\),
\[
||u(\tilde{x}_{aug}) - u(\tilde{x}_{aug})|| = ||u(\tilde{x}_{aug}) - u(\tilde{x}_{aug} + e)|| \leq L ||e||,
\]
where \(L\) is the Lipschitz non-negative real constant.

**Assumption 2:** Global Lipschitz continuity of the closed-loop system with respect to the state \(\tilde{x}_{aug}\) and to the gap \(\varepsilon\).

**Lemma 1:** Suppose that Assumptions 1, 2 hold. Then the gap between the HJI with the continuous-sample controller given by (17) and the HJI with the event-triggered controller given by (18) is
\[
H(\tilde{x}_{aug}, u^*(\tilde{x}_{aug}), d^*(\tilde{x}_{aug}), \frac{\partial V^*(\tilde{x}_{aug})}{\partial \tilde{x}_{aug}}) = R_s(u^*(\tilde{x}_{aug}) - u^*(\tilde{x}_{aug})), \forall \tilde{x}_{aug},
\]  
(19)
where
\[
R_s(u^*(\tilde{x}_{aug}) - u^*(\tilde{x}_{aug})) = \frac{1}{2} \left( u^*(\tilde{x}_{aug}) - u^*(\tilde{x}_{aug}) \right)^T (\theta + \varepsilon) \left( u^*(\tilde{x}_{aug}) - u^*(\tilde{x}_{aug}) \right).
\]
Proof. The required result can be proved by taking the difference between (17) and (18) and completing the squares.

The result of Lemma 1 motivates the selection of the triggering instants \(r_j\) based on a triggering rule that will be defined in the following Theorem. A scheme of the event-triggered controller is shown in Figure 2.

Fig. 2. The scheme of the sampled-data control system.

From [26], we know that an equivalent formulation of the (17) that does not involve the dynamics and produces a finite cost is given as,
\[
V(\tilde{x}_{aug}(t - T)) = \int_{t-T}^{t} \left( R_s(u^* - \gamma^2 |d^*|^2 + \tilde{x}_{aug}^T \tilde{Q} \tilde{x}_{aug}) \right) d\tau + V(\tilde{x}_{aug}(t)),
\]
for any time \(t \geq 0\) and time interval \(T > 0\).

**Theorem 2:** Suppose that Assumptions 1-2 and Lemma 1 hold, and that there exists a continuously differentiable, positive definite function \(V\) that satisfies the following HJI inequality \(\forall \tilde{x}_{aug}\),
\[
\int_{t-T}^{t} \left( \tilde{V}(\tilde{x}_{aug}) + R_s(u^* - \gamma^2 |d^*|^2 + \tilde{x}_{aug}^T \tilde{Q} \tilde{x}_{aug}) \right) d\tau \leq 0, \forall t, \forall \tilde{x}_{aug},
\]  
(20)
with \(V(0) = 0\) and a constant \(\gamma \in \mathbb{R}^+\). The closed-loop system with event-triggered control policy given by,
\[
u(\tilde{x}_{aug}) = u^*(\tilde{x}_{aug}) := -\frac{1}{2} \tilde{g}(\tilde{x}_{aug})^T \frac{\partial V(\tilde{x}_{aug})}{\partial \tilde{x}_{aug}},
\]  
(21)
for \(t \in \{r_j, r_{j+1}\}\) and all \(j \in \mathbb{N}\) and disturbance given by
\[
d(\tilde{x}_{aug}) = d^*(\tilde{x}_{aug}) := \frac{1}{2\gamma^2} \left[ K \right]^T \frac{\partial V(\tilde{x}_{aug})}{\partial \tilde{x}_{aug}}
\]  
(22)
is asymptotically-stable given that the following triggering condition holds \(\forall t \geq 0\),
\[
R_s(L ||e||) \leq (1 - \beta^2) \Delta(\tilde{Q}) ||\tilde{x}_{aug}||^2 + R_s(u(\tilde{x}_{aug})),
\]  
(23)
for some user defined \(\beta \in [0, 1]\). Moreover, the control (21) and (22) lead to a cost of, \(J^*(\cdot; u^*, d^*) = V^*(\tilde{x}_{aug}(0)) + \int_{0}^{T} R_s(u^*(\tilde{x}_{aug}) - u^*(\tilde{x}_{aug})) dt\).

