

# STOCHASTIC HYBRID SYSTEMS WITH RENEWAL TRANSITIONS: MOMENT ANALYSIS WITH APPLICATION TO NETWORKED CONTROL SYSTEMS WITH DELAYS

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**Abstract.** We consider Stochastic Hybrid Systems (SHSs) for which the lengths of times that the system stays in each mode are independent random variables with given distributions. We propose an approach based on a set of Volterra equations to compute any statistical moment of the state of the SHS. Moreover, we provide a method to compute the Lyapunov exponents of a given degree, i.e., the exponential rate of decrease or increase at which statistical moments converge to zero or to infinity, respectively. We also discuss how, by computing the statistical moments, one can provide information about the probability distribution of the state of the SHS. The applicability of the results is illustrated in the analysis of a networked control problem with independently distributed intervals between data transmissions and delays.

**Key words.** Networked Control Systems, Volterra Integral Equations, semi-Markov processes

**AMS subject classifications.** 60K30, 45D05, 60K15

**1. Introduction and examples.** Stochastic Hybrid Systems (SHSs) combine continuous dynamics and discrete logic. The execution of a SHS is specified by the dynamic equations of the continuous state, a set of rules governing the transitions between discrete modes, and reset maps determining jumps of the state at transition times. As surveyed in [1], various models of SHSs [2], [3], [4] have been proposed, mostly differing on the way randomness enters the different equations governing the evolution of the system. See also [5], [6], [7], [8].

In the present work, we consider SHSs with linear dynamics, linear reset maps, and for which the lengths of times that the system stays in each mode are independent *arbitrarily distributed* random variables, whose distributions may depend on the discrete mode. The process that combines the transition times and the discrete mode is typically called a Markov renewal process [9], which motivated us to refer to these systems as *Stochastic Hybrid Systems with Renewal Transitions*. This class of systems can be viewed as a special case of the SHS model in [5], which in turn is a special case of a Piecewise Deterministic Process [4], and also a special case of a State-Dependent Jump-Diffusion [6, Sec. 5.3]. Alternatively, SHSs with renewal transitions can be viewed as a generalization of a Markov Jump Linear System (MJLS) [8], in which the lengths of times that the system stays in each mode follow an exponential distribution, or as a generalization of an Impulsive Renewal System [10], in which there is only one discrete mode and one reset map.

Our contribution concerns the analyses of the transitory and asymptotic behavior of the statistical moments of a SHS with renewal transitions, and is summarized in two main results, made possible due to the special structure of SHSs with renewal

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transitions. In the first main result, we provide expressions for the moments of the SHS at a given time  $t$  in terms of a set of Volterra equations. Since [8, Ch. 2] presents a method to obtain the statistical moments of a MJLS, based on the solution of linear differential equations, that does not appear to generalize to SHS with renewal transitions (cf. [8, Sec. 8.2]), this first main result shows that for this latter class of systems, one can still compute statistical moments by solving Volterra equations, instead of differential equations. We also discuss how, by computing the moments using these Volterra Equations, one can provide information about the probability distribution of the state of the SHS, and highlight the advantages of this approach with respect to more general methods available in the literature [4], [5], [6]. The second main result provides a method to obtain the Lyapunov exponents of degree  $m$ , which are defined, in accordance to [8, p. 41], as the exponential rate of decrease or increase at which the expected value of the  $m$ th power of the norm of the state of the SHS, tends to zero or to infinity, respectively. We show that when  $m$  is even, Lyapunov exponents can be efficiently determined by finding the zeros of monotonic functions. As a corollary, we provide necessary and sufficient conditions for mean exponential stability, which is defined in terms of the exponential convergence to zero of the expected value of the squared norm of the state of the SHS.

The applicability of our theoretical results is illustrated in a networked control problem, with independently distributed intervals between data transmissions of the control signal in a feedback-loop, which is a reasonable assumption in networked control systems utilizing CSMA communication protocols [10]. The impulsive renewal systems considered in [10] did not permit us to consider the effect of network induced delays, which is now possible with SHSs with renewal transitions.

A preliminary version of the results presented here appeared in the conference paper [11]. However, in this paper we have been able to reduce significantly the dimension of the Volterra equations needed to compute the statistical moments of the SHS. In fact, the dimension of the Volterra equations in the present paper grows polynomially with  $m$ , whereas in [11] it grows exponentially with  $m$ . This enables the use of our results to compute statistical moments of much larger order. An additional novelty of the results presented here is that the asymptotic analysis extends both [10] and [11], where the asymptotic analysis is restricted to a special moment, i.e., the squared norm of the state.

The remainder of the paper is organized as follows. SHSs with renewal transitions are defined in Section 2. In Section 3, we establish and discuss our main results. Section 4 addresses the applicability of the results to a networked control example. In Section 5 we draw final conclusions.

*Notation and Preliminaries:* The Kronecker product is denoted by  $\otimes$ . The notation  $x(t_k^-)$  indicates the limit from the left of a function  $x(t)$  at the point  $t_k$ . For vectors  $u_1, \dots, u_m$  in  $\mathbb{R}^n$ , we define  $(u_1, \dots, u_m) := [u_1^\top \dots u_m^\top]^\top$ . The spectral radius of a matrix  $A$  is denoted by  $r_\sigma(A)$ . We denote by  $e_i$  the canonical vector in  $\mathbb{R}^n$ , i.e., the component  $j$  of  $e_i \in \mathbb{R}^n$  equals 1 if  $j = i$  and 0 otherwise. The indicator function of a set  $A$  is denoted by  $\chi_{x \in A}$ , which equals 1 if  $x \in A$  and 0 otherwise. The number of elements of a finite set  $C$  is denoted by  $|C|$ .

**2. SHS with Renewal Transitions.** A linear SHS with renewal transitions, is defined by (i) a linear differential equation

$$\dot{x}(t) = A_{q(t)}x(t), \quad x(0) = x_0, q(0) = q_0, t_0 = 0, \quad (2.1)$$

where  $x(t) \in \mathbb{R}^n$  and  $q(t) \in \mathcal{Q} := \{1, \dots, n_q\}$ ; (ii) a family of  $n_\ell$  *discrete transition/reset maps*

$$(q(t_k), x(t_k)) = (\xi_\ell(q(t_k^-)), J_{q(t_k^-), \ell} x(t_k^-)), \quad \ell \in \mathcal{L} := \{1, \dots, n_\ell\}, \quad (2.2)$$

where  $\xi_\ell$  is a map from  $\mathcal{Q}$  to  $\mathcal{Q}$  and the matrix  $J_{q(t_k^-), \ell}$  belongs to a given set  $\{J_{i, \ell} \in \mathbb{R}^{n \times n}, i \in \mathcal{Q}, \ell \in \mathcal{L}\}$ ; and (iii) a family of *reset-time measures*

$$\mu_{i, \ell}, \quad i \in \mathcal{Q}, \ell \in \mathcal{L}. \quad (2.3)$$

Between *transition times*, denoted by  $t_k$ , the discrete mode  $q$  remains constant whereas the continuous state  $x$  flows according to (2.1). At transition times, the continuous state and discrete mode of the SHS are reset according to (2.2). The intervals between transition times are independent random variables determined by the reset-time measures (2.3) as follows. A reset-time measure can be either a probability measure or identically zero. In the former case,  $\mu_{i, \ell}$  is the probability measure of the random time that transition  $\ell \in \mathcal{L}$  takes to trigger in the state  $q(t) = i \in \mathcal{Q}$ . The next transition time is determined by the minimum of the triggering times of the transitions associated with state  $q(t) = i \in \mathcal{Q}$ . When  $\mu_{i, \ell}([0, s]) = 0, \forall s \geq 0$ , the transition  $\ell$  does not trigger in state  $i \in \mathcal{Q}$ , which allows for some reset maps not to be active in some states.

The construction of a sample path of the SHS with renewal transitions can then be described as follows.

1. Set  $k = 0, t_0 = 0, (q(t_k), x(t_k)) = (q_0, x_0)$ .
2. For every  $j \in \mathcal{L}$ , obtain  $\bar{h}_k^j$  as a realization of a random variable distributed according to  $\mu_{q(t_k), j}$ , if  $\mu_{q(t_k), j}$  is not identically zero, and set  $\bar{h}_k^j = \infty$  otherwise.
3. Take  $h_k = \min\{\bar{h}_k^j, j \in \mathcal{L}\}$  and set the next transition time to  $t_{k+1} = t_k + h_k$ . The state of the SHS in the interval  $t \in [t_k, t_{k+1})$  is given by  $(q(t), x(t)) = (q(t_k), e^{A_{q(t_k)}(t-t_k)} x(t_k))$ .
4. If  $t_{k+1} < \infty$  (otherwise stop), let  $l_k$  denote the index of the transition that achieves the minimum in step 3, i.e.,  $l_k = j : h_k = \bar{h}_k^j$  and update the state according to  $(q(t_{k+1}), x(t_{k+1})) = (\xi_{l_k}(q(t_{k+1}^-)), J_{q(t_{k+1}^-), l_k} x(t_{k+1}^-))$ . Set  $k = k + 1$  and repeat the construction from the step 2.

