Chapter 1
A Constant Factor Approximation Algorithm for Event-Based Sampling

Randy Cogill, Sanjay Lall, and João P. Hespanha

Abstract We consider a control system in which sensor data is transmitted from the plant to a receiver over a communication channel, and the receiver uses the data to estimate the state of the plant. Using a feedback policy to choose when to transmit data, the goal is to schedule transmissions to balance a trade-off between communication rate and estimation error. Computing an optimal policy for this problem is generally computationally intensive. Here we provide a simple algorithm for computing a suboptimal policy for scheduling state transmissions which incurs a cost within a factor of six of the optimal achievable cost.

1.1 Introduction

We consider a control system in which sensor data is transmitted from the plant to a receiver over a communication channel, and the receiver uses the data to estimate the state of the plant. Sending data more frequently leads to increased use of limited communication resources, but also allows the average

Randy Cogill
Department of Systems and Information Engineering
University of Virginia, Charlottesville, VA 22904, U.S.A.
e-mail: rcogill@virginia.edu

Sanjay Lall
Department of Aeronautics and Astronautics
Stanford University, Stanford, CA 94305, U.S.A.
e-mail: lall@stanford.edu

João P. Hespanha
Department of Electrical and Computer Engineering
University of California, Santa Barbara, CA 93106, U.S.A.
e-mail: hespanha@ece.ucsb.edu

estimation error to be reduced. Conversely, of course we may reduce the use of the channel if we are willing to allow larger estimation errors.

We consider feedback policies for choosing when to transmit data. That is, instead of simply choosing a transmission rate, at the plant measurements are used to decide whether to transmit data to the controller. This type of measurement is called *Lebesgue* or *event-based* sampling in [2]. Several other authors have considered both control and filtering problems using such sampling schemes, in particular [2, 8, 28, 7, 26, 18, 9].

The plant is modeled by a discrete-time linear system, and at each time step the channel allows exact transmission of the state. The cost function of interest in this problem is a weighted sum of the estimation error and the transmission rate. The optimal controller for a given weight then lies on the Pareto-optimal trade-off curve, and choosing the weight allows one to select the trade-off between rate and error.

For this cost function, the problem of finding the optimal policy was considered in [27], where the authors show that the problem of computing an optimal scheduling policy can be addressed in the framework of Markov decision processes, and consequently the value iteration algorithm can be used to compute an optimal policy. Although this provides an algorithm for computing an optimal policy, the computation required to compute such a policy quickly becomes prohibitive as the system's state dimension increases.

Since the optimal policy is very difficult to compute, we consider *approximately optimal* policies. Specifically, the main result of this paper is to give a simple algorithm for computing a policy, and show that this policy is guaranteed to achieve a cost within a factor of six of the optimal achievable cost. This result is Theorem 1 below.

Approximation algorithms have been widely used for addressing computationally intractable problems. While some NP-hard problems may be approximated to arbitrary accuracy, others may not be approximated within any constant factor. It is therefore extremely promising that the particular problem of rate-error trade-off considered in this paper is approximable within a constant factor of six. It is not currently known whether policies achieving better approximation ratios may be efficiently obtained.

Finally, due to space constraints, all proofs have been ommitted from this paper. Proofs of the theorems in this paper can be found in [6].

1.2 Problem formulation

Here we will present the problem that will be considered throughout this paper. In the following subsection, it will be shown how this problem is a generalization of the problem of networked estimation.