**Remark 1:** It is worth noting that if one needs to approach the performance of the continuous sampled controller we need to make the term \(R_s(u^*(\tilde{x}_{aug}) - u^*(\tilde{x}_{aug}))\) as close to zero as possible by adjusting the parameter \(\beta\) of the triggering condition given in (23). This means that when \(\beta\) is close to 1 one samples more frequently whereas when \(\beta\) is close to zero, the intersampling periods become longer and the performance will be far from the continuous sampled optimal controller.

**Remark 2:** Since the closed-loop system is Lipschitz according to Assumption 2, it has been shown in [17] that
subtracting zero from (27) by using the equation (19) and

and by solving the time triggered HJI (20) for

Proof.

and (21)-(22) is given by,

The orbital derivative of

as,

and hence (26) can be upper bounded

By substituting (25) into (24) and after noting that

By using the Assumption 1, we can write,

and hence (26) can be upper bounded as,

V = \frac{1}{4\gamma^2} \tilde{\mathcal{V}}(\tilde{x}(0)) + R_s(u(\tilde{x}(0))) - R_s(\tilde{x}(0))

and by setting \( u(\tilde{x}(0)) = u^*(\tilde{x}(0)) \) in (28), leads to,

Thus, the Nash equilibrium condition (7) follows directly from (29)-(30).

V. LEARNING ALGORITHM

where the ideal weights are denoted by \( W_1^* \in \mathbb{R}^h_1 \), \( W_u^* \in \mathbb{R}^h_2 \), \( W_d^* \in \mathbb{R}^h_3 \), which are bounded as \( \| W_1^* \| \leq W_{1\text{max}} \), \( \| W_u^* \| \leq W_{u\text{max}} \), \( \| W_d^* \| \leq W_{d\text{max}} \) respectively. The basis functions for the critic \( \phi_c \) are picked in a way that as \( h_1 \to \infty \), \( V^* \) is uniformly approximated and the basis functions are bounded and continuously differentiable (i.e. \( \| \phi_c \| \leq \phi_{\text{max}} \) and \( \| \nabla \phi_c \| \leq \phi_{\text{max}} \)). The residual error \( \epsilon_c \) for (31) is assumed to satisfy \( \sup_{x,a} \| \epsilon_c \| \leq \epsilon_{\text{max}} \) and \( \sup_{x,a} \| \nabla \epsilon_c \| \leq \epsilon_{\text{max}} \). Similarly for the two actors (32)-(33) the basis functions (picked so that as \( h_2 \to \infty \), \( h_3 \to \infty \) the optimal control and worst case disturbance are uniformly approximated) and residual errors are assumed to be upper bounded as \( \sup_{x,a} \| \epsilon_u \| \leq \epsilon_{\text{max}} \), \( \| \phi_u \| \leq \phi_{\text{max}} \) and respectively \( \sup_{x,a} \| \epsilon_d \| \leq \epsilon_{\text{max}} \), \( \| \phi_d \| \leq \phi_{\text{max}} \).