We assume that each transition probability measure  $\mu_{i, \ell}$  can be decomposed as  $\mu_{i, \ell} = \mu_{i, \ell}^c + \mu_{i, \ell}^d$ , with  $\mu_{i, \ell}^c([0, t]) = \int_0^t f_{i, \ell}(s) ds$ , for some density function  $f_{i, \ell}(s) \geq 0$ , and  $\mu_{i, \ell}^d$  is a discrete measure that captures possible point masses  $\{b_{i, \ell}^r, r \geq 1\}$  such that  $\mu_{i, \ell}^d(\{b_{i, \ell}^r\}) = w_{i, \ell}^r$ . The integral with respect to the measure  $\mu_{i, \ell}$  is then defined as

$$\int_0^t W(s) \mu_{i, \ell}(ds) = \int_0^t W(s) f_{i, \ell}(s) ds + \sum_{r: b_{i, \ell}^r \in [0, t]} w_{i, \ell}^r W(b_{i, \ell}^r). \quad (2.4)$$

When the discrete measures  $\mu_{i, \ell}^d$  are non-zero measures, different transitions may be triggered at the same time with non-zero probability, leading to an ambiguity in choosing the next state. To avoid ambiguity, we assume the following.

ASSUMPTIONS 2.1.

- (i) For every  $i \in \mathcal{Q}$ , the reset-time measures  $\mu_{i, \ell}, \ell \in \mathcal{L}$  for mode  $i$  have a finite number of point masses;

(ii) For every  $i \in \mathcal{Q}$ , no two measures  $\mu_{i,\ell_1}$ ,  $\mu_{i,\ell_2}$ , for mode  $i$  have common point masses, i.e.,  $b_{i,\ell_1}^j \neq b_{i,\ell_2}^k$  for  $\ell_1 \neq \ell_2, \ell_1 \in \mathcal{L}, \ell_2 \in \mathcal{L}, \forall_{j \geq 1, \kappa \geq 1}$ ;

Due to the Assumption 2.1.(ii), there is zero probability that in step 3 of the construction of a sample path of the SHS the minimum is achieved by two or more indices  $j$ .

**3. Main results.** Consider a general  $m$ -th degree uncentered moment of the state of the SHS with renewal transitions, i.e.,

$$\mathbb{E}[x_1(t)^{i_1} x_2(t)^{i_2} \dots x_n(t)^{i_n}], \quad \sum_{j=1}^n i_j = m, i_j \geq 0. \quad (3.1)$$

We provide a method to compute (3.1) in subsection 3.1, we obtain the Lyapunov exponents of special moments of the SHS in Subsection 3.2, and we discuss how by computing the moments one can reconstruct probability distributions in Subsection 3.3. The proofs of the main results are deferred to Subsection 3.4.

**3.1. Moment computation.** It is easy to see that there are

$$p := \frac{(m+n-1)!}{m!(n-1)!} \quad (3.2)$$

different monomials of degree  $m$ , and hence  $p$  different moments (3.1) of degree  $m$ . Let

$$\{\rho(\kappa) = [i_1(\kappa), \dots, i_n(\kappa)], 1 \leq \kappa \leq p\} \quad (3.3)$$

be an enumeration of the indices  $i_1 \dots, i_n$  uniquely characterizing such monomials, e.g., for  $m = n = 2$ , one such enumeration is  $\rho(1) = [1 \ 1]$ ,  $\rho(2) = [2 \ 0]$ ,  $\rho(3) = [0 \ 2]$ . Then, we use the notation,

$$x^{[\kappa]} := x_1^{i_1} \dots x_n^{i_n}, \text{ for } \kappa : \rho(\kappa) = [i_1, \dots, i_n], \quad 1 \leq \kappa \leq p,$$

and define the map  $\Gamma^m : \mathbb{R}^{n \times n} \mapsto \mathbb{R}^{p \times p}$  as

$$\Gamma^m(A) = B, \quad (3.4)$$

where  $B = [B_{\ell,\kappa}]$  is *uniquely* determined by

$$(Ax)^{[\ell]} = \sum_{\kappa=1}^p B_{\ell,\kappa} x^{[\kappa]}, \quad 1 \leq \ell, \kappa \leq p. \quad (3.5)$$

Note that, indeed the left hand side of (3.5) is a linear combination of monomials of degree  $m$ , with *unique* coefficients  $B_{\ell,\kappa}$ . The following theorem provides a method to compute any moment of the state of the SHS. Let

$$\gamma_\kappa := \frac{m!}{(i_1! \dots i_n!)}, \quad \text{for } \kappa : \rho(\kappa) = [i_1, \dots, i_n], \quad \sum_{j=1}^n i_j = m, \quad (3.6)$$

and define the following operator

$$\Theta(u^\kappa(t)) := (\Theta_1(u^\kappa(t)), \dots, \Theta_{n_q}(u^\kappa(t))),$$

where  $u^\kappa(t) := (u^{\kappa,1}(t), \dots, u^{\kappa,n_q}(t))$  and each  $\Theta_i$  is a convolution operator defined by

$$\Theta_i(u^\kappa(t)) := \sum_{\ell=1}^{n_\ell} \int_0^t \Gamma^m(e^{A_i^\top s} J_{i,\ell}^\top) u^{\kappa,\xi_\ell(i)}(t-s) \frac{r_i(s)}{r_{i,\ell}(s)} \mu_{i,\ell}(ds), \quad (3.7)$$

where  $i \in \mathcal{Q}$ ,  $r_i(s) := \prod_{\ell=1}^{n_\ell} r_{i,\ell}(s)$ ,  $r_{i,\ell}(s) := \mu_{i,\ell}((s, \infty])$ .

**THEOREM 3.1.** *A moment of degree  $m$  (3.1) of the SHS (2.1)-(2.3), indexed by  $\kappa : \rho(\kappa) = [i_1 \dots i_n]$ , can be computed as*

$$\mathbb{E}[x(t)^{[\kappa]}] = \sum_{\iota=1}^p x_0^{[\iota]} u_t^{\kappa, q_0}(t), \quad (3.8)$$

where the  $u^\kappa = (u^{\kappa,1}, \dots, u^{\kappa,n_q})$ ,  $u^{\kappa,i} \in \mathbb{R}^p$ , are the unique solution to the following Volterra equation

$$u^\kappa(t) = \Theta(u^\kappa(t)) + h^\kappa(t), \quad t \geq 0, \quad (3.9)$$

where  $h^\kappa(t) := (h^{\kappa,1}(t), \dots, h^{\kappa,n_q}(t))$ ,

$$h^{\kappa,i}(t) = \Gamma^m(e^{A_i^\top t}) e_\kappa \frac{r_i(t)}{\gamma_\kappa}, \quad i \in \mathcal{Q}, \quad (3.10)$$

where  $e_\kappa$  is the canonical vector in  $\mathbb{R}^p$  with non-zero component  $\kappa$ . □

An explicit solution to (3.9) takes the form

$$u^\kappa(t) = \sum_{k=0}^{\infty} \Theta^k(h^\kappa(t)) \quad (3.11)$$

where  $\Theta^k(h^\kappa(t))$  denotes composition, i.e., e.g.,  $\Theta^2(h^\kappa(t)) = \Theta(\Theta(h^\kappa(t)))$  and  $\Theta^0(h^\kappa(t)) = h^\kappa(t)$ . However, in practice, a numerical method is generally preferable to solve the Volterra equation (3.9). We discuss next the computation complexity of one such method comparing it with the complexity of previous methods.