We have dynamics

$$e_{t+1} = (1 - a_t)Ae_t + w_t \qquad e_0 = 0, \tag{1.1}$$

where $A \in \mathbb{R}^{n \times n}$, and for each $t \in \mathbb{N}$ the state is $e_t \in \mathbb{R}^n$ and the action is $a_t \in \{0, 1\}$. Here $w_0, w_1, \ldots$ is a sequence of independent identically distributed Gaussian random vectors, with $w_t \sim \mathcal{N}(0, \Sigma)$, where $\Sigma \succ 0$. Define the function $r : \mathbb{R}^n \times \{0, 1\} \to \mathbb{R}$ to be the cost at time t, given by

$$r(e_t, a_t) = (1 - a_t)e_t^T Q e_t + \lambda a_t \tag{1.2}$$

where $Q \succ 0$ and $\lambda > 0$. We would like to choose a state-feedback control policy $\mu : \mathbb{R}^n \to \{0, 1\}$ to make the average cost incurred by the policy μ small. Here the average cost J is defined as

$$J(\mu) = \limsup_{N \to \infty} \frac{1}{N} \sum_{t=0}^{N-1} \mathbf{E}\big(r(e_t, \mu(e_t))\big) \tag{1.3}$$

See [1] for background on this choice of cost. Here, each a_t is determined according to the static state-feedback policy $a_t = \mu(e_t)$, and then the sequence $e_0, e_1, \ldots$ is a Markov process. Therefore, the problem of choosing a policy which minimizes the cost J is can be addressed using the theory of Markov decision processes. The cost J given by equation (1.3) is called the *average per-period cost*, and we focus specifically on the problem of choosing a policy to minimize this. For convenience, define the space of policies

$$\mathcal{P} = \{\, f : \mathbb{R}^n \to \{0, 1\} \mid f \text{ is measurable} \,\}$$

Then the above problem can be stated as follows.

Problem 1 (Rate-Error Trade-off). Given A, $\Sigma \succ 0$, $Q \succ 0$, $\lambda > 0$, and $\gamma > 0$, find a state feedback policy $\mu \in \mathcal{P}$ such that

$$J(\mu) \le \gamma$$

Minimizing the cost J balances a trade-off between the average size of e_t, as measured by the quadratic form defined by Q, and the frequency with which e_t is reset to the level of the noise by setting $a_t = 1$. The problem of computing an optimal policy was considered in [27], and a numerical procedure for finding such a policy was given. However, the computation required to compute an optimal policy increases rapidly with the state dimension. In the following section we present an easily computable and easily implementable policy for this problem which incurs a cost within a provable bound of the optimal achievable cost. Specifically, we focus our attention on the set of problem instances where Q and A are such that $A^T Q A - Q \preceq 0$ and $Q \succ 0$. In particular, this implies that $\rho(A) \le 1$ and the system is therefore at least marginally stable. We show that in this case there is a simple policy which always achieves a cost within a factor of six of the optimal cost. It is worth noting that, in general, both the policy which always transmits and the policy which never transmits may achieve cost arbitrarily far from optimal.

1.2.1 Application to Networked Estimation

Suppose we have the dynamical system

$$\begin{aligned} x_{t+1} &= Ax_t + w_t \quad x_0 = 0 \\ y_t &= a_t x_t \end{aligned}$$

where for each $t \in \mathbb{N}$ the state $x_t \in \mathbb{R}^n$ and $a_t \in \{0,1\}$. As above, $w_0, w_1, \dots$ is a sequence of independent identically distributed zero mean Gaussian random vectors with covariance $\Sigma \succ 0$. We have a per-period cost of

$$c(x_t, a_t, b_t) = (1 - a_t)(x_t - b_t)^T Q(x_t - b_t) + \lambda a_t \tag{1.4}$$

and we would like to choose two controllers. The first is the function $\mu : \mathbb{R}^n \to \{0,1\}$, and the second is the sequence of functions ϕ_t indexed by t where $\phi_t : \{0,1\}^t \times \mathbb{R}^{nt} \to \mathbb{R}^n$. These are connected according to

$$\begin{aligned} a_t &= \mu(x_t) \\ b_t &= \phi_t(a_0, \dots, a_{t-1}, y_0, \dots, y_{t-1}) \end{aligned}$$