A. Actor/Critic Structure

Since the ideal weights for the three approximators (31), (32) and (33), are not available, we use weight estimates for their representation and then tune them appropriately,

\[
V(\tilde{x}) = \hat{W}_c \phi_c(\tilde{x}), \quad \forall \tilde{x},
\]

\[
J(\cdot; u, d) = \int_0^\infty R_s(u(\tilde{x}(0))) - \gamma^2 \| d(\tilde{x}(0)) \|^2 dt + V^*([\tilde{x}(0)])
\]

\[
J(\cdot; u, d) = \int_0^\infty (R_s(u(\tilde{x}(0))) - u^*(\tilde{x}(0))) - \gamma^2 \| d(\tilde{x}(0)) - d^*(\tilde{x}(0)) \|^2 dt + V^*([\tilde{x}(0)])
\]

subtracting zero from (27) by using the equation (19) and completing the squares yields,

\[
V^*([\tilde{x}(0)]) = \int_0^\infty R_s(u^*(\tilde{x}(0))) + u^*(\tilde{x}(0)) - u^*(\tilde{x}(0)) dt.
\]
Now we shall follow techniques from adaptive control [13] to design tuning laws for the aforementioned approximators.

In order to find the tuning laws for the critic we define the following error signal \( e_c \) in \( \mathbb{R} \) that is desired to be driven to zero, \( e_c = W_c^T \Delta \phi - \hat{\pi} \), where \( \Delta \phi = \phi(\hat{x}_{\text{aug}}(t)) - \phi(x_{\text{aug}}(t - T)) \) and \( \hat{\pi} = \int_{t-T}^t (R_a(\hat{u}) - \gamma^2 \|\hat{d}\|^2 + \hat{\gamma}_T^T \hat{Q} \hat{x}_{\text{aug}}) \, dt \). Similarly, to find the tuning laws for the two actors we use the following error signals \( e_u, e_d \in \mathbb{R} \), \( e_u = W_u^T \phi_u(\hat{x}_{\text{aug}}) + \theta(\frac{1}{2} \hat{g}(\hat{x}_{\text{aug}})^T \frac{\partial \phi_u}{\partial \hat{x}_{\text{aug}}} \hat{W}_c) \), \( e_d = W_d^T \phi_d(\hat{x}_{\text{aug}}) - \frac{1}{2\gamma^2} \begin{bmatrix} K \end{bmatrix}^T \frac{\partial \phi_d}{\partial \hat{x}_{\text{aug}}} \hat{W}_c \).

The tuning laws for the critic and the two actors are obtained using gradient descent to drive the errors \( e_c, e_u, e_d \) to zero,

\[
\dot{W}_c = -\alpha_c \frac{\Delta \phi}{(\|\Delta \phi\|^2 + 1)^2} (\Delta \phi^T W_c + \hat{\pi}), \quad \forall t \geq 0
\]  

(37)

\[
\dot{W}_u = 0, \quad \forall t \leq t_j + 1, j \in \mathbb{N},
\]  

(38)

\[
W_u^+ = W_u(t) - \alpha_u \phi_u(\hat{x}_{\text{aug}}(t)) \left( W_u^T \phi_u(\hat{x}_{\text{aug}}(t)) + \theta \left( \frac{1}{2} \hat{g}(\hat{x}_{\text{aug}})^T \frac{\partial \phi_u}{\partial \hat{x}_{\text{aug}}} \hat{W}_c \right) \right)^T, \quad \forall t \in \mathbb{R}
\]  

(39)

\[
W_d = -\alpha_d \phi_d(\hat{x}_{\text{aug}}) \left( W_d^T \phi_d(\hat{x}_{\text{aug}}) - \frac{1}{2\gamma^2} \begin{bmatrix} K \end{bmatrix}^T \frac{\partial \phi_d}{\partial \hat{x}_{\text{aug}}} \hat{W}_c \right)^T.
\]  

(40)

Since the control \( u = V_{\text{dc}} \) is obtained by connecting \( V_{\text{dc}} \) in Figure 1 to either \( +V_{\text{DC}} \) or \( -V_{\text{DC}} \) we must have \( u \in \{-V_{\text{DC}}, +V_{\text{DC}}\} \), and for that reason one should pass the control input through a hard limiter that limits its output to \( \pm V_{\text{DC}} \), i.e., \( a_{\text{dc}} = V_{\text{DC}} \theta_{\text{lim}}(0); \theta_{\text{lim}}(u) \begin{cases} 1, & \text{if } u > 0 \\ -1, & \text{if } u < 0 \end{cases} \).