**3.1.1. Numerical computation.** One numerical method to solve the Volterra equation (3.9) consists in choosing a set of integration nodes  $\{a_l \in [0, t]\}$ , and obtaining  $u^\kappa(a_l)$  by iteratively replacing the integrals in (3.9) by quadrature formulas at the nodes  $\{a_j : a_j \in [0, a_l]\}$  (cf. [12]). Suppose that  $a_l = l\bar{h}$ ,  $l \in \{0, \dots, \bar{L}\}$  are nodes in an interval of interest  $t \in [0, \bar{L}\bar{h}]$ , where  $\bar{L}$  is the number of nodes and  $\bar{h}$  is the spacing between two consecutive nodes. Assuming, for simplicity, that the distributions  $\mu_{i,\ell}$  have no atom points, we have

$$\bar{u}^{\kappa,i}(\bar{h}) = \sum_{\ell=1}^{n_\ell} \sum_{k=0}^l \bar{c}_{i,k} \Gamma^m(e^{A_i^\top(k\bar{h})} J_{i,\ell}^\top) \bar{u}^{\kappa,\xi_\ell(i)}((l-k)\bar{h}) \frac{r_i(k\bar{h})}{r_{i,\ell}(k\bar{h})} f_{i,\ell}(k\bar{h}) + \Gamma^m(e^{A_i^\top l\bar{h}}) e_\kappa \frac{r_i(\bar{h})}{\gamma_\kappa}, \quad (3.12)$$

for  $i \in \mathcal{Q}$ , where the  $\bar{c}_{i,k}$  denote the weights of the quadrature formula and  $\bar{u}^{\kappa,i}(\bar{h})$  approximates  $u^{\kappa,i}(\bar{h})$ . Considering possible atom points of  $\mu_{i,\ell}$  would add additional additive terms on the right hand side of (3.12) taking into account the atom points in the interval  $b_{i,\ell}^* \in [0, \bar{h}]$  which would not significantly impact in our discussion. The

number of matrix multiplications  $\Gamma^m(e^{A_i^\top(k\bar{h})} J_{i,\ell}^\top) u^{\kappa, \xi_\ell(i)}((l-k)\bar{h})$  in (3.12) is given by

$$\sum_{\ell=1}^{n_\ell} |\{k \in \{0, \dots, l\} : f_{i,\ell}(k\bar{h}) > 0 \text{ and } r_i(k\bar{h}) > 0\}| \quad (3.13)$$

which is bounded by  $ln_\ell$ . Each of these matrix multiplications requires in general  $p^2$  scalar multiplications. Moreover, if  $f_{i,\ell}(0) \neq 0$  for some  $i \in \mathcal{Q}$  and some  $\ell \in \mathcal{L}$ , a linear system of at most  $n_qp$  equations must in general be solved to obtain  $u^\kappa(l\bar{h})$  from the set of equations (3.12),  $i \in \mathcal{Q}$ . Thus, the number of total multiplication needed to iterate (3.12) from  $l = 0$  to  $l = \bar{L} - 1$  is bounded by

$$p^2 n_q n_\ell \frac{\bar{L}(\bar{L} + 1)}{2} + \bar{L}G(n_qp) \quad (3.14)$$

where  $G(a) = \frac{a^3}{3} + a^2 - \frac{a}{3}$  is the number of multiplication required by the standard Gaussian method to solve a system with  $a \in \mathbb{N}$  variables. Note that the matrices

$$\Gamma^m(e^{A_i^\top(l\bar{h})} J_{i,\ell}^\top), \quad \Gamma^m(e^{A_i^\top lh}), \quad l \in \{0, \dots, \bar{L}\}, \quad (3.15)$$

can be computed and stored in memory *before* iterating (3.12). The computational complexity of computing these at most  $2(\bar{L} + 1)$  matrices by symbolically manipulating monomials is no larger than the computational complexity of iterating (3.12), which is bounded by (3.14). In fact, suppose that we wish to compute  $C = \Gamma^m(D)$  for  $D \in \mathbb{R}^{n \times n}$  with row vector  $d^i \in \mathbb{R}^{1 \times n}$ ,  $i \in \{1, \dots, n\}$ . To compute a row  $\iota$  of  $C$  such that  $\rho(\iota) = [i_1 \dots i_n]$  we recursively compute  $\zeta(y) = (d^1 y)^{i_1} \times (d^2 y)^{i_2} \dots (d^n y)^{i_n}$ ,  $y \in \mathbb{R}^n$ , according to  $\zeta(y) = \xi_m(y)$ ,

$$\xi_{k+1}(y) = (d^{r(k)} y) \xi_k(y), \quad \xi_1(y) = 1, \quad k = 1, \dots, m-1, \quad (3.16)$$

where  $r(k) = j$  if  $\sum_{s=0}^{j-1} i_s \leq k \leq \sum_{s=0}^j i_s$ ,  $j \in \{1, \dots, n\}$ ,  $k \in \{1, \dots, m\}$ . Since  $\xi_k(x)$  is a sum of monomials with dimension  $k$  and the number of such monomials can be obtained from (3.2), we have that (3.16) requires  $n \times \sum_{k=1}^{m-1} \frac{(k+n-1)!}{k!(n-1)!}$  multiplications to compute each row of  $C$ , which is bounded by  $nmp$ . Thus, the total number of multiplications to compute  $C$  is bounded by  $nmp^2$  and the number of multiplications needed to compute (3.15) is bounded by

$$2(\bar{L} + 1)nmp^2.$$

A similar result to Theorem 3.1 was given in a preliminary version of the present paper [11]. However, there exists redundancy both in the state of the Volterra equation proposed in [11] and in the state of the differential equation proposed in [8], that is eliminated by Theorem 3.1. In fact, given a SHS with state-space dimension  $n$  and  $n_q$  discrete modes, in [11] one needs to solve a Volterra equation with  $n_q \times n^m$  unknown functions, to obtain a moment of degree  $m$ . This is also the dimension of the linear differential equation presented in [8] to compute the moments in the special case of exponential reset-time distributions. The Volterra equation (3.9) has dimension  $n_q \times p = n_q \times \frac{(m+n-1) \times \dots \times (m+1)}{(n-1)!}$ , i.e., the number of unknown functions in (3.9) grows polynomially with  $m$ , instead of exponentially.

From a practical point of view our approach (3.12) is feasible in a current personal computer if  $p$  is in the order of tens of thousands. For example, in Section 4.1 we

shall consider  $n = 4$ , for which we can compute up to about  $m = 10$  moments of the state in which case  $p = 10296$ . Note that with previous approaches [11], [8] this would require to manipulate Volterra equations with dimension two orders of magnitude larger  $4^{10} = 1048576$ , which would clearly not be feasible in a current personal computer.

**3.2. Lyapunov Exponents.** The following definition of Lyapunov exponent of degree  $m$  is adapted from [8, p. 41]<sup>1</sup>. Recall the definition of  $L_p$  norm, for  $p \in \mathbb{N}$ , of a vector  $x(t)$

$$\|x(t)\|_p := \left( \sum_{i=1}^n |x_i(t)|^p \right)^{1/p}, \quad p \geq 1. \quad (3.17)$$

DEFINITION 3.2. *Suppose that  $\mathbb{E}[\|x(t)\|_m^m] \neq 0, \forall t \geq 0$  and that for every  $x_0 \neq 0$  the following limit exists*

$$\lambda_L^m(x_0) := \lim_{t \rightarrow \infty} \frac{1}{t} \log \mathbb{E}[\|x(t)\|_m^m]. \quad (3.18)$$

Then the Lyapunov exponent  $\lambda_L^m$  of degree  $m$  for the SHS (2.1)- (2.3) is defined as

$$\lambda_L^m := \sup_{x_0 \in \mathbb{R}^n} \lambda_L^m(x_0).$$

Moreover, if  $\exists b > 0 : \mathbb{E}[\|x(t)\|_m^m] = 0, \forall t > b$  then  $\lambda_L^m := -\infty$ . □

Let

$$\hat{\Theta}^m(z) := \begin{bmatrix} \hat{\Theta}_{1,1}^m(z) & \dots & \hat{\Theta}_{1,n_q}^m(z) \\ \vdots & \ddots & \vdots \\ \hat{\Theta}_{n_q,1}^m(z) & \dots & \hat{\Theta}_{n_q,n_q}^m(z) \end{bmatrix},$$

where

$$\hat{\Theta}_{i,j}^m(z) := \sum_{\ell: \xi_\ell(i)=j} \int_0^\infty \Gamma^m(e^{A^\top s} J_{i,\ell}^\top) e^{-zs} \frac{r_i(s)}{r_{i,\ell}(s)} \mu_{i,\ell}(ds).$$

For  $a \in \mathbb{R}$ , we say that  $\hat{\Theta}^m(a)$  converges absolutely if for every  $i, j \in \mathcal{Q}$

$$\sum_{\ell: \xi_\ell(i)=j} \int_0^\infty \|\Gamma^m(e^{A^\top s} J_{i,\ell}^\top)\| e^{-as} \frac{r_i(s)}{r_{i,\ell}(s)} |\mu_{i,\ell}(ds)| < \infty$$

where for a matrix  $X$ ,  $\|X\|^2 := \text{tr}(X^\top X)$  and  $|\mu_{i,\ell}(ds)|$  is the total variation norm of the measure  $\mu_{i,\ell}$  (cf. [13]).

**THEOREM 3.3.** *Suppose that all the distributions  $\mu_{i,\ell}$  have finite support and let  $b := \inf\{a : \hat{\Theta}^m(a) \text{ converges absolutely}\}$ . Then, if  $m$  is even, the spectral radius  $r_\sigma(\hat{\Theta}^m(a))$  of  $\hat{\Theta}^m(a)$  is a monotone non-increasing function of  $a$  for  $a > b$  and the*

<sup>1</sup>The definition in [8, p. 41] does not specify which norm to take for  $\|x(t)\|$  (we choose to take (3.17)). Note that in fact (3.18) does not change with the choice of the norm, due to the equivalence of norms in finite-dimensional vector spaces.