Again, we are interested in the cost

$$J(\mu, \phi_0, \phi_1, \dots) = \limsup_{N \to \infty} \frac{1}{N} \sum_{t=0}^{N-1} \mathbf{E}\big(r(x_t, a_t, b_t)\big)$$

The interpretation is shown in Figure 1.1, where the linear dynamics $x_{t+1} = Ax_t + w_t$ is denoted by G. The dashed lines indicate a communication channel. At each time t the transmitter μ chooses whether to transmit the signal x_t to the receiver ϕ. Each transmission costs λ. The receiver would like to estimate the state x_t of G, and choose b_t to minimize the error $x_t - b_t$ as measured by the quadratic form Q. The cost r is used to compute the trade-off, parametrized by λ, of estimation error against frequency of transmissions.

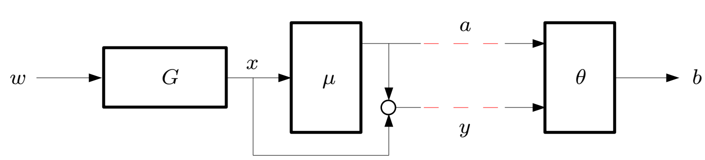

Fig. 1.1 Networked Estimation.

The estimator ϕ considered in Xu and Hespanha [27] is as follows. Let $b_t = \phi_t(a_0, \dots, a_{t-1}, y_0, \dots, y_{t-1})$, and define ϕ by the realization

$$b_{t+1} = (1 - a_t)Ab_t + a_t Ax_t \quad b_0 = 0$$

If the random variables $a_0, a_1, \dots$ are independent of $x_0, x_1, \dots$ then this is the time-varying Kalman filter, and b_t is the minimum mean square error estimate of x_t given measurements $y_0, \dots, y_{t-1}$.

We now have the dynamics

$$\begin{bmatrix} x_{t+1} \\ b_{t+1} \end{bmatrix} = \begin{bmatrix} A & 0 \\ a_t A & (1 - a_t)A \end{bmatrix} \begin{bmatrix} x_t \\ b_t \end{bmatrix} + \begin{bmatrix} I \\ 0 \end{bmatrix} w_t$$

We change coordinates to

$$\begin{bmatrix} e_t \\ f_t \end{bmatrix} = \begin{bmatrix} I & -I \\ 0 & I \end{bmatrix} \begin{bmatrix} x_t \\ b_t \end{bmatrix}$$

to give

$$\begin{bmatrix} e_{t+1} \\ f_{t+1} \end{bmatrix} = \begin{bmatrix} (1 - a_t)A & 0 \\ a_t A & A \end{bmatrix} \begin{bmatrix} e_t \\ f_t \end{bmatrix} + \begin{bmatrix} I \\ 0 \end{bmatrix} w_t$$

In these coordinates, the cost c specified in equation (1.4) is exactly equal to the cost (1.2), and e evolves according to the dynamics (1.1). With this choice of ϕ therefore the optimal choice of μ is found by solving the Rate-Error Trade-off problem.

1.3 Main results

In this section we present the main result of this paper, which is that for a slightly restricted version of the Rate-Error Tradeoff problem, there is a simple policy which achieves cost within a constant factor of optimal. Define for convenience

$$J_{\text{opt}} = \inf_{\mu \in \mathcal{P}} \left(\liminf_{N \to \infty} \frac{1}{N} \sum_{t=0}^{N-1} \mathbf{E}\big(r(e_t, \mu(e_t))\big) \right)$$

The policy that we consider is a simple quadratic threshold policy. The main result of this paper is as follows.