We shall define the following notations for the weight estimation errors, \( W_c := W_{\text{c}}^* - W_c \), \( W_u := W_{\text{u}}^* - W_u \) and \( W_d := W_{\text{d}}^* - W_d \). The weight estimation error dynamics are given for the critic as,

\[
\dot{\hat{W}}_c = -\alpha_c \frac{\Delta \phi \Delta \phi^T}{(\|\Delta \phi\|^2 + 1)^2} \hat{W}_c + \alpha_c \frac{\Delta \phi}{(\|\Delta \phi\|^2 + 1)^2} \hat{W}_c, \quad \forall t \geq 0
\]  

(41)

The weight estimation error dynamics for the controller actor are given as, \( \hat{W}_u = 0, \) for \( r_j < t < r_{j+1}, j \in \mathbb{N} \), and, \( \hat{W}_u^+ = W_u(t) - \alpha_u \phi_u(\hat{x}_{\text{aug}}(t)) \left( W_u^T \phi_u(\hat{x}_{\text{aug}}(t)) + \frac{1}{2} \hat{g}(\hat{x}_{\text{aug}}(t))^T \frac{\partial \phi_u}{\partial \hat{x}_{\text{aug}}} \hat{W}_c \right)^T, \quad \forall t \geq 0 \) \( \hat{W}_d = -\alpha_d \phi_d(\hat{x}_{\text{aug}}) \left( W_d^T \phi_d(\hat{x}_{\text{aug}}) + \frac{1}{2\gamma^2} \begin{bmatrix} K \end{bmatrix}^T \frac{\partial \phi_d}{\partial \hat{x}_{\text{aug}}} \hat{W}_c \right)^T, \quad \forall t \geq 0 \),

(42)

where \( \epsilon_c = \frac{\Delta \phi}{\|\Delta \phi\|^2 + 1} \). The following main theorem is proved in the next subsection and states that the impulsive closed-loop is UUB \([8],[16]\).

**Theorem 3:** Suppose that Assumptions 1,2 hold and that the signal \( \Delta := \frac{\Delta \phi}{\|\Delta \phi\|^2 + 1} \) is persistently exciting over the interval \([t, t + T_{\text{PE}}]\) with \( T_{\text{PE}} \in \mathbb{R}^+ \) and \( \frac{\Delta \phi}{\|\Delta \phi\|^2 + 1} \Delta \phi^T \geq \beta I \) with \( \beta \in \mathbb{R}^+ \) and \( I \) an identity matrix of appropriate dimensions. Consider the system dynamics given by (11), the critic approximator given by (34), the event-triggered control given by (35) and the worst case disturbance given by (36). The tuning law for the weights of the critic is given by (37), the tuning law for the actor weights are given by the impulsive system (38)-(39) for the event-triggered controller and by (40) for the disturbance. Then the equilibrium point of the impulsive system \( \psi := \left[ \hat{x}_{\text{aug}}^T \hat{x}_{\text{aug}}^T W_c^T W_u^T W_d^T \right]^T \) for all initial conditions \( \psi(0) \) in the compact set is UUB given that the following condition holds \( \forall t \geq 0 \),

\[
R_s(\|\psi\|) \leq (1 - \beta^2) \Delta(\hat{Q}) \|\hat{x}_{\text{aug}}\|^2 + R_s(W_d^T \phi_u(\hat{x}_{\text{aug}})), \quad \forall t \geq 0,
\]  

(43)

and the following inequalities,

\[
\alpha_c \lambda \left( \frac{\Delta \phi \Delta \phi^T}{(\|\Delta \phi\|^2 + 1)^2} \right) - \frac{1}{2\alpha_c} > \frac{\phi_{\text{dmax}}^2}{4\gamma^2},
\]  

(44)

\[
\phi_{\text{dmax}} > \sqrt{\frac{3}{4(1 - \frac{1}{4\gamma^2})}}; \quad \gamma > \frac{1}{2},
\]  

(45)

\[
0 < \alpha_u < 2,
\]  

(46)

hold.