Lyapunov exponent  $\lambda_L^m$  for the SHS (2.1)-(2.3) is the (unique) real root  $a$  to the equation

$$r_\sigma(\hat{\Theta}^m(a)) = 1$$

if this equation has a real root and  $-\infty$  otherwise.  $\square$

As we shall see in Section 3.4, the proof of Theorem 3.3 relies on establishing that  $\hat{\Theta}^m(z)$  for real  $z$  is a positive operator (cf. [14]). This only holds if  $m$  is even and this is the main reason for restricting Theorem (3.3) to this case. We do not address in the present paper the case in which  $m$  is odd.

We say that the SHS with renewal transitions is mean exponentially stable if there exists constants  $c > 0$  and  $\alpha > 0$  such that for every  $(x_0, q_0)$ ,

$$\mathbb{E}[x(t)^\top x(t)] \leq ce^{-\alpha t} x_0^\top x_0, \forall t \geq 0,$$

Theorem 3.3 allows us to establish necessary and sufficient conditions for the stability of a SHS with renewal transitions defined as follows.

**COROLLARY 3.4.** *Suppose that all the distributions  $\mu_{i,\ell}$  have bounded support. Then the SHS with renewal transitions is mean exponentially stable if and only if*

$$r_\sigma(\hat{\Theta}^2(0)) < 1. \quad (3.19)$$

$\square$

**3.3. Probability distribution.** Let  $\mu$  be the probability distribution of the state of the SHS at time  $t$ , i.e.,  $\int_E \mu(ds) := \text{Prob}[x(t) \in E \subseteq \mathbb{R}^n]$ . In this subsection we address the problem of obtaining information on  $\mu$ , based on its moments (3.1). It is well known that when the moments are finite and known, the measure  $\mu$  can be uniquely determined [15], while in the case where only a finite number of moments are known, this reconstruction is naturally not unique. In this latter case, an elegant procedure to obtain an explicit expression for an approximating distribution to  $\mu$ , can be found in the literature (we follow closely [15]; see also [16], [17]). Assume, for simplicity, that  $\mu$  has bounded support, which without loss of generality (by proper scaling) can be assumed to be contained in the interval  $D := [0, 1]^n = [0, 1] \times \dots \times [0, 1]$  (see [15] for the case of unbounded support). Given a continuous function  $f : D \mapsto \mathbb{R}$  define the higher dimensional Bernstein polynomials as:

$$B_r(f)(y) = \sum_{k: 0 \leq k_i \leq r_i} f\left(\frac{k_1}{r_1}, \dots, \frac{k_n}{r_n}\right) P_{r,k}(y), \quad y \in \mathbb{R}^n$$

where  $r = (r_1, \dots, r_n)$  is a vector of integers, and  $P_{r,k}(y) := \prod_{i=1}^n \binom{r_i}{k_i} y_i^{k_i} (1 - y_i)^{r_i - k_i}$ . It is shown in [15] that  $B_r(f)(y)$  converges uniformly to  $f(y)$  as  $r_i \rightarrow \infty, \forall 1 \leq i \leq n$ . Thus, if we define  $I_{r,k} := \int_D P_{r,k}(y) \mu(ds)$ , we have that

$$\sum_{0 \leq k_i \leq r_i} f\left(\frac{k_1}{r_1}, \dots, \frac{k_n}{r_n}\right) I_{r,k} \quad (3.20)$$

is an approximation to  $\mathbb{E}[f(x(t))] = \int_E f(s) \mu(ds)$  that converges to  $\mathbb{E}[f(x(t))]$  as  $r_i \rightarrow \infty, \forall 1 \leq i \leq n$ . This means that the measure  $\mu$  can be approximated by the discrete measure  $\mu_r$  defined by

$$\sum_{0 \leq k_i \leq r_i} I_{r,k} \delta\left(\left[\frac{k_1}{r_1}, \dots, \frac{k_n}{r_n}\right]\right), \quad (3.21)$$

where  $\delta(y)$  denotes the Dirac measure at  $y \in \mathbb{R}^n$ . Since Bernstein polynomials are a linear combinations of monomials, we can compute the  $I_{r,k}$  through a linear combination of moments, which can be computed as proposed in Section 3.1.

It is important to mention that the problem of recovering a probability distribution from its moments is typically a numerically ill-conditioned problem; and small numerical errors in the moments may lead to large errors in the coefficients  $I_{r,k}$  that define the approximate discrete measure in (3.21). Thus, it is especially important to have a computationally efficient method to compute the moments, as provided by Theorem 3.1, so that one can obtain a good numerical precision in this procedure of obtaining an approximating probability distribution.

One way to interpret the procedure just described is that each integral  $I_{r,n}$  approximates the probability that  $x$  lies in a small neighborhood of  $(\frac{k_1}{r_1}, \dots, \frac{k_n}{r_n})$ . Instead of obtaining an approximation, one can use similar ideas to obtain an upper bound on the probability that  $x$  belongs to a given region. That is, suppose, for example, that we wish to determine a bound on the probability that a random variable  $z = g(x(t))$  is greater than a given value. Assume that  $z$  takes values in a compact set, which, without loss of generality, is assumed to be  $[0, 1]$ . Then we can choose a polynomial  $p_n : [0, 1] \rightarrow \mathbb{R}_{\geq 0}$  such that  $p_n(w) \geq \chi_{[a,1]}(w)$ ,  $w \in [0, 1]$ ,  $a < 1$ , and obtain the following bound

$$\text{Prob}[z \geq a] \leq \mathbb{E}[p_n(z)] \quad (3.22)$$

where the right-hand side of (3.22) can be computed using Theorem 3.1, when  $g(x(t))$  is a sum of monomial functions of the state. Note that when  $p_n(w) = \frac{w}{a}$  and  $p_n(w) = \frac{w^2}{a^2}$ , (3.22) corresponds to the Markov and Chebychev inequalities, respectively. This procedure is used in the networked control example of Section 4.

Alternative approaches to estimate the probability distribution of the state of the SHS include those based on the Focker-Plank equation and Dynkin's formula. The Focker-Plank equation of the SHS with renewal transitions can be obtained by specializing the expressions provided in [5], [6, Sec. 5.3], to this class of systems, and can be shown to be integro-partial differential equations. Besides the numerical difficulty and computational burden associated with solving these equations, the derivation of these Focker-Plank equations requires the map  $y \mapsto Jy$ ,  $y \in \mathbb{R}^n$  to be invertible (equivalent to matrix  $J$  invertible). The approach based on the Dynkin's formula can be found in [4, Ch.3, Sec. 32.2] where a numerical method is provided to compute the expected value of a given function  $\mathbb{E}[f(x(t))]$ , and a fortiori estimating the probability distribution from the relation  $\mathbb{E}[\chi_E(x(t))] = \text{Prob}[x(t) \in E]$ , for given  $E \subseteq D$ . Note that the approximation (3.20), which can be obtained with the methods derived in the present paper, provides an alternative to the method in [4, Ch.3, Sec. 32.2] to approximate the expected value of continuous functions of the state. Although we omit the derivations here, it is possible to prove that when specialized to SHSs with renewal transitions and to the case where  $f$  is a monomial, the recursive method provided in [4, Ch.3, Sec. 32.2] is equivalent to providing an approximation to (3.8) at each iteration  $n_r$  of the recursive algorithm, taking the form,

$$\sum_{\iota=1}^p x_0^{[\iota]} v_\iota^{\kappa, q_0}(t), \quad (3.23)$$

with

$$v^\kappa(t) = \sum_{k=0}^{n_r} \Theta^k(h^\kappa(t)). \quad (3.24)$$

This last equation converges to (3.11), and hence (3.23) converges to (3.8), but this it is clearly an inefficient method to obtain the solution to the Volterra equation (3.9), when compared to the method described in Subsection 3.1 (cf. [12]). Thus, by exploiting linearity and the special structure of the SHS with renewal transitions, our approach provides an insight that allows to compute moments more efficiently, than when seeing the SHS with renewal transitions as a piecewise deterministic process and specializing the approach proposed in [4, Ch.3, Sec. 32.2].

**3.4. Proofs of Theorems 3.1 and 3.3.** We start by establishing four preliminary facts stated in the form of propositions.

Let  $A^{(m)}$ ,  $m > 0$ , denote the  $m$ -th fold Kronecker product of a matrix or a vector  $A$  with itself, i.e.,

$$A^{(m)} := \underbrace{A \otimes A \cdots \otimes A}_m,$$

and recall that

$$(A \otimes B)(C \otimes D) = (AC) \otimes (BD) \quad (3.25)$$

(cf. [18]).