Theorem 1. *Suppose $A \in \mathbb{R}^{n \times n}$, $Q \succ 0$, $\Sigma \succ 0$, and $A^T QA - Q \preceq 0$. Then there exists a unique matrix $M \in \mathbb{S}^n$ satisfying*

$$\frac{1}{1 + \operatorname{trace}(\Sigma M)} A^T MA - M + \frac{Q}{\lambda} = 0 \tag{1.5}$$

Furthermore, define the policy μ *by*

$$\mu(e) = \begin{cases} 0 & \textit{if } e^T M e \le 1 \\ 1 & \textit{otherwise} \end{cases} \tag{1.6}$$

For this policy, the cost satisfies

$$J(\mu) \le 6 J_{opt} \tag{1.7}$$

Proof. The result follows immediately from Theorems 2 and 3 which are proved below.

Note that implementation of the policy μ requires an algorithm for computing the unique solution M of equation (1.5). It is easily shown that this equation can be solved by performing a bisection search and solving a sequence of Lyapunov equations.

1.4 Bounds for the Communication Cost

1.4.1 Upper bounds

We are now ready to the upper bound on $J(\mu)$ is obtained, where μ is the policy in (1.6). The following lemma provides the upper bound and also shows that one may use semidefinite programming, combined with a line search, to find policies that minimize this upper bound.

Lemma 1. *Suppose* $M \succeq 0$ *and* $H \succeq 0$ *are symmetric positive semidefinite matrices, and* $\alpha \in \mathbb{R}$*. If*

$$\begin{aligned} A^T H A - H + Q - \alpha M &\preceq 0 \\ (\lambda - \alpha) M - H &\preceq 0 \\ \alpha - \lambda &\le 0 \\ \alpha &\ge 0 \end{aligned} \tag{1.8}$$

Then the policy

$$\mu(e) = \begin{cases} 0 & \textit{if } e^T M e \le 1 \\ 1 & \textit{otherwise} \end{cases}$$

achieves a cost which satisfies

$$J(\mu) \le \operatorname{trace}(\Sigma H) + \alpha$$

We now make use of this result to provide an explicit upper bound.

Theorem 2. *Suppose* $A \in \mathbb{R}^{n \times n}$, $Q \succ 0$, $\Sigma \succ 0$ *and* $A^T Q A - Q \preceq 0$. *Let* M *be the unique solution to*

$$\frac{1}{1 + \operatorname{trace}(\Sigma M)} A^T M A - M + Q/\lambda = 0$$

Then the policy

$$\mu(e) = \begin{cases} 0 & \text{if } e^T M e \leq 1 \\ 1 & \text{otherwise} \end{cases}$$

achieves

$$J(\mu) \leq \frac{2\lambda \operatorname{trace}(\Sigma M)}{1 + \operatorname{trace}(\Sigma M)}$$

1.4.2 Lower bounds

For the class of instances of RATE-ERROR TRADEOFF with A and Q satisfying $A^T Q A - Q \preceq 0$, we can show that the policy μ of equation (1.6) achieves a cost within a constant factor of optimal. To complete the presentation of the main result of this paper, we now determine a lower bound on J_{opt} which guarantees that for this class of instances,

$$J(\mu) \leq 6 J_{\text{opt}}$$

This result can be established using the lemmas below, the proofs of which can be found in [6].

Lemma 2. *Suppose* $Y \succeq 0$ *and* $q \in \mathbb{R}^n$, *and* $w \sim \mathcal{N}(0, \Sigma)$ *is a Gaussian random vector. Let* f *be the random variable*

$$f = (q + w)^T Y (q + w)$$

Then

$$\mathbf{E} f = q^T Y q + \operatorname{trace}(\Sigma Y) \tag{1.9}$$

$$\begin{aligned} \mathbf{E}(f^2) = (q^T Y q)^2 + 4 q^T Y \Sigma Y q + \big(\operatorname{trace}(\Sigma Y)\big)^2 \\ + 2 \operatorname{trace}(\Sigma Y \Sigma Y) + 2 q^T Y q \operatorname{trace}(\Sigma Y) \end{aligned} \tag{1.10}$$

and further

$$\mathbf{E}(f^2) \leq (q^T Y q)^2 + 6 q^T Y q \operatorname{trace}(\Sigma Y) + 3\big(\operatorname{trace}(\Sigma Y)\big)^2$$