**Corollary 1:** Suppose that the hypotheses and the statements of theorem 3 hold. Then, the policies \( \hat{u} \) given by (35) and \( \hat{d} \) given by (36) form an approximate Nash equilibrium (saddle point).

**Proof.** Follows from theorem 3. 

\[ \square \]

**B. Stability Analysis based on an Impulsive Approach**

**Assumption 3:** The function \( \hat{g} \) is uniformly upper bounded on \( \Omega \) by \( 1/2 \).

In order to prove stability we have to combine the continuous and the discrete time dynamics under the framework of impulsive systems, inspired by the work of \([8],[10]\). Hence, the time derivative of \( \psi := \left[ \hat{x}_{\text{aug}}^T \hat{x}_{\text{aug}}^T W_c^T W_u^T W_d^T \right]^T \) for \( t \in (r_j, r_{j+1}], j \in \mathbb{N} \) can be written as,

\[
\dot{\psi} = \begin{bmatrix} \hat{f}(W_c^* - \hat{W}_c)^T \phi_u + \begin{bmatrix} K \end{bmatrix}^T (W_d^* - \hat{W}_d)^T \phi_d \begin{bmatrix} 0 \end{bmatrix} \begin{bmatrix} 0 \end{bmatrix} - \alpha_c \frac{\Delta \phi \Delta \phi^T}{(\|\Delta \phi\|^2 + 1)^2} \hat{W}_c + \alpha_c \frac{\Delta \phi}{(\|\Delta \phi\|^2 + 1)^2} \epsilon_c \hat{W}_c \begin{bmatrix} 0 \end{bmatrix} \alpha \end{bmatrix}
\]  

(42)
where $\Lambda = -\sigma_c \phi_u(\hat{x}_{aug}) \left( W_u^T \phi_d(\hat{x}_{aug}) + \epsilon_d^T \phi_u(\hat{x}_{aug}) - \frac{1}{2 \sigma_c^2} \begin{bmatrix} K & 0 \\ 0 & 0 \end{bmatrix}^T \frac{\partial \phi_u}{\partial \hat{x}_{aug}} W_c + \frac{1}{2} \begin{bmatrix} K & 0 \\ 0 & 0 \end{bmatrix}^T \frac{\partial \phi_d}{\partial \hat{x}_{aug}} \right)^T$ and the jump dynamics for $t = r_j, j \in \mathbb{N}$, are given by:

$$\psi(t) + \begin{cases} 0 & \epsilon(t) \\ 0 & \epsilon(t) \end{cases}, \quad t = r_j, j \in \mathbb{N}, \quad \text{where} \quad \Gamma_j = \begin{cases} 0 & 0 \\ 0 & 0 \end{cases}$$

$$\dot{W}_u(t) - \alpha_u \phi_u(\hat{x}_{aug}(t)) \left( \hat{x}_{aug}(t) \right) + c^T_0 \phi_u(\hat{x}_{aug}(t)) + \theta \left( \frac{1}{2} \hat{g}(\hat{x}_{aug}(t))^T \frac{\partial \phi_u}{\partial \hat{x}_{aug}} W_c + \theta \left( \frac{1}{2} \hat{g}(\hat{x}_{aug}(t))^T \frac{\partial \phi_d}{\partial \hat{x}_{aug}} \right) \right)^T.$$ 

**Proof of Theorem 3**

We consider the continuous and jump dynamics separately. First, in order to prove stability for the continuous part we will start with following Lyapunov function $V : \mathbb{R}^2 \times \mathbb{R}^2 \times \mathbb{R}^{21} \times \mathbb{R}^{28} \times \mathbb{R}^{33} \rightarrow \mathbb{R}$,

$$V(\psi) = V^*(\hat{x}_{aug}) + V^*(\hat{x}_{aug}) + \frac{1}{2} \|\hat{W}_c\|^2 + \frac{1}{2 \alpha_u} \|\hat{W}_u\|^2 + \frac{1}{2 \alpha_d} \|\hat{W}_d\|^2.$$  