PROPOSITION 3.5. *The following holds*

$$\mathbb{E}[x_1(t)^{i_1} x_2(t)^{i_2} \dots x_n(t)^{i_n}] = \mathbb{E}[(x(t)^\top)^{(m)} c_\kappa], \quad (3.26)$$

where

$$c_\kappa := e_1^{(i_1)} \otimes \dots \otimes e_n^{(i_n)}, \text{ for } \rho(\kappa) = [i_1, \dots, i_n]. \quad (3.27)$$

□

*Proof.* Follows directly by using (3.25) □

Let  $\mathcal{T}(m, n)$  be the set of symmetric tensors, i.e., multilinear functions  $R$  on the  $m$ -fold  $\mathbb{R}^n \times \dots \times \mathbb{R}^n$  (cf. [19, Ch. 4]) such that

$$R(w_1, w_2, \dots, w_m) = R(w_{\sigma(1)}, w_{\sigma(2)}, \dots, w_{\sigma(m)})$$

for every  $w_i \in \mathbb{R}^n$ ,  $1 \leq i \leq m$ , and every one-to-one permutation of indices  $\sigma : \{1, 2, \dots, m\} \mapsto \{1, 2, \dots, m\}$ . We note that there is a natural identification between monomials of degree  $m$  in  $\mathbb{R}^n$  and  $\mathcal{T}(m, n)$ . In fact, to every  $m$ -degree monomial  $x^{[\kappa]} = x_1^{i_1} x_2^{i_2} \dots x_n^{i_n}$ , indexed by  $\kappa$ , as in (3.3), we can associate an element  $R^\kappa$  of the following orthogonal basis of symmetric tensors defined by

$$R^\kappa(w_1, w_2, \dots, w_m) := b^\kappa w_1 \otimes w_2 \otimes \dots \otimes w_m$$

where

$$b^\kappa := \sum_{j \in \mathcal{J}_\kappa} (e_{j_1}^\top) \otimes (e_{j_2}^\top) \otimes \dots \otimes (e_{j_m}^\top), \quad 1 \leq \kappa \leq p, \quad (3.28)$$

and the vector  $j$  belongs a set of  $\gamma_\kappa$  permutations of indices defined as follows

$$\mathcal{J}_\kappa := \{j = (j_1, \dots, j_m) : x_{j_1} x_{j_2} \dots x_{j_m} = x^{[\kappa]}, \forall x \in \mathbb{R}^n\} \quad (3.29)$$

where  $\kappa : \rho(\kappa) = [i_1, \dots, i_n]$ .

PROPOSITION 3.6. *The following holds*

$$(x(t)^{(m)})^\top c_\kappa = (x(t)^{(m)})^\top d_\kappa, \quad (3.30)$$

where  $d_\kappa := \frac{(b^\kappa)^\top}{\gamma_\kappa}$  and

$$x(t)^{(m)} = \sum_{\kappa=1}^p x^{[\kappa]} (b^\kappa)^\top, \quad (3.31)$$

where  $p$  is given by (3.2). □

*Proof.* Follows directly from definitions (3.28) and (3.29). □

Let  $T_i(t)$  denote the transition matrix of the SHS starting at the discrete mode  $q_0 = i$ , i.e.,  $x(t) = T_{q_0}(t)x_0$  where

$$T_i(t) = e^{A_{q(t)}(t-t_r)} \dots J_{\xi_{l_0}(i), l_1} e^{A_{\xi_{l_0}(i)}(t_1-t_0)} J_{i, l_0} e^{A_i t_1},$$

$r = \max\{k \in \mathbb{Z}_{\geq 0} : t_k \leq t\}$ .

PROPOSITION 3.7. *The following holds*

$$\mathbb{E}[(x(t)^\top)^{(m)} c_\kappa] = (x_0^\top)^{(m)} w^{\kappa, q_0}(t). \quad (3.32)$$

where the  $w^{\kappa, i}(t)$ ,  $i \in \mathcal{Q}$  are defined as

$$w^{\kappa, i}(t) := \mathbb{E}[(T_i(t)^\top)^{(m)} d_\kappa], \quad i \in \mathcal{Q}. \quad (3.33)$$

□

*Proof.* Using (3.25), (3.26), and (3.30) we obtain

$$\mathbb{E}[(x(t)^\top)^{(m)} c_\kappa] = \mathbb{E}[(x(t)^\top)^{(m)} d_\kappa] = (x_0^\top)^{(m)} \mathbb{E}[(T_{q_0}(t)^\top)^{(m)} d_\kappa] = (x_0^\top)^{(m)} w^{\kappa, q_0}(t).$$

□

For a matrix  $A \in \mathbb{R}^{n \times n}$ , we define a map  $T_A : \mathbb{R}^p \mapsto \mathbb{R}^p, y \mapsto v = T_A(y)$  by

$$\sum_{\kappa_2=1}^p v_{\kappa_2} (b^{\kappa_2})^\top = A^{(m)} \sum_{\kappa_2=1}^p y_{\kappa_2} (b^{\kappa_2})^\top, \quad (3.34)$$

PROPOSITION 3.8. *For a matrix  $A \in \mathbb{R}^{n \times n}$ , the map  $T_A : \mathbb{R}^p \mapsto \mathbb{R}^p, y \mapsto v = T_A(y)$  defined by (3.34) can be described by*

$$v = \Gamma^m(A)y \quad (3.35)$$

where  $\Gamma^m$  is given by (3.4). □

*Proof.* If we denote the elements and row of  $A$  by  $A_{ij}$  and  $a^i$ , respectively, i.e.,  $A = [A_{ij}] = [(a^1)^\top (a^2)^\top \dots (a^n)^\top]^\top$ , then using (3.25), (3.28), (3.29), and (3.31), we have that, for  $\kappa_1 : \rho(\kappa_1) = [i_1 i_2, \dots i_n]$ ,

$$\begin{aligned} (Ax)^{[\kappa_1]} &= (a^1)^{(i_1)} \otimes (a^2)^{(i_2)} \otimes \dots \otimes (a^n)^{(i_n)} x^{(m)} \\ &= \frac{1}{\gamma_{\kappa_1}} \sum_{j \in \mathcal{J}_{\kappa_1}} a^{j_1} \otimes a^{j_2} \otimes \dots \otimes a^{j_m} x^{(m)} \\ &= \frac{1}{\gamma_{\kappa_1}} \sum_{j \in \mathcal{J}_{\kappa_1}} a^{j_1} \otimes a^{j_2} \otimes \dots \otimes a^{j_m} \sum_{\kappa_2=1}^p (b^{\kappa_2})^\top x^{[\kappa_2]} \\ &= \sum_{\kappa_2=1}^p B_{\kappa_1 \kappa_2} x^{[\kappa_2]} \end{aligned}$$

where

$$B_{\kappa_1 \kappa_2} = \frac{1}{\gamma_{\kappa_1}} \sum_{j \in \mathcal{J}_{\kappa_1}} \sum_{i \in \mathcal{J}_{\kappa_2}} A_{j_1 i_1} A_{j_2 i_2} \dots A_{j_n i_n}. \quad (3.36)$$

Applying the tensors  $b^{\kappa_1}$ ,  $1 \leq \kappa_1 \leq p$  on both sides of (3.34) one also obtains that the linear map (3.34) is described by (3.36) in the basis  $(b^j)^\top$ ,  $1 \leq j \leq p$ .  $\square$

*Proof.* (of Theorem 3.1) We start by showing that

$$w^\kappa(t) := (w^{\kappa,1}(t), \dots, w^{\kappa,n_q}(t)), \quad (3.37)$$

where the  $w^{\kappa,i}$  are defined in (3.33), satisfies a Volterra equation.

Consider an initial condition  $q_0 = i$ ,  $i \in \mathcal{Q}$  and a given time  $t$  and partition the probability space into the events  $[t_1 \leq t] \cup [t_1 > t]$ . We can further partition  $[t_1 \leq t]$  into  $[t_1 \leq t] = \cup_{\ell=1}^{n_\ell} B_\ell(t) \cup B_0(t)$ , where  $B_0(t)$  is the event of two transitions triggering at the same time in the interval  $[0, t]$ , which has probability zero due to Assumption 2.1, and  $B_\ell(t)$  is the event of the transition  $\ell$  being the first to trigger in the interval  $[0, t]$ , i.e.,  $B_\ell(t) = [\min\{\bar{h}_0^j, j \in \mathcal{L}\} = \bar{h}_0^\ell = t_1 \leq t] \wedge [\bar{h}_0^j > \bar{h}_0^\ell, \ell \neq j]$ , where  $\wedge$  denotes intersection. Notice that, since the initial state is  $q_0 = i$ ,  $\bar{h}_0^j$  is distributed according to  $\mu_{i,j}$ , for a given  $j \in \mathcal{L}$  for which  $\mu_{i,j}$  is the non zero measure. When transition  $j$  does not trigger in state  $q_0 = i$ , the event  $B_j(t)$  is empty in which case  $\mu_{i,j}$  is the zero measure. Using this partition we can write

$$\mathbb{E}[(T_i(t)^\top)^{(m)} d_\kappa] = \mathbb{E}[(T_i(t)^\top)^{(m)} d_\kappa \chi_{[t_1 > t]}] + \sum_{\ell=1}^{n_\ell} \mathbb{E}[(T_i(t)^\top)^{(m)} d_\kappa \chi_{B_\ell(t)}] \quad (3.38)$$