Lemma 3. *Suppose there exists a positive semidefinite matrix* $C \succeq 0$ *and* $s \in \mathbb{R}$ *such that*

$$\begin{aligned} \big(s - 6\,\mathrm{trace}(C\Sigma)\big)A^TCA - sC + Q &\succeq 0 \\ s^2 &\leq 4\lambda \\ A^TCA - C &\preceq 0 \end{aligned} \tag{1.11}$$

Then for all policies $\mu \in \mathcal{P}$

$$J(\mu) \geq s\,\mathrm{trace}(C\Sigma) - 3\big(\mathrm{trace}(C\Sigma)\big)^2$$

Lemma 4. *Suppose there exists $M \succeq 0$ such that*

$$\begin{aligned} \frac{1}{1+\mathrm{trace}(\Sigma M)}A^TMA - M + Q/\lambda &= 0 \\ A^TMA - M &\preceq 0 \end{aligned}$$

Then for all policies $\mu \in \mathcal{P}$ we have

$$J(\mu) \geq \frac{\lambda\,\mathrm{trace}(\Sigma M)}{3\big(1+\mathrm{trace}(\Sigma M)\big)}$$

Lemma 5. *Suppose $Q \succ 0$ and $A^TQA - Q \preceq 0$, and $\alpha \in \mathbb{R}$ satisfies $0 \leq \alpha < 1$. Then there exists a unique $M \in \mathbb{S}^n$ such that*

$$\alpha A^TMA - M + Q = 0 \tag{1.12}$$

and the matrix M is positive definite and satisfies

$$A^TMA - M \preceq 0$$

Finally, the lemmas above can be combined to obtain the following theorem.

Theorem 3. *Suppose $A \in \mathbb{R}^{n\times n}$, $Q \succ 0$, $\Sigma \succ 0$ and $A^TQA - Q \preceq 0$. Let M be the unique solution to*

$$\frac{1}{1+\mathrm{trace}(\Sigma M)}A^TMA - M + Q/\lambda = 0$$

Then for all policies $\mu \in \mathcal{P}$ we have

$$J(\mu) \geq \frac{\lambda\,\mathrm{trace}(\Sigma M)}{3\big(1+\mathrm{trace}(\Sigma M)\big)}$$

1.5 Conclusions

In this paper we considered a simple, yet fundamental estimation problem involving balancing the trade-off between communication rate and estimation error in networked linear systems. This paper extended work of [27], where it was shown that this problem can be posed as a Markov decision process. Here we show that there is a simple, easily computable suboptimal policy for scheduling state transmissions which incurs a cost within a factor of six of the optimal achievable cost.