(47)

Now after taking the time derivative of (47) and by noting that $\frac{dV^*(\hat{x}_{aug})}{dt} = 0$ and $\frac{d}{dt} \begin{bmatrix} \hat{W}_u \\ \hat{W}_d \end{bmatrix} = 0$ we have between the jumps,

$$\dot{V} = \frac{\partial V^*(\hat{x}_{aug})}{\partial \hat{x}_{aug}} \dot{\hat{x}}_{aug} + W_u^T \dot{W}_c + \frac{1}{\alpha_d} W_d^T \dot{W}_d,$$

(48)

which after substituting (11), (41), (42) in (48), grouping the similar terms together and using the condition (43) yields,

$$\dot{V} \leq -\beta^2 \Delta(Q) \|\hat{x}_{aug}\|^2 - \alpha_c \Delta \left( \frac{\Delta \phi_d \phi_d^T}{(\|\phi_d\|^2 + 1)^2} \right) - \frac{1}{2 \alpha_u} \|\hat{W}_c\|^2 - \left( \frac{\phi_d^2_{cmax}}{4 \gamma^2} \right) \|\hat{W}_c\|^2 + \theta \left( \frac{1}{2} \hat{g}(\hat{x}_{aug}(t))^T \frac{\partial \phi_d}{\partial \hat{x}_{aug}} \right)^T$$

(49)

where $\mu_d = \frac{1}{2 \alpha_u} \|\hat{W}_c\|^2 + \frac{1}{\alpha_c} \frac{2 \phi_d^2_{cmax}}{4 \gamma^2} + \frac{\phi_d^2_{cmax}}{4 \gamma^2}$. Given the inequalities, (44)-(45), there exists a class-$K$ function $k_1$ to write (49) as

$$\dot{V} \leq -k_1(\|\psi\|) + \mu_d$$

(50)

from which we can conclude that $\dot{V} \leq 0$ whenever the state $\psi$ lies outside the set $\Omega_1 = \{ \psi : k_1(\|\psi\|) \leq \mu_d \}$ from which UUB of the continuous closed-loop signals follows.

For the jump dynamics for $t = r_j, j \in \mathbb{N}$, we will consider the following difference Lyapunov function,

$$\Delta V(\psi) = V^*(\hat{x}_{aug}(t)) - V^*(\hat{x}_{aug}(t)) + V^*(\hat{x}_{aug}(t)) - V^*(\hat{x}_{aug}(t)) + V_c(W_c^T) + \left( \frac{d}{dt} W_c(t) + V_d(W_d(t)) + \frac{1}{2 \alpha_u} \|\hat{W}_c\|^2 + \frac{1}{2 \alpha_d} \|\hat{W}_d\|^2 \right) - \frac{1}{2 \alpha_u} \|\hat{W}_c\|^2 - \frac{1}{2 \alpha_d} \|\hat{W}_d\|^2$$