where we denote by  $\chi_{x \in A}$  the characteristic function of a set  $A$ , i.e.,  $\chi_{x \in A}$  equals 1 if  $x \in A$  and 0 otherwise. The first term on the right hand side of (3.38) is given by

$$\begin{aligned} \mathbb{E}[(T_i(t)^\top)^{(m)} d_\kappa \chi_{[t_1 > t]}] &= (e^{A_i^\top t})^{(m)} d_\kappa \mathbb{E}[\chi_{[t_1 > t]}] \\ &= (e^{A_i^\top t})^{(m)} d_\kappa r_i(t), \\ &= (e^{A_i^\top t})^{(m)} \frac{(b^\kappa)^\top}{\gamma_\kappa} r_i(t) \end{aligned} \quad (3.39)$$

where we used the fact that  $\mathbb{E}[\chi_{[t_1 > t]}] = \text{Prob}([t_1 > t]) = \prod_{j=1}^{n_\ell} \text{Prob}[h_0^j > t] = \prod_{j=1}^{n_\ell} r_{i,j}(t) = r_i(t)$ . To obtain an expression for the second term on the right hand side of (3.38), notice first that for a function of the first jump time  $G(t_1)$ ,

$$\begin{aligned} \mathbb{E}[G(t_1) \chi_{B_\ell(t)}] &= \int_0^t \mathbb{E}[G(s) \chi_{[h_0^j > s, j \neq \ell]} | \bar{h}_0^\ell = s] \mu_{i,\ell}(ds) \\ &= \int_0^t G(s) \prod_{j=1, j \neq \ell}^{n_\ell} r_{i,j}(s) \mu_{i,\ell}(ds) = \int_0^t G(s) \frac{r_i(s)}{r_{i,\ell}(s)} \mu_{i,\ell}(ds). \end{aligned}$$

Notice also that  $T_i(t) = \hat{T}_{\xi_\ell(i)}(t - t_1)(E_{i,\ell}(t_1))$  when the transition  $\ell$  is first triggered, where  $\hat{T}_{\xi_\ell(i)}(t - t_1)$  is the transition matrix of the SHS from  $t_1$  to  $t$  starting the process at  $q_1 = \xi_\ell(i)$  and  $E_{i,\ell}(s) := J_{i,\ell} e^{As}$ ,  $s \geq 0$ . Each of the terms of the summation on the right hand side of (3.38) can then be expressed as

$$\mathbb{E}[(T_i(t)^\top)^{(m)} d_\kappa \chi_{B_\ell(t)}] = \int_0^t (E_{i,\ell}(s)^\top)^{(m)} \mathbb{E}[(\hat{T}_{\xi_\ell(i)}(t - s)^\top)^{(m)} d_\kappa] \frac{r_i(s)}{r_{i,\ell}(s)} \mu_{i,\ell}(ds). \quad (3.40)$$

By construction of the process (or in other words using the Markov property of the process at transition times)  $\mathbb{E}[(\hat{T}_{\xi_\ell(i)}(t-s)^\top)^{(m)} d_\kappa] = \mathbb{E}[(T_{\xi_\ell(i)}(t-s)^\top)^{(m)} d_\kappa] = w^{\kappa, \xi_\ell(i)}(t-s)$ . Replacing (3.39) and (3.40) in (3.38) and noticing that  $q_0 = i \in \mathcal{Q}$  is arbitrary we obtain that

$$w^{\kappa, i}(t) = W_i(w^\kappa(t)) + (e^{A_i^\top t})^{(m)} \frac{(b^\kappa)^\top}{\gamma_\kappa} r_i(t), \quad i \in \mathcal{Q}. \quad (3.41)$$

where

$$W_i(w^\kappa(t)) := \sum_{\ell=1}^{n_\ell} \int_0^t (E_{i,\ell}(\tau)^\top)^{(m)} w^{\kappa, \xi_\ell(i)}(t-\tau) \frac{r_i(\tau)}{r_{i,\ell}(\tau)} \mu_{i,\ell}(d\tau).$$

An explicit solution to the set of equations (3.41) takes the form

$$w^\kappa(t) = \sum_{k=0}^{\infty} W^k((e^{A_i^\top t})^{(m)} \frac{(b^\kappa)^\top}{\gamma_\kappa} r_i(t)) \quad (3.42)$$

where  $w^\kappa = (w^{\kappa,1}, \dots, w^{\kappa, n_q})$ ,  $W = (W_1, \dots, W_{n_q})$ , and  $W^k$  denotes composition. From (3.42) it is possible to conclude that  $w^{\kappa, i}(t)$  belongs to the dual vector space of the symmetric tensors for every  $i \in \mathcal{Q}$ , and therefore can be written as

$$w^{\kappa, i}(t) = \sum_{j=1}^p u_j^{\kappa, i}(t) (b^j)^\top \quad (3.43)$$

Replacing (3.43) in (3.41) we get

$$\begin{aligned} \sum_{j=1}^p u_j^{\kappa, i}(t) (b^j)^\top &= (e^{A_i^\top t})^{(m)} (b^\kappa)^\top \frac{1}{\gamma_\kappa} r_i(t) \\ &+ \sum_{\ell=1}^{n_\ell} \int_0^t (E_{i,\ell}(\tau)^\top)^{(m)} \sum_{j=1}^p u_j^{\kappa, \xi_\ell(i)}(t-\tau) (b^j)^\top \frac{r_i(\tau)}{r_{i,\ell}(\tau)} \mu_{i,\ell}(d\tau), \quad i \in \mathcal{Q}. \end{aligned} \quad (3.44)$$

Multiplying both sides of (3.44) by  $b^\iota$ ,  $\iota \in \{1, \dots, p\}$ , and using the fact that the maps (3.34) and (3.35) are equivalent, we obtain the set of equations (3.9).  $\square$

*Proof.* (of Theorem 3.3)

Similarly to (3.26) we can write

$$\mathbb{E}\left[\sum_{j=1}^n (x_j(t))^m\right] = \sum_{j=1}^n \mathbb{E}[(x(t)^\top)^{(m)} e_j^{(m)}], \quad (3.45)$$

Each term of the summation  $\mathbb{E}[(x(t)^\top)^{(m)} e_j^{(m)}]$  can be obtained from Theorem 3.1, where the index  $\kappa$  in (3.8), (3.9), (3.10), should be taken as  $\kappa_j$ , defined by

$$\kappa_j : \rho(\kappa_j) = m e_j, \quad 1 \leq j \leq n,$$

where  $e_j$  is a canonical vector in  $\mathbb{R}^n$ . Due to the linearity of the Volterra equation we can obtain

$$\sum_{j=1}^n \mathbb{E}[(x_j(t))^m] = \sum_{\iota=1}^p x_0^{[\iota]} u_\iota^{m, q_0}(t), \quad (3.46)$$

where  $u^m = (u^{m,1}, \dots, u^{m,n_q})$  is uniquely determined by the Volterra equation

$$u^m(t) = \Theta(u^m(t)) + h^m(t), \quad t \geq 0, \quad (3.47)$$

where  $h^m(t) := (h^{m,1}(t), \dots, h^{m,n_q}(t))$ ,

$$h^{m,i}(t) = \Gamma^m(e^{A_i^\top t}) \sum_{j=1}^n e_{\kappa_j} \frac{r_i(t)}{\gamma_{\kappa}}, \quad i \in \mathcal{Q}.$$

and  $e_{\kappa_j} \in \mathbb{R}^p$ . It is then clear that the limit  $\lambda_L(x_0) = \lim_{t \rightarrow \infty} \frac{1}{t} \log(\mathbb{E}[\|x(t)\|_m^m])$  exists for every  $x_0$  if and only if the limit

$$\lambda_V := \lim_{t \rightarrow \infty} \frac{1}{t} \log(\|u^m(t)\|) \quad (3.48)$$

exists, and  $\lambda_V = \sup_{x_0} \lambda_L(x_0) = \lambda_L$ . Note also that  $\mathbb{E}[\|x(t)\|_m^m] = 0, \forall x_0, t > b > 0$  if and only if  $u^m(t) = 0, t > b > 0$  in which case both  $\lambda_L$  and  $\lambda_V$  equal  $-\infty$  according to our definitions.