References

1. A. Arapostathis, V. S. Borkar, E. Fernandez-Gaucherand, M. K. Ghosh, and S. I. Marcus. Discrete-time controlled markov processes with average cost criterion: a survey. *Optimization*, 31(2):282–344, 1993.
2. K. J. Åström and B. M. Bernhardsson. Comparison of Riemann and Lebesgue sampling for first order stochastic systems. In *Proceedings of the IEEE Conference on Decision and Control*, 2002.
3. A. Cervin and K.J. Åström. On limit cycles in event-based control systems. *Proceedings of the 46th IEEE Conference on Decision and Control*, pages 3190–3195, 2007.
4. A. Cervin and E. Johannesson. Sporadic control of scalar systems with delay, jitter and measurement noise. *Proceedings of the 17th IFAC World Congress*, 2008.
5. R. Cogill and S. Lall. Suboptimality bounds in stochastic control: A queueing example. *Proceedings of the 2006 American Control Conf.*, pages 1642 – 1647, 2006.
6. R. Cogill, S. Lall, and J.P. Hespanha. A constant factor approximation algorithm for optimal estimation subject to communication costs. *Proceedings of the 2007 American Control Conference*, 2007.
7. J.P. Hespanha, A. Ortega, and L. Vasudevan. Towards the control of linear systems with minimum bit-rate. *Proc. of the Int. Symp. on the Mathematical theory of Networks and Sys.*, August 2002.
8. D. Hristu-Varsakelis and P. R. Kumar. Interrupt-based feedback control over a shared communication medium. In *Proceedings of the IEEE Conference on Decision and Control*, 2002.
9. O. C. Imer and T. Başar. Optimal estimation with limited measurements. In *Proceedings of the IEEE Conference on Decision and Control*, pages 1029–1034, 2005.
10. O. C. imer and T. Başar. Optimal estimation with scheduled measurements. *Appl. Comput. Math.*, 4(2):92–101, 2005.
11. O. C. Imer and T. Başar. Optimal control with limited controls. In *Proceedings of the American Control Conference*, 2006.
12. O. C. Imer and T. Başar. To measure or to control: Optimal control with scheduled measurements and controls. In *Proceedings of the American Control Conference*, 2006.
13. M. Mazo Jr. and P. Tabuada. On event-triggered and self-triggered control over sensor/actuator networks. *Proceedings of the 47th IEEE Conference on Decision and Control*, 2008.
14. E. Kofman and J. Braslavsky. Level crossing sampling in feedback stabilization under data-rate constraints. *Proceedings of the 45th IEEE Conference on Decision and Control*, pages 4423–4428, 2006.
15. J.S. Baras M. Rabi, G.V. Moustakides. Adaptive sampling for linear state estimation. *Submitted to SIAM Journal on Control and Optimization*, 2008.

16. M. Miskowicz. Bandwidth requirements for event driven observations of continuous time variable. *Proc. of IAFC Workshop on Discrete-Event Systems WODES04*, pages 475–480, 2004.
17. M. Miskowicz. Efficiency of level-crossing sampling for bandlimited gaussian random processes. *2006 IEEE International Workshop on Factory Communication Systems*, pages 137–142, 2006.
18. M. Rabi and J. S. Baras. Sampling of diffusion processes for real-time estimation. In *Proceedings of the IEEE Conference on Decision and Control*, 2004.
19. M. Rabi and J.S. Baras. Level-triggered control of a scalar linear system. *Proceedings of the 2007 Mediterranean Conference on Control and Automation*, 2007.
20. M. Rabi, J.S. Baras, and G. Moustakides. Multiple sampling for estimation on a finite horizon. *Proceedings of the 45th IEEE Conference on Decision and Control*, pages 1351–1357, 2006.
21. M. Rabi and K.H. Johansson. Event-triggered strategies for industrial control over wireless networks. *Proceedings of the Fourth Annual Wireless Interned Conference*, 2008.
22. P. Tabuada. Event-triggered real-time scheduling of stabilizing control tasks. *IEEE Trans. Automatic Control*, 52(9):1680–1685, 2007.
23. K. Triantafyllopoulos. On the central moments of the multidimensional Gaussian distribution. *The Mathematical Scientist*, 28:125–128, 2003.
24. X. Wang and M. Lemmon. Self-triggered feedback control systems with finite-gain stability. *To appear in IEEE Trans. Automatoc Control*, 2009.
25. X. Wang and M.D. Lemmon. Event-triggered broadcasting across distributed networked control systems. *Proceedings of the 2008 American Control Conference*, 2008.
26. Y. Xu and J. P. Hespanha. Communication logics for networked control systems. In *Proceedings of the American Control Conference*, 2004.
27. Y. Xu and J. P. Hespanha. Optimal communication logics in networked control systems. *Proceedings of the IEEE Conference on Decision and Control*, pages 3527–3532, 2004.
28. J. K. Yook, D. M. Tilbury, and N. R. Soparkar. Trading computation for bandwidth: Reducing communication in distributed control systems using state estimators. *IEEE Trans. Contr. Sys. Technol.*, 4(10):503–518, July 2002.