where $V_c = \frac{1}{2} \hat{W}_c^T \hat{W}_c$, $V_d = \frac{1}{2 \alpha_u} \|\hat{W}_c\|^2$, and $V_d = \frac{1}{2 \alpha_d} \|\hat{W}_d\|^2$. And from the continuous-time dynamics we have that $\forall \Delta V(\hat{x}_{aug}(t)) \leq \hat{V}(\hat{x}_{aug}(t)), V_c(W_c^T) \leq V_c(W_c(t))$, and $V_d(W_d^T) \leq V_d(W_d(t))$ for $t = r_j, j \in \mathbb{N}$. Now since the states $\hat{x}_{aug}$ are UUB from (50) then since $\hat{x}_{aug} = \hat{x}_{aug}(t)$ for $t = r_j, j \in \mathbb{N}$ it is true that $V^*(\hat{x}_{aug}) \leq V^*(\hat{x}_{aug})$ which can lead us to write $\Delta V(\hat{x}_{aug}) \leq -k_2(\|\hat{x}_{aug}\|)$ with $k_2$ a class-$K$ function. After some algebra in the error actor jump dynamics (similarly to the disturbance weight error in the continuous dynamics) we can write the Lyapunov difference as, $\Delta V(\psi) \leq -\frac{\mu_d}{\sigma_c^2} \|\hat{x}_{aug}\|^2 + \mu_u$, where $\mu_u = 2 \phi_u^2 \|\hat{W}_d\| + \frac{1}{2} \phi_u^2 \|\hat{W}_d\|$. Finally using the result from (46) we can conclude that the jump dynamics are UUB as long as $\psi$ lies outside the set $\Omega(\hat{x}) = \{ \psi : k_2(\|\psi\|) \leq \mu_u \}$. Hence the results of the theorem follow.

VI. SIMULATION

Consider the configuration of the off-grid VSI as shown in Figure 1 with $L = 1.1 mH$, $C_{up} = 50 \mu F$, $R = 0.2 \Omega$ and disturbance matrix given as $K = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$. The user defined matrices are defined as $Q = \begin{bmatrix} 0.1 & 0 \\ 0 & 0 \end{bmatrix}$, $Q_r = I_2$ and $\hat{Q}$ is picked according to (6) and $\gamma = 5$. The control input is $\pm V_{dc}$ with $V_{dc} = 170$, the observer gain is $L_0 = \begin{bmatrix} 500 \\ 800 \end{bmatrix}$, the observer weight parameters are given as $B_1 = \begin{bmatrix} 6 \times 10^5 \\ 0 \end{bmatrix}$, $B_2 = \begin{bmatrix} 5 \times 10^4 \\ 0 \end{bmatrix}$, $\alpha_1 = \alpha_2 = 2 \times 10^{-4}$, the tuning gains for the actor/critic approximators are $\alpha_c = 30$, $\alpha_u = 1$ and $\alpha_d = 1$. The parameter $\beta$ in (43) is picked as $\beta = 0.5$, such that the performance and the controller bandwidth are “balanced” in a way. For the load we will consider sudden changes. Hence for $0 \leq t < 0.263$ second we will not connect a load, from $0.263 \leq t < 0.5$ second we will connect a linear load and then at $t = 0.5$ second we will add a nonlinear load corresponding to a fluorescent lamp [4]. Figure 3 shows the evolution of the current at the load, and the output voltage waveform while the load is changing. In order to check the efficiency we will compute the Total Harmonic Distortion (THD) (in relation to the fundamental frequency of the power grid source) of the output voltage and this is found to be THD = 1.48% which is well below the IEEE Standards 519 – 1992 (a standard developed for utility companies and their customers in order to limit harmonic content and provide all users with better power quality) voltage distortion limits. The very small THD shows that the proposed algorithm is robust to load and parameter changes. Figure 4 shows the inter-event times $T := r_j - r_{j-1}, j \in \mathbb{N}^+$ and the evolution of the gap and the threshold for the first 0.2 second. A bandwidth improvement of almost 84.1% is achieved, while attenuating disturbances.

VII. CONCLUSIONS AND FUTURE WORK

We proposed an observer-based optimal adaptive event-triggered control algorithm for a completely unknown off-grid VSI in order to attenuate the disturbance at the current and produce the desired AC behavior. Since the dynamics are not known, we use an adaptive observer to reconstruct the states based only at the output voltage. The control mechanism is constructed based on an impulse actor/critic framework.
that approximates the optimal cost, the event-triggered optimal controller and the worst case disturbance. Rigorous stability proofs and simulation results are presented to show the efficacy of the proposed approach. Future work will be concentrated on extending the results to three-phase VSIs and connected in parallel.

**References**