Let

$$\mathcal{U} : \{T \in \mathcal{T}(m, n) : T(x, x, \dots, x) \geq 0, \quad \forall x \in \mathbb{R}^n\}$$

and for  $y \in \mathbb{R}^p$

$$T_y(w_1, w_2, \dots, w_m) = \left( \sum_{\iota=1}^p y_\iota b^\iota \right) (w_1 \otimes w_2 \otimes \dots \otimes w_m),$$

and consider the set

$$\mathcal{K} : \{y \in \mathbb{R}^p : T_y \in \mathcal{U}\},$$

where the  $b^\iota$  are described by (3.28). The set  $\mathcal{K}$  is a cone, i.e.,  $\mathcal{K}$  is a closed convex set such that if  $y, z \in \mathcal{K}$  then  $\alpha_1 y + \alpha_2 z \in \mathcal{K}$ , for  $\alpha_1 \geq 0, \alpha_2 \geq 0$  and is such that the set  $-\mathcal{K} := \{-y : y \in \mathcal{K}\}$  intersects  $\mathcal{K}$  only at the zero vector. Moreover  $\mathcal{K}$  is a solid cone, which in finite-dimensional spaces is equivalent to being reproducing, i.e., any element  $y \in \mathbb{R}^p$  can be written as  $y = y_1 - y_2$  where  $y_1, y_2 \in \mathcal{K}$  (cf. [14, p. 10]). In fact, let  $z \in \mathcal{K}$  be defined as  $\sum_{\iota=1}^p \beta_\iota b^\iota$ , where  $\beta_\iota = \delta$  if  $\iota : \rho(\iota) = m e_i, e_i \in \mathbb{R}^n$ , for some  $i \in \{1, \dots, n\}$ , and  $\beta_\iota = 0$  otherwise. Then, from (3.28),  $z = \delta \sum_{j=1}^n (e_j)^{(m)}$ . Given any  $y \in \mathbb{R}^p$ , then  $y = y_1 - y_2$ , for  $y_1 = z + y$  and  $y_2 = z$ , which belong to  $\mathcal{K}$  for sufficiently large  $\delta$ . Likewise one can also prove that  $\mathcal{K}^{n_q} := \mathcal{K} \times \dots \times \mathcal{K} \subset \mathbb{R}^p \times \mathbb{R}^p \times \dots \times \mathbb{R}^p$ ,  $y = (y_1, \dots, y_{n_q}) \in \mathcal{K}^{n_q}$  if and only if  $y_i \in \mathcal{K}, \forall 1 \leq i \leq n_q$  is a solid cone. We prove that the Volterra equation (3.47) has a positive kernel with respect to the solid cone  $\mathcal{K}^{n_q}$ , in the sense of [10], and therefore we can directly apply [10, Th. 13] to conclude that,  $r_\sigma(\hat{\Theta}^m(a))$  is non-decreasing and that the root with largest real part of  $\det(I - \hat{\Theta}^m(z))$  is real and coincide with the unique value  $a$  such that  $r_\sigma(\hat{\Theta}(a)^m) = 1$ . As stated in [10, Th. 13], the zero  $a$  equals  $\lambda_V$ , given by (3.48), which in turn equals the Lyapunov exponent  $\lambda_L$ , provided this zero is not a removable singularity of a given complex function. The proof of this latter statement follows similar steps to the ones provided in [10, Th. 4] and is therefore omitted.

To prove that the kernel of the Volterra equation (3.47) is a positive operator, we need to prove (cf. [10, Sec. IV.B]) that if  $y = (y^1, \dots, y^{n_q}) \in \mathcal{K}^{n_q}, y^i \in \mathbb{R}^p, 1 \leq i \leq n_q$

then  $M(s)y \in \mathcal{K}^{n_q}$ , for every  $s > 0$ , where

$$M(s) := \begin{bmatrix} M_{1,1}(s) & \dots & M_{1,n_q}(s) \\ \vdots & \ddots & \vdots \\ M_{n_q,1}(s) & \dots & M_{n_q,n_q}(s) \end{bmatrix},$$

and

$$M_{i,j}(s) := \sum_{\ell: \xi_\ell(i)=j} \Gamma^m(e^{A^\top s} J_{i,\ell}^\top) \frac{r_i(s)}{r_{i,\ell}(s)}. \quad (3.49)$$

Note that if  $(z^1, z^2, \dots, z^{n_q}) = M(s)y$  then for a given  $s$ , each  $z^i \in \mathbb{R}^p$ ,  $1 \leq i \leq n_q$  can be written as a sum of terms taking the form  $\Gamma^m(e^{A^\top s} J_{i,\ell}^\top) y^j$  multiplied by positive scalars. Thus to prove that  $z^i \in \mathcal{K}$  and hence  $z \in \mathcal{K}^{n_q}$ , it suffices to prove that  $w = \Gamma^m(C)y \in \mathcal{K}$  for an arbitrary  $1 \leq j \leq n_q$ ,  $y \in \mathcal{K}$ , and  $C \in \mathbb{R}^{n \times n}$ . To this effect, using the fact that the map (3.34) can be written as (3.35), we have that

$$w : \sum_{\iota=1}^p w_\iota(b^\iota)^\top = C^{(m)} \sum_{\iota=1}^p y_\iota(b^\iota)^\top$$

belongs to  $\mathcal{K}$  because  $\sum_{\iota=1}^p w_\iota(b^\iota)$  belongs to  $\mathcal{U}$  since

$$\begin{aligned} \sum_{\iota=1}^p w_\iota(b^\iota) x^{(m)} &= \sum_{\iota=1}^p y_\iota(b^\iota) (C^\top)^{(m)} x^{(m)} \\ &= \sum_{\iota=1}^p y_\iota(b^\iota) (C^\top x)^{(m)} \geq 0 \end{aligned}$$

where in the last equality we used the fact that  $\sum_{\iota=1}^p y_\iota(b^\iota) \in \mathcal{U}$ .  $\square$

**4. Application to Networked Control.** We consider the following simplified version of the networked control set-up in [10]: suppose that we wish to control a linear plant

$$\dot{x}_P(t) = A_P x_P(t) + B_P \hat{u}(t), \quad (4.1)$$

using a state feedback controller taking the form  $K_C x_P(t)$  that needs to be implemented digitally and suppose that the actuation is held constant  $\hat{u}(t) = \hat{u}(s_\kappa)$ ,  $t \in [s_\kappa, s_{\kappa+1})$  between actuation update times denoted by  $\{s_\kappa, \kappa \geq 0\}$ .

The controller has direct access to the state measurements, but communicates with the plant actuators through a network possibly shared by other users. The controller attempts to do periodic transmissions of data, at a desired sampling period  $T_s$  but these regular transmissions may be perturbed by the medium access protocol. For example, users using CSMA for medium access, may be forced to back-off for a typically random amount of time until the network becomes available. We assume these random back-off times to be i.i.d. and denote by  $\mu_s$  the associated measure.

We consider two different cases:

*Case I:* After waiting to obtain network access, the controller (re)samples the sensor, computes the control law and transmits this most recent data. Assuming that

the transmission delays are negligible, and defining  $x := (x_P, \hat{u})$ , we have

$$\begin{aligned} \dot{x} &= Ax, \quad A = \begin{bmatrix} A_P & B_P \\ 0 & 0 \end{bmatrix}, \\ x(s_k) &= Jx(s_k^-), \quad J = \begin{bmatrix} I & 0 \\ K_C & 0 \end{bmatrix}. \end{aligned} \tag{4.2}$$

Since the intervals  $\{s_{k+1} - s_k, k \geq 0\}$  result from the controller waiting a fixed time  $T_s$  plus a random amount of time with a measure  $\mu_s(s)$ , these intervals are independent and identically distributed according to

$$\mu([0, \tau)) = \begin{cases} \mu_s(\tau - T_s), & \text{if } \tau \geq T_s \\ 0, & s \in [0, T_s). \end{cases}$$

Note that the system (4.2) is a special case of a SHS with a single state and a single reset map.

*Case II:* After waiting to obtain access to the network, the controller does not re-sample the sensor and simply transmits the data that it had collected at the time of the first attempt to transmit the sensor data. We model this by a SHS with the following two discrete-modes ( $n_q = 2$ ),

- State  $q(t) = 1$  - The controller waits for a fixed time  $T_s$ .
- State  $q(t) = 2$  - The controller waits a random time to gain access to the network;

Let  $r_k = s_k + T_s$ ,  $x := (x_P, \hat{u}, v)$  where  $v(t) := u_\kappa, t \in [r_\kappa, r_{\kappa+1})$  is a variable that holds the last computed control value. The transitions between the two discrete modes can be modeled by a single transition ( $n_\ell = 1$ ) which is a function of the two discrete modes and takes the form (2.2), specified as follows. When in state 1 the SHS transits to state 2 ( $\xi_1(1) = 2$ ) at times  $r_\kappa$ . The corresponding state jump models the update of the variable  $v(r_\kappa) = u_\kappa$  that holds the last computed control value and is described by

$$x(r_\kappa) = J_{1,1}x(r_\kappa^-), \quad J_{1,1} = \begin{bmatrix} I & 0 & 0 \\ 0 & I & 0 \\ K_C & 0 & 0 \end{bmatrix}.$$

When in state 2 the SHS transits to state 1 ( $\xi_1(2) = 1$ ) at actuation update times  $s_\kappa$ . The state jump models the actuation update  $\hat{u}(s_\kappa) = v(s_\kappa^-)$  and is described by

$$x(s_\kappa) = J_{2,1}x(s_\kappa^-), \quad J_{2,1} = \begin{bmatrix} I & 0 & 0 \\ 0 & 0 & I \\ 0 & 0 & I \end{bmatrix}.$$

The reset-time measures are given by

- $\mu_{1,1}(\tau) = \delta(\tau - T_s)$  is a discrete measure that places all mass  $w_i = 1$  at time  $T_s$ .
- $\mu_{2,1}(\tau) = \mu_s(\tau)$ .

In both discrete modes, the continuous-time dynamics are described by  $\dot{x} = A_i x$ ,  $i \in \{1, 2\}$ ,  $A_1 = A_2 = A$  where

$$A = \begin{bmatrix} A_P & B_P & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

**4.1. Numerical Example.** Suppose that the plant (4.1) is described by

$$A_P = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad B_P = \begin{bmatrix} 0 \\ 1 \end{bmatrix},$$

which by properly scaling the state and input can be viewed as a linearized model of a damp-free inverted pendulum. Moreover, suppose that the measure  $\mu_s$  is uniform with support on the interval  $[0, \tau]$ , and fix  $T_s = 0.1s$  and  $x_P(0) = [1 \ 0]^\top$ . A continuous-time state feedback controller is synthesized using LQR and is given by  $\hat{u}(t) = K_C x(t)$ ,  $K_C = -[1 + \sqrt{2} \quad 1 + \sqrt{2}]$ , which is the solution to the problem  $\min_{\hat{u}(t)} \int_0^\infty [x_P(t)^\top x_P(t) + \hat{u}(t)^2] dt$ , yielding the following closed-loop eigenvalues  $\lambda_i(A_P + B_P K_C) = \{-1, -\sqrt{2}\}$ . We wish to investigate the stability and performance of the closed-loop when instead of the ideal networked-free case we consider the scenarios of Cases I and II. To this effect we define the quantity

$$e(t) = x_P(t)^\top x_P(t) + \hat{u}(t)^2,$$

which can be written as  $e(t) = x^\top P x$ , where (i) in the network-free case  $P = I_2 + K_C^\top K_C$ , and  $x = x_P$ ; (ii) in case I,  $P = I_3$  and  $x = (x_P, \hat{u})$ ; (iii) and in case II,  $P = \text{diag}(I_2, 1, 0)$ , and  $x = (x_P, \hat{u}, v)$ . Note that, in the network-free case,  $e(t)$  is the quantity whose integral is minimized by LQR control synthesis and  $e(t)$  decreases exponentially fast at a rate  $\alpha = 2$ , since the dominant closed-loop eigenvalue equals  $\lambda_i(A_P + B_P K_C) = -1$ . In cases I and II,  $\mathbb{E}[e(t)]$  converging to zero is equivalent to MSS, which is equivalent to SS and MES since the reset-time measures have finite support ( $\tau$  and  $\tau_s$  are finite). Corollary 3.4 can be used to determine whether or not the closed-loop is MSS in cases I and II. Moreover, when the closed-loop is MSS, we can determine the exponential decay constant of  $\mathbb{E}[e(t)]$  from the Theorem 3.3. The results are summarized in Table 4.1, for different values of the support  $\tau$  of the uniform measure  $\mu_s$  of the back-off time.

TABLE 4.1  
Exponential decay rates  $\mathbb{E}[e(t)] \leq ce^{-\alpha t}$

$\tau$	0.4	0.6	0.8	1.0	1.2	> 1.211
$\alpha$	2.000	2.000	1.969	0.477	$7.63 \times 10^{-5}$	NOT MSS

(a) Case I

$\tau$	0.1	0.2	0.3	0.4	0.5	> 0.521
$\alpha$	2.000	2.000	2.000	0.849	0.118	NOT MSS

(b) Case II

The fact that closed-loop stability is preserved for larger values of  $\tau$  in Case I, confirms what one would expect intuitively, i.e., Case I is more appropriate when transmitting dynamic data, since the most recent sampling information is sent through the network.

Using the state moment expressions provided by Theorems 3.1, we can perform a more detailed analysis by plotting the moments of  $e(t)$ , which can be expressed in terms of the moments of the state. For example, the two first moments take the form

$$\begin{aligned} \mathbb{E}[e(t)] &= \mathbb{E}[x(t)^\top P x(t)] = \mathbb{E}[(x(t)^\top)^{(2)}] \nu(P), \\ \mathbb{E}[e(t)^2] &= \mathbb{E}[(x(t)^\top P x(t))^2] = \mathbb{E}[(x(t)^\top)^{(4)}] (\nu(P) \otimes \nu(P)). \end{aligned}$$

In Figure 4.1, we plot the expected value of the error  $\mathbb{E}[e(t)]$  and its  $2\text{-}\sigma$  confidence interval  $\mathbb{E}[e(t)] \pm 2\mathbb{E}[(e(t) - \mathbb{E}[e(t)])^2]^{1/2}$  for a network measure support  $\tau = 0.4$ . Note

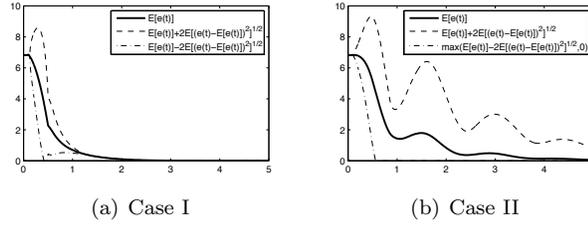


FIG. 4.1. Plot of  $\mathbb{E}[e(t)]$ , where  $e(t)$  is quadratically state dependent. For a fixed  $t$ ,  $\mathbb{E}[e(t)]$  lies between the dashed curves with probability  $> \frac{3}{4}$ .

that, from the Chebyshev inequality, we conclude that

$$\text{Prob}[|e(t) - \mathbb{E}[e(t)]| > a(t)] \leq \frac{\mathbb{E}[(e(t) - \mathbb{E}[e(t)])^2]}{a(t)^2},$$

and therefore one can guarantee that for a fixed  $t$ ,  $e(t)$  lies between the curves  $\mathbb{E}[e(t)] \pm a(t)$ ,  $a(t) = 2\mathbb{E}[(e(t) - \mathbb{E}[e(t)])^2]^{1/2}$  with a probability greater than  $\frac{3}{4}$ . The numerical method used to compute the solution of the Volterra equation is the one described in Section 3.1 for which we used a trapezoidal rule for the integration method. In case I, the expected value of the quadratic state function  $e(t)$  tends to zero much faster, and with a much smaller variance than in case II, confirming once again that case I is more appropriate when transmitting dynamic data.

The plots of Figure (4.1) can be confirmed through Monte Carlo simulation, although this latter method is in general more computationally demanding. For illustration, in Table 4.2 we compare the numerical error  $\sum_{k=0}^{\underline{L}} \mathbb{E}[|e(k\underline{h}) - \hat{e}(k\underline{h})|]$ ,  $\underline{L} = 100$ ,  $\underline{h} = 0.01$ , obtained with our proposed method (with trapezoidal rule for integration) and the Monte Carlo method, for Case I, where  $e(k\underline{h})$  is a very accurate approximation of  $e(t)$  at time  $k\underline{h}$  and  $\hat{e}(k\underline{h})$  is the approximation obtained with one of the methods presented in Table 4.2. For the Monte Carlo method, we show the numerical errors obtained with different numbers of Monte Carlo runs, whereas for our proposed method we show the numerical errors obtained with different discretization steps, or equivalently with different number of discretization points  $\bar{L} + 1$  equally spaced in the interval  $[0, 1]$ . The computational times were obtained with a computer with the following specifications: Intel Core 2 Duo, Processor Speed: 3 GHz, 3.5GB RAM memory. From the values in Table 4.2 it is clear that our proposed method provides a better accuracy with less computations.

TABLE 4.2  
Numerical errors  $\sum_{k=0}^{\underline{L}} \mathbb{E}[|e(k\underline{h}) - \hat{e}(k\underline{h})|]$  for case I,  $\underline{L} = 100$ ,  $\underline{h} = 0.01$

Number of discretization points $L$	500	1000	2000	4000	8000	16000
Computational Time (sec)	6.04	7.44	12.17	29.37	103.19	347.52
Num. Error	5.0187	2.4482	1.1796	0.5493	0.2352	0.0783

(a) Proposed method based on Volterra Equations

Number of Monte Carlo runs	300	600	1200	2400	4800	9600
Computational Time (sec)	10.03	19.47	42.90	90.80	235.64	701.22
Num. Error	3.73	2.26	3.07	2.02	0.79	0.77

(b) Monte Carlo Simulations

**5. Conclusions and Future Work.** We proposed an approach based on Volterra renewal-type equations to analyze SHSs for which the lengths of times that the system

stays in each mode are independent random variables with given distributions. We showed that any statistical  $m$ -th degree moment of the state can be computed using this approach, and provided a number of results characterizing the asymptotic behavior of a second-degree moment of the system. Due to the large number of problems that fit the stochastic hybrid systems framework, finding more applications where the results can be applied is a topic for future work.

